ENSLAPP–A–579/96
THU–96/05
hep–th/9604003

TODA FIELDS OF SO(3) HYPER–KAHLER METRICS AND FREE FIELD REALIZATIONS

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ABSTRACT

The Eguchi–Hanson, Taub–NUT and Atiyah–Hitchin metrics are the only complete non–singular $SO(3)$–invariant hyper–Kahler metrics in four dimensions. The presence of a rotational $SO(2)$ isometry allows for their unified treatment based on solutions of the 3–dim continual Toda equation. We determine the Toda potential in each case and examine the free field realization of the corresponding solutions, using infinite power series expansions. The Atiyah–Hitchin metric exhibits some unusual features attributed to topological properties of the group of area preserving diffeomorphisms. The construction of a descending series of $SO(2)$-invariant 4–dim regular hyper–Kahler metrics remains an interesting question.

March 1996

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1 Introduction

Hyper–Kahler manifolds in four dimensions have been studied extensively in general relativity over the last twenty years, in connection with the theory of gravitational instantons [1], and their algebraic generalizations through Penrose's non–linear graviton theory [2] and the heavenly equations [3]. They have also been of primary interest in supersymmetric field theories [4], where the presence of an extended $N = 4$ supersymmetry imposes powerful constraints that account for various non–renormalization theorems, and their finiteness at the ultra–violet. More recently it was found that moduli spaces arising in topological field theories and other areas of mathematical physics possess hyper–Kahler structures, a notable example being the moduli space of BPS magnetic monopoles [5]. Similar structures are also encountered in superstring theory, and in many other moduli problems of physical interest in quantum gravity.

The classification of all complete, regular, hyper–Kahler manifolds remains an open question to this date, and even for some known examples the explicit form of the metric has been difficult or impossible to determine so far. The dimensionality and the boundary conditions imposed on the manifold are clearly very important for such an investigation; recall that in four dimensions the only compact (simply connected) space of this type is K_3, while there is an infinite class of non–compact hyper–Kahler manifolds. It is a well–known fact that all asymptotically locally Euclidean (ALE) hyper–Kahler 4–manifolds admit an A–D–E classification [6], where the boundary at infinity is S^3/R_k with R_k being any discrete subgroup of $SU(2)$. Similar results exist for asymptotically locally flat (ALF) hyper–Kahler 4–manifolds, although a complete catalogue is not yet available for them.

In this paper we consider the algebraic description of 4–dim non–compact hyper–Kahler manifolds that possess (at least) one abelian isometry and address the issue of their classification, without really focusing on any particular applications. The translational or the rotational character of the corresponding Killing vector fields is a decisive factor in our investigation. A Killing vector field K_μ satisfies $\nabla_{(\nu}K_{\mu)} = 0$ by definition, while the self–duality of the anti–symmetric part $\nabla_{[\nu}K_{\mu]}$ provides the relevant distinction [7]: K_μ will be called translational if it satisfies the condition

$$\nabla_\nu K_\mu = \pm\frac{1}{2}\sqrt{\det g}\ \epsilon_{\nu\mu}{}^{\kappa\lambda}\nabla_\kappa K_\lambda \ , \tag{1.1}$$

otherwise it will be called rotational Killing vector field. The latter is also called non–tri–holomorphic to distinguish from the former whose action is holomorphic with respect to all three Kahler structures [8]. The $\pm$ sign in (1.1) is chosen according to the self–dual or the anti–self–dual nature of the underlying 4–dim metric $g_{\mu\nu}$.

We consider 4–dim metrics of the general form

$$ds^2 = V(d\tau + \omega_i dx^i)^2 + V^{-1}\gamma_{ij}dx^i dx^j \ , \tag{1.2}$$

where V, ω_i and γ_{ij} are all independent of τ, and hence $K^\mu = (1, 0, 0, 0)$. It will be convenient in these adapted coordinates to use the notation $x = x^1$, $y = x^2$ and $z = x^3$.

For translational Killing vector fields K_μ, we may choose without loss of generality a coordinate system so that

$$\partial_i(V^{-1}) = \pm\frac{1}{2}\epsilon_{ijk}(\partial_j\omega_k - \partial_k\omega_j) \ , \qquad \gamma_{ij} = \delta_{ij} \ . \tag{1.3}$$

Then, for hyper–Kahler metrics (1.2), the self–duality (or anti–self–duality) condition is equivalent to the 3–dim Laplace equation [1]

$$(\partial_x^2 + \partial_y^2 + \partial_z^2)V^{-1} = 0 \tag{1.4}$$

with respect to the flat metric δ_{ij}. Localized solutions of the form

$$V^{-1} = \epsilon + \sum_{i=1}^{n} \frac{m}{\mid \vec{x} - \vec{x}_{0,i} \mid} \tag{1.5}$$

describe non–singular multi–center 4–metrics with moduli parameters m and $\vec{x}_{0,i}$, provided that τ is periodic. The (multi)–Eguchi–Hanson or (multi)–Taub–NUT metrics are obtained for $\epsilon = 0$ or $\epsilon \neq 0$ (in which case ϵ can be normalized to 1) respectively; $\epsilon = 0$ corresponds to the A–series of ALE spaces, while $\epsilon = 1$ to ALF spaces [1].

For rotational Killing vector fields K_μ, we may choose without loss of generality a coordinate system (1.2) so that

$$V^{-1} = \partial_z\Psi \ , \qquad \omega_1 = \mp\partial_y\Psi \ , \qquad \omega_2 = \pm\partial_x\Psi \ , \qquad \omega_3 = 0 \tag{1.6}$$

and the diagonal γ–metric

$$\gamma_{11} = \gamma_{22} = e^{\Psi} \ , \qquad \gamma_{33} = 1 \ , \tag{1.7}$$

in terms of a single scalar function $\Psi(x, y, z)$. Then, the self–duality (or anti–self–duality) condition of the metric is equivalent to the 3–dim continual Toda equation [7]

$$(\partial_x^2 + \partial_y^2)\Psi + \partial_z^2(e^{\Psi}) = 0 \ . \tag{1.8}$$

This equation can be thought as arising in the large N limit of the 2–dim Toda theory based on the group $SU(N)$, where the Dynkin diagram becomes a continuous line parametrized by the third space variable z [9, 10]. The main problem that arises in this context is to determine the solutions of (1.8) that correspond to complete, regular, 4–dim metrics, without worrying at the moment about the specific form of the boundary conditions. We note that although there exists a classification of the 4–dim hyper–Kahler metrics with (at least) one translational Killing symmetry, as was described above, a similar result is not yet available for metrics with rotational isometries.

There are a few results scattered in the literature about self–dual solutions with rotational Killing symmetries. Two of these examples are provided by the Eguchi–Hanson and the Taub–NUT metrics, which admit a much larger group of isometries, namely $SO(3) \times SO(2)$. For the first one, $SO(3)$ acts as a translational isometry (in the sense of (1.1)) and $SO(2)$ as rotational, while the situation is reversed for the other [8]. In either case, there exists a translational Killing symmetry that accounts for their occurence in the known list of ALE or ALF spaces respectively. The only example known to this date that is purely rotational, without exhibiting any translational isometries, is the Atiyah–Hitchin metric on the moduli space M_2^0 of BPS $SU(2)$ monopoles of magnetic charge 2 [5]. These three examples exhaust the list of $SO(3)$–invariant, complete, regular, hyper–Kahler 4–metrics [8], and they will provide the basis for our investigation in the following.

It is worth stressing that not every solution of the continual Toda equation (1.8) has a good space–time interpretation, since the corresponding metrics might be incomplete with singularities. Furthermore, those ones that arise as large N limit solutions of the $SU(N)$ Toda theory do not exhaust the list of all candidate metrics, in contrary to the naive expectation from large N limit considerations. As we will see later, there are solutions of the continual Toda equation that do not admit a free field specialization according to the standard group theoretical formula [9]. We will point out the relevance of such solutions, showing that the Atiyah–Hitchin metric is precisely of this type. Since the Atiyah–Hitchin metric admits only rotational isometries, it may be regarded as the simplest representative from a series of purely rotational hyper–Kahler 4–metrics. The explicit construction of a new series of metrics will certainly require a better and more systematic understanding of the unusual features that arise in the continual limit of the Toda field equations. The Atiyah–Hitchin metric, which has so far been regarded as an isolated example with respect to all known series, could provide the means for exploring further this direction.

The present paper is an attempt to summarize the known results on the subject, including several new, and set up the framework for exploring the construction of a Toda–like series of hyper–Kahler metrics in four dimensions. In section 2, we review the classification of $SO(3)$–invariant hyper–Kahler metrics using their Bianchi IX formulation [11, 5, 8]. Although these metrics have more symmetry than it is required on general grounds, they provide the only non–trivial examples known to this date with (at least) one rotational isometry. In section 3, we formulate the Eguchi–Hanson, Taub–NUT and Atiyah–Hitchin metrics in the adapted rotational coordinates (1.6), (1.7) and determine the Toda potential Ψ in each case. In section 4, we consider the free field realization of solutions of the Toda field equations [12, 9] and identify the free field configurations associated with the Eguchi–Hanson and Taub–NUT metrics. In section 5 we argue that a new class of solutions exist in the continual limit, where the Atiyah–Hitchin metric fits quite naturally. Their occurence is attributed to topological properties of the group of area preserving diffeomorphisms, whose commutation relations descibe $SU(\infty)$ as a continual Lie algebra with distributional structure constants [9] (see also [13]), but they are not shared by $SU(N)$ for any finite N. Finally in section 6, we present our conclusions, and discuss further the framework for possible generalizations.

