POLFIS-TH 02/96 NSF-ITP-96-11
CERN-TH/95-350 hep-th/xx9603004

General Matter Coupled N=2 Supergravity

L. Andrianopoli[1], M. Bertolini[2 †], A. Ceresole[2,4]
R. D'Auria[2], S. Ferrara[3−5] and P. Fré[6]

[1] Dipartimento di Fisica, Universitá di Genova, via Dodecaneso 33, I-16146 Genova, Italy

[2] Dipartimento di Fisica, Politecnico di Torino,
Corso Duca degli Abruzzi 24, I-10129 Torino, Italy

[3] CERN, CH 1211 Geneva 23, Switzerland

[4] Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA

[5] Department of Physics, University of California, Los Angeles, CA 90024, USA

[6] Dipartimento di Fisica Teorica, Università di Torino, via P. Giuria, 1 I-10125 Torino, Italy

Abstract

The general form of $N = 2$ supergravity coupled to an arbitrary number of vector multiplets and hypermultiplets, with a generic gauging of the scalar manifold isometries is given. This extends the results already available in the literature in that we use a coordinate independent and manifestly symplectic covariant formalism which allows to cover theories difficult to formulate within superspace or tensor calculus approach. We provide the complete lagrangian and supersymmetry variations with all fermionic terms, and the form of the scalar potential for arbitrary quaternionic manifolds and special geometry, not necessarily in special coordinates. Our results can be used to explore properties of theories admitting $N = 2$ supergravity as low energy limit.

† Fellow by Ansaldo Ricerche srl, C.so Perrone 24, I-16152 Genova

* Supported in part by DOE grant DE-FGO3-91ER40662 Task C., EEC Science Program SC1*CT92-0789, NSF grant no. PHY94-07194, and INFN.

1 Introduction

Impressive results over the last year on non perturbative properties of $N = 2$ supersymmetric Yang-Mills theories[1, 2] and their extension to string theory[3, 4, 5, 6] through the notion of string-string duality[7], have used the deep underlying mathematical structure of these theories and its relation to algebraic geometry.

In the case of $N = 2$ vector multiplets, describing the effective interactions in the Abelian (Coulomb) phase of a spontaneously broken gauge theory, Seiberg and Witten [1] have shown that positivity of the metric on the underlying moduli space identifies the geometrical data of the effective $N = 2$ rigid theory with the periods of a particular torus.

In the coupling to gravity it was conjectured by some of the present authors [3, 4] and later confirmed by heterotic-Type II duality [8, 9, 10, 11], that the very same argument based on positivity of the vector multiplet kinetic metric identifies the corresponding geometrical data of the effective $N = 2$ supergravity with the periods of Calabi-Yau threefolds.

On the other hand, when matter is added, the underlying geometrical structure is much richer, since $N = 2$ matter hypermultiplets are associated with quaternionic geometry[12, 13, 14], and charged hypermultiplets are naturally associated with the gauging of triholomorphic isometries of these quaternionic manifolds [15].

It is the aim of this paper to complete the general form of the $N = 2$ supergravity lagrangian coupled to an arbitrary number of vector multiplets and hypermultiplets in presence of a general gauging of the isometries of both the vector multiplets and hypermultiplets scalar manifolds. Actually this extends results already obtained years ago by some of us [15], that in turn extended previous work by Bagger and Witten on ungauged general quaternionic manifolds coupled to $N = 2$ supergravity[12], by de Wit, Lauwers and Van Proeyen on gauged special geometry and gauged quaternionic manifolds obtained by quaternionic quotient in the tensor calculus framework [16], and by Castellani, D'Auria and Ferrara on covariant formulation of special geometry for matter coupled supergravity [17].

This paper firstly provides in a geometrical setting the full lagrangian with all the fermionic terms and the supersymmetry variations. Secondly, it uses a coordinate independent and manifestly symplectic covariant formalism which in particular does not require the use of a prepotential function $F(X)$. Whether a prepotential $F(X)$ exists or not depends on the choice of a symplectic gauge[4]. Moreover, some physically interesting cases are precisely instances where $F(X)$ does not exist[4].

Of particular relevance is the fact that we exhibit a scalar potential for arbitrary quaternionic geometries and for special geometry not necessarily in special coordiantes. This allows us to go beyond what is obtainable with the tensor calculus (or superspace) approach. Among many applications, our results allow the study of general conditions for spontaneous supersymmetry breaking in a manner analogous to what was done for $N = 1$ matter coupled supergravity [18]. Many examples of supersymmetry breaking studied in the past are then reproduced in a unified framework.

Recently the power of using simple geometrical formulae for the scalar potential was exploited while studying the breaking of half supersymmetries in a particular simple model, using a symplectic basis where $F(X)$ is not defined[19]. The method has potential applications in string theory to study non perturbative phenomena such as conifold transitions [20], p-forms condensation [21] and Fayet-Iliopoulos terms [19, 22].

