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SOFTLY BROKEN $N=2$ QCD

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Abstract

We analyze the possible soft breaking of $N=2$ supersymmetric Yang-Mills theory with and without matter flavour preserving the analyticity properties of the Seiberg-Witten solution. For small supersymmetry breaking parameter with respect to the dynamical scale of the theory we obtain an exact expression for the effective potential. We describe in detail the onset of the confinement transition and some of the patterns of chiral symmetry breaking. If we extrapolate the results to the limit where supersymmetry decouples, we obtains hints indicating that perhaps a description of the QCD vacuum will require the use of Lagrangians containing simultaneously mutually non-local degrees of freedom (monopoles and dyons).

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1 Introduction

In two remarkable papers [1, 2], Seiberg and Witten obtained exact information on the dynamics of $N = 2$ supersymmetric gauge theories in four dimensions with gauge group $SU(2)$ and $N_f \leq 4$ flavour multiplets. Their work was extended to other groups in [3]. One of the crucial advantages of using $N = 2$ supersymmetry is that the low-energy effective action in the Coulomb phase up to two derivatives (*i.e.* the Kähler potential, the superpotential and the gauge kinetic function in $N = 1$ superspace language) are determined in terms of a single holomorphic function called the prepotential [4]. In references [1, 2], the exact prepotential was determined using some plausible assumptions and many consistency conditions. For $SU(2)$ the solution is neatly presented by associating to each case an elliptic curve together with a meromorphic differential of the second kind whose periods completely determine the prepotential. For other gauge groups [3] the solution is again presented in terms of the period integrals of a meromorphic differential on a Riemann surface whose genus is the rank of the group considered. It was also shown in [1, 2] that by soft breaking $N = 2$ down to $N = 1$ (by adding a mass term for the adjoint $N = 1$ chiral multiplet in the $N = 2$ vector multiplet) confinement follows due to monopole condensation [5].

For $N = 1$ theories exact results have also been obtained [6] using the holomorphy properties of the superpotential and the gauge kinetic function, culminating in Seiberg's non-abelian duality conjecture [7].

With all this new exact information it is also tempting to obtain exact information about ordinary QCD. The obvious problem encountered is supersymmetry breaking. A useful avenue to explore is soft supersymmetry breaking. The structure of soft supersymmetry breaking in $N = 1$ theories has been known for some time [8]. In [9, 10] soft breaking terms are used to explore $N = 1$ supersymmetric QCD (SQCD) with gauge group $SU(N_c)$ and N_f flavours of quarks, and to extrapolate the exact results in [6] concerning the superpotential and the phase structure of these theories in the absence of supersymmetry. This leads to expected and unexpected predictions for non-supersymmetric theories which may eventually be accessible to lattice computations. In some cases however (for instance when $N_f \geq N_c$) it is known in the supersymmetric case that the origin of moduli space is singular, and therefore some of the assumptions made about the Kähler potential for meson and baryon operators are probably too strong. Since the methods of [1, 2] provide us with the effective action up to two derivatives, the kinetic and potential term for all low-energy fields are under control, and therefore in this paper we prefer to explore in which way we can softly break $N = 2$ SQCD directly to $N = 0$ while at the same time preserving the analyticity properties of the Seiberg-Witten solution. This is a very strong constraint and there is, essentially, only one way to accomplish this task: we make the dynamical scale Λ of the $N = 2$ theory a function of an $N = 2$ vector multiplet which is then frozen to become a spurion whose F and D-components break softly $N = 2$ down to $N = 0$. If we want to interpret physically the spurion, one can recall the string derivation of the Seiberg-Witten solution in [11, 12] based on type II-heterotic duality. In the field theory limit in the heterotic side (in order to decouple string and gravity loops) the natural scaling is taken to be $Me^{iS} = \Lambda$, where M is the Planck mass, S is the dilaton (in the low-energy theory $S = \theta/2\pi + 4\pi i/g^2$, with g the gauge coupling constant and θ the CP-violating phase), and Λ the dynamical scale of the gauge theory which is kept fixed while $M \to \infty$ and $iS \to \infty$. Since the dilaton sits in a vector multiplet of $N = 2$ when the heterotic string is compactified on $K3 \times T_2$, this is precisely the field we want to make into a spurion, and we show later that this procedure is compatible with the Seiberg-Witten monodromies. In this way we obtain a theory at $N = 0$ with a more restricted structure that those used in [9, 10]. As a consistency check, we start along the lines of [11, 12] with the theory coupled to $N = 2$ supergravity with a simple superpotential which breaks spontaneously supersymmetry through an auxiliary field associated to the graviphoton, which also gives vacuum expectation values to the auxiliaries in the dilaton multiplet. At low-energies one obtains a theory with all the allowed soft breakings, however in the scaling limit mentioned previously, the only surviving soft terms are those one would obtain had we worked from the beginning with the rigid $N = 2$ theory plus the dilaton spurion. As soon as the soft breaking terms are turned on monopole condensation appears, and we get a unique ground state (near the massless monopole point of [1, 2]). Furthermore, in the Higgs region we can compute the effective potential, and we can verify that this potential drives the theory towards the region where condensation takes place. When the supersymmetry breaking parameter is increased, the minimum displaces to the right along the real u-axis. At the same time, the region in the u-plane in which the monopole condensate is energetically-favoured expands. Near the massless dyon point of [1, 2], we find that dyon condensation is energetically favourable but, unlike monopole condensation, it is not sufficiently-strong an effect to lead to another minimum of the effective potential. Eventually, when the soft supersymmetry breaking parameter is made sufficiently large, the regions where monopole and dyon condensation are favoured begin to overlap. At this point, it is clear that our methods break down, and new physics is needed to describe the dynamics of these mutually-nonlocal degrees of freedom.

One advantage of this method of using the dilaton spurion to softly break supersymmetry from $N = 2$ to $N = 0$ is its universality. It works for any gauge group and any number of massive or massless quarks. As a further example we consider the theory with two hypermultiplets of massless quarks. The global symmetry is $O(4) \times SU(2)_R \times U(1)_R$, where $SU(2)_R$ is the R-symmetry associated to $N = 2$ supersymmetry. Monopole condensation leads to a peculiar pattern of chiral symmetry breaking. Writing $SO(4) = SU(2)_l \times SU(2)_r$, we find that near the massless monopole region $SU(2)_r$ breaks completely while $SU(2)_l$ remains intact. Due to the properties of the $N = 2$ solution in [1, 2] we can compute the low-energy Goldstone boson Lagrangian reliably at least for small supersymmetry breaking parameter. We also find two Higgs branches corresponding to the two Higgs phases described in [2]. As one would expect, they are smoothly connected to the confining phase.

The organization of this paper is as follows: In section two we collect some useful formulæ summarizing the main features on [1, 2] which are needed in later sections. In section three we analyze the effective action once the dilaton spurion is included. The modular transformations of the action and coupling constants will be derived, agreeing with the general results derived in [13] concerning the modification of the symplectic transformations of special geometry in the presence of background $N = 2$ vector superfields. There are some interesting consequences of the modular transformations related to the fact that in the moduli space of the $N = 2$ theory we have to use different effective actions in different patches such that the light fields in different patches are not mutually local. In section four we derive the same action starting with the $N = 2$ supergravity theory and spontaneous breaking of supersymmetry. Section five presents the detailed analysis of the low-energy effective action, the onset of monopole condensation and the numerical results. In section six we extend our results to the case of $SU(2)$ with two massless quark hypermultiplets. Finally in section seven we present the conclusions and outlook.

2 The Seiberg-Witten Solution

We will concentrate for simplicity on the case of $SU(2)$ with $N_f = 0, 2$ flavours of quarks. Because of the different normalization of the charge generator in [1] and [2] due to the presence of flavours, the elliptic curve in these two cases is the same, and most of the analytic and numerical computations are exactly the same. In the $N_f = 0$ case the classical theory is described by a quadratic prepotential

$$\mathcal{F}^{\rm cl} = \frac{1}{2}\tau^{\rm cl}(A^a)^2 \tag{2.1}$$

$$\tau^{\rm cl} = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \tag{2.2}$$

where A^a, $a = 1, 2, 3$, are the $N = 2$ vector multiplets associated to the generators of $SU(2)$. In terms of $N = 1$ multiplets A^a contains a vector multiplet (A^a_μ, λ^a), and a chiral multiplet (ψ^a, ϕ^a). Hence it describes a vector, two Majorana fermions and a complex scalar; all in the adjoint representation. $N = 2$ supersymmetry does not allow a superpotential for the theory and therefore the scalar potential is purely D-term:

$$V(\phi) = \frac{1}{g^2}{\rm Tr}[\phi, \phi^\dagger]^2 \tag{2.3}$$

There is a moduli space of vacua. The minima of (5.3) can be taken to be of the form $\phi = \frac{1}{2}a\sigma^3$ with a complex. A gauge invariant description of this moduli space is provided by the variable $u = {\rm Tr}\phi^2 = \frac{1}{2}a^2$ at the classical level. Each point in this moduli space represents a different theory. For $a \neq 0$ the charged multiplets acquire a mass $M = \sqrt{2}|a|$, and $SU(2)$ is spontaneously broken to $U(1)$, and at $a = 0$ the full $SU(2)$ symmetry is restored. Away from the origin we can integrate out the massive multiplets and obtain a low-energy effective theory which depends only on the "photon" multiplet. The theory is fully described in terms of a prepotential $\mathcal{F}(A)$. The lagrangian in $N = 1$ superspace is

$$\mathcal{L} = \frac{1}{4\pi}{\rm Im}\Big[\int d^4\theta \frac{\partial F}{\partial A}\overline{A} + \frac{1}{2}\int d^2\theta \frac{\partial^2 F}{\partial A^2} W_\alpha W^\alpha\Big]. \tag{2.4}$$

The Kähler potential and gauge kinetic functions are given in general by:

$$K(a,\bar{a}) = \frac{1}{4\pi}\mathrm{Im} a_{D,i}\bar{a}^i,$$

$$\tau_{ij} = \frac{1}{2}\frac{\partial^2\mathcal{F}}{\partial a^i \partial a^j},$$

$$a_{D,i} \equiv \frac{\partial \mathcal{F}}{\partial a^i}. \tag{2.5}$$

In perturbation theory $\mathcal{F}$ only receives one-loop contributions. The important thing is to determine the non-perturbative corrections. This was done in [1, 2]. Some of the properties of the exact solution are:

i) The $SU(2)$ symmetry is never restored. The theory stays in the Coulomb phase throughout the u-plane.

ii) The moduli space has a symmetry $u \to -u$ (the non-anomalous subset of the $U(1)_R$ group), and at the points $u = \Lambda^2$, $-\Lambda^2$ singularities in $\mathcal{F}$ develop. Physically they correspond respectively to a massless monopole and dyon with charges $(q_e, q_m) = (0, 1)$, $(-1, 1)$. Hence near $u = \Lambda^2$, $-\Lambda^2$ the correct effective action should include together with the photon vector multiplet monopole or dyon hypermultiplets.