2 SO(3) hyper-Kahler 4-metrics

Metrics with $SO(3)$ isometry can be written using the Bianchi IX formalism,

$$ds^2 = f^2(t)dt^2 + a^2(t)\sigma_1^2 + b^2(t)\sigma_2^2 + c^2(t)\sigma_3^2 \ , \tag{2.1}$$

where

$$\begin{aligned} \sigma_1 &= \frac{1}{2}(\sin\psi d\theta - \sin\theta\cos\psi d\phi) \ , \\ \sigma_2 &= -\frac{1}{2}(\cos\psi d\theta + \sin\theta\sin\psi d\phi) \ , \\ \sigma_3 &= \frac{1}{2}(d\psi + \cos\theta d\phi) \end{aligned} \tag{2.2}$$

are the invariant 1–forms of $SO(3)$, which are normalized so that $\sigma_i \wedge \sigma_j = \frac{1}{2}\epsilon_{ijk}d\sigma_k$. The Euler angles θ, ψ, ϕ range as usual, while $f(t)$ can always be chosen as

$$f(t) = \frac{1}{2}abc \ , \tag{2.3}$$

by appropriate reparametrization in t.

Self–dual metrics with $SO(3)$ isometry were studied some time ago [11], where it was found that the corresponding second–order differential equations in t can be integrated once to yield the following first–order system:

$$\begin{aligned} 2\,\frac{a'}{a} &= b^2 + c^2 - 2\lambda_1 bc - a^2 \ , \\ 2\,\frac{b'}{b} &= c^2 + a^2 - 2\lambda_2 ca - b^2 \ , \\ 2\,\frac{c'}{c} &= a^2 + b^2 - 2\lambda_3 ab - c^2 \ . \end{aligned} \tag{2.4}$$

The derivation of these equations assumes the choice (2.3) for the coordinate t. Also, depending on the conventions for having a self–dual or anti–self–dual metric, there is an ambiguity in the overall sign of (2.4); the two cases are related to each other by letting $t \to -t$.

We essentially have two distinct cases, depending on the values of the parameters λ_1, λ_2, λ_3. The first case is described by $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and corresponds to the Eguchi–Hanson metric, while for $\lambda_1 = \lambda_2 = \lambda_3 = 1$ we obtain the Taub–NUT metric. These two solutions, apart from an $SO(3)$ isometry, also exhibit an additional $SO(2)$ symmetry that arises from equating two of the metric coefficients, say $a = b$ [11]. A third possibility was found a few years later, also having $\lambda_1 = \lambda_2 = \lambda_3 = 1$, but with unequal metric coefficients a, b, c for generic values of the parameter t [5]. This solution is known as the Atiyah–Hitchin metric, and it has been used for studying the geometry and dynamics of $SU(2)$ monopoles with magnetic charge 2. It was subsequently shown that these three cases provide the only non–trivial hyper–Kahler 4–metrics with $SO(3)$ isometry that are complete and non–singular [8].

Before discussing explicit solutions, we present the expressions for the three Kahler forms for the general $SO(3)$–invariant hyper–Kahler metric (2.1)–(2.4). They are given by [14]

$$F_i = \left\{ \begin{array}{ccc} K_i & \text{if} & \lambda_1 = \lambda_2 = \lambda_3 = 0 \\ C_{ij}K_j & \text{if} & \lambda_1 = \lambda_2 = \lambda_3 = 1 \end{array} \right\} , \tag{2.5}$$

where

$$K_i = 2e_0 \wedge e_i + \epsilon_{ijk} e_j \wedge e_k \ , \tag{2.6}$$

with the tetrads defined as $e_0 = f dt$, $e_1 = a\sigma_1$, $e_2 = b\sigma_2$ and $e_3 = c\sigma_3$. The matrix (C_{ij}) defines the adjoint representation of $SO(3)$

$$C_{ij} = \frac{1}{2} Tr(\sigma_i g \sigma_j g^{-1}) \ , \qquad g = e^{\frac{i}{2}\phi\sigma_3} e^{\frac{i}{2}\theta\sigma_2} e^{\frac{i}{2}\psi\sigma_3} \ . \tag{2.7}$$

Of course, one should make a distinction between the invariant 1–forms of $SO(3)$ and the Pauli matrices used above.

(i) Eguchi–Hanson metric : The solution of the differential equations (2.4) with $a = b$ and $\lambda_1 = \lambda_2 = \lambda_3 = 0$ can be easily found. Letting $t \to -t$ for convenience, which amounts to flipping the sign in (2.4), we have

$$a^2 = b^2 = m^2 \coth(m^2 t) \ , \qquad c^2 = \frac{2m^2}{\sinh(2m^2 t)} \ , \tag{2.8}$$

where m is the moduli parameter of the Eguchi–Hanson metric. In this case, the metric coefficient $f(t)$ in (2.1) is given by (2.3). The more standard description of the Eguchi–Hanson metric follows by introducing r,

$$r^2 = m^2 \coth(m^2 t) \ , \tag{2.9}$$

which yields the line element

$$ds^2 = \frac{dr^2}{1 - \left(\frac{m}{r}\right)^4} + r^2 \left(\sigma_1^2 + \sigma_2^2 + \left(1 - \left(\frac{m}{r}\right)^4 \right) \sigma_3^2 \right) \ . \tag{2.10}$$

It is well–known that the $SO(3)$ isometry of this metric acts in a translational (or tri–holomorphic) way, while the Killing vector field $\partial/\partial\psi$, which arises from the equation $a = b$, acts as a rotational isometry [8].

(ii) Taub–NUT metric : The solution of the differential equations (2.4) with $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $a = b$ is given by

$$a^2 = b^2 = \frac{1 + 4m^2 t}{4m^2 t^2} \ , \qquad c^2 = \frac{4m^2}{1 + 4m^2 t} \ , \tag{2.11}$$

describing the Taub–NUT metric with a moduli parameter m. A more conventional description is obtained by introducing r,

$$r = m + \frac{1}{2mt} \ , \tag{2.12}$$

which yields the line element

$$ds^2 = \frac{1}{4}\frac{r+m}{r-m}dr^2 + (r^2 - m^2)(\sigma_1^2 + \sigma_2^2) + 4m^2\frac{r-m}{r+m}\sigma_3^2 \ . \tag{2.13}$$

An alternative formulation is also given in terms of $\tilde{r} = r - m$. This metric also admits an additional isometry generated by the Killing vector field $\partial/\partial\psi$, which now turns out to be translational. On the contrary, the $SO(3)$ isometry of the Taub–NUT metric is rotational with respect to all of its generators.

(iii) Atiyah–Hitchin metric : The Taub–NUT metric is the only globally defined solution of the system (2.4) with $\lambda_1 = \lambda_2 = \lambda_3 = 1$, provided that a, b, c are all positive. Global considerations in this case force the existence of an additional $SO(2)$ isometry. The Atiyah–Hitchin metric on the other hand has $c < 0$, and moreover a, b, c are all different from each other for generic values of t. Equivalently one may consider all a, b and c positive and choose $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 1$.

The metric components are given in terms of the complete elliptic integrals of the first and second kind

$$K(k) = \int_0^{\pi/2} \frac{d\gamma}{\sqrt{1 - k^2 \sin^2\gamma}} \ , \quad E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\gamma} \ d\gamma \ . \tag{2.14}$$

In particular, using the parametrization

$$ds^2 = \frac{1}{4}a^2b^2c^2 \left(\frac{dk}{kk'^2K^2}\right)^2 + a^2(k)\sigma_1^2 + b^2(k)\sigma_2^2 + c^2(k)\sigma_3^2 \ , \tag{2.15}$$

the solution obtained by Atiyah and Hitchin is given as function of k [5, 15],

$$\begin{aligned} ab &= -K(k)(E(k) - K(k)) \ , \\ bc &= -K(k)(E(k) - k'^2K(k)) \ , \\ ac &= -K(k)E(k) \ . \end{aligned} \tag{2.16}$$

Here, $0 < k < 1$ and $k'^2 = 1 - k^2$.

In the limit $k \to 1$, the metric becomes exponentially close to Taub–NUT. This can be easily seen if we define

$$k' = \sqrt{1 - k^2} \simeq 4 \exp\left(\frac{1}{\gamma}\right) \tag{2.17}$$

for $\gamma \to 0^-$. It follows from standard expansions of the elliptic integrals [16] that in this vicinity

$$a^2 \simeq b^2 \simeq \frac{1+\gamma}{\gamma^2} \ , \quad c^2 \simeq \frac{1}{1+\gamma} \ , \tag{2.18}$$

which is the Taub–NUT configuration (2.11) with $t = \gamma$ and $m^2 = 1/4$; in fact, as $k \to 1$, one obtains the Taub–NUT metric with a negative mass parameter $m = -1/2$.

The coordinate t in the parametrization (2.1) and (2.3) is given by the change of variables

$$t = -\frac{2K(k')}{\pi K(k)} \ , \tag{2.19}$$

up to an additive constant. Yet another convenient parametrization of the metric components is given in terms of the coordinate

$$r = 2K(k) \ , \qquad \pi < r < \infty \ , \tag{2.20}$$

in which case the Taub–NUT limit is attained asymptotically as $r \to \infty$. In this parametrization it becomes more clear that asymptotically the Taub-NUT parameter is negative [5].

The generators of the $SO(3)$ isometry act as rotational vector fields in this case [8] and, therefore, this metric provides the only known example of a purely rotational hyper–Kahler 4–metric. The Atiyah–Hitchin metric has many similarities with the Taub–NUT, but the absence of an additional (translational) isometry in the former has profound implications in monopole physics, where slowly moving monopoles can be converted into dyons [5].