Although the supersymmetric Lagrangian and the transformation rules look quite involved, all the couplings, the mass matrices and the vacuum energy are completely fixed and organized in terms of few geometrical data, such as the choice of a gauge group G, and of a special Kähler $\mathcal{SK}(n_V)$ and of a Quaternionic manifold $\mathcal{Q}(n_H)$ describing the self-interactions of the n_V vector and n_H hypermultiplets respectively, whose direct product yields the full scalar manifold of the theory

$$\mathcal{M} \equiv \mathcal{SK}(n_V) \otimes \mathcal{Q}(n_H) . \tag{1.1}$$

If G is non-abelian, it must be a subgroup of the isometry group of the scalar manifold $\mathcal{SK}(n_V)$ with a block diagonal immersion in the symplectic group $Sp(2n_V + 2, \mathbb{R})$ of electric-magnetic duality rotations.

An expanded version of this paper, with particular attention to the geometrical properties of the scalar manifolds, the rigidly supersymmetric version and further related issues is given in [23].

2 Resumé and Glossary of Special and Quaternionic Geometry

Here we collect some useful formulae for special and quaternionic geometry, following closely the conventions of [24]. The n_V complex scalar fields z^i of $N = 2$ vector multiplets are coordinates of a special Kähler manifold, that is a Kähler-Hodge manifold $\mathcal{SK}(n_V)$ with the additional constraint on the curvature

$$R_{ij^\star kl^\star} = g_{ij^\star} g_{kl^\star} + g_{il^\star} g_{kj^\star} - C_{ikp} C_{j^\star l^\star p^\star} g^{pp^\star} , \tag{2.1}$$

where $g_{ij^\star} = \partial_i \partial_{j^\star} K$ is the Kähler metric, K is the Kähler potential and C_{ikp} is a completely symmetric covariantly holomorphic tensor. We remind that the Levi-Civita connection one form and the Riemann tensor are given by

$$\Gamma^i_j = \Gamma^i_{kj} dz^k \ , \ \Gamma^i_{kj} = g^{il^\star} \partial_j g_{kl^\star} \ , \ R^i{}_{jk^\star l} = \partial_{k^\star} \Gamma^i_{jl} \ . \tag{2.2}$$

A Kähler-Hodge manifold has the property that there is a $U(1)$ bundle $\mathcal{L}$ whose first Chern class coincides with the Kähler class. This means that locally the $U(1)$ connection Q can be written as

$$Q = -\frac{i}{2}(\partial_i K dz^i - \partial_{i^\star} K d\overline{z}^{i^\star}) . \tag{2.3}$$

The covariant derivative of a generic field ψ^i, that under a Kähler transformation $K \rightarrow K + f + \overline{f}$ transforms as $\psi^i \rightarrow exp[-\frac{1}{2}(pf + \overline{p}\overline{f})]\psi^i$ is given by

$$\begin{aligned} D_i \psi^j &= \partial_i \psi^j + \Gamma^j_{ik} \psi^k + \frac{p}{2} \partial_i K \psi^j , \\ D_{i^\star} \psi^j &= \partial_{i^\star} \psi^j + \frac{\overline{p}}{2} \partial_{i^\star} K \psi^j . \end{aligned} \tag{2.4}$$

(In the following we always have $\overline{p} = -p$). Note that $\overline{\psi}^{j^\star}$ has weight $(-p, -\overline{p})$. Since C_{ikp} is covariantly holomorphic and has weight $p = 2$, it satisfies $D_{q^\star} C_{ikp} = (\partial_{q^\star} - \partial_{q^\star} K) C_{ikp} = 0$.

A more intrinsic and useful definition of a special Kähler manifold can be given by constructing a flat $2n_V+2$-dimensional symplectic bundle over the Kähler -Hodge manifold whose generic sections (with weight $p=1$)

$$V=(L^\Lambda, M_\Lambda) \qquad \Lambda=0,\ldots,n_V \ , \tag{2.5}$$

are covariantly holomorphic

$$D_{i^\star}V=(\partial_{i^\star}-\frac{1}{2}\partial_{i^\star}K)V=0 \tag{2.6}$$

and satisfy the further condition

$$i<V,\overline{V}>=i(\overline{L}^\Lambda M_\Lambda-\overline{M}_\Lambda L^\Lambda)=1 \ , \tag{2.7}$$

where $<\ ,\ >$ denotes a symplectic inner product with metric chosen to be $\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$.

Defining $U_i=D_iV=(f_i^\Lambda, h_{i\Lambda})$, and introducing a symmetric three-tensor C_{ijk} by

$$D_iU_j=iC_{ijk}g^{kk^\star}\overline{U}_{k^\star} \ , \tag{2.8}$$

one can show that the symplectic connection

$$\begin{aligned} D_iV &= U_i \\ D_iU_j &= iC_{ijk}g^{kk^\star}\overline{U}_{k^\star} \\ D_iU_{j^\star} &= g_{ij^\star}\overline{V} \\ D_i\overline{V} &= 0 \end{aligned} \tag{2.9}$$

is flat, provided the constraint (2.1) is verified. Furthermore, the Kähler potential can be computed as a symplectic invariant from eq. (2.7). Indeed, introducing also the holomorphic sections

$$\begin{aligned} \Omega &= e^{-K/2}V=e^{-K/2}(L^\Lambda, M_\Lambda)=(X^\Lambda, F_\Lambda) \\ \partial_{i^\star}\Omega &= 0 \end{aligned} \tag{2.10}$$

eq. (2.7) gives

$$K=-\ln i<\Omega,\overline{\Omega}>=-\ln i(\overline{X}^\Lambda F_\Lambda-X^\Lambda\overline{F}_\Lambda) \ . \tag{2.11}$$