iii) The function $\mathcal{F}(a)$ is holomorphic. It is better to think in terms of the vector ${}^t v = (a_D, a)$ which defines a flat $SL_2(\mathbf{Z})$ vector bundle over the moduli space $\mathcal{M}_u$ (the u-plane). Its properties are determined by the singularities and the monodromies around them. Since $\partial^2\mathcal{F}/\partial a^2$ or $\partial a_D/\partial a$ is the coupling constant, these data are obtained from the β-function in the three patches: large-u, the Higgs phase, the monopole and the dyon regions. From the BPS mass formula [14, 15] the mass of a BPS state of charge (q_e, q_m) (with q_e, q_m coprime for the charge to be stable) is:

$$M = \sqrt{2}|q_e a + q_m a_D|. \tag{2.6}$$

If at some point u_0 in $\mathcal{M}_u$, $M(u_0) = 0$, the monodromy around this point is given by [1, 2, 3]

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to M(q_e, q_m) \begin{pmatrix} a_D \\ a \end{pmatrix}, \tag{2.7}$$

$$M(q_e, q_m) = \begin{pmatrix} 1 + 2q_e q_m & 2q_e^2 \\ -2q_m^2 & 1 - 2q_e q_m \end{pmatrix}. \tag{2.8}$$

Also for large u, $\mathcal{F}$ is dominated by the perturbative one loop contribution, obtained from the one loop β-function:

$$\mathcal{F}_{1-\text{loop}}(a) = \frac{i}{2\pi} a^2 \ln \frac{a^2}{\Lambda} \tag{2.9}$$

Hence we also have monodromy at infinity. The three generators of the monodromy are therefore:

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \quad M_{\Lambda^2} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad M_{-\Lambda^2} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}; \tag{2.10}$$

and they satisfy:

$$M_\infty = M_{\Lambda^2} M_{-\Lambda^2}. \tag{2.11}$$

These matrices generate the subgroup $\Gamma_2 \subset SL_2(\mathbf{Z})$ of 2×2 matrices congruent to the unit matrix modulo 2.

We learn from (2.6)-(2.7) that in the Higgs, monopole and dyon patches, the natural independent variables to use are respectively $a^{(h)} = a$, $a^{(m)} = a_D$, $a^{(d)} = a_D - a$. Thus in each patch we have a different prepotential:

$$\mathcal{F}^{(h)}(a), \quad \mathcal{F}^{(m)}(a^m), \quad \mathcal{F}^{(d)}(a^d). \tag{2.12}$$

iv) The explicit form of $a(u)$, $a_D(u)$ is given in terms of the periods of a meromorphic differential of the second kind on a genus one surface described by the equation:

$$y^2 = (x^2 - \Lambda^4)(x - u), \tag{2.13}$$

describing the double covering of the plane branched at $\pm\Lambda^2$, u, ∞. We choose the cuts $\{-\Lambda^2, \Lambda^2\}$, $\{u, \infty\}$. The correctly normalized meromorphic 1-form is:

$$\lambda = \Lambda \frac{\sqrt{2}}{2\pi} \frac{dx\sqrt{x - u/\Lambda^2}}{\sqrt{x^2 - 1}}. \tag{2.14}$$

Then:

$$a(u) = \Lambda \frac{\sqrt{2}}{\pi} \int_{-1}^{1} \frac{dt\sqrt{u/\Lambda^2 - t}}{\sqrt{1 - t^2}}; \tag{2.15}$$

$$a_D(u) = \Lambda \frac{\sqrt{2}}{\pi} \int_{1}^{u/\Lambda^2} \frac{dt\sqrt{u/\Lambda^2 - t}}{\sqrt{1 - t^2}}. \tag{2.16}$$

Using the hypergeometric representation of the elliptic functions [16]:

$$K(k) = \frac{\pi}{2}F(1/2,1/2,1;k^2); \quad K'(k) = K(k');$$

$$E(k) = \frac{\pi}{2}F(-1/2,1/2,1;k^2); \quad E'(k) = E(k'), \quad k'^2 + k^2 = 1, \tag{2.17}$$

we obtain :

$$k^2 = \frac{2}{1+u/\Lambda^2}, \quad k'^2 = \frac{u-\Lambda^2}{u+\Lambda^2}, \tag{2.18}$$

$$a(u) = \frac{4\Lambda}{\pi k}E(k), \quad a_D(u) = \frac{4\Lambda}{i\pi}\frac{E'(k)-K'(k)}{k}. \tag{2.19}$$

Using the elliptic function identities:

$$\frac{dE}{dk} = \frac{E-K}{k}, \quad \frac{dK}{dk} = \frac{1}{kk'^2}(E - k'^2 K), \tag{2.20}$$

$$\frac{dE'}{dk} = -\frac{k}{k'^2}(E'-K'), \quad \frac{dK'}{dk} = -\frac{1}{kk'^2}(E' - k^2 K'), \tag{2.21}$$

the coupling constant becomes:

$$\tau_{11} = \frac{\partial a_D}{\partial a} = \frac{da_D/dk}{da/dk} = \frac{iK'}{K}, \tag{2.22}$$

which is indeed the period matrix of the curve (2.13).

Finally, to determine the prepotential $\mathcal{F} = \mathcal{F}(a)$, we have to invert $a = a(u)$, to write $u = u(a)$, and then integrate $a_D = \partial\mathcal{F}/\partial a$.

Before closing this section, we derive the modular transformation properties of $\mathcal{F}(a)$. If $\Gamma \in SL_2(\mathbf{Z})$, $\Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then $a_D^\Gamma = \alpha a_D + \beta a$, $a^\Gamma = \gamma a_D + \delta a$. We want to express $\mathcal{F}_\Gamma(a^\Gamma)$ in terms of $\mathcal{F}(a)$. Since

$$\begin{aligned} \frac{\partial \mathcal{F}_\Gamma(a^\Gamma)}{\partial a} &= \frac{\partial a^\Gamma}{\partial a}\frac{\partial \mathcal{F}_\Gamma(a^\Gamma)}{\partial a^\Gamma} = \Big(\gamma\frac{\partial a_D}{\partial a} + \delta\Big) a_D^\Gamma \\ &= (\alpha\mathcal{F}' + \beta)(\gamma\mathcal{F}'' + \delta), \end{aligned} \tag{2.23}$$

using $\alpha\delta - \beta\gamma = 1$ we obtain:

$$\mathcal{F}_\Gamma(a_\Gamma) = \frac{1}{2}\beta\delta a^2 + \frac{1}{2}\alpha\gamma a_D^2 + \beta\gamma a a_D + \mathcal{F}(a). \tag{2.24}$$

In particular, under the two generators T, S of $SL_2(\mathbf{Z})$:

$$\Gamma = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{F}_T(a_T) = \tfrac{1}{2}a^2 + \mathcal{F}(a),$$
$$\Gamma = S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{F}_S(a_S) = -aa_D + \mathcal{F}(a). \tag{2.25}$$

When there are flavours similar results apply in the Coulomb phase [1, 2], the solution of the model is presented in terms of an elliptic curve and $a(u)$, $a_D(u)$ are given by period integrals. We will recall some details in section 6.

3 Breaking $N = 2$ with a Dilaton Spurion

We now would like to break $N = 2$ supersymmetry preserving the holomorphy properties of the Seiberg-Witten solution. In the theory without flavours we want to introduce another $N = 2$ vector multiplet s in the prepotential $\mathcal{F}(a, s)$ in such a way that s, $s_D = \partial\mathcal{F}/\partial s$ be monodromy invariant. We can then freeze the scalar and auxiliary components of this superfield to be constants to generate soft breaking of $N = 2$. Since the only free parameter in the Seiberg-Witten solution is Λ, the simplest choice is to make Λ a function of a background vector superfield. The scale Λ is related to the coupling constant and θ-parameter by $\Lambda^4 \sim \exp(-\frac{8\pi}{g^2} + i\theta)$, it is then natural to include a dilaton field S such that $\Lambda \sim e^{iS}$, Im $S \sim 1/g^2$, Re $S \sim \theta$. This is the correct choice if we think of the embedding of the $N = 2$ $SU(2)$ theory in the heterotic string compactified on $K3 \times T_2$ [11, 12] where the dilaton is part of a vector multiplet. If we can show that $\partial\mathcal{F}/\partial s$ is invariant under the Seiberg-Witten monodromy, the addition of this extra superfield does not change any of the holomorphic properties of the solution presented in section 2. In each region of the moduli space we can write a prepotential adapted to the local coordinates of the form:

$$\mathcal{F} = a^2 f(a/\Lambda). \tag{3.1}$$

A simple consequence of the modular transformation properties of a, a_D and $\mathcal{F}$ (2.24) imply that

$$\mathcal{F} - \frac{1}{2}aa_D \tag{3.2}$$

is modular invariant. Hence (3.2) is only a function of the moduli. To determine this function it suffices to note that the periods $a_D(u)$, $a(u)$ satisfy a second order differential equation (they are hypergeometric functions), the Picard-Fuchs equation for the curve (2.13):

$$\frac{d^2\omega}{du^2} + \frac{1}{u^2 - \Lambda^4}\omega = 0. \tag{3.3}$$

The absence of a first derivative term in (3.3) implies that the Wronskian of the two independent solutions $a_D(u)$, $a(u)$: $a da_D/du - a_D da/du$ is a constant, whose value can be determined by evaluating it in the weak coupling (large u) region. Integrating the wronskian with respect to u leads to

$$\mathcal{F} - \frac{1}{2}a a_D = -\frac{i}{\pi}u. \tag{3.4}$$

This relation was first derived in [17] and further explored in [18]. Once the spurion field S is introduced, we have two vector multiplets $a^0 \equiv s$, $a^1 \equiv a$, and a 2×2 matrix of couplings:

$$\tau_{11} = \frac{\partial^2 \mathcal{F}}{\partial^2 a}, \quad \tau_{01} = \frac{\partial^2 \mathcal{F}}{\partial s \partial a}, \quad \tau_{00} = \frac{\partial^2 \mathcal{F}}{\partial^2 s}, \tag{3.5}$$

whose modular properties and explicit representation we would like to determine. From (3.1) plus the identification $\Lambda = \mathrm{e}^{iS}$, we obtain ($\partial/\partial s = i\Lambda\partial/\partial\Lambda$):

$$\begin{aligned} a_D &= 2af + \tfrac{a^2}{\Lambda}f', \quad \tau_{11} = 2f + 4\tfrac{a}{\Lambda}f' + \tfrac{a^2}{\Lambda^2}f'', \\ \tau_{01} &= -\tfrac{3ia^2}{\Lambda}f' - \tfrac{ia^3}{\Lambda^2}f'', \quad \tau_{00} = -\tfrac{a^3}{\Lambda}f' - \tfrac{a^4}{\Lambda^2}f''; \end{aligned} \tag{3.6}$$

and from (3.6) we obtain

$$\begin{aligned} \tau_{01} &= i(a_D - a\tau_{11}), \qquad \frac{\partial \tau_{01}}{\partial a} = -ia\frac{\partial \tau_{11}}{\partial a}; \\ \frac{\partial \tau_{00}}{\partial a} &= i\tau_{01} - a^2\frac{\partial \tau_{11}}{\partial a}. \end{aligned} \tag{3.7}$$