We also note for completeness that there have been attempts in the past to construct exact self–dual metrics of Bianchi IX type with all directions unequal, which generalize the Eguchi–Hanson metric. These solutions have $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and although asymptotically Euclidean they contain singularities [11, 17]. A particularly simple parametrization of them is given by

$$ds^2 = \frac{1}{4}P^{-1/2}dx^2 + P^{1/2}\left(\frac{\sigma_1^2}{x-x_1} + \frac{\sigma_2^2}{x-x_2} + \frac{\sigma_3^2}{x-x_3}\right) \ , \tag{2.21}$$

where

$$P(x) = (x-x_1)(x-x_2)(x-x_3) \tag{2.22}$$

and x_1, x_2, x_3 are the relevant moduli parameters. We also mention that the $SO(3)$ isometry acts as a translational symmetry in this case; the coordinate transformation that brings (2.21) into the form (1.2)–(1.4) is also known [17]. In this regard, the Atiyah–Hitchin metric is the unique non–singular solution with $SO(3)$ symmetry having unequal coefficients a, b and c.

3 Toda potential of the metrics

There is an alternative algebraic description of the translational or the rotational character of a Killing vector field based on the notion of the nut potential. Using the adapted coordinate system (1.2) for a 4–dim metric with a Killing vector field $\partial/\partial\tau$, we define the nut potential b_{nut} associated with this isometry as follows [1]:

$$\partial_i b_{nut} = \frac{1}{2} V^2 \sqrt{\det\gamma}\ \epsilon_i{}^{jk} (\partial_j \omega_k - \partial_k \omega_j) \ . \tag{3.1}$$

The compatibility condition for the system (3.1) is provided by the vacuum Einstein equations for the metric (1.2), and hence b_{nut} exists only on–shell.

Consider now the case of self–dual or anti–self–dual metrics and define respectively the field

$$S_{\pm} = b_{nut} \pm V \ . \tag{3.2}$$

We also define the characteristic quadratic quantity

$$\Delta S_{\pm} = \gamma^{ij} (\partial_i S_{\pm})(\partial_j S_{\pm}) \ , \tag{3.3}$$

which is clearly ≥ 0. It is a well–known theorem [7] that for translational Killing vector fields (1.1) $\Delta S_{\pm} = 0$, while for rotational $\Delta S_{\pm} > 0$. In the former case this means that $S_{\pm} = 0$, up to an additive constant, while in the latter $S_{\pm}$ is a coordinate dependent configuration. In fact, for rotational isometries, the passage to the special coordinate system (1.6)–(1.8) can be achieved by considering $S_{\pm}$ as the z–coordinate, since in this case $\Delta S_{\pm} = 1$. In general we may choose

$$z = \frac{S_{\pm}}{\sqrt{\Delta S_{\pm}}} \ , \tag{3.4}$$

and then determine the transformation of the remaining coordinates that brings the metric into the desired Toda form. This is precisely the prescription that will be followed to determine the Toda potential Ψ of the three $SO(3)$–invariant hyper–Kahler metrics in question. The computation of the nut potential is clearly very important for performing the required coordinate transformation in each case.

We choose our conventions in the following so that the relevant variable is the field S_+ instead of S_-.

The Toda frame formulation of the $SO(3)$ hyper–Kahler metrics is also equivalent to the problem of determining the explicit form of the Kahler potential and the corresponding Kahler coordinates, using the method of Boyer and Finley [7]. Moreover, the three complex structures form an $SO(2)$–doublet and a singlet with respect to every rotational isometry. The three Kahler forms can be written explicitly in the Toda frame formulation [18]; we have

$$\begin{pmatrix} F_1 \\ \\ F_2 \end{pmatrix} = e^{\frac{1}{2}\Psi} \begin{pmatrix} \cos\frac{\tau}{2} & \sin\frac{\tau}{2} \\ \\ \sin\frac{\tau}{2} & -\cos\frac{\tau}{2} \end{pmatrix} \begin{pmatrix} f_1 \\ \\ f_2 \end{pmatrix} \tag{3.5}$$

for the doublet, where

$$\begin{aligned} f_1 &= (d\tau + \omega_2 dy) \wedge dx - V^{-1} dz \wedge dy \ , \\ f_2 &= (d\tau + \omega_1 dx) \wedge dy + V^{-1} dz \wedge dx \ , \end{aligned} \tag{3.6}$$

while for the singlet we have

$$F_3 = (d\tau + \omega_1 dx + \omega_2 dy) \wedge dz + V^{-1} e^{\Psi} dx \wedge dy \ . \tag{3.7}$$

We also note that all generators of the $SO(3)$ isometry act either in a translational or in a rotational fashion, in which case the three complex structures form either singlets or an $SO(3)$–triplet [8].

(i) Eguchi–Hanson metric : Starting from the Eguchi–Hanson metric in the form (2.10), we note that $\partial/\partial\psi$ is a Killing vector field thanks to the equality $a^2 = b^2 = r^2$. It is relatively easy to calculate the nut potential associated with this isometry, using the defining relations (3.1). In this parametrization we find

$$V = \frac{1}{4} r^2 \left(1 - \frac{m^4}{r^4}\right) \ , \qquad b_{nut} = \frac{1}{4} r^2 \left(1 + \frac{m^4}{r^4}\right) \ , \tag{3.8}$$

and hence $S_+ = b_{nut} + V = r^2/2$. The result is consistent with the fact that $\partial/\partial\psi$ is rotational. We also find that $\Delta S_+ = 4$ and therefore, according to the previous discussion, the first step in transforming the Eguchi–Hanson metric into the Toda frame consists of the choice $z = r^2/4$. The remaining change of coordinates, as well as the identification of the Toda potential, is easily done by rescaling the metric (2.10) with an overall factor of 2. The complete transformation can be summarized as follows:

$$\begin{aligned} x &= 2\sqrt{2} \cos\phi \tan\frac{\theta}{2} \ , \qquad y = 2\sqrt{2} \sin\phi \tan\frac{\theta}{2} \ , \\ z &= \frac{1}{4} r^2 \ , \qquad \tau = 2(\psi + \phi) \ . \end{aligned} \tag{3.9}$$

The ϕ–shift in the definition of τ is used in order to set $\omega_3 = 0$ for convenience. Then, the metric assumes the form (1.2), with (1.6) and (1.7) satisfied, where

$$e^{\Psi} = \frac{z^2 - \alpha^2}{2\left(1 + \frac{1}{8}(x^2 + y^2)\right)^2} \ , \qquad z^2 \geq \alpha^2 \ , \tag{3.10}$$

with $4\alpha = m^2$. The potential Ψ we have determined clearly obeys the continual Toda equation (1.8) (see also [7, 10]).

The Killing vector field $\partial/\partial\phi$ is translational, as it can be verified directly by computing the corresponding nut potential, which yields $S_+ = 0$. The explicit transformation to the translational frame (1.3)–(1.5) is known in this case [19]; it simply reads,

$$\vec{x} = \frac{1}{8}\left(\sqrt{r^4 - m^4} \sin\theta \cos\psi, \ \sqrt{r^4 - m^4} \sin\theta \sin\psi, \ r^2 \cos\theta\right), \qquad \tau = 2\phi, \tag{3.11}$$

with

$$\begin{aligned} V^{-1} &= \frac{1}{r_+} + \frac{1}{r_-} \ ; \qquad r_\pm^2 = x^2 + y^2 + z_\pm^2 \ , \qquad z_\pm = z \pm \frac{m^2}{8} \ , \\ \vec{\omega} &= \frac{1}{x^2 + y^2} \left(\frac{z_+}{r_+} + \frac{z_-}{r_-}\right) (-y, \ x, \ 0) \ . \end{aligned} \tag{3.12}$$

(ii) Taub–NUT metric : Here, the Killing vector field $\partial/\partial\psi$ is translational, and the explicit transformation to the coordinate system (1.3)–(1.5) is given by

$$\vec{x} = (r-m)(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)\ , \qquad \tau = \frac{m}{2}\psi\ , \tag{3.13}$$

with

$$V^{-1} = \frac{1}{4}\left(1 + \frac{2m}{|\,\vec{x}\,|}\right), \quad \vec{\omega} = \frac{mz}{2\,|\,\vec{x}\,|\,(x^2+y^2)}(-y,\ x,\ 0)\ . \tag{3.14}$$

The isometry generated by $\partial/\partial\phi$, however, is rotational and it will be used to determine the Toda potential of the Taub–NUT metric. Starting from the Bianchi IX formulation (2.1)–(2.4), we calculate first the nut potential of the corresponding isometry. The explicit form (2.11) of the coefficients $a = b$ and c is not required for the computation, as they may be substituted only in the final expression. We obtain,

$$b_{nut} = \frac{1}{4}c\left(2a - c - (a-c)\sin^2\theta\right), \quad V = \frac{1}{4}\left(c^2 + (a^2-c^2)\sin^2\theta\right)\ . \tag{3.15}$$

We also consider $S_+ = b_{nut} + V$ and find $\Delta S_+ = 4$. Hence, we define the z–coordinate as

$$z = \frac{1}{8}a\left(2c + (a-c)\sin^2\theta\right) \;\equiv\; \frac{1}{4t}\left(1 + \frac{1}{8m^2t}\sin^2\theta\right)\ . \tag{3.16}$$

Since the metric γ_{ij} is diagonal, the x and y coordinates can be easily found to be

$$x = \psi\ , \quad y = \frac{c-a}{c}\cos\theta + \log\left(\tan\frac{\theta}{2}\right) \;\equiv\; -\frac{1}{4m^2t}\cos\theta + \log\left(\tan\frac{\theta}{2}\right)\ , \tag{3.17}$$

and moreover,

$$\tau = 2\phi\ . \tag{3.18}$$

Like the Eguchi–Hanson metric, it is also convenient here to rescale the Bianchi IX metric with a factor of 2, and in the process we find the Toda potential (see also [20]),

$$e^{\Psi} = \frac{1}{16}a^2c^2\sin^2\theta \;\equiv\; \frac{1}{16t^2}\sin^2\theta\ . \tag{3.19}$$

We note that the Toda potential of the Taub–NUT metric is independent of x, satisfying a reduced 2–dim continual Toda equation. This is merely a reflection of the additional $SO(2)$ isometry that arises from the identification $a^2 = b^2$ in this case. As we will see later, the Atiyah–Hitchin metric, which generalizes Taub–NUT with $a^2 \neq b^2$, does not share this feature. Also, one of the main differences with the Eguchi–Hanson result is that the Ψ above can only be implicitly written as a function of y and z, because the coordinate transformation that is involved, $(t,\theta) \to (y,z)$, can not be inverted in closed form. For this reason we expect that the Toda potential of the Atiyah–Hitchin metric will be an even more complicated implicit function of the coordinates x, y and z.