From eqs. (2.9), (2.7) we have

$$<V,U_i> \;=\; 0 \;\rightarrow\; X^\Lambda\partial_iF_\Lambda-\partial_iX^\Lambda F_\Lambda=0 \ , \tag{2.12}$$

$$\begin{aligned} <\overline{V},U_i> &= 0 \ , \\ \mathrm{Im}\ \mathcal{N}_{\Lambda\Sigma}L^\Lambda\overline{L}^\Sigma &= -\frac{1}{2} \;\rightarrow\; K=-\ln -2(\overline{X}^\Lambda \mathrm{Im}\ \mathcal{N}_{\Lambda\Sigma}X^\Sigma) \ , \end{aligned} \tag{2.13}$$

where the complex symmetric $(n_V+1)\times(n_V+1)$ matrix $\mathcal{N}_{\Lambda\Sigma}$ is defined through the relations

$$M_\Lambda=\mathcal{N}_{\Lambda\Sigma}L^\Sigma \quad , \quad h_{i^\star\Lambda}=\mathcal{N}_{\Lambda\Sigma}f^\Sigma_{i^\star} \ . \tag{2.14}$$

The Kähler metric and Yukawa couplings C_{ijk} can be written in a manifestly symplectic invariant form as

$$g_{ij^\star} = -i < U_i, U_{j^\star} >= -2 f_i^\Lambda \mathrm{Im}\, \mathcal{N}_{\Lambda\Sigma} f_{j^\star}^\Sigma \ , \tag{2.15}$$

$$C_{ijk} = < D_i U_j, U_k > \ . \tag{2.16}$$

It is also useful to define

$$U^{\Lambda\Sigma} \equiv f_i^\Lambda g^{ij^\star} f_{j^\star}^\Sigma = -\frac{1}{2} (\mathrm{Im}\, \mathcal{N})^{-1\Lambda\Sigma} - \overline{L}^\Lambda L^\Sigma \ , \tag{2.17}$$

which is the inverse relation of eq. (2.15). Under coordinate transformations, the sections Ω transform as

$$\tilde{\Omega} = e^{-f_S(z)} \mathcal{S} \Omega \ , \tag{2.18}$$

where $\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is an element of $Sp(2n_V+2, \mathrm{I\!R})$,

$$A^T D - C^T B = 1\!\!1 \ , \quad A^T C - C^T A = B^T D - D^T B = 0 \ , \tag{2.19}$$

and the factor $e^{-f_S(z)}$ is a $U(1)$ Kähler transformation. We also note that

$$\widetilde{\mathcal{N}}(\widetilde{X}, \widetilde{F}) = (C + D\mathcal{N}(X,F))(A + B\mathcal{N}(X,F))^{-1} \ , \tag{2.20}$$

a relation that simply derives from its definition, eq. (2.14) .

Note that under Kähler transformations $K \to K + f + \overline{f}$ and $\Omega \to \Omega e^{-f}$. Since $X^\Lambda \to X^\Lambda e^{-f}$, this means that we can regard, at least locally, the X^Λ as homogeneous coordinates on $\mathcal{SK}(\backslash_\mathcal{V})$[25], provided the matrix

$$e_i^a(z) = \partial_i (X^a / X^0) \quad a = 1, \ldots, n_V \tag{2.21}$$

is invertible[4]. In this case, we may set

$$F_\Lambda = F_\Lambda(X) \tag{2.22}$$

and then eq. (2.12) implies the integrability condition

$$\frac{\partial F_\Sigma}{\partial X^\Lambda} - \frac{\partial F_\Lambda}{\partial X^\Sigma} = 0 \quad \to \quad F_\Lambda = \frac{\partial F(X)}{\partial X^\Lambda} \tag{2.23}$$

with

$$X^\Sigma \partial_\Sigma F = 2F \ . \tag{2.24}$$

$F(X(z))$ is the prepotential of $N = 2$ supergravity vector multiplet couplings [25] and "special coordinates" correspond to a coordinate choice for which

$$e_i^a = \partial_i (X^\Lambda / X^0) = \delta_i^a \ . \tag{2.25}$$

This means $X^0 = 1$, $X^i = z^i$. Since $F = \frac{1}{2} X^\Lambda F_\Lambda$, under symplectic transformations the prepotential transforms as

$$\widetilde{F}(\widetilde{X}) = F(X) + X^\Lambda (C^T B)_\Lambda^\Sigma F_\Sigma + \frac{1}{2} X^\Lambda (C^T A)_{\Lambda\Sigma} X^\Sigma + \frac{1}{2} F_\Lambda (D^T B)^{\Lambda\Sigma} F_\Sigma \ , \tag{2.26}$$

where

$$\widetilde{X} = (A + B\mathcal{F})X \quad , \quad \mathcal{F} = F_{\Lambda\Sigma} = \frac{\partial^2 F}{\partial X^\Lambda \partial X^\Sigma} \ . \tag{2.27}$$