In particular:

$$\frac{\partial \mathcal{F}}{\partial s} = 2i\Big(\mathcal{F} - \frac{1}{2}a a_D\Big) = \frac{2}{\pi}u \tag{3.8}$$

and:

$$\tau_{01} = \frac{2}{\pi}\frac{\partial u}{\partial a},$$

$$\tau_{00} = \frac{2i}{\pi}\Big(2u - a\frac{\partial u}{\partial a}\Big). \tag{3.9}$$

The last equation in (3.9) is obtained by integrating $\partial\tau_{00}/\partial a$ using (3.8). A lesson we draw from (3.8) is the monodromy invariance of $s_D = \partial\mathcal{F}/\partial s$, although we will obtain this result from a more indirect procedure later. Finally in writing τ_{00} in (3.9) we have set to zero an integration constant depending only on s. This is the result we would have obtained had we started with the Seiberg-Witten solution and compute τ_{00} as $-(\Lambda\partial/\partial\Lambda)^2\mathcal{F}$. As an application of (3.7)-(3.9) we can compute the couplings τ_{ij} in the Higgs and monopole region.

i) Higgs region:

$$a_D^{(h)} = \frac{4\Lambda}{i\pi}\frac{E'-K'}{k}, \quad a^{(h)} = \frac{4\Lambda}{\pi k}E(k),$$
$$\tau_{11}^{(h)} = \frac{iK'}{K}, \quad \tau_{01}^{(h)} = \frac{2\Lambda}{kK}, \quad \tau_{00}^{(h)} = -\frac{8i\Lambda^2}{\pi}\Big(\frac{E-K}{k^2K}+\frac{1}{2}\Big). \tag{3.10}$$

ii) Monopole region:

$$a_D^{(m)} = \frac{4\Lambda}{\pi k}E(k), \quad a^{(m)} = -\frac{4\Lambda}{i\pi}\frac{E'-K'}{k},$$
$$\tau_{11}^{(m)} = \frac{iK}{K'}, \quad \tau_{01}^{(m)} = \frac{2i\Lambda}{kK'}, \quad \tau_{00}^{(m)} = \frac{8i\Lambda^2}{\pi}\Big(\frac{E'}{k^2K'}-\frac{1}{2}\Big). \tag{3.11}$$

Between (3.10) and (3.11) we find an apparent puzzle. If we compute the difference between $\tau_{00}^{(m)}$ and $\tau_{00}^{(h)}$ the result is not zero as one might naïvely expect:

$$\tau_{00}^{(m)} - \tau_{00}^{(h)} = \frac{4i\Lambda^2}{k^2KK'}. \tag{3.12}$$

Before we showed that $\partial\mathcal{F}/\partial s$ is a monodromy invariant, thus one would be tempted to believe that $\partial^2\mathcal{F}/\partial s^2$ is also invariant and that it should take the same values in the Higgs and monopole region. The reason for this apparent mismatch has to do with the fact that the light fields in the two regions are not mutually local, and $\mathcal{F}$ is written in each region in terms of the light fields. We can compute the difference (3.12) on general grounds as follows. In a region where the coordinate describing the light fields is a_Γ (Γ an element of $SL_2(\mathbf{Z})$), the prepotential is:

$$\mathcal{F}_\Gamma = \mathcal{F}_\Gamma(a_\Gamma, s) = a_\Gamma^2 f_\Gamma(a_\Gamma/\Lambda), \tag{3.13}$$

with couplings:

$$\tau_{ij}^{\Gamma} = \frac{\partial^2 \mathcal{F}_\Gamma}{\partial a_\Gamma^i \partial a_\Gamma^j}. \tag{3.14}$$

As $a_\Gamma = a_\Gamma(a,s)$, we must be careful in computing the derivatives (as in Thermodynamics). This will give us the transformation rules of τ_{ij}^{Γ}. Since $a_\Gamma = a_\Gamma(a,s) = \gamma a_D(a,s) + \delta a$, $\Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, we have:

$$\begin{pmatrix} \frac{\partial a_\Gamma}{\partial a} & \frac{\partial a_\Gamma}{\partial s} \\ \frac{\partial s}{\partial a} & \frac{\partial s}{\partial s} \end{pmatrix} = \begin{pmatrix} \gamma\tau_{11} + \delta & \gamma\tau_{01} \\ 0 & 1 \end{pmatrix}, \tag{3.15}$$

with inverse

$$\begin{pmatrix} \frac{\partial a}{\partial a_\Gamma} & \frac{\partial a}{\partial s} \\ \frac{\partial a}{\partial a_\Gamma} & \frac{\partial s}{\partial s} \end{pmatrix} = \frac{1}{\gamma\tau_{11} + \delta} \begin{pmatrix} 1 & -\gamma\tau_{01} \\ 0 & \gamma\tau_{11} + \delta \end{pmatrix}. \tag{3.16}$$

In particular,

$$\begin{aligned} \left(\frac{\partial}{\partial a_\Gamma}\right)_{\Gamma-\mathrm{basis}} &= \frac{1}{\gamma\tau_{11} + \delta}\frac{\partial}{\partial a}, \\ \left(\frac{\partial}{\partial s}\right)_{\Gamma-\mathrm{basis}} &= \frac{\partial}{\partial s} - \frac{\gamma\tau_{01}}{\gamma\tau_{11} + \delta}\frac{\partial}{\partial a}; \end{aligned} \tag{3.17}$$

and together with the transformation rules for $\mathcal{F}_\Gamma$ (2.24), (3.17) leads to:

$$\begin{aligned} \tau_{11}^{\Gamma} &= \frac{\alpha\tau_{11} + \beta}{\gamma\tau_{11} + \delta}, \qquad \tau_{01}^{\Gamma} = \frac{\tau_{01}}{\gamma\tau_{11} + \delta}, \\ \tau_{00}^{\Gamma} &= \tau_{00} - \frac{\gamma\tau_{01}^2}{\gamma\tau_{11} + \delta}. \end{aligned} \tag{3.18}$$

The Γ-transformations which change τ_{00} are those for which $\gamma \neq 0$, but these are precisely the ones mixing non-trivially the electric and magnetic fields. With the explicit formulæ (3.10) and (3.11) it is easy to verify that (3.12) follows from (3.18). Furthermore, to check that $\partial\mathcal{F}/\partial s$ is modular invariant it suffices to prove that $(\partial\mathcal{F}_\Gamma/\partial s)_{\Gamma-\mathrm{basis}} = \partial\mathcal{F}/\partial s$; a straightforward consequence of the previous equations. Similarly, but with some more algebra, one can verify:

$$K_\Gamma = \mathrm{Im} A^\Gamma_{D,i}\overline{A}^{\Gamma i} = \mathrm{Im}\Big(\frac{\partial\mathcal{F}_\Gamma}{\partial s}\Big|_{\Gamma-\mathrm{basis}}\bar{s} + \frac{\partial\mathcal{F}_\Gamma}{\partial a_\Gamma}\Big|_{\Gamma-\mathrm{basis}}\bar{a}^\Gamma\Big) = K(A,S). \quad (3.19)$$

An illuminating way to obtain the transformation (3.18) when $\Gamma = S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is to start with the $N=1$ superspace action:

$$\frac{1}{4\pi}\mathrm{Im}\int(\frac{1}{2}\tau_{11}W_1W_1 + \tau_{01}W_0W_1 + \frac{1}{2}\tau_{00}W_0W_0). \quad (3.20)$$

S-duality follows by adding

$$\frac{1}{4\pi}\mathrm{Im}\int W_DW_1 \quad (3.21)$$

to (3.20) and integrating out W_1. This yields the dual action

$$\frac{1}{4\pi}\mathrm{Im}\int\Big(-\frac{1}{2\tau_{11}}W_DW_D + \frac{\tau_{01}}{\tau_{11}}W_0W_D + \frac{1}{2}(\tau_{00} - \frac{\tau_{01}^2}{\tau_{11}})W_0W_0\Big), \quad (3.22)$$

in exact agreement with (3.18). These transformation rules also agree with the general formulæ in [13].

Now we have all the ingredients to write the low-energy effective action including the spurion. To analyze the vacuum structure we also need to include in the monopole (and dyon) region the coupling to the monopole hypermultiplets. In rigid $N=2$ supersymmetry the scalar components of a hypermultiplet take values in a hyperkähler manifold [19]. If we denote by m, $\widetilde{m}$ the complex scalar components of the monopole multiplet, the $SU(2)_R$-symmetry of $N=2$ supersymmetry implies that $(m,\widetilde{m})$ form a doublet under this symmetry. $(m,\widetilde{m})$ have opposite $U(1)$ charges. Hence the hyperkähler manifold has complex dimension two and must have an isometry group $SU(2)\times U(1)$. If we knew some properties of the theory for large values of $(m,\widetilde{m})$ we could determine the asymptotic structure of the monopole manifold. Assuming no global identifications at large values of m, $\widetilde{m}$, the only two natural choices would be flat space and the Taub-Nut instanton. In four dimensions hyperkähler manifolds are equivalent to gravitational instantons with self-dual connections. With the given isometry group we can identify flat space, Eguchi-Hanson and Taub-Nut. However in the Eguchi-Hanson instanton the space is asymptotically $S^3/\mathbf{Z}_2$, and in the Taub-Nut case it looks asymptotically like S^3 but in a distorted form: it is given by the Hopf fibration of S^3 over S^2, where the S^1-fibre reaches a constant asymptotic value whereas the radius of the S^2-base goes to infinity. It does not seem physically reasonable to impose such behaviour for large monopole fields. However one should not extrapolate the effective action to that region. We will assume that the hyperkähler manifold is $\mathbf{C}^2$. For small fields this is a good approximation. Since the monopoles come in a hypermultiplet, in a heterotic string they do not couple to the dilaton in the first two terms in the effective action. Therefore the monopole Lagrangian will be taken to be:

$$\mathcal{L}_M = \int d^4\theta (M^* \mathrm{e}^{2V_D} M + \widetilde{M}^* \mathrm{e}^{-2V_D} \widetilde{M}) + \Big(\int d^2\theta \sqrt{2} A_D M \widetilde{M} + \text{h.c.} \Big) \quad (3.23)$$

where A_D is the chiral multiplet in the $N = 2$ vector multiplet of the dual photon [1, 2]. Its scalar component is a_D, a good coordinate in the $u = \Lambda^2$ region of the moduli space where the monopole becomes massless. The full lagrangian is given by adding up (2.4) and (3.23). Here we should be careful with the prepotential $\mathcal{F}(A, S)$ that is included in (2.4). The exact solution (2.15), (2.16), (2.22) describes the Wilsonian effective action where all states but the photon multiplet are integrated out, in particular the monopoles. Near $u = \Lambda^2$, where the monopole becomes massless in the $N = 2$ theory, we have to include (3.23) in the effective action and we should be careful in not overcounting the monopole contribution in $\mathcal{F}(A)$.