It is also worth clarifying the meaning of the Toda frame in the case that the moduli parameter of the Eguchi–Hanson or the Taub–NUT metric is chosen so that the flat space limit is attained. Taking $m \to 0$ for the Eguchi–Hanson metric (2.10) or $m \to \infty$ for the Taub–NUT metric (2.11) we obtain the flat space metric

$$ds^2 = dR^2 + R^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \ , \tag{3.20}$$

with $R = r$ and $R^2 = 1/t$ respectively. In these limiting cases both vector fields $\partial/\partial\psi$ and $\partial/\partial\phi$ generate translational isometries: although S_+ remains non–trivial, $S_- \to 0$ in both cases, and since flat space is self–dual as well as anti–self–dual the rotational character of $\partial/\partial\psi$ or $\partial/\partial\phi$ disappears. Then, the transformation to the Toda frame can only be regarded as a change of coordinates in flat space, with no reference to any rotational isometries. Note that in this limit the change of variables (3.16) and (3.17) for Taub–NUT can be inverted so that

$$z = \frac{1}{4t} \ , \qquad \sin\theta = \frac{1}{\cosh y} \ , \qquad e^{\Psi} = \frac{z^2}{\cosh^2 y} \ . \tag{3.21}$$

This example, together with the $\alpha \to 0$ limit of (3.10) demonstrate explicitly that not every solution of the continual Toda equation corresponds to rotational isometries. Furthermore, algebraically inequivalent solutions could yield 4–dim hyper–Kahler metrics that are simply related by diffeomorphisms.

(iii) Atiyah–Hitchin metric : The starting point here is also the Bianchi IX formulation (2.1)–(2.4) of the metric. The isometry generated by the Killing vector field $\partial/\partial\phi$ is again rotational and the corresponding nut potential can be calculated without knowing the explicit form of the coefficients a, b and c, but only their differential equations in t. These coefficients are implicit functions of t through the defining relation (2.19). We have determined that

$$\begin{aligned} b_{nut} &= \frac{1}{4}\Big(c(a+b-c) - (a-c)(a+c-b)\sin^2\theta + (a-b)(a+b-c)\sin^2\theta\sin^2\psi\Big) \ , \\ V &= \frac{1}{4}\Big(c^2 + (a^2-c^2)\sin^2\theta - (a^2-b^2)\sin^2\theta\sin^2\psi\Big) \ . \end{aligned} \tag{3.22}$$

The field $S_+ = b_{nut} + V$ satisfies $\Delta S_+ = 4$ as before, and hence one of the coordinates in the desired Toda frame is

$$\begin{aligned} z &= \frac{1}{8}\Big(c(a+b) + b(a-c)\sin^2\theta - c(a-b)\sin^2\theta\sin^2\psi\Big) \\ &\equiv \frac{1}{8}K^2(k)\left(k^2\sin^2\theta + k'^2(1+\sin^2\theta\sin^2\psi) - 2\frac{E(k)}{K(k)}\right) . \end{aligned} \tag{3.23}$$

Notice that if $a = b$ the above expressions reduce to (3.15) and (3.16) respectively for the Taub–NUT metric.

The appropriate transformation for determining x, y and τ (with the choice $\omega_3 = 0$) is rather difficult to perform, but fortunately Olivier has already provided the result [15]. It is convenient to rescale the metric by a factor of 2 and introduce the quantities

$$\mu = \frac{2ab - c(a+b)}{c(b-a)} \equiv \frac{1+k^2}{1-k^2} \ , \qquad \nu = 2\left(\log\left(\tan\frac{\theta}{2}\right) + i\psi\right) \ . \tag{3.24}$$

Then, using our conventions, and substituting the expressions (2.16) for the metric coefficients, one finds that the coordinates y and x coincide with the real and the imaginary parts of the complex variable

$$y + ix = K(k)\sqrt{1 + k'^2 \sinh^2\frac{\nu}{2}} \left(\cos\theta + \frac{\tanh\frac{\nu}{2}}{K(k)} \int_0^{\pi/2} d\gamma \frac{\sqrt{1 - k^2 \sin^2\gamma}}{1 - k^2 \tanh^2\frac{\nu}{2} \sin^2\gamma} \right) , \qquad (3.25)$$

and also

$$\tau = 2 \left(\phi + \arg\left(1 + k'^2 \sinh^2\frac{\nu}{2} \right) \right) . \qquad (3.26)$$

Furthermore, the Toda potential in this case turns out to be

$$e^\Psi = \frac{1}{32} c(b - a) \sin^2\theta \mid \mu + \cosh\nu \mid \equiv \frac{1}{16} K^2(k) \sin^2\theta \mid 1 + k'^2 \sinh^2\frac{\nu}{2} \mid . \qquad (3.27)$$

Notice that as $a \to b$ the Toda potential of the metric becomes

$$e^\Psi \to \frac{1}{16} a(a - c) \sin^2\theta , \qquad (3.28)$$

which at first sight seems to disagree with the expression (3.19) for the Taub–NUT metric. There is no contradiction, however, because it can be readily verified using the Taub–NUT coefficients (2.11) that $a^2c^2 = a(a - c)$ provided that the moduli parameter $m^2 = 1/4$. This is precisely the case of interest in the limit, where the Atiyah–Hitchin metric can be approximated with the Taub–NUT metric. Similarly, expanding $K(k)$ as $k \to 1$, and using the parametrization (2.17), we find that Olivier's coordinate transformation yields the result we have already described for the Taub–NUT metric. In the physical context of monopole dynamics, the limit $k \to 1$ corresponds to configurations where the two monopoles are very far away from each other.

Another interesting limit is $k \to 0$, for which we find

$$e^\Psi \to \frac{\pi^2}{64} (1 - \sin^2\theta \sin^2\psi) \equiv -\frac{1}{2} z , \qquad (3.29)$$

and

$$y \to 0 , \quad x \to \frac{\pi}{2} \sin\theta \sin\psi . \qquad (3.30)$$

In this limit, the 3–dim orbit of $SO(3)$ collapses to a 2–sphere, and there is a coordinate singularity of bolt type [5]. In the physical context of monopole dynamics, this limit corresponds to configurations where the two monopoles coincide.

4 Free field realizations

We are primarily interested in understanding the algebraic differences of the Atiyah–Hitchin metric from the other two metrics, in view of possible generalizations that this configuration may have in purely rotational hyper–Kahler geometry. The results we have included so far are not very illuminating in their present form, although they have been described directly in the Toda frame that unifies all metrics in question. Next, we will consider the free field realization of the general solution of the continual Toda equation, as it is derived by taking the large N limit of the 2–dim $SU(N)$ Toda systems [9]. The Toda potential of the three metrics will be considered from this particular point of view, which turns out to be most appropriate for understanding the qualitative differences we are looking for the Atiyah–Hitchin metric.

Consider the continual Toda equation (1.8) and supply the non–linearity with a parameter λ^2,

$$(\partial_x^2 + \partial_y^2)\Psi + \lambda^2 \partial_z^2 (e^{\Psi}) = 0 \ . \tag{4.1}$$

A solution admits a free field realization if Ψ has a power series expansion,

$$\Psi = \Psi_0 + \lambda^2 \Psi_1 + \lambda^4 \Psi_2 + \cdots \ , \tag{4.2}$$

where Ψ_0 is z–dependent and satisfies the 2–dim free field equation $(\partial_x^2 + \partial_y^2)\Psi_0 = 0$. Then, Ψ_1, Ψ_2, ... can be determined recursively order by order in λ^2 from the field equation (4.1). In this regard $\lambda \neq 0$ is a book–keeping parameter, since it can be absorbed by rescaling x and y, or equivalently shifting Ψ by $2 \log \lambda$; it organizes the free field expansion of a given solution and it should be set equal to 1 at the end. The advantage of this realization is that Ψ_0 has a much simpler form compared to exact solutions of the full non-linear problem. For this reason we will search for the free field configurations that correspond to the Eguchi–Hanson, Taub–NUT and Atiyah–Hitchin metrics, in order to examine more transparently the differences of the latter from the former two solutions. Although the free field realization of a given solution may not be uniquely determined, as it can be seen in Liouville theory whose general solution is invariant under fractional $SL(2)$ transformations of the building holomorphic and anti–holomorphic field variables, the existence of free fields simplifies considerably the description of the solutions of the underlying non–linear problem.