In terms of the special coordinates $t^a = \frac{X^a}{X^0}$, one has $F(X^\Lambda) = (X^0)^2 f(t^a)$, and the Kähler potential and the metric are expressed by

$$\begin{aligned} K(t,\bar{t}) &= -\ln\, i\left[2f - 2\overline{f} + (\bar{t}^a - t^a)(f_a + \overline{f}_a)\right] \\ G_{a\bar{b}} &= \partial_a \partial_{\bar{b}} K(t,\bar{t}) \ . \end{aligned} \tag{2.28}$$

Eq. (2.27) shows that the tranformation $X \to \widetilde{X}$ can be actually singular, thus implying the non existence of the prepotential $F(X)$, depending on the choice of symplectic gauge[4]. On the other hand, some physically interesting cases, such as the $N = 2 \to N = 1$ supersymmetry breaking [19], are precisely instances where $F(X)$ does not exist. On the contrary the prepotential $F(X)$ seems to be a necessary ingredient in the tensor calculus constructions of $N = 2$ theories that for this reason are not completely general. This happens because tensor calculus uses special coordinates from the very start.

Next we turn to the hypermultiplet sector of an $N = 2$ theory. $N = 2$ hypermultiplets are field representations of $N = 2$ supersymmetry which contain a pair of left-handed fermions and a quadruple of real scalars. $N = 2$ supergravity requires that the $4n_H$ scalars q^u of n_H hypermultiplets be the coordinates of a quaternionic manifold[12].

Supersymmetry requires the existence of a principal $SU(2)$–bundle $\mathcal{SU}$ that plays for hypermultiplets the same role played by the the line–bundle $\mathcal{L}$ in the case of vector multiplets.

A quaternionic manifold is a $4n_H$-dimensional real manifold endowed with a metric h:

$$ds^2 = h_{uv}(q) dq^u \otimes dq^v \quad ; \quad u,v = 1,\ldots,4n_H \tag{2.29}$$

and three complex structures J^x, $(x = 1,2,3)$ that satisfy the quaternionic algebra

$$J^x J^y = -\delta^{xy}\,\mathbb{1} \,+\, \epsilon^{xyz} J^z \tag{2.30}$$

and such that the metric is hermitian with respect to the three complex structures:

$$h\left(J^x \mathbf{X}, J^x \mathbf{Y}\right) = h\left(\mathbf{X},\mathbf{Y}\right) \qquad (x = 1,2,3) \tag{2.31}$$

where $\mathbf{X},\mathbf{Y}$ are generic tangent vectors. The triplet of two-forms K^x

$$K^x = K^x_{uv} dq^u \wedge dq^v \ ; \ K^x_{uv} = h_{uw} (J^x)^w_v \tag{2.32}$$

that provides the generalization of the concept of Kähler form occurring in the complex case, is covariantly closed with respect to an $SU(2) \simeq Sp(2)$ connection ω^x

$$\nabla K^x \equiv dK^x + \epsilon^{xyz}\omega^y \wedge K^z \,=\, 0 \tag{2.33}$$

with curvature given by

$$\Omega^x \,\equiv\, d\omega^x + \frac{1}{2}\epsilon^{xyz}\omega^y \wedge \omega^z = \lambda K^x \tag{2.34}$$

where λ is a real parameter related to the scale of the quaternionic manifold. Supersymmetry, together with appropriate normalizations for the kinetic terms in the lagrangian fixes it to the value $\lambda = -1$. Introducing a physical normalization one can set $\lambda = M_P^{-2}\widehat{\lambda}$, M_P being the Planck mass, so that the limit of rigid supersymmetry $M_P \to +\infty$ can be identified with $\lambda \to 0$[23]. In that case eq. (2.34) defines a flat $SU(2)$ connection relevant to the hyperKähler manifolds of rigid supersymmetry, treated extensively in [23]).

The holonomy group of a generic quaternionic manifold is in $Sp(2) \times Sp(2n_H)$. Introducing flat indices $\{A, B, C = 1,2\}$, $\{\alpha, \beta, \gamma = 1,.., 2n_H\}$ that run, respectively, in the fundamental representations of $SU(2)$ and $Sp(2n_H, \mathbb{R})$, we can introduce a vielbein 1-form

$$\mathcal{U}^{A\alpha} = \mathcal{U}^{A\alpha}_u(q) dq^u \tag{2.35}$$

such that

$$h_{uv} = \mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v \mathbb{C}_{\alpha\beta} \epsilon_{AB} \tag{2.36}$$

where $\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}$, $\mathbb{C}^2 = -\mathbb{1}$, $\epsilon_{AB} = -\epsilon_{BA}$, $\epsilon^2 = -\mathbb{1}$ are, respectively, the flat $Sp(2n_H)$ and $Sp(2) \sim SU(2)$ invariant metrics.