We have already integrated out the quantum fluctuations of the monopole; they are already represented in (2.4). What appears in (3.23) is the *classical* monopole field. In order to find the vacuum, we still need to extremize with respect to it. In fact, as Lorentz-invariance is unbroken, we really need only concern ourselves with the constant mode of the monopole field. Our task, then will be to minimize the effective potential with respect to the classical monopole field.

One way to think about this is that, in obtaining the Wilsonian effective action (2.4) at low energies, we have integrated out all of the nonzero-momentum modes of the monopole field, but we have not (yet) integrated out the constant mode. Since, in the softly-broken case (as we shall see) all of the scalars are massive, there is, essentially, no difference between the

Wilsonian and 1PI effective actions. The latter, for the constant modes of the fields is just the usual effective potential, $V_{\rm eff}$ [20].

What we will find is that, over most of the u-plane, including the monopole has no effect on $V_{\rm eff}(u)$. The extremum occurs at zero monopole VEV. However, there will be a region, near $u=\Lambda^2$, where a nonzero monopole VEV is favoured and the effect of including (3.23) is to *lower* the energy.

Therefore, to determine the vacuum structure in this region, we *must* add up (3.23) with

$$\mathcal{L}=\frac{1}{4\pi}{\rm Im}\Big[\int d^4\theta\frac{\partial F}{\partial A^i}\overline{A}^i+\frac{1}{2}\int d^2\theta\frac{\partial^2 F}{\partial A^i\partial A^j}W^i_\alpha W^{\alpha j}\Big],$$
$$i=0,1;\quad A^0=S,\quad A^1=A, \tag{3.24}$$

using the complete prepotential in the Seiberg-Witten solution. We read off the potential by keeping non-derivative terms and auxiliary fields. S is frozen to be a constant. Its lowest component fixes the scale Λ but we also freeze its auxiliaries F_0, D_0 (from the chiral and the $N=1$ vector multiplets, respectively). Eliminating the auxiliary fields F_m, $F_{\widetilde{m}}$ and F_a we obtain a potential:

$$\begin{aligned}V &= \frac{1}{2b_{11}}(|m|^2+|\widetilde{m}|^2)^2+2|a|^2(|m|^2+|\widetilde{m}|^2)\\ &+ \frac{1}{b_{11}}\Big(\sqrt{2}b_{01}(\overline{F}_0 m\widetilde{m}+F_0\overline{m}\overline{\widetilde{m}})+b_{01}D_0(|m|^2-|\widetilde{m}|^2)\Big)\\ &- \frac{\det\ b_{ij}}{b_{11}}\Big(\frac{1}{2}D_0^2+|F_0|^2\Big),\end{aligned} \tag{3.25}$$

where

$$b_{ij}\equiv\frac{1}{4\pi}{\rm Im}\ \tau_{ij}=\frac{1}{4\pi}{\rm Im}\frac{\partial^2\mathcal{F}}{\partial a^i\partial a^j}. \tag{3.26}$$

m, $\widetilde{m}$ are, as before, the scalar components of M, $\widetilde{M}$; in the same way a is taken as the scalar component of A, and $\mathcal{F}$ is the exact solution of Seiberg and Witten. For small values of F_0, D_0 with respect to Λ (3.25) is the exact expression including supersymmetry breaking. Note that in (3.25) not all allowed soft breaking terms from the $N=1$ point of view appear. We do not have for instance a diagonal mass for m, $\widetilde{m}$, $B(|m|^2+|\widetilde{m}|^2)$, or the trilinear term $A(am\widetilde{m}+\bar{a}\overline{m}\overline{\widetilde{m}})$, but we have a μ-term $\sim m\widetilde{m}+$c.c. and a cosmological term. If we look at the fermion terms there are also gluino masses induced, for both sets of spinors associated to the vector multiplet. The terms in V that remain after $D_0, F_0 \to 0$ are $SU(2)_R$ invariant as expected. More important, V contains the contribution for the metric coming from the Kähler potential. This information is missing when we only consider soft breaking in $N=1$ theories where one may hope to control the superpotential but not the kinetic terms. This is an important advantage of starting with $N=2$ SQCD, the disadvantage is the presence of an extra adjoint chiral multiplet. Using the monodromy transformations of the couplings (3.18) one can see that det b_{ij}/b_{11} is a monodromy invariant. To prove it, it is sufficient to check the invariance under the generators S, T of the modular group. Under T it is obvious, and for S it can be done with a little algebra. This tells us that in the vacuum energy we are taking into account the quantum fluctuations in the right way for different patches.

In section five we analyze in detail the potential (3.25). In the next section we derive the same action (3.23) plus (3.24) starting from the spontaneously broken theory coupled to $N=2$ supergravity. The same set of soft breaking terms is obtained in the flat limit, including the cosmological term. This reassures us that we are not missing any important term. The reader not interested in this derivation can skip directly to section five.

4 A Brief Foray into $N=2$ Supergravity

In order to give a physical meaning to the soft breaking terms it is necessary to justify their origin in a more fundamental theory in which the $N=2$ supersymmetry is spontaneously broken with zero (or almost zero) cosmological term. This requirement implies that supersymmetry must be local and that the two gravitini will become massive via an $N=2$ superhiggs phenomenon [4, 21]. Thus, our starting point must be an $N=2$ supergravity coupled to (n_v+1)-vector multiplets in which the desired superhiggs breaking takes place with vanishing vacuum energy at the classical level [21]. It is interesting that the structure of the $N=2$ supergravity theories with the above properties are quite restricted and are based on a prepotential which has the following form [21]:

$$\mathcal{F} = \frac{1}{x_0} d_{abc} x^a x^b x^c, \tag{4.1}$$

where x^a $a = 1, 2, \ldots, n_v$ are the matter vector multiplets and x^0 is an extra auxiliary vector multiplet in association with the graviphoton of the $N = 2$ gravitational multiplet. In this section $\mathcal{F}$ denotes the prepotential in $N = 2$ supergravity, not to be confused with the Seiberg-Witten prepotential.

The above choice of the prepotential defines a particular class of Kähler potential of the no-scale type [22, 21]:

$$K = -\log\, Y, \tag{4.2}$$

with

$$\begin{aligned} Y &= i(x^I \overline{\mathcal{F}}_I - \bar{x}^I \mathcal{F}_I) \\ &= -i\Big(2(\mathcal{F} - \overline{\mathcal{F}}) - (x^a - \bar{x}^a)(\mathcal{F}_a + \overline{\mathcal{F}}_a)\Big) \\ &= -i d_{abc}(x^a - \bar{x}^a)(x^b - \bar{x}^b)(x^c - \bar{x}^c), \end{aligned} \tag{4.3}$$

where the subscripts indicate differentiation with respect to the corresponding variable. In the above equations we denote by $x^I = (x^0\ , x^a)$ and after the algebraic operations we choose the gauge $x^0 = 1$. The breaking of supersymmetry implies the existence of a superpotential for the vector multiplets, $W_v(X^I)|_{x^0=1}$. The form of W is restricted by $N = 2$ supersymmetry to be a homogeneous function of degree one in x^I [21, 23]:

$$W = g_I x^I - f^I \mathcal{F}_I. \tag{4.4}$$

An interesting subclass of models are those in which the prepotential is given by:

$$F = \frac{1}{x^0} s(z^2 - y_i^2). \tag{4.5}$$

In that case the Kähler manifold has an interesting structure, namely the scalars of the vector multiplets are coordinates of the coset

$$\left[\frac{SL(2,R)}{U(1)}\right]_s \times \left[\frac{SO(2,\ n_v - 1)}{SO(2) \times SO(n_v - 1)}\right]_{z,y}. \tag{4.6}$$

This is precisely the structure which emerges in heterotic strings with $N = 2$ spacetime supersymmetry [24, 25]. The s–field is the string dilaton–axion vector multiplet with a $U(1)_s$ gauge field. The other abelian gauge symmetries are the $U(1)_{x^0}$ associated to the graviphoton of the supergravity multiplet and the $U(1)_z$ of the z–vector multiplet. The remaining gauge group in association with the y^i–vector multiplets can be a non–abelian gauge group at particular points of the y^i–moduli–space. Observe that the $U(1)_z$ cannot have a non abelian extension at any point of the z–moduli–space as soon as $y^i \neq 0$. In terms of the usual string notation, z and y correspond respectively to the $T + U$ and $T - U$ combinations. The non–abelian extension happens in some special points of the y^i moduli space, e.g. the $SU(2)_y$ extension when $y^1 = 0$ and $z \neq 2e^{2i\pi/3}$. Working in the large z–regime we can avoid in string theory, as well as in the effective field theory limit, the extension of the $U(1)_z \times U(1)_y$ to $SU(3)$ which happens at the point $y = 0,\ z = 2e^{i2\pi/3}$ of the moduli space. Thus in the large z–regime the only non–abelian extensions happen for special values of $y^i = 0$.

We are now in a position to define in a consistent way the Seiberg–Witten theory in a supergravity model where supersymmetry is spontaneously broken. The minimal set of vector superfields at the classical level are x^0, s, z, and $y^a, a = 1, 2, 3$, where a is the adjoint index of $SU(2)_y$. The remaining gauge group consists of abelian factors $U(1)_{x^0} \times U(1)_s \times U(1)_z$. Neglecting gravitational corrections but including perturbative and non-perturbative gauge corrections, the $N = 2$ supergravity prepotential become:

$$\mathcal{F} = \frac{sz^2}{x^0} - y^2 \Phi\Big(\frac{y}{x^0}, \frac{s}{x^0}\Big). \tag{4.7}$$

The justification for the above expression follows from the fact that $U(1)_{x^0} \times U(1)_s \times U(1)_z$ does not receive corrections in the limit where we neglect the gravitational interactions. On the other hand, the y^2 part receives perturbative and non-perturbative $SU(2)$ corrections similar to those in global supersymmetry. Obviously, one can do much better in the context of string theory where the gravitational corrections (at least the perturbative ones) can be also be included [26, 27, 28]. For our purposes however this is not necessary since, in the end, we will take the limit in which the gravitational interactions are neglected, keeping only the soft breaking terms.

Concerning supersymmetry breaking, we must specify our choice for the superpotential $W_v(x^I)$. Although there are several possibilities, our choice must be consistent with the stability of the scalar potential at the classical level, *i.e.* with the existence of a perturbative vacuum in the large s-limit. One consistent choice is when $W_v = cx^0 \mid_{x^0=1}$.

Finally, we must specify the remaining interactions among the vector multiplets and the monopole-dyon hypermultiplets. Using the $N = 1$ language these interactions are given in terms of an effective superpotential W_m and the usual D-terms. W_m is restricted by $N = 2$ supersymmetry to have the following form [23]:

$$W_m = \gamma(m_I x^I - n^I \mathcal{F}_I) M\widetilde{M} \tag{4.8}$$

To recover the results of the global case it is necessary to choose the m^I, n_I coefficients to be non–zero only when I is taken in the y-direction. The total superpotential is then

$$W_t \; = \; W_v + W_m \tag{4.9}$$

The remaining interactions are given by the usual D-terms. The normalization γ of W_m is fixed by $N = 2$ supersymmetry (see below).