For $SU(N)$ Toda theories all the solutions admit a free field realization, which can be systematically described using highest weight representations. It is convenient to introduce the system of Weyl generators H_i, $X_i^{\pm}$ $(1 \leq i \leq N-1)$,

$$[H_i, X_j^{\pm}] = \pm K_{ij} X_j^{\pm} \ , \quad [X_i^+, X_j^-] = \delta_{ij} H_j \ , \quad [H_i, H_j] = 0 \ , \tag{4.3}$$

where K is the Cartan matrix of $SU(N)$, and define matrices $M_{\pm}(q_{\pm})$ as follows,

$$\partial_{\pm} M_{\pm}(q_{\pm}) = M_{\pm}(q_{\pm}) \left(\lambda \sum_{i=1}^{N-1} e^{\psi_i^{\pm}(q_{\pm})} X_i^{\pm} \right); \qquad q_{\pm} = \frac{1}{2}(y \pm ix) \ . \tag{4.4}$$

The fields $\psi_i^{\pm}(q_{\pm})$ are $N-1$ arbitrary holomorphic and anti–holomorphic functions so that $\psi_i^+(q_+) + \psi_i^-(q_-)$ satisfy the 2–dim free wave equation. We also consider highest weight states $| \, j >$,

$$X_i^+ \, | \, j >= 0 \ , \qquad < j \, | \, X_i^- = 0 \ , \quad H_i \, | \, j >= \delta_{ij} \, | \, j > \ , \tag{4.5}$$

and formulate the general solution of the 2–dim Toda field equation

$$\partial_+\partial_-\psi_i(q_+,q_-) = \lambda^2 \sum_{j=1}^{N-1} K_{ij} e^{\psi_j(q_+,q_-)} \ . \tag{4.6}$$

It is given by the well–known expression [12],

$$\psi_i(q_+,q_-) = \psi_i^+(q_+) + \psi_i^-(q_-) - \sum_{j=1}^{N-1} K_{ij} \log < j \mid M_+^{-1} M_- \mid j > \ , \tag{4.7}$$

and hence it clearly admits a power series expansion in λ^2 around the free field configurations $\psi_i^+(q_+) + \psi_i^-(q_-)$ that characterize every exact solution.

The matrices $M_\pm$ that follow by integration of (4.4) are path–ordered exponentials of $SU(N)$ Lie algebra elements and therefore they can be regarded as one–parameter family subgroups of the corresponding gauge group with values in $SU(N)$; the free fields in their exponentiated form play the role of the canonical coordinates around the identity element of the group. Furthermore, for N finite, $\exp(-\psi_i)$ admits a power series expansion that terminates at a finite order in λ, as it can be easily seen by considering the terms

$$D_j^{\{i_1,\cdots,i_n;i'_1,\cdots,i'_n\}} =< j \mid X_{i_1}^+ X_{i_2}^+ \cdots X_{i_n}^+ X_{i'_n}^- \cdots X_{i'_2}^- X_{i'_1}^- \mid j > \tag{4.8}$$

in the power series expansion of the path–ordered exponentials in (4.7); only terms with $n < N$ are non–zero and contribute. The algebra of Lie–Backlund transformations is finite dimensional in this case, and it is not possible to have additional terms contributing to the general form of the solutions. We also note that the solutions of the Toda field equations have the following interpretation in 2–dim Minkowski space, which is obtained by analytic continuation $ix \to x$: the free field configurations $\psi_i^\pm(q_\pm)$ are the boundary values of the Toda field $\psi_i(q_+,q_-)$ on the light–cone, and if the reference light–cone point is taken at 0 the corresponding free fields have to be normalized by substructing the constant value $\psi_i(0)/2$. In 2–dim Euclidean space, which is relevant for our geometrical problem, the solution (4.7) of the boundary value problem for the Toda field equation is determined via the solution of the Laplace equation that corresponds to the free theory, namely the asymptotic values of the non–interacting free fields $\psi_i^\pm(q_\pm)$.

For large N we obtain the continual Toda equation as a limit, where $\Psi(q_+,q_-,z)$ is a master field for all $\psi_i(q_+,q_-)$, and $K_{ij} \to -\partial_z^2\delta(z-z')$, $\delta_{ij} \to \delta(z-z')$. The introduction of the new coordinate z turns summations over i and j into integrations that can be easily performed using the δ–functions. Taking the continual version of the algebraic data (4.3)–(4.5), as it was done in [9, 10], we obtain solutions of the continual Toda equation of the form

$$\Psi(q_+,q_-,z) = \Psi_0(q_+,q_-,z) + \partial_z^2 \left(\log < z \mid M_+^{-1} M_- \mid z >\right) \ , \tag{4.9}$$

where $\Psi_0(q_+,q_-,z) = \Psi^+(q_+,z) + \Psi^-(q_-,z)$ is the decomposition of the free field configuration into holomorphic and anti–holomorphic parts, and

$$M_\pm(q_\pm,z) = \mathrm{P}\exp\left(\lambda \int^{q_\pm} dq'_\pm \int^z dz' e^{\Psi^\pm(q'_\pm,z')} X^\pm(z')\right) . \tag{4.10}$$

In this case all the terms (4.8) contribute for generic free field configurations, and the power series expansion of $< z \mid M_+^{-1} M_- \mid z >$ is infinite and could be divergent. Moreover, it is not clear at this point whether every solution of the continual Toda field equation can be obtained in this fashion or whether there exist solutions that do not admit a realization in terms of 2–dim free fields according to (4.9). We will see later that the Atiyah–Hitchin metric precisely corresponds to such unusual solutions.

We analyse first the free field realization of the Toda potential for the Eguchi–Hanson and Taub–NUT metrics.

(i) Eguchi–Hanson metric : Introducing $q_\pm = (y \pm ix)/2$ the asymptotic expansion of the Toda potential (3.10) is easily obtained,

$$e^\Psi = \frac{2(z^2 - \alpha^2)}{q_+^2 q_-^2} \left(1 - \frac{4}{q_+ q_-} + \frac{12}{q_+^2 q_-^2} - \frac{32}{q_+^3 q_-^3} + \cdots \right) . \tag{4.11}$$

Therefore, we may identify the free field configuration $\Psi^+(q_+, z) + \Psi^-(q_-, z)$ of the Eguchi–Hanson metric choosing

$$\Psi^\pm(q_\pm, z) = \log \frac{\sqrt{2(z^2 - \alpha^2)}}{q_\pm^2} . \tag{4.12}$$

It is fairly straightforward to verify that the expansion (4.11) coincides with (4.9) for this particular free field configuration, as it should. For this we compute first

$$\begin{aligned}
D_z^{\{z_1;z_1'\}} &= \delta(z, z_1)\delta(z_1, z_1') , \\
D_z^{\{z_1,z_2;z_1',z_2'\}} &= \delta(z, z_1)\delta(z_1, z_1')\delta(z_2, z_2') \left(2\delta(z_1, z_2) + \partial_{z_1}^2 \delta(z_1, z_2)\right) , \\
D_z^{\{z_1,z_2,z_3;z_1',z_2',z_3'\}} &= \delta(z, z_1)\delta(z, z_2)\delta(z_1, z_3')\delta(z_2, z_1')\delta(z_3, z_2') \left(2\delta(z_1, z_3) + \partial_{z_1}^2 \delta(z_1, z_3)\right) + \\
&\delta(z, z_1)\delta(z_1, z_1')\delta(z_2, z_3')\delta(z_3, z_2') \left(\delta(z_1, z_2) + \partial_{z_1}^2 \delta(z_1, z_2)\right) \left(2\delta(z_1, z_3) + \partial_{z_1}^2 \delta(z_1, z_3)\right) + \\
&\delta(z, z_1)\delta(z_1, z_1')\delta(z_2, z_2')\delta(z_3, z_3') \left(2\delta(z_1, z_2) + \partial_{z_1}^2 \delta(z_1, z_2)\right) \cdot \\
&\left(2\delta(z_1, z_3) + \partial_{z_1}^2 \delta(z_1, z_3) + \partial_{z_2}^2 \delta(z_2, z_3)\right) ,
\end{aligned} \tag{4.13}$$

etc. Then, using (4.12) we find the following result for the expansion,

$$\begin{aligned}
< z \mid M_+^{-1} M_- \mid z > &= 1 - (z^2 - \alpha^2) \frac{2}{q_+ q_-} + \frac{1}{2}(z^2 - \alpha^2)(z^2 - \alpha^2 + 1) \left(\frac{2}{q_+ q_-}\right)^2 - \\
&\frac{1}{6}(z^2 - \alpha^2)(z^2 - \alpha^2 + 1)(z^2 - \alpha^2 + 2) \left(\frac{2}{q_+ q_-}\right)^3 + \cdots \equiv \frac{1}{\left(1 + \frac{2}{q_+ q_-}\right)^{z^2 - \alpha^2}} ,
\end{aligned} \tag{4.14}$$

which indeed yields (4.11) when it is substituted into the exponential form of the master equation (4.9).