The vielbein $\mathcal{U}^{A\alpha}$ satisfies the metric postulate, i.e. it is covariantly closed with respect to the $SU(2)$-connection ω^z and to some $Sp(2n_H, \mathbb{R})$-Lie Algebra valued connection $\Delta^{\alpha\beta} = \Delta^{\beta\alpha}$ of the holonomy group:

$$\nabla \mathcal{U}^{A\alpha} \equiv d\mathcal{U}^{A\alpha} + \frac{i}{2}\omega^x (\epsilon \sigma_x \epsilon^{-1})^A{}_B \wedge \mathcal{U}^{B\alpha} + \Delta^{\alpha\beta} \wedge \mathcal{U}^{A\gamma} \mathbb{C}_{\beta\gamma} = 0 \tag{2.37}$$

where $(\sigma^x)_A{}^B$ are the standard Pauli matrices. Furthermore, because of the reality of h, $\mathcal{U}^{A\alpha}$ satisfies the reality condition:

$$\mathcal{U}_{A\alpha} \equiv (\mathcal{U}^{A\alpha})^* = \epsilon_{AB} \mathbb{C}_{\alpha\beta} \mathcal{U}^{B\beta} \tag{2.38}$$

A stronger version of eq. (2.36) is

$$\begin{aligned} (\mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v + \mathcal{U}^{A\alpha}_v \mathcal{U}^{B\beta}_u) \mathbb{C}_{\alpha\beta} &= h_{uv} \epsilon^{AB} \\ (\mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v + \mathcal{U}^{A\alpha}_v \mathcal{U}^{B\beta}_u) \epsilon_{AB} &= h_{uv} \frac{1}{n_H} \mathbb{C}^{\alpha\beta} . \end{aligned} \tag{2.39}$$

We have also the inverse vielbein $\mathcal{U}^u_{A\alpha}$ defined by the equation

$$\mathcal{U}^u_{A\alpha} \mathcal{U}^{A\alpha}_v = \delta^u_v . \tag{2.40}$$

Let us consider the Riemann tensor

$$\mathcal{R}^{A\alpha}_{uv}{}^{B\beta} = \Omega^x_{uv} \frac{i}{2} (\epsilon^{-1} \sigma_x)^{AB} \mathbb{C}^{\alpha\beta} + \mathbb{R}^{\alpha\beta}_{uv} \epsilon^{AB} \tag{2.41}$$

where $\mathbb{R}^{\alpha\beta}_{uv}$ is the field strength of the $Sp(2n_H)$ connection:

$$d\Delta^{\alpha\beta} + \Delta^{\alpha\gamma} \wedge \Delta^{\delta\beta} \mathbb{C}_{\gamma\delta} \equiv \mathbb{R}^{\alpha\beta} = \mathbb{R}^{\alpha\beta}_{uv} dq^u \wedge dq^v . \tag{2.42}$$

The Ω^x and $\mathbb{R}^{\alpha\beta}$ curvature satisfy the following relations

$$\begin{aligned} \Omega^x_{A\alpha, B\beta} &= \Omega^x_{uv} \mathcal{U}^u_{A\alpha} \mathcal{U}^v_{B\beta} = -i\lambda \mathbb{C}_{\alpha\beta} (\sigma^x \epsilon)_{AB} \\ \mathbb{R}^{\alpha\beta}_{uv} &= \frac{\lambda}{2} \epsilon_{AB} (\mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v - \mathcal{U}^{A\alpha}_v \mathcal{U}^{B\beta}_u) + \mathcal{U}^{A\gamma}_u \mathcal{U}^{B\delta}_v \epsilon_{AB} \mathbb{C}^{\alpha\rho} \mathbb{C}^{\beta\sigma} \Omega_{\gamma\delta\rho\sigma} , \end{aligned} \tag{2.43}$$

where $\Omega_{\gamma\delta\rho\sigma}$ is a completely symmetric tensor. The previous equations imply that the quaternionic manifold is an Einstein space with Ricci tensor given by

$$\mathcal{R}_{uv} = \lambda(2 + n_H) h_{uv} . \tag{2.44}$$

3 The Gauging

The problem of gauging matter coupled $N = 2$ supergravity theories consists in identifying the gauge group G as a subgroup, at most of dimension $n_V + 1$ of the isometries of the product space

$$\mathcal{M} = \mathcal{SK}(n_V) \otimes \mathcal{Q}(n_H) \; . \tag{3.1}$$

Here we shall mainly consider two cases even if more general situations are possible. The first is when the gauge group G is non abelian, the second is when it is the abelian group $G = U(1)^{n_V+1}$. In the first case supersymmetry requires that G be a subgroup of the isometries of $\mathcal{M}$, since the scalars (more precisely, the sections L^Λ) must belong to the adjoint representation of G. In such case the hypermultiplet space will generically split into[26]

$$n_H = \sum_i n_i R_i + \frac{1}{2} \sum_l n_l^P R_l^P \tag{3.2}$$

where R_i and R_l^P are a set of irreducible representations of G and R_l^P denote pseudoreal representations.