In the spirit of references [11, 12] we would like to derive the softly broken action of the previous section starting with a spontaneously broken $N = 2$ supergravity theory inspired by an $N = 2$ compactification of the heterotic string. From the geometrical point of view this is related to the question of how to obtain rigid special geometry from local special geometry [29]. One problem with the prepotential in [12] is it does not admit a straightforward flat limit. A further change of variables is required to go to a system of coordinates analogous to the Calabi-Visentini variables [30, 32]. We take a different route. Together with the dilaton and the other multiplets in the non-gravitational part of the theory we include the graviphoton in the local prepotential. String theory suggests to start with a prepotential of the form (4.7):

$$\mathcal{F} = sz^2 - F(y, s). \tag{4.10}$$

The scaling limit we will take involves writing $y = a/M$, $|z| \sim 1$ and $M\mathrm{e}^{iS} = \Lambda$ as $M, S \to \infty$, where $F(y, s)$ becomes $\frac{1}{M^2} F_{\mathrm{SW}}(a, \Lambda)$, F_{SW} is the Seiberg-Witten prepotential. The Kähler potential in local special geometry is constructed from (4.3) and (4.10) as:

$$\begin{aligned} i\mathrm{e}^{-K} = 2(\mathcal{F} - \overline{\mathcal{F}}) \;\; & - \;\; (s - \bar{s})(\mathcal{F}_s + \overline{\mathcal{F}}_s) \\ & - \;\; (z - \bar{z})(\mathcal{F}_z + \overline{\mathcal{F}}_z) - (y - \bar{y})(\mathcal{F}_y + \overline{\mathcal{F}}_y). \end{aligned} \tag{4.11}$$

We also include a contribution to K coming from the monopoles of the form:

$$\delta K = \alpha(|m|^2 + |\widetilde{m}|^2), \tag{4.12}$$

where we will have to work out the scaling properties of α. Finally the simplest superpotential breaking supersymmetry spontaneously is (4.9):

$$W_t = c + \sqrt{2} A_D M \widetilde{M} \equiv c + w. \tag{4.13}$$

In the Higgs region we would simply take the constant term. There are more general choices for W, but (4.13) is the simplest one. Supersymmetry breaking is primarily done by the graviphoton sector which then communicates it through gravity to the other sectors of the theory. Defining the G-function as:

$$G = K + \ln|W|^2, \tag{4.14}$$

the scalar potential, after the auxiliary fields are eliminated, is given by:

$$\begin{aligned} V &= \mathrm{e}^G \Big(G_{\bar{i}} (G^{-1})^{\bar{i}j} G_j - 3 \Big) + D\text{–terms}, \\ G_{\bar{i}} &= \partial_{\bar{i}} G, \quad G_j = \partial_j G, \quad G_{\bar{i}j} = \partial_{\bar{i}} \partial_j G. \end{aligned} \tag{4.15}$$

In (4.10) the first term in the right-hand side is much bigger than the second; hence we expand in powers of $\frac{1}{M}$ (the Planck mass):

$$\mathrm{e}^{-K} = i\Sigma Z^2 \left(1 - \frac{1}{Z^2} \Big(F_s + \overline{F}_s + \frac{y\overline{F}_y - \bar{y}F_y - i(F_s + \overline{F}_s)}{\Sigma} \Big) \right), \tag{4.16}$$

with

$$\Sigma \equiv s - \bar{s}, \quad Z \equiv z - \bar{z}. \tag{4.17}$$

To second order in $1/Z$ we have:

$$K = -\log i\Sigma - 2\log Z + \frac{1}{\Sigma^2}\phi(s, y),$$

$$\phi(s, y) = F_s + \overline{F}_s + \frac{y\overline{F}_y - \bar{y}F_y - i(F_s + \overline{F}_s)}{\Sigma}. \tag{4.18}$$

It is now a long and tedious algebraic computation to evaluate (4.15) to leading order. The answer is:

$$\begin{aligned}
&\tfrac{1}{i\Sigma\phi_{y\bar{y}}}|\partial_y W|^2+\tfrac{1}{i\Sigma\phi_{y\bar{y}}Z^2}\Big(\phi_y W\overline{\partial_y W}+\phi_{\bar{y}}\overline{W}\partial_y W\Big)\\
&\quad+\tfrac{1}{i\Sigma Z^2}\Big(\alpha^{-1}|\partial_m W|^2+\alpha^{-1}|\partial_{\widetilde{m}}W|^2\\
&\quad+\alpha|c|^2(|m|^2+|\widetilde{m}|^2)+2c(w+\bar{w})\Big)\\
&\quad+\tfrac{|c|^2}{i\Sigma Z^4\phi_{y\bar{y}}}\Big(\Sigma(\phi_{y\bar{y}}\phi_s-\phi_{y\bar{y}}\phi_{\bar{s}})-2\phi_{y\bar{y}}\phi+\phi_y\phi_{\bar{y}}\\
&\quad+\Sigma^2\phi_{s\bar{s}}\phi_{y\bar{y}}-\Sigma^2\phi_{s\bar{y}}\phi_{y\bar{s}}+\Sigma\phi_{\bar{y}}\phi_{y\bar{s}}-\Sigma\phi_y\phi_{\bar{s}y}\Big)\\
&\quad+\text{l.o.t},
\end{aligned}\tag{4.19}$$

a slightly unwieldy expression. The l.o.t. stand for lower order terms in M. It is also important to consider the kinetic term for y, $\bar{y}$ to correctly normalize the low-energy fields. From (4.18) we obtain:

$$\begin{aligned}
\phi_s &= F_{ss}+\frac{1}{\Sigma}\Big((y-\bar{y})F_{ys}-2F_s\Big)-\frac{1}{\Sigma^2}\Big((y-\bar{y})(F_y+\overline{F}_y)-2F+2\overline{F}\Big),\\
\phi_y &= F_{sy}+\frac{1}{\Sigma}\Big(\overline{F}_y-F_y+(y-\bar{y})F_{yy}\Big),\\
\phi_{s\bar{y}} &= -\frac{1}{\Sigma}F_{ys}-\frac{1}{\Sigma^2}\Big(\overline{F}_y-F_y+(y-\bar{y})\overline{F}_{yy}\Big),\\
\phi_{y\bar{y}} &= \frac{1}{\Sigma}(\overline{F}_{yy}-F_{yy}),\\
\phi_{s\bar{s}} &= \frac{1}{\Sigma^2}\Big((y-\bar{y})(F_{ys}-\overline{F}_{ys})-2F_s-2\overline{F}_s\Big)\\
&\quad-\frac{2}{\Sigma^3}\Big((y-\bar{y})(F_y+\overline{F}_y)-2F+2\overline{F}\Big).
\end{aligned}\tag{4.20}$$

Inserting (4.20) into (4.19) and keeping leading order terms we obtain:

$$\begin{aligned}
&\tfrac{1}{i\Sigma\phi_{y\bar{y}}}|\partial_y W|^2-\tfrac{c}{i\Sigma\phi_{y\bar{y}}Z^2}(F_{ys}+\overline{F}_{ys})(\partial_y W+\overline{\partial}_y W)\\
&\quad+\tfrac{|c|^2}{i\Sigma Z^4\phi_{y\bar{y}}}\Big((\overline{F}_{yy}-F_{yy})(F_{ss}-\overline{F}_{ss})+(F_{ys}+\overline{F}_{ys})^2\Big)\\
&\quad+\tfrac{1}{i\Sigma Z^2}\Big(\alpha^{-1}|\partial_m W|^2+\alpha^{-1}|\partial_{\widetilde{m}}W|^2+\alpha|c|^2(|m|^2+|\widetilde{m}|^2)\\
&\quad 2c(w+\bar{w})\Big)+D\text{−terms}.
\end{aligned}\tag{4.21}$$

To determine the scaling limit we want to scale $y\sim a/M$, hence $F\sim\frac{1}{M^2}$, $F_y\sim\frac{1}{M}$, $F_{yy}\sim 1$. From the kinetic term of y we learn that $i\Sigma Z^2\sim 1$. As in section 3 we define $\tau_{ij} = \partial^2_{ij} F$ for the a, s variables. The scaling inside $F_{\rm SW}$ is then:

$$M e^{iS} = \Lambda, \tag{4.22}$$

with Λ fixed. Since we want to recover the purely supersymmetric terms in the potential, this fixes $\alpha \sim 1/M^2$. Finally the second and third terms in (4.21) define the scaling behaviour of c:

$$c \frac{i\Sigma}{M} = m_{3/2}. \tag{4.23}$$

$m_{3/2}$ is the gravitino mass and Λ is fixed. From (2.14) we learn that $i\Sigma = 2 \ln \frac{M}{\Lambda}$. For $M \sim M_{\rm Pl}$, $\Lambda \sim 1$ GeV, $i\Sigma \sim 10^2$. The last two terms in (4.21) become:

$$\frac{m^2_{3/2}}{(i\Sigma)^2}(|m|^2 + |\widetilde{m}|^2) + \frac{2m_{3/2}}{i\Sigma}(w + \bar{w}). \tag{4.24}$$

In the formal limit $i\Sigma \to \infty$, $M \to \infty$ with $M^2 e^{i\Sigma} = \Lambda^2$ fixed, these two terms disappear; if, however, we take $M \sim M_{\rm Pl}$, they stay but with very small coefficients with respect to the other soft-breaking terms in (4.21). If we were to consider the full potential, the higher order corrections are of two types. First those suppressed by powers of $1/\Sigma$, $1/\Sigma^2$, and those suppressed by powers of $1/M$. The latter can be ignored, while the former can be neglected in a first approximation. Notice that (4.21) is equivalent to (3.25) in the $i\Sigma \to \infty$ limit with $F_0 \sim m_{3/2}$, and similarly for D_0. Although we have not presented here the explicit computation of the D-terms in supergravity, they also lead to the same term in (3.25).

The conclusion we draw from this computation is that the soft-breaking terms included in (3.25) are precisely those which are induced from a spontaneously broken $N = 2$ supergravity theory in the flat limit, and although some soft-breaking terms like (4.24) also appear, they are suppressed with respect to the leading order ones in (3.25). Therefore, to analyze the vacuum structure, (3.25) contains all the relevant terms and we are not missing any essential ingredient. This is additional support for the procedure we are following.