The Toda potential of the Eguchi–Hanson metric is special in that it belongs to the subclass of solutions of the continual Toda equation given by an ansatz with factorized

z–dependence

$$e^{\Psi(q_+,q_-,z)} = (z^2 + \beta z + \gamma) e^{\varphi_L(q_+,q_-)} \ , \tag{4.15}$$

provided that φ_L solves the Liouville equation

$$\partial_+ \partial_- \varphi_L + 2e^{\varphi_L} = 0 \ . \tag{4.16}$$

According to (4.7), and taking into account the sign difference that appears in the coupling constant between (4.6) (with $N = 2$) and (4.16), we write down the general solution of the latter in free field realization,

$$e^{\varphi_L(q_+,q_-)} = \frac{e^{\varphi_L^+(q_+) + \varphi_L^-(q_-)}}{\left(1 + \int^{q_+} dq'_+ e^{\varphi_L^+(q'_+)} \int^{q_-} dq'_- e^{\varphi_L^-(q'_-)}\right)^2} , \tag{4.17}$$

which is indeed described by two arbitrary functions of q_+ and q_-, namely $\int^{q_+} dq'_+ e^{\varphi_L^+(q'_+)}$ and $\int^{q_-} dq'_- e^{\varphi_L^-(q'_-)}$ respectively. We find from this point of view that

$$\varphi_L^\pm(q_\pm) = -2 \log q_\pm + \log \sqrt{2} \tag{4.18}$$

provides the relevant solution that yields (4.12) using the ansatz (4.15) with $\beta = 0$ and $\gamma = -\alpha^2$. This metric resides entirely in the $SU(2)$ subalgebra of $SU(\infty)$, and hence is the simplest one to consider.

(ii) Taub–NUT metric : The Toda potential of this metric satisfies the dimensionally reduced equation

$$\partial_y^2 \Psi(y,z) + \partial_z^2 (e^{\Psi(y,z)}) = 0 \ , \tag{4.19}$$

and therefore the asymptotic behaviour of the Toda field should be linear in y,

$$\Psi(y,z) \to \mp a(z) y + b(z) \ ; \qquad y \to \pm\infty \ , \tag{4.20}$$

with $a(z) > 0$. We will determine the coefficients of the corresponding free field configuration by analysing the solution (3.19) in the limit $y \to -\infty$, but a similar analysis can also be performed around $+\infty$. As soon as $a(z)$ and $b(z)$ have been extracted, the exact solution will be described as a power series expansion in the usual way. The analysis is a little bit more involved than before, because in this case the Toda potential is only implicitly described as a function of y and z. In the process of analyzing the asymptotic form of $\Psi(y,z)$ a power series expansion of θ and t will also emerge in terms of the y and z coordinates. Throughout this process we keep t arbitrary, but never equal to zero although it can be asymptotically close to it with exponentially small corrections.

A close inspection of the equations (3.17) and (3.19) shows that the limit $y \to -\infty$ is achieved as $\theta \to 0$, in which case $\exp \Psi(y,z) \to 0$, as it should in the asymptotic free field limit. In this limit we may approximate $\theta \approx 2e^{y+z/m^2}$ and $z \approx 1/4t$ to lowest order in e^y. Then, the Toda potential approaches zero with leading behaviour given by $e^\Psi \approx 4z^2 e^{2y+2z/m^2}$, and consequently the asymptotic free field configuration (4.20) is given by

$$\Psi_0(y,z) = 2y + \log 4z^2 + \frac{2z}{m^2} \ . \tag{4.21}$$

Having determined $a(z) = 2$ and $b(z) = \log 4z^2 + 2z/m^2$ we may proceed in this case calculating the form of the subleading terms of the expansion, in order to compare them with those following from the general solution of the continual Toda equation.

It is convenient to introduce as an expansion parameter the following quantity

$$F(y,z) = e^{2y+2z/m^2} \equiv \frac{1}{4z^2} e^{\Psi_0} \ , \tag{4.22}$$

and use it for inverting the coordinate transformation $(t,\theta) \to (y,z)$ in the asymptotic region. Explicit calculation yields the following result,

$$\cos\theta = 1 - 2F + 2F^2 \left(1 + \frac{2z}{m^2}\right)^2 - 2F^3 \left(1 + 12\frac{z}{m^2} + 48\frac{z^2}{m^4} + 64\frac{z^3}{m^6} + 24\frac{z^4}{m^8}\right) + \cdots \ , \tag{4.23}$$

and

$$\frac{1}{t} = 4z \left(1 - 2F\frac{z}{m^2} + 4F^2\frac{z}{m^2}\left(1 + \frac{4z}{m^2} + \frac{2z^2}{m^4}\right) + \cdots\right), \tag{4.24}$$

where the dots denote higher order corrections in F. Using these expansions we also find that $\Psi(y,z)$ admits a power series expansion around the free field configuration (4.21),

$$\Psi = \Psi_0 - 2F\left(1 + \frac{4z}{m^2} + \frac{2z^2}{m^4}\right) + F^2\left(1 + 28\frac{z}{m^2} + 92\frac{z^2}{m^4} + 96\frac{z^3}{m^6} + 30\frac{z^4}{m^8}\right) + \cdots \ . \tag{4.25}$$

It can be independently verified that this expression is consistent with (4.19) order by order in the expansion parameter F (or $\lambda^2 F$, if we had supplied the non–linearity with a parameter λ^2 as in (4.1)).

The free field (4.21) can be applied in (4.9) and (4.10) in order to reformulate the expansion of $\Psi(y,z)$ in a systematic way, as well as use it as a cross–check of the validity of the series (4.25). For this purpose we have to find the general form of the solution of the continual Toda equation in the dimensionally reduced sector, where all the dependence is on $q_+ + q_- = y$ and not on $q_+ - q_-$. The correct way of doing this is to decompose Ψ_0 close to $-\infty$ as

$$\Psi_0(y,z) = \Psi^+(q_+,z) + \Psi^-(q_-,z) \ ; \qquad \Psi^\pm = a(z)q_\pm + b_\pm(z) \ , \tag{4.26}$$

so that $b_+(z) + b_-(z) = b(z)$; in fact $b(z)$ is not physically important since it can be set equal to zero by shifting y. Then, the quantity $< z \mid M_+^{-1} M_- \mid z >$ in (4.9) can be calculated by expanding the path–ordered exponentials in a straightforward way [9], which for $a(z) = 2$ (independent of z) yields the result

$$< z \mid M_+^{-1} M_- \mid z > = 1 - \frac{1}{4} e^{\Psi_0} + \frac{1}{32} e^{2\Psi_0} + \frac{1}{64} e^{\Psi_0} \partial_z^2 (e^{\Psi_0}) + \cdots \ , \tag{4.27}$$

where the dots denote third and higher order corrections in $e^{\Psi_0} \equiv 4z^2 F$. Taking the logarithm and expanding it in power series we may determine $\Psi(y,z)$ following the general formula (4.9),

$$\Psi = \Psi_0 - \frac{1}{4}\partial_z^2 (e^{\Psi_0}) + \frac{1}{64}\partial_z^2\left(e^{\Psi_0}\partial_z^2(e^{\Psi_0})\right) + \cdots \ . \tag{4.28}$$

This expansion is identical to (4.25), as it can be readily verified using the definition (4.22). It is systematically summarized by $< z \mid M_+^{-1} M_- \mid z >$, which encodes all the information for $\Psi(y, z)$.

The Lie–Backlund transformation that relates the free field configuration Ψ_0 with Ψ can be regarded as a classical analogue of the half–S–matrix connecting the asymptotic value of the field at $-\infty$ with its value at y. From this point of view the expansion in terms of free fields arises quite naturally in the general theory of Toda systems. One could have similarly considered analyzing the asymptotic behaviour at $+\infty$, which for the Taub–NUT metric would lead to the free field configuration $\Psi_0 = -2y + \log 4z^2 + 2z/m^2$.

Concluding this section we note that our results have a smooth limit as the moduli parameter of the Taub–NUT metric $m \to \infty$. The coordinate transformations (4.23) and (4.24) contract to $\cos\theta = -\tanh y$ and $t = 1/4z$ respectively, in agreement with (3.21). Also the series (4.25) essentially provides the polynomial expansion of $1/\cosh^2 y$ in powers of e^{2y}. We also note that the Toda field configuration in this case belongs to the special class of solutions given by the ansatz (4.15), provided that the corresponding Liouville field is chosen to be

$$e^{\varphi_L(y)} = \frac{1}{\cosh^2 y} \ . \tag{4.29}$$

An alternative description of this is given by the formula (4.17) making the choice $\varphi_L^{\pm}(q_{\pm}) = 2q_{\pm}$ as in (4.26). Therefore, in the limit $m \to \infty$ the solution can be thought as residing entirely in the $SU(2)$ subalgebra of $SU(\infty)$, and it "spreads" into the whole of $SU(\infty)$ by including all the $1/m^2$ corrections. Furthermore, close to the nut located at $r = m$ (or equivalently at $t \to \infty$) we find that the Toda potential of the Taub–NUT metric (with finite m) is also given by $e^{\Psi} \approx z^2/\cosh^2 y$; in this case, however, z approaches zero and the complete description of the metric requires the power series corrections in z appearing in the expansion (4.25). These two remarks complete our understanding of the way that the full $SU(\infty)$ algebra is employed for the description of this metric.