In the abelian case, the special manifold is not required to have any isometry and if the hypermultiplets are charged with respect to the $n_V + 1$ $U(1)$'s, then the $\mathcal{Q}$ manifold should at least have $n_V + 1$ abelian isometries.

The gauging proceeds by introducing $n_V + 1$ Killing vectors generically acting on $\mathcal{M}$

$$\begin{aligned} z^i &\rightarrow z^i + \epsilon^\Lambda k^i_\Lambda(z) \\ q^u &\rightarrow q^u + \epsilon^\Lambda k^u_\Lambda(q) \; . \end{aligned} \tag{3.3}$$

The Kähler and quaternionic structures of the factors of $\mathcal{M}$ imply that k^i_Λ, k^u_Λ can be determined in terms of "Killing prepotentials" which generalize the so called "D-terms" of $N = 1$ supersymmetric gauge theories. The analysis of such prepotentials was carried out in ref. [15], and we now briefly summarize the main results.

For the $\mathcal{SK}$ manifold, the Killing prepotential is a real function $\mathcal{P}_\Lambda$ satisfying

$$k^i_\Lambda = i \; g^{ij^\star} \partial_{j^\star} \mathcal{P}_\Lambda \; , \tag{3.4}$$

with inverse formula

$$i\mathcal{P}_\Lambda = \frac{1}{2}(k^i_\Lambda \partial_i K - k^{i^\star}_\Lambda \partial_{i^\star} K) = k^i_\Lambda \partial_i K = -k^{i^\star}_\Lambda \partial_{i^\star} K \; . \tag{3.5}$$

Formulae (3.4), (3.5) are the results of $N = 1$ supergravity. For special geometry we have the stronger constraint that the gauge group, in the non abelian case, should be imbedded in the symplectic group $Sp(2n_V + 2, \mathbb{R})$, and thus one must have

$$\mathcal{L}_\Lambda V \equiv k^i_\Lambda \partial_i V + k^{i^\star}_\Lambda \partial_{i^\star} V = T_\Lambda V + f_\Lambda V \tag{3.6}$$

for some $T_\Lambda \in Sp(2n_V + 2, \mathbb{R})$ Lie algebra, and $f_\Lambda(z)$ corresponding to an infinitesimal Kähler transformation ($\mathcal{L}_\Lambda$ is the Lie derivative acting on the symplectic section V). We consider here the case where $f_\Lambda = 0$ and G is a subgroup of the classical isometries of $\mathcal{SK}$ which have a diagonal embedding in the symplectic group. Since $f_\Lambda = 0$, this means that $\mathcal{L}_\Lambda K = 0$ and $\mathcal{P}_\Lambda$ satisfies eq. (3.5). Moreover we have

$$k^i_\Lambda U_i = T_\Lambda V + i\mathcal{P}_\Lambda V \tag{3.7}$$

and taking the symplectic scalar product with $\overline{V}$ we get

$$\mathcal{P}_\Lambda = - < \overline{V}, T_\Lambda V >= -e^K < \overline{\Omega}, T_\Lambda \Omega > \ . \tag{3.8}$$

By using the property

$$T_\Lambda = \begin{pmatrix} f^\Sigma_{\Lambda\Delta} & 0 \\ 0 & -f^\Sigma_{\Lambda\Delta} \end{pmatrix} \ , \tag{3.9}$$

we finally get the explicit expression

$$\mathcal{P}_\Lambda = e^K (F_\Delta f^\Delta_{\Lambda\Sigma} \overline{X}^\Sigma + \overline{F}_\Delta f^\Delta_{\Lambda\Sigma} X^\Sigma) \tag{3.10}$$

in terms of the holomorphic sections $\Omega = (X^\Lambda, F_\Lambda)$.

For quaternionic manifolds, eq. (3.4) is replaced by

$$k^u_\Lambda \Omega^x_{uv} = -\nabla_v \mathcal{P}^x_\Lambda = -(\partial_v \mathcal{P}^x_\Lambda + \epsilon^{xyz} \omega^y_v \mathcal{P}^z_\Lambda) \ , \tag{3.11}$$

where $\mathcal{P}^x_\Lambda$ is a triplet of real zero-form prepotentials. Eq. (3.11) can be solved for the Killing vectors in terms of the prepotentials as follows

$$k^u_\Lambda = \frac{1}{6\lambda^2} \sum_{x=1}^{3} h^{vw} (\nabla_v \mathcal{P}^x \Omega^x_{wt}) h^{tu} \ , \tag{3.12}$$

where h^{uv} is the inverse quaternionic metric. If the gauge group G has structure constants $f^\Lambda_{\Sigma\Delta}$ the Killing vectors k^i_Λ, k^u_Λ satisfy the following relations

$$\begin{aligned} i g_{ij^\star} (k^i_\Lambda k^{j^\star}_\Sigma - k^i_\Sigma k^{j^\star}_\Lambda) &= f^\Gamma_{\Lambda\Sigma} \mathcal{P}_\Gamma \ , \\ K^x_{uv} k^u_\Lambda k^v_\Sigma - \frac{\lambda}{2} \epsilon^{xyz} \mathcal{P}^y_\Lambda \mathcal{P}^z_\Sigma &= \frac{1}{2} f^\Delta_{\Lambda\Sigma} \mathcal{P}^x_\Gamma \ . \end{aligned} \tag{3.13}$$

Eq. (3.13) can be derived from the group relations of the Killing vectors

$$[k_\Lambda, k_\Sigma] = f^\Delta_{\Lambda\Sigma} k_\Delta \tag{3.14}$$

together with their relation to the prepotential functions.