5 Vacuum structure

We now turn to the analysis of the potential (3.25). We will make two additional technical simplifications. The first one is to ignore the small terms in (4.24). The second one is to set $D_0 = 0$. This makes the algebraic structure simpler but the conclusions remain the same. In minimizing the effective potential (3.25) we proceed in two stages: first we minimize with respect to the monopoles m, $\widetilde{m}$; and then we look graphically for the minima with respect to the dual photon a. The explicit formulæ are those in (3.11) for the monopole region.

$$\frac{\partial V}{\partial \overline{m}} = \frac{1}{b_{11}}(|m|^2 + |\widetilde{m}|^2)m + 2|a|^2 m + \frac{\sqrt{2}}{b_{11}} b_{01} F_0 \overline{\widetilde{m}} = 0, \tag{5.1}$$

$$\frac{\partial V}{\partial \overline{\widetilde{m}}} = \frac{1}{b_{11}}(|m|^2 + |\widetilde{m}|^2)\widetilde{m} + 2|a|^2 \widetilde{m} + \frac{\sqrt{2}}{b_{11}} b_{01} F_0 \overline{m} = 0. \tag{5.2}$$

Multiplying (5.1) by $\widetilde{m}$, (5.2) by m and subtracting we obtain:

$$\frac{\sqrt{2}}{b_{11}} b_{01} F_0 (|m|^2 - |\widetilde{m}|^2) = 0, \tag{5.3}$$

hence $|m|^2 = |\widetilde{m}|^2$. Writing

$$m = \rho e^{i\alpha}, \quad \widetilde{m} = \rho e^{i\beta}, \quad F_0 = f_0 e^{i\gamma}; \tag{5.4}$$

we can fix the gauge so that $\alpha = 0$, and absorb γ in β; then $e^{i(\gamma-\beta)}$ must be real. This implies that we can choose:

$$m = \rho, \quad \widetilde{m} = \epsilon\rho, \quad \epsilon = \pm 1, \quad F_0 = f_0, \tag{5.5}$$

without loss of generality. Substituting (5.5) in (5.1) leads to:

$$\frac{1}{b_{11}} \rho \Big(\rho^2 + b_{11}|a|^2 + \frac{b_{01}\epsilon f_0}{\sqrt{2}} \Big) = 0, \tag{5.6}$$

with two possibilities:

$$\text{i)} \;\; \rho = 0, \tag{5.7}$$

$$\text{ii)} \;\; \rho^2 = -b_{11}|a|^2 + \frac{b_{01}\epsilon f_0}{\sqrt{2}} > 0. \tag{5.8}$$

To determine whether (5.7) or (5.8) is favored we need to compute the full potential. Note however that $b_{11} = \frac{1}{4\pi}\mathrm{Im}\ \tau_{11}$ is always positive, and therefore (5.8) determines a region in the u-plane where the monopoles acquire a vacuum expectation value (VEV). Depending on the sign of b_{01} we choose the sign of ϵ. In fact we can replace (5.8) by:

$$\rho^2 = -b_{11}|a|^2 + \frac{1}{\sqrt{2}}|b_{01}|f_0 > 0 \tag{5.9}$$

and f_0 is always measured in units of Λ. Thus for the numerical plots we set $\Lambda = 1$. Inserting (5.8) into (3.25) we obtain:

$$V = -\frac{2}{b_{11}}\rho^4 - \frac{\det b}{b_{11}}f_0^2 \tag{5.10}$$

This is good news. It implies that the region where the monopoles acquire a VEV is energetically favored, and we have confinement. Depending on the sign of b_{01}, m and $\widetilde{m}$ are either aligned or antialigned. The $SU(2)_R$ symmetry of $N = 2$ supersymmetry is broken by the explicit off-diagonal term $b_{01}m\widetilde{m}/b_{11}$ in (3.25) and by the VEV $\rho \neq 0$.

Where $\rho^2 \to 0$, the potential maps smoothly onto the potential for the Higgs region,

$$V^{(h)} = -\frac{\det b^{(h)}}{b_{11}^{(h)}}f_0^2, \tag{5.11}$$

since, we recall, $\det b/b_{11}$ is monodromy-invariant. In the monopole region, a nonzero monopole VEV is favoured, and the effective potential is given by (5.10) and written in terms of magnetic variables:

$$V^{(m)} = -\frac{2}{b_{11}^{(m)}}\rho^4 - \frac{\det b^{(m)}}{b_{11}^{(m)}}f_0^2 \tag{5.12}$$

where $b^{(h)}$, $b^{(m)}$ are given in (3.10), (3.11), (3.26).

In the Higgs region, the effective potential is given by (5.11) and we plot it in fig. 1. It has no minimum outside the monopole region near $u = \Lambda^2$ (where, as we shall see, the energy can be further lowered by giving the monopoles a VEV). One sees that the shape of the potential makes the fields roll towards the monopole region. In fig. 2, we plot slices of the potential $V^{(h)}$ along the real u-axis and parallel to the imaginary u-axis with $\mathrm{Re}(u) = \Lambda^2$.

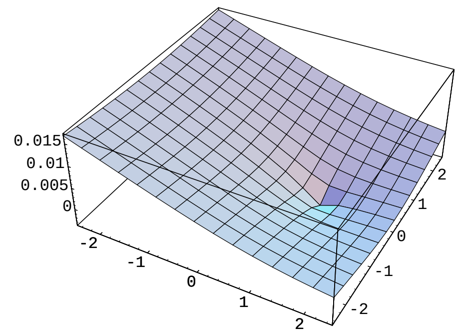

Figure 1: Effective potential, $V^{(h)}$, (5.11).

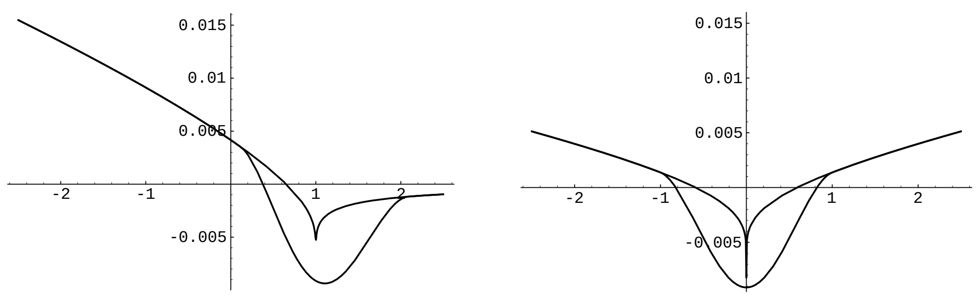

Figure 2: Effective potential, $V^{(h)}$, (5.11) (top) and, $V^{(m)}$, (5.12) (bottom) along the real axis (left) and for $u = \Lambda^2(1 + iy)$ (right). Both are plotted for $f_0 = 0.3\Lambda$.

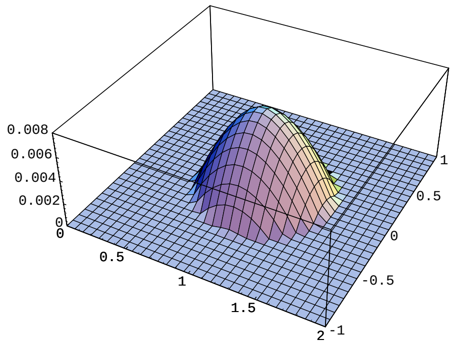

Figure 3: Monopole expectation value ρ^2 for $f_0 = 0.1\Lambda$ on the u-plane.

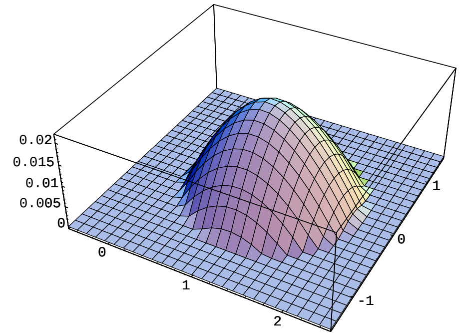

Figure 4: Monopole expectation value ρ^2 for $f_0 = 0.3\Lambda$ on the u-plane.

For comparison, we also plot $V^{(m)}$. Note that they agree in the Higgs region (where the monopole VEV vanishes), and that $V^{(m)}$ lowers the energy (and smooths out the cusp in $V^{(h)}$ at $u = \Lambda^2$) in the monopole region.

Next we look at the monopole region (5.9). a (*i.e.* $a^{(m)}$) is a good coordinate in this region vanishing at $u = \Lambda^2$. As soon as f_0 is turned on monopole condensation and confinement occur. In figs. 3,4 we plot ρ^2 in the u-plane for values of $f_0 = 0.1\Lambda$, 0.3Λ; and in figs. 5,6 the effective potential (5.10) for the same values of the supersymmetry breaking parameter f_0.

One can see that the minimum is stable and that the size of the monopole VEV is $\sim f_0$. There are two features worth noticing. The first is that the absolute minimum occurs along the real u-axis. This is seen numerically and also as a consequence of the reality properties of the elliptic functions. Second, as f_0 is increased, the region where (5.9) holds becomes wider. This is seen in fig. 7, where ρ^2 is plotted along the real u-axis as a function of f_0. Accordingly, the minimum of the effective potential moves to the right along the real u-axis, as one can see in fig. 8, where $V^{(m)}/f_0^2$ is plotted for three increasing values of f_0 (we have divided by f_0^2 to fit the three potentials on the same graph).

Finally, we turn to the dyon region. To understand what happens in the dyon region, we study the transformation rules of the τ_{ij} couplings under the residual $\mathbf{Z}_8 \subset U(1)_R$ symmetry whose generator acts on the u-plane as $u \mapsto -u$. The reason why we need to analyze in general the behavior under $\mathbf{Z}_8$

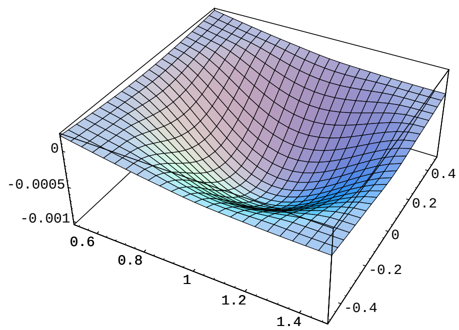

Figure 5: Effective potential (5.12) for $f_0 = 0.1\Lambda$.

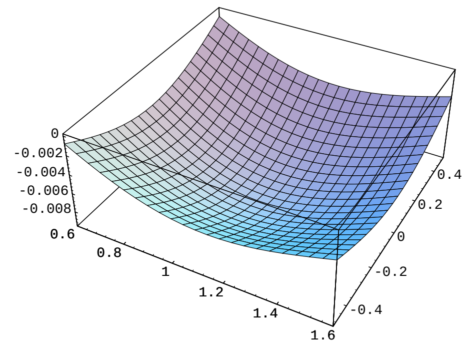

Figure 6: Effective potential (5.12) for $f_0 = 0.3\Lambda$.

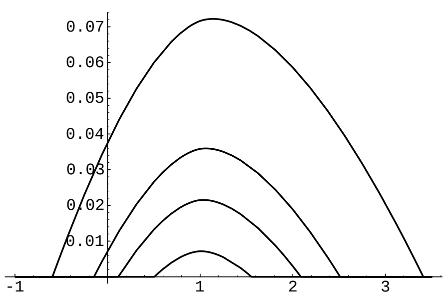

Figure 7: Plot of ρ^2 along the real u-axis, for $f_0/\Lambda =$ (from bottom to top) 0.1, 0.3, 0.5, 1.0.