5 Analysis of the Atiyah-Hitchin metric

Searching for the free field realization of a given metric requires first to locate the zeros of e^{Ψ}. Straightforward calculation shows that for the Atiyah–Hitchin metric, and for generic values of $0 < k < 1$, the zeros of (3.27) occur at

$$\cos^2\theta = k^2 \ ; \quad \psi = \frac{\pi}{2} \ \text{or} \ \frac{3\pi}{2} \ . \tag{5.1}$$

These four points are actually not distinct from each other, due to certain identifications imposed by discrete symmetries on the angle variables $0 \leq \theta \leq \pi$, $0 \leq \psi \leq 2\pi$ and $0 \leq \phi \leq 2\pi$. Explicitly, we consider the symmetry operations

$$\theta \to \pi - \theta \ , \quad \psi \to 2\pi - \psi \ , \quad \phi \to \pi + \phi \ , \tag{5.2}$$

and

$$\theta \to \theta \ , \qquad \psi \to \pi + \psi \ , \qquad \phi \to \phi \ , \tag{5.3}$$

which are clearly invariances of σ_1^2, σ_2^2 and σ_3^2. Note that under (5.2), which exchanges the position of monopoles so that they are treated as identical particles, the point $(\cos\theta, \psi) = (k, \ \pi/2)$ is mapped to $(-k, \ 3\pi/2)$, and $(k, \ 3\pi/2)$ is mapped to $(-k, \ \pi/2)$. Similarly, under (5.3) the point $(k, \ \pi/2)$ is mapped to $(k, \ 3\pi/2)$, and $(-k, \ \pi/2)$ is mapped to $(-k, \ 3\pi/2)$. The discrete transformations (5.2) and (5.3) are invariances of the Toda potential (3.27), but they have the following effect on the coordinates ν, given by (3.24), and $q = y + ix$, given by (3.25):

$$\nu \to -\nu + 4\pi i \ , \qquad q \to -q \tag{5.4}$$

$$\nu \to \nu + 2\pi i \ , \qquad q \to q \tag{5.5}$$

respectively. In the Atiyah–Hitchin metric we always assume the identifications imposed by (5.2), and hence there is a folding (5.4) that results to a cone–like structure around $q = 0$, but eventually we also impose (5.3) when considering the metric on the moduli space of $SU(2)$ BPS monopoles M_2^0 (for details see [5]). Thus, up to a double covering implied by (5.3), it suffices to consider

$$\cos\theta = k \ , \qquad \psi = \frac{\pi}{2} \tag{5.6}$$

as describing the unique free field point, and let $0 \leq \theta \leq \pi/2$ and $0 \leq \psi \leq \pi$.

The moduli space $M_2^0 = \tilde{M}_2^0/Z_2$ has non–trivial homotopy $\pi_1(M_2^0) = Z_2$, and hence its double covering $\tilde{M}_2^0$ is simply connected and universal [5]. In the ν–plane there are two free field points, say $Q_0^{\pm}$ that correspond to $(\cos\theta, \ \psi) = (k, \ \pi/2)$ and $(k, \ 3\pi/2)$ in the center of the "in" and "out" regions, while q and e^{Ψ} are invariant under the Z_2 symmetry (5.3) and so they are well–defined on the monopole moduli space. Our analysis essentially takes place in the simply connected space $\tilde{M}_2^0$, choosing to work with (5.6). We assume in the following the free field realization of Toda field configurations as in flat space. The multi–valuedness of the transformation (3.25) for $\theta = 0$, which yields $q = 0$ for all values of the angle variable ψ (also Ψ is independent of ψ for $\theta = 0$), will be regarded as a coordinate singularity analogous to the origin of a spherical coordinate system. We also note that at the free field point (5.6) the coordinates θ and k are related to each other, whereas in the Taub–NUT limit we chose earlier $\theta = 0$ independently of the other coordinates. Hence, we do not expect the free field analysis of the Atiyah–Hitchin metric to yield smoothly the Taub–NUT free field configuration for $k \to 1$, although the line elements themselves become exponentially close to each other.

In principle, we could have considered points where $e^{\Psi} \to az + b$, assuming that a and b are z–independent constants, since the non–linear term $\partial_z^2 e^{\Psi}$ would have also been zero there. A typical example that illustrates the relevance of this possibility is the solution $e^{\Psi} = z/2$, independent of q and $\bar{q}$ everywhere, which describes the flat space metric in polar coordinates after a suitable change of variables. The Toda field is identical to the free field in this case, and a simple calculation shows that the logarithmic term in (4.9) is linear in z, thus giving zero contribution when ∂_z^2 acts on it. We have not considered this possibility for the Atiyah–Hitchin metric because there already exists the point (5.6) for which $e^{\Psi} = 0$.

Next, in analogy with the analysis done for the Taub–NUT metric, we have to invert the transformation (3.25) in the vicinity of the free field point (5.6), and try to express directly the corresponding Toda potential in terms of q and $\bar{q}$. It is convenient to parametrize this region by two real variables ϵ_1 and ϵ_2, so that

$$\cos\theta = k - \frac{1}{2}k'^2\epsilon_1 \ , \quad \psi = \frac{\pi}{2} + \frac{1}{2}\epsilon_2 \ , \tag{5.7}$$

and also introduce the complex variable $\epsilon = \epsilon_1 + i\epsilon_2$. Then, to lowest order in ϵ, we have

$$\nu = \nu_0 + \epsilon \ ; \qquad \nu_0 = \log\frac{1-k}{1+k} + i\pi \ , \tag{5.8}$$

and

$$1 + k'^2\sinh^2\frac{\nu}{2} = k\epsilon \ , \quad k\tanh\frac{\nu}{2} = -\left(1 + \frac{k'^2}{2k}\epsilon\right) \ . \tag{5.9}$$

Therefore, the Toda potential admits around ν_0 the expansion

$$e^{\Psi} = \frac{1}{16}kk'^2K^2(k) \mid \epsilon \mid + \mathcal{O}(\epsilon^2) \ . \tag{5.10}$$

We also find that the coordinate z given by (3.23) has the expansion

$$z = \frac{1}{4}K(k)(k'^2K(k) - E(k)) + \mathcal{O}(\epsilon) \ . \tag{5.11}$$

The expansion of the coordinate q around ν_0 is a bit trickier because the elliptic integral in (3.25) diverges as $\epsilon \to 0$, while $\sqrt{1 + k'^2\sinh^2\nu/2}$ tends to zero. It is convenient to rewrite

$$\int_0^{\pi/2} d\gamma \frac{\sqrt{1 - k^2\sin^2\gamma}}{1 - k^2\tanh^2\frac{\nu}{2}\sin^2\gamma} = \frac{K(k)}{\tanh^2\frac{\nu}{2}} - \frac{1}{\sinh^2\frac{\nu}{2}}\Pi(k^2\tanh^2\frac{\nu}{2}, \ k) \ , \tag{5.12}$$

where

$$\Pi(n^2, k) = \int_0^{\pi/2} \frac{d\gamma}{(1 - n^2\sin^2\gamma)\sqrt{1 - k^2\sin^2\gamma}} \tag{5.13}$$

is the complete elliptic integral of the third kind, and study its expansion close to $\mid n \mid = 1$. Note that (5.13) is well–defined for $\mid n \mid < 1$, while it diverges on the real axis for all $\mid n \mid \geq 1$, and the Cauchy principal value of the integral representation has to be taken into account there [16]. In fact, for $\psi = \pi/2$, $n^2 < 1$ corresponds to $\cos\theta < k$, and $n^2 \geq 1$ to $\cos\theta \geq k$.

The correct definition of the complete elliptic integral of the third kind, for all complex values of n, is provided by the analytic continuation in terms of theta functions. We parametrize $n^2 = k^2 \mathrm{sn}^2\alpha$, using the Jacobi elliptic function of a complex variable α, and arrive at the expression [16]

$$\Pi(k^2 \mathrm{sn}^2\alpha,\ k) = K(k) + \frac{\pi}{2k'} \frac{\theta_4'\left(\frac{\pi\alpha}{2K(k)}\right)\theta_1\left(\frac{\pi\alpha}{2K(k)}\right)}{\theta_2\left(\frac{\pi\alpha}{2K(k)}\right)\theta_3\left(\frac{\pi\alpha}{2K(k)}\right)} \ , \tag{5.14}$$

where the modulus of the theta functions is $iK(k')/K(k)$, which is proportional to the coordinate variable t in (2.19). Then, (5.13) admits the following expansion

$$\Pi(n^2, k) = K(k) - \frac{E(k)}{k'^2} + \frac{\pi(1 + k'^2 - k^2 n^2)}{4k'^3\sqrt{1-n^2}} + \mathcal{O}(1-n^2) \ , \tag{5.15}$$

which yields

$$\Pi(k^2 \tanh^2\frac{\nu}{2},\ k) = K(k) - \frac{E(k)}{k'^2} + \frac{\pi\sqrt{k}}{2k'^2\sqrt{-\epsilon}} + \mathcal{O}(\sqrt{-\epsilon}) \tag{5.16}$$

using the parametrization (5.9) around ν_0. This result, which is actually valid for all complex values of the parameters, leads to the expansion of the coordinate q

$$q = q_0 + \frac{E(k) - k'^2 K(k)}{\sqrt{k}}\sqrt{\epsilon} + \mathcal{O}(\epsilon) \ , \tag{5.17}$$

where $q_0 = i\pi/2$. There is an overall sign ambiguity in evaluating q that depends on which branch we choose for $\sqrt{-1} = \pm i$. However, thanks to the discrete symmetry (5.4) that has been imposed on the coordinate system of the Atiyah–Hitchin metric this ambiguity is not relevant. Putting everything together we obtain the desired expansion of the Toda potential

$$e^{\Psi} = \frac{1}{16}\left(\frac{kk'K(k)}{E(k) - k'^2 K(k)}\right)^2 \mid q - q_0 \mid^2 + \mathcal{O}(\mid q - q_0 \mid^4) \ . \tag{5.18}$$

The k-dependent coefficient of $\mid q - q_0 \mid^2$ term can be expressed as a function of z, say $e^{f(z)}$, using (5.11) to lowest order, although there is no closed formula for it.