An important observation comes from the possible existence of Fayet- Iliopoulos terms in $N=2$ supergravity. This corresponds to a constant shift in the prepotential functions

$$\begin{aligned} \mathcal{P}_\Lambda &\rightarrow \mathcal{P}_\Lambda + \mathcal{C}_\Lambda \\ \mathcal{P}^x_\Lambda &\rightarrow \mathcal{P}^x_\Lambda + \xi^x_\Lambda \ . \end{aligned} \tag{3.15}$$

It is important to observe that in $N=2$ special geometry the Killing vectors and prepotentials satisfy the relation

$$k^i_\Lambda L^\Lambda = k^{i\star}_\Lambda \overline{L}^\Lambda = \mathcal{P}_\Lambda L^\Lambda = \mathcal{P}_\Lambda \overline{L}^\Lambda = 0 \tag{3.16}$$

This implies that $\mathcal{C}_\Lambda = 0$. However, a similar relation does not exist for the quaternionic Killing vectors so that a F-I term in that case is possible, subject to the constraint (3.13). This implies that in a pure abelian theory with only neutral hypermultiplets we can still have $\xi^x_\Lambda \neq 0$ provided

$$\epsilon^{xyz} \xi^x_\Lambda \xi^y_\Sigma = 0 \tag{3.17}$$

holds. Models of this sort, breaking $N = 2 \to N = 0$ with vanishing cosmological constant, were constructed in ref. [27] and will be discussed in the last section.

The gauging procedure can now be performed through the following steps [28]. One first defines gauge covariant differentials

$$\begin{aligned} \nabla z^i &= dz^i + gA^\Lambda k^i_\Lambda(z) \\ \nabla \overline{z}^{i^\star} &= d\overline{z}^{i^\star} + gA^\Lambda k^{i^\star}_\Lambda(\overline{z}) \\ \nabla q^u &= dq^u + gA^\Lambda k^u_\Lambda(q) \ . \end{aligned} \tag{3.18}$$

Secondly, one gauges the composite connections by modifying them by means of killing vectors and prepotential functions.

$$\begin{aligned} \Gamma^i{}_j \equiv \Gamma^i{}_{jk} dz^k \quad &\to \quad \widehat{\Gamma}^i{}_j = \Gamma^i{}_{jk}\nabla z^k + g\, A^\Lambda\, \partial_j k^i_\Lambda \\ Q \equiv -\frac{\mathrm{i}}{2}(\partial_i K dz^i - \partial_{i^\star} K d\overline{z}^{i^\star}) \quad &\to \quad \widehat{Q} = -\frac{\mathrm{i}}{2}(\partial_i K \nabla z^i - \partial_{i^\star} K \nabla \overline{z}^{i^\star}) + g\, A^\Lambda\, \mathcal{P}^0_\Lambda \\ \omega^x \equiv \omega^x_u dq^u \quad &\to \quad \widehat{\omega}^x = \omega^x_u \nabla q^u + g\, A^\Lambda\, \mathcal{P}^x_\Lambda \\ \Delta^{\alpha\beta} \equiv \Delta^{\alpha\beta}_u dq^u \quad &\to \quad \widehat{\Delta}^{\alpha\beta} = \Delta^{\alpha\beta}_u \nabla q^u + g\, A^\Lambda\, \partial_u k^v_\Lambda\, \mathcal{U}^{u\alpha A}\, \mathcal{U}^\beta_{vA} \ , \end{aligned} \tag{3.19}$$

where for simplicity we have assumed a single coupling constant. If there are many coupling constants corresponding to various factors of the gauge group the formulas are obviously modified. Correspondingly, the gauged curvatures are:

$$\begin{aligned} \widehat{R}^i{}_j &= R^i{}_{j\ell^\star k}\, \nabla \overline{z}^{\ell^\star} \wedge \nabla z^k \,+\, g\, F^\Lambda\, \partial_j k^i_\Lambda \\ \widehat{\mathcal{K}} \equiv d\widehat{Q} &= \mathrm{i} g_{ij^\star}\, \nabla \overline{z}^i \wedge \nabla z^{j^\star} \,+\, g\, F^\Lambda\, \mathcal{P}^0_\Lambda \\ \widehat{\Omega}^x &= \Omega^x_{uv}\, \nabla q^u \wedge \nabla q^v \,+\, g\, F^\Lambda\, \mathcal{P}^x_\Lambda \\ \widehat{\mathbb{R}}^{\alpha\beta} &= \mathbb{R}^{\alpha\beta}_{uv}\, \nabla q^u \wedge \nabla q^v \,+\, g\, A^\Lambda\, \partial_u k^v_\Lambda\, \mathcal{U}^{u|\alpha A}\, \mathcal{U}^\beta_{v|A} \ . \end{aligned} \tag{3.20}$$

4 The Complete N=2 Supergravity Theory

We are finally ready to write the supersymmetric invariant action and supersymmetry transformation rules for a completely general $N = 2$ supergravity.