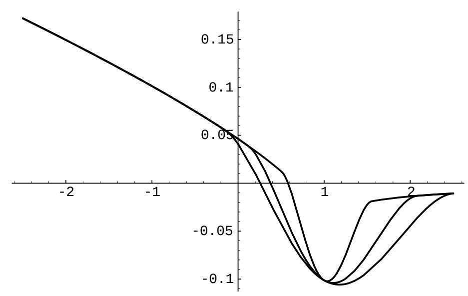

Figure 8: $V^{(m)}/f_0^2$ along the real u-axis for $f_0 = 0.1\Lambda$ (top), 0.5Λ (middle) and Λ (bottom).

is because the representation we have chosen for the Seiberg-Witten solution in sections 2,3 is well adapted to study the monopole region. Naively applying them to the dyon region, we may encounter some discontinuities due to the position of the cuts. Outside the curve of marginal stability one can write the prepotential as [1]:

$$\mathcal{F} = \frac{i}{2\pi} a^2 \log \frac{a^2}{\Lambda^2} + a^2 \sum_{k\geq 1} c_k \Big(\frac{\Lambda}{a}\Big)^{4k}. \tag{5.13}$$

If $\omega = e^{2\pi i/8}$ is the generator of the $\mathbf{Z}_8$ symmetry, it is easy to show that the couplings τ_{ij} transform according to[2]:

$$a \mapsto ia, \qquad a_D \mapsto i(a_D - a),$$

$$\tau_{11} \mapsto \tau_{11} - 1, \qquad \tau_{01} \mapsto i\tau_{01}, \qquad \tau_{00} \mapsto -\tau_{00}. \tag{5.14}$$

So the relation between the dyon and monopole variables is:

$$a^{(d)}(u) = ia^{(m)}(-u), \qquad a_D^{(d)}(u) = i\left(a_D^{(m)}(-u) - a^{(m)}(-u)\right), \tag{5.15}$$

$$\tau_{11}^{(d)}(u) = \tau_{11}^{(m)}(-u) - 1, \qquad \tau_{01}^{(d)}(u) = i\tau_{01}^{(m)}(-u), \quad \tau_{00}^{(d)}(u) = -\tau_{00}^{(m)}(-u).$$

Using the expressions for the monopole couplings in (3.11), which are well-behaved near $u = \Lambda^2$, we obtain expressions for the dyon couplings which are well-behaved near $u = -\Lambda^2$. The analysis of (5.9) changes crucially once these rules are implemented. Near the monopole region $a^{(m)} \sim i(u - \Lambda^2)$, hence $\tau_{01}^{(m)} \sim i$ is purely imaginary. In (5.9) although b_{11} diverges at $u = \Lambda^2$ the divergence is cancelled by the vanishing of $a^{(m)}$ at the same point. Since $\mathrm{Im}\tau_{01}^{(m)} > 0$ as soon as $f_0 \neq 0$ the monopoles condense. Using (5.15), however, we see that $a^{(d)} \sim (u+\Lambda^2)$ with a real coefficient. Thus $\mathrm{Im}\tau_{01}^{(d)} = 0$ at $u = -\Lambda^2$ and we conclude from (5.9) that the dyon condensate *vanishes* along the real u-axis. Nevertheless, a dyon condensate *is* energetically favoured in a pair of complex-conjugate regions in the u-plane centered about $u = -\Lambda^2$. We

[2]There is one more aspect of the $\mathbf{Z}_8$ transformation rules worth noticing. If we implement these rules we find that the condensate moves to the dyon region, and one might be tempted to conclude that with this choice it is the dyon that condenses. This is not the case. Using the one-loop β-function, we know that $\Lambda^4 \sim \exp(-\frac{8\pi^2}{g^2} + i\theta)$. The action of $\mathbf{Z}_8$ amounts to the change $\Lambda \mapsto i\Lambda$ or what is the same, $\theta \mapsto \theta + 2\pi$. Using the relation found in [31], when we make this change the massless state at $u = -\Lambda^2$ (before supersymmetry breaking) has zero electric charge, while the state at $u = \Lambda^2$ acquires charge one. Thus we find again a monopole condensate, in a way consistent with the $\mathbf{Z}_2$-symmetry.

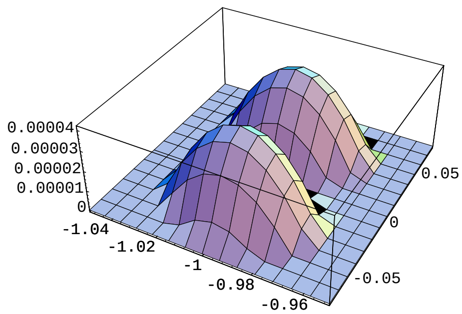

Figure 9: Dyon expectation value $\rho^2_{(d)}$ for $f_0 = 0.3\Lambda$ on the u-plane.

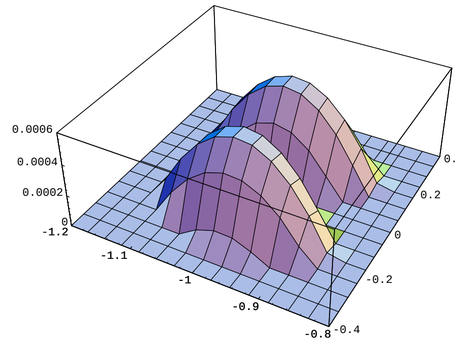

Figure 10: Dyon expectation value $\rho^2_{(d)}$ for $f_0 = \Lambda$ on the u-plane.

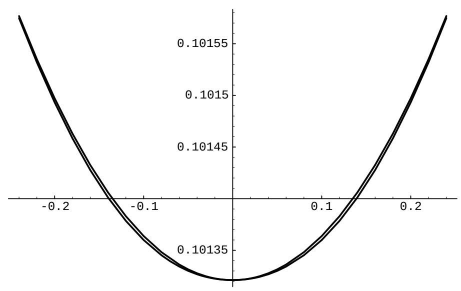

Figure 11: Plot of $V^{(h)}(u)$ (top) and $V^{(d)}(u)$ (bottom) versus $\mathrm{Im}(u)$ for $\mathrm{Re}(u) = -\Lambda^2$ and $f_0 = \Lambda$.

plot $\rho^2_{(d)}$, for two different values of f_0 in figs. 9,10.

Unlike the monopole VEV, the magnitude of the dyon VEV is *tiny* on the scale of $V^{(h)}$. It therefore makes an all-but-negligible contribution to the effective potential (fig. 11). In particular, $V^{(d)}$ does *not* have a minimum in the dyon region. The only minimum of the full effective potential is the one we previously found in the monopole region.

Given that the expectation value of the dyons are about two orders of magnitude smaller than the monopole expectation value, one might worry that small corrections to the potential may erase the dyon VEV altogether. In particular we can consider the two extra soft breaking terms appearing in (4.24) in the decoupling of supergravity. Identifying $m_{3/2}$ with f_0, and taking into account that $i\Sigma \approx 10^2$, it is not difficult to include these effects in our equations for the VEV's or monopoles and dyons. What we find is that the effect is rather small and that the expectation values remain essentially the same. This means that within our approximations, the two types of VEV do not change significantly once these extra soft breaking terms are included.

As we have already noted, the monopole region (in which $\rho^2_{(m)} \neq 0$) expands as f_0 is increased. Eventually, for $f_0 \sim 1.3\Lambda$, it reaches the dyon region (in which $\rho^2_{(d)} \neq 0$). At this point, it is clear that our whole approximation of including *just* the monopole field (or *just* the dyon field) in the effective action breaks down.

What are the other limitations of our approximations? First, we have neglected certain soft supersymmetry breaking terms which arise in the supergravity action. As discussed in section 4, these scale to zero in the rigid limit, that is, they are suppressed by powers of $\log \frac{\Lambda}{M_{\rm Pl}}$ or $\frac{\Lambda}{M_{\rm Pl}}$ and, for our purposes are negligible. We have also neglected higher-spinor-derivative corrections to the Seiberg-Witten effective action. These clearly cannot affect the vacuum structure in the supersymmetric limit. They also, *by definition* must be supersymmetric; otherwise they lead to explicitly hard supersymmetry breaking terms, which is an entirely different matter from the soft supersymmetry breaking we are considering. Nevertheless, once supersymmetry is broken, they can, in principle, lead to corrections to the scalar potential suppressed by higher powers of f_0^2/Λ^2. For the moderate values of f_0 that we are considering, these corrections are numerically rather small, and do not affect the qualitative features of the solutions we have found. *A priori*, if the higher spinor derivative terms in the Seiberg-Witten effective action were known, we could systematically improve our approximations by going to higher order in f_0^2/Λ^2.

However, the fundamental obstacle to pushing our approximation to larger values of the soft supersymmetry breaking parameters would remain. The mutual non-locality of the monopoles and dyons leads to our inability to calculate the effective potential where the monopole and dyon regions overlap. Since this is, at least initially, far from the monopole vacuum, we expect that the monopole vacuum persists, at least as metastable minimum, even beyond the critical value of f_0. But we do not know when (or if) a new, lower minimum develops once the monopole and dyon regions overlap. If a new vacuum does appear there, then we would have a first order phase transition to this new confining phase[3]. This raises the exciting possibility that the correct description of the QCD vacuum requires the introduction of mutually non-local monopoles and dyons. Phases of this nature have been shown to arise in the $N=2$ moduli space for gauge group $SU(3)$ (see the paper by Argyres and Douglas in [3]). Perhaps the way to approach the true QCD vacuum in the correct phase is to start with one of these $N=2$-superconformal field theories and turn on a relevant, soft supersymmetry-breaking perturbation.

Although we have illustrated our method of supersymmetry breaking so far only for pure $SU(2)$, the fact that the soft breaking terms are all produced by making Λ a function of the spurion makes this procedure quite universal, and similar results can be obtained for other gauge groups with and without quark hypermultiplets with arbitrary masses. One example is illustrated in the next section where we include two doublets of massless quarks.