The main point of this analysis is to demonstrate explicitly that there is no free field realization of the Atiyah–Hitchin metric along the lines of the previous section. Indeed, if we consider

$$\Psi_0 = \log \mid q - q_0 \mid^2 + f(z) \tag{5.19}$$

as the corresponding free field extracted from the expansion (5.18), then the path–ordered exponentials (4.10) will depend only on the combination $\mid q - q_0 \mid$, and so the complete Toda potential (4.9) will depend only on $\mid q - q_0 \mid$ as well. We realize, therefore, that if the Toda potential of the Atiyah–Hitchin metric were to admit a free field realization according to (4.9), the metric would exhibit an additional $U(1)$ isometry, which is manifest in $\mid q - q_0 \mid$, and hence commute with $SO(3)$. Since this is not geometrically allowed, we conclude that this metric corresponds to a new class of solutions of the continual Toda equation that can accomodate the breaking of the additional $U(1)$ isometry. The essential ingredient here is the factorization of the coordinate variables q and z implied by (5.19) in the "would be" free field configuration. If such a factorization were not present, it would have been impossible to translate the reality condition of the corresponding Ψ_0 into a manifest $U(1)$ isometry of the metric by suitable conformal transformation of the coordinates q and $\bar{q}$; in Liouville theory the reality condition of the solutions in Euclidean space always yields a manifest $U(1)$ isometry, but in the continual Toda theory the situation is different, unless the z–dependence factorizes, because of the operator ∂_z^2 that acts on e^{Ψ}.

We have no systematic description of such new solutions generalizing (4.9). One may think that there is a possible resolution by introducing suitable prefactors in (4.10). The particular choice that was made for (4.10) corresponds to some fixed boundary condition, but in general we could have integrated the continual analogue of the linearized equations (4.4) to

$$\tilde{M}_{\pm} = M_{\pm}^{(0)} \mathrm{P} \exp \left(\lambda \int^{q_{\pm}} dq'_{\pm} \int^{z} dz' e^{\Psi^{\pm}(q'_{\pm}, z')} X^{\pm}(z') \right) \tag{5.20}$$

with constant group elements $M_{\pm}^{(0)}$. In doing that we should consider solutions (4.9) with

$$< z \mid \tilde{M}_{+}^{-1} \tilde{M}_{-} \mid z >=< z \mid M_{+}^{-1} g_0^{(0)} M_{-} \mid z > \ , \tag{5.21}$$

where

$$g_0^{(0)} = (M_{+}^{(0)})^{-1} M_{-}^{(0)} \ . \tag{5.22}$$

So the solutions are characterized in reality by the free field specialization as well as the choice of a constant group element $g_0^{(0)}$. This point is very important in affine Toda theory because soliton solutions have zero free fields, but non–trivial $g_0^{(0)}$ [21]. We have analysed whether this modification can change the previous conclusions. We have verified that there is no consistent choice of $g_0^{(0)}$, which is the exponential of Lie algebra elements, that breaks the unwanted $U(1)$ isometry while preserving the reality of the metric.

We think that responsible for this situation are held the topological properties of the group of area preserving diffeomorphisms, whose algebra describe $SU(N)$ in the continual large N limit. In ordinary Toda theory one uses the fact that the exponential of an algebra is a Lie group that admits the Gauss decomposition [12]. For $SU(\infty)$, however, the exponentiation of the algebra of diffeomorphisms is not a Lie group in the topological sense (see for instance [22]), and hence the existence of a Gauss type decomposition is a topological assumption that one makes to arrive eventually to (4.9).

Since the Atiyah–Hitchin metric provides the first example of such non–trivial configurations in Toda theory, it deserves further study in view of possible generalizations in rotational hyper–Kahler geometry. It is rather remarkable that the purely rotational character of this metric, due to the absence of an additional $U(1)$ translational isometry, has profound implications in monopole scattering, where slowly moving monopoles can be converted into dyons [5]. The absence of this isometry is also responsible for the special features of the corresponding Toda field configuration, but we do not understand very well whether there exists a deeper relation between the particle–like approximation of monopole scattering (given by the Taub–NUT limit of the metric) and the possibility to have free field realizations of the Toda potential. Also, the discrete Z_2 symmetry (5.3) has no effect on the presence or the absense of an additional continuous $U(1)$ isometry in the metric, and so our analysis should be insensitive to it.

6 Conclusions and discussion

We have presented a systematic study of the Toda field formulation of 4–dim hyper–Kahler metrics with a rotational isometry. The only examples known to this date are the Eguchi–Hanson, Taub–NUT and Atiyah–Hitchin metrics, which actually exhibit a bigger group of isometries containing $SO(3)$. We found that the qualitative differences between them can be better understood in algebraic terms by considering the free field realization of the corresponding Toda field configurations. The solutions of the continual Toda equation that were obtained by taking the large N limit of the $SU(N)$ Toda theory seem to be inadequate for describing the Atiyah–Hitchin metric. For the other two metrics, however, the 2-dim free field specialization of Saveliev's group theoretical formula exists and it has been given explicitly.

The Atiyah–Hitchin metric can be regarded as the simplest representative from a class of 4–dim purely rotational hyper–Kahler metrics. We note, however, that such a generalized series of regular metrics (if they exist) can have only one rotational isometry, and no additional isometries of either type. Recall the observation of Boyer and Finley that there are no real hyper–Kahler manifolds with two rotational isometries whose algebra closes upon itself [7],

$$[K_1,\ K_2] = \alpha K_1 + \beta K_2\ . \tag{6.1}$$

If one insists to have two isometries satisfying (6.1) the structure constants can be chosen without loss of generality so that $\alpha = 0$ and β is purely imaginary, and the self–duality condition of the metric forces one of the Killing isometries to be translational. Moreover, it can also be easily shown that the generators of rotational isometries either form an $SO(3)$ algebra or there is only one of them, since otherwise their algebra would contain a two–dimensional solvable subalgebra. According to the classification of $SO(3)$ hyper–Kahler metrics only the Atiyah–Hitchin metric defines a complete and regular space of the first kind, with no additional translational isometries, and as a result we expect that a descending series of purely rotational metrics in four dimensions will necessarily exhibit only one isometry. Hence, it might be difficult to obtain candidate new metrics in closed form.

We mention finally that certain generalizations of the Atiyah–Hitchin metric have been already proposed [5], using a different method. The moduli space of BPS $SU(2)$ monopoles of magnetic charge 2 is associated to the space of rational functions

$$f(z) = \frac{uz+v}{z^2+w}\ ; \qquad v^2 + wu^2 \neq 0\ , \tag{6.2}$$

and the equation $v^2 + wu^2 = 1$ defines a manifold in C^3 which is a double covering of it. This equation can be regarded as a special case of the more general equation $v^2 + wu^2 = w^{k-1}$ in C^3, which leads to a series of hyper–Kahler metrics associated to the orthogonal groups $SO(2k)$ (D–series). This algebraic description is analogous to the A–series generalization of the Taub–NUT metric, where the multi–Taub–NUT solutions have a characteristic equation $uv = w^k$ related to $SU(k)$ (for details on polygon constructions see [6]). Recently, there has been a discussion of this in the literature, using the twistor space interpretation of the Legendre transform construction of hyper–Kahler metrics [23]. In that regard there are certain similarities with the Toda frame formulation of rotationally invariant metrics. Our approach, however, is quite different emphasizing the presence of a rotational isometry in the (proposed) series of metrics that descend from Atiyah–Hitchin; whereas in the D–series generalization, it is unlike that any isometries will be at all present.

In any case, it will be interesting to explore further all these possibilities, and answer the question whether the Atiyah–Hitchin metric is the unique purely rotational complete hyper–Kahler 4–metric or it is indeed the first representative from a new series of metrics. One drawback at this stage is the local nature of our method, which makes rather delicate the issue of the completeness of the metrics that can be obtained from different solutions of the continual Toda equation.

Acknowledgements

One of us (K.S.) wishes to thank the Theory Division of CERN for hospitality during the course of this work. We also thank Elias Kiritsis and Misha Saveliev for useful remarks on the manuscript. The work of K.S. was carried out with the financial support of the European Union research program "Training and Mobility of Researchers" and is part of the project "String Duality Symmetries and Theories with Space–Time Supersymmetry".

Note Added

An interesting series of papers came recently to our attention, which prove the existence of at least two families of new non–singular 4–dim hyper–Kahler manifolds with rotational $SO(2)$ isometry.

The first family describes a 1–parameter deformation of the Atiyah–Hitchin space as complex hypersurface in C^3, $v^2 + wu^2 = 1 + imu$. The 4-dim metric was implicitly described by Dancer [24] starting from a 12–dim moduli space of solutions to Nahm's equations and then performing suitable hyper–Kahler quotients of it. For $m = 0$ we obtain the Atiyah–Hitchin space, but for $m \neq 0$ the resulting manifold has only one purely rotational isometry. It is interesting that these spaces arise naturally as moduli spaces of the supersymmetric 3–dim $SU(2)$ gauge theory with $N_f = 1$ and a bare mass $\sim m$ [25].

The second family corresponds to the hypersurface in C^3, $v^2 + wu^2 = w + ikuv + lv$, which is non-singular if and only if $k \neq \pm l$ [26]. It also turns out that the resulting 4–dim hyper–Kahler manifolds exhibit no isometries, with the only exception of Im$l = 0$ that gives rise to spaces with one rotational isometry. A more intuitive geometric construction was apparently proposed by Page a number of years ago through a limiting procedure from the Einstein metric on K_3 [27]. In this case we have a 3–parameter family of purely rotational hyper–Kahler manifolds although the explicit form of the metric is also not known. It is conceivable that suitable deformations of the more general hypersurfaces $v^2 + wu^2 = w^{k-1}$ may exist as well that yield purely rotational hyper–Kahler manifolds for appropriate limits of the relevant deformation parameters.

These additional results supplement nicely our point of view, thus making more urgent the question of the explicit metrics and the construction of the corresponding Toda field configurations. Perhaps there is a more direct relation between $SU(\infty)$ Toda theory, moduli spaces of Nahm's equations and the deformations of hypersurfaces in C^3 that were considered above.

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