6 Including Two Massless Quark Multiplets

When N_f massless hypermultiplets of quarks are included the global flavour symmetry is $O(2N_f)$, because the $\mathbf{2}$ and the $\bar{\mathbf{2}}$ representations of $SU(2)$ are equivalent. The full group of global symmetries is $O(2N_f)\times SU(2)_R\times U(1)_R$. In [2] Seiberg and Witten have given the exact form of the low-energy effective action when $N_f\leq 4$ with and without masses. When $N_f=2$ and the masses are set to zero, the global symmetry is $O(4)\times SU(2)_R\times U(1)_R$, and the elliptic curve is exactly (2.13):

$$y^2=(x^2-\Lambda^4)(x-u). \tag{6.1}$$

The reason is that the normalizations of [1] and [2] are different. In [1] the charge operator is normalized so that the $W^{\pm}$-boson has charge ± 1, while in [2] the quarks are taken to have charges ± 1 and hence $W^{\pm}$ has charge ± 2. In the conventions of [2] the curve associated to the $N_f=0$ case is:

$$y^2=x^2(x-u)+\frac{1}{4}\Lambda^4 x, \tag{6.2}$$

[3]In theories with matter, as discussed in section 6, this phase transition would change the exotic pattern of chiral symmetry realized in the monopole and dyon vacua into the standard pattern expected in the true QCD vacuum.

and the monodromy group is contained in $\Gamma_0(4)$. Using the curve (6.1) with the conventions of [2] most of the formulæ of sections 2, 3 still apply. There are two singularities in the moduli space, at $u = \pm\Lambda^2$. However now the monopoles and dyons behave respectively as $(\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{2})$ with respect to the global $O(4)$ group. In the monopole region the full global symmetry group is $SU(2)_r \times SU(2)_R \times U(1)_R$ with $SU(2)_r \subset O(4)$, and similarly in the dyon region changing $SU(2)_r \longleftrightarrow SU(2)_l$. We can arrange the scalar monopoles in a 2×2 matrix:

$$\Phi = \begin{pmatrix} m_1 & \widetilde{m}_1^* \\ m_2 & \widetilde{m}_2^* \end{pmatrix}, \tag{6.3}$$

which under $SU(2)_r \times SU(2)_R$ transforms according to:

$$\Phi \rightarrow g_r \Phi g_R^{-1}. \tag{6.4}$$

Making Λ a function of the spurion S, and again for simplicity setting $D_0 = 0$, we obtain a potential analogous to (3.25):

$$\begin{aligned} V &= \frac{1}{2b_{11}}(|m|^2 - |\widetilde{m}|^2)^2 + 2|a|^2(|m|^2 + |\widetilde{m}|^2) \\ &+ \frac{1}{b_{11}}\left|b_{01}f_0 + \sqrt{2}m \cdot \widetilde{m}\right|^2 - b_{00}f_0^2, \end{aligned} \tag{6.5}$$

where $|m|^2 = |m_1|^2 + |m_2|^2$, $m \cdot \widetilde{m} = m_1\widetilde{m}_1 + m_2\widetilde{m}_2$, and phases have been chosen to make f_0 real. If we use the identity:

$$\sum_i (\text{Tr } \sigma_i A)^2 = 2\text{Tr } A^2 - (\text{Tr } A)^2, \tag{6.6}$$

which holds for any 2×2 matrix of the form $A = a_0 + a^i\sigma_i$, with σ_i the Pauli matrices, the potential V can be written in a more transparent form:

$$\begin{aligned} V &= \frac{1}{2b_{11}}\Big(2\text{Tr}(\Phi^\dagger\Phi)^2 - (\text{Tr } \Phi^\dagger\Phi)^2\Big) + 2|a|^2\text{Tr } \Phi^\dagger\Phi \\ &+ \frac{\sqrt{2}b_{01}}{b_{11}}f_0\text{Tr } \sigma_1\Phi^\dagger\Phi - \frac{\det b}{b_{11}}f_0^2. \end{aligned} \tag{6.7}$$

Varying V with respect to $\Phi^\dagger$ leads to:

$$\frac{1}{b_{11}}\Big(2\Phi\Phi^\dagger\Phi - (\text{Tr }\Phi^\dagger\Phi)\Phi\Big) + 2|a|^2\Phi + \frac{\sqrt{2}b_{01}}{b_{11}}f_0\Phi\sigma_1 = 0. \tag{6.8}$$

Multiply (6.8) by $\Phi^\dagger$, and let $A \equiv \Phi^\dagger\Phi$:

$$\frac{1}{b_{11}}\Big(2A^2 - (\text{Tr }A)A\Big) + 2|a|^2 A + \frac{\sqrt{2}b_{01}}{b_{11}}f_0 A\sigma_1 = 0. \tag{6.9}$$

There are several solutions to (6.9).

i) If $m_i = 0$, $\Phi^\dagger\Phi = \begin{pmatrix} 0 & 0 \\ 0 & |\widetilde{m}|^2 \end{pmatrix}$, and (6.9) implies $\widetilde{m}_i = 0$. The same conclusion applies if $\widetilde{m}_i = 0$. Hence $\Phi = 0$ and the monopoles do not get a VEV. As in section 5, this phase has higher energy.

ii) If the matrix Φ is invertible, so is A. We can left-multiply by A^{-1} in (6.9), and then take the trace. This implies that $a = 0$. In the monopole region this means $u = \Lambda^2$. At this particular point (6.9) implies:

$$|m|^2 = |\widetilde{m}|^2, \qquad m \cdot \widetilde{m} = -b_{01}f_0/\sqrt{2}. \tag{6.10}$$

In this branch the monopole acquire a VEV, but their auxiliary fields do not, while the auxiliary fields of a get a VEV. When (6.10) is inserted in (6.7) we obtain:

$$V = -b_{00}f_0^2, \tag{6.11}$$

where b_{00} is evaluated at $a = 0$. We will comment on this branch later.

iii) Finally, Φ may not be invertible. Ignore the cases $m_i = \widetilde{m}_i = 0$ covered in i); m_i and $\widetilde{m}_i$ must be proportional:

$$m_i = \lambda^{-1}\widetilde{m}_i, \qquad \lambda \neq 0. \tag{6.12}$$

Thus,

$$\Phi = \begin{pmatrix} m_1 & \lambda m_1 \\ m_2 & \lambda m_2 \end{pmatrix}, \qquad \Phi^\dagger\Phi = |m|^2 \begin{pmatrix} 1 & \lambda \\ \lambda^* & |\lambda|^2 \end{pmatrix}. \tag{6.13}$$

(6.9) now implies that $\lambda = \epsilon = \pm 1$; and

$$\frac{2}{b_{11}}|m|^4 + 2|a|^2|m|^2 + \sqrt{2}\frac{b_{01}\epsilon f_0}{b_{11}}|m|^2 = 0. \tag{6.14}$$

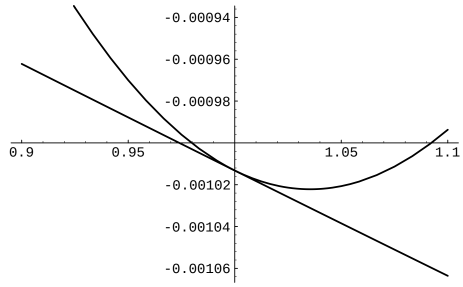

Figure 12: Plot of (5.12) (top) and (6.11) (bottom) near $u = \Lambda^2$ for $f_0 = 0.1\Lambda$.

Note however that we have already encountered (6.14) in the previous section (see (5.6), (5.9)), and we will not repeat the analysis here. Away from $u = \Lambda^2$ we have one ground state. The symmetry of (6.7) is not all of $SU(2)_r \times SU(2)_R$ because the term $\mathrm{Tr}\sigma_1\Phi^\dagger\Phi$ breaks explicitly $SU(2)_R$ to the $U(1)$ subgroup commuting with σ_1. If we had also included a D_0 soft breaking term, $SU(2)_R$ would be completely broken. However with only $f_0 \neq 0$ the global group $SU(2)_r \times U(1)_R$ breaks to $U(1)$ because of the VEV for the monopoles. With $D_0 \neq 0$ we would have $SU(2)_r$ breaking completely while $SU(2)_l$ remains intact. If we restrict the computation to regions where $f_0/\Lambda < 1$ we can use the effective action to obtain the Goldstone boson effective lagrangian up to two derivatives, including the non-perturbative corrections. Once quark masses are included this may be an interesting ground to test many ideas about the computation of the low-energy chiral lagrangian in terms of QCD.

To obtain the standard pattern of chiral symmetry breaking, in which $SU(2)_l \times SU(2)_r \to SU(2)_V$, we presumably need to be in the phase, alluded to in the previous section, where both monopoles and dyons condense.

The phase ii) is analogous to the the two Higgs phases in the $N_f = 2$ case described in [2]. In the purely supersymmetric setting at the classical level, there are together with the Coulomb phase two Higgs phases meeting at the origin of the classical moduli space. In the quantum theory these two phases meet the Coulomb phase at different points. This is precisely what is found in solution ii): there are two analogues of the Higgs phase attached to either $u = \Lambda^2$ or $u = -\Lambda^2$. These two Higgs branches lie on a flat direction of the effective potential, where V takes the constant value given by (6.11). Notice that (5.12), when evaluated in $a = 0$, equals (6.11) (as one can see in fig. 12

for $f_0 = 0.1\Lambda$). Hence there are no discontinuities in the vacuum energy and both phases are smoothly connected, as one should expect in a theory with matter fields in the fundamental representation [33]. As the minimum of (5.12) lies on $\text{Re}u > 1$, $\text{Im}u = 0$ for any non-zero f_0, the phase in iii) is energetically favoured.

7 Conclusions

In this paper we have shown that there is a general procedure to softly break $N = 2$ down to $N = 0$ theories without losing the holomorphic properties of the Seiberg-Witten solution [1, 2]. When the supersymmetry breaking scale is small compared to the dynamical scale Λ, this leads to an analytic determination of the low-energy effective action including non-perturbative effects. The advantage of breaking softly using a dilaton spurion is its universality: it applies to any of the generalizations of [1, 2] in [3], and in particular to theories with massive quarks.

We have exhibited two applications to $N = 2$ theories with gauge group $SU(2)$ and $N_f = 0, 2$, exhibiting some details of their phase structure and patterns of symmetry breaking. We have also shown that the structure of the soft-breaking terms induced can be derived from a spontaneously broken $N = 2$ supergravity theory. One could envisage more complicated ways of achieving similar results. The basic idea is to have an extra $N = 2$ vector multiplet invariant under the Seiberg-Witten monodromy. Thus we could consider embedding the $SU(2)$ moduli space into the $SU(3)$ moduli space, and determine the $SU(3)$ vector multiplet in the low-energy theory with this property; and then declare this multiplet to become the spurion. While feasible, this is not straightforward due to the subtleties in embedding the Seiberg-Witten moduli space inside the $SU(3)$ or higher moduli spaces.

It is intriguing that the clear breakdown of our approach is associated with the coalescence of the two regions in which, respectively, the monopoles and dyons condense. Though monopole condensation is clearly the mechanism of confinement for small values of the soft supersymmetry-breaking parameters, it appears likely that nature of the QCD vacuum in the decoupling limit is more complicated, involving, perhaps, the condensation of both monopoles and dyons.

There are many issues in quantum field theory which we believe can be explored with this method. In particular one can obtain the dependence in quark masses in the low-energy Goldstone boson Lagrangian (for the time being with a non-QCD-like pattern of symmetry breaking), and one can analyze the non-perturbative ambiguities appearing in the Operator Product Expansion associated to renormalon problems. It would also be interesting to study the large-N limit. In $N = 2$ Yang-Mills theories the large-N limit is very rich and by including N in our scaling relations it may be possible to reach reliably more realistic scenarios. We plan to return to these issues in the future.

Some years ago it was almost inconceivable to expect analytic control on fully interacting four-dimensional gauge theories. After Seiberg and Witten's big leap, we hope this work is a small step towards the real world.

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