Such a theory includes

1. *the gravitational multiplet*
$$(V^a_\mu, \psi^A, \psi_A, A^0) \tag{4.1}$$
described by the vielbein one-form V^a, $(a = 0,1,2,3)$ (together with the spin connection one-form ω^{ab}), the $SU(2)$ doublet of gravitino one-forms ψ^A, ψ_A $(A = 1,2$ and the upper or lower position of the index denotes right, respectively left chirality, namely $\gamma_5\psi_A = -\gamma_5\psi^A = 1)$, and the graviphoton one-form A^0

2. n_V *vector multiplets*
$$(A^I, \lambda^{iA}, \lambda^{i^\star}_A, z^i) \tag{4.2}$$
containing a gauge boson one-form A^I $(I = 1, \ldots, n_V)$, a doublet of gauginos (zero-form spinors) λ^{iA}, $\lambda^{i^\star}_A$ of left and right chirality respectively, and a complex scalar field (zero-form) z^i $(i =, 1, \ldots, n_V)$. The scalar fields z^i can be regarded as arbitrary coordinates on the special manifold $\mathcal{SK}$ of complex dimension n_V.

3. n_H *hypermultiplets*

$$(\zeta_\alpha, \zeta^\alpha, q^u) \tag{4.3}$$

formed by a doublet of zero-form spinors, that is the hyperinos $\zeta_\alpha\, \zeta^\alpha$ ($\alpha = 1, \dots, 2n_H$ and here the lower or upper position of the index denotes left, respectively right chirality), and four real scalar fields q^u ($u = 1, \dots, 4n_H$), that can be regarded as arbitrary coordinates of the quaternionic manifold $\mathcal{Q}$, of real dimension $4n_H$. As already mentioned, any quaternionic manifold has a holonomy group:

$$\mathcal{H}ol\,(\mathcal{Q}) \subset SU(2) \otimes Sp(2n_H, \mathbb{R}) \tag{4.4}$$

and the index α of the hyperinos transforms in the fundamental representation of $Sp(2n_H, \mathbb{R})$.

The definition of curvatures in the gravitational sector is given by:

$$\begin{aligned}
T^a &\equiv dV^a - \omega^a{}_b \wedge V^b - \mathrm{i}\,\overline{\psi}_A \wedge \gamma^a \psi^A \\
\rho_A &\equiv d\psi_A - \frac{1}{4}\gamma_{ab}\,\omega^{ab} \wedge \psi_A + \frac{\mathrm{i}}{2}\widehat{Q} \wedge \psi_A + \widehat{\omega}_A{}^B \wedge \psi_B \equiv \nabla\psi_A \\
\rho^A &\equiv d\psi^A - \frac{1}{4}\gamma_{ab}\,\omega^{ab} \wedge \psi^A - \frac{\mathrm{i}}{2}\widehat{Q} \wedge \psi^A + \widehat{\omega}^A{}_B \wedge \psi^B \equiv \nabla\psi^A \\
R^{ab} &\equiv d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb}\ ,
\end{aligned} \tag{4.5}$$

where $\omega_A{}^B = \frac{\mathrm{i}}{2}\omega^x(\sigma_x)_A^B$ and $\omega^A{}_B = \epsilon^{AC}\epsilon_{DB}\omega_C{}^D$. In all the above formulae the pull–back on space–time through the maps

$$z^i \ :\ M_4 \longrightarrow \mathcal{SK} \quad ; \quad q^u \ :\ M_4 \longrightarrow \mathcal{Q} \tag{4.6}$$

is obviously understood. In this way the composite connections become one-forms on space-time.

In the vector multiplet sector the curvatures and covariant derivatives are:

$$\begin{aligned}
\nabla z^i &\equiv dz^i + g\, A^\Lambda\, k^i_\Lambda(z) \\
\nabla \overline{z}^{i^\star} &\equiv d\overline{z}^{i^\star} + g\, A^\Lambda\, k^{i^\star}_\Lambda(\overline{z}) \\
\nabla\lambda^{iA} &\equiv d\lambda^{iA} - \frac{1}{4}\gamma_{ab}\,\omega^{ab}\lambda^{iA} - \frac{\mathrm{i}}{2}\widehat{Q}\lambda^{iA} + \widehat{\Gamma}^i{}_j\lambda^{jA} + \widehat{\omega}^A{}_B \wedge \lambda^{iB} \\
\nabla\lambda^{i^\star}_A &\equiv d\lambda^{i^\star}_A - \frac{1}{4}\gamma_{ab}\,\omega^{ab}\lambda^{i^\star}_A + \frac{\mathrm{i}}{2}\widehat{Q}\lambda^{i^\star}_A + \widehat{\Gamma}^{i^\star}{}_{j^\star}\lambda^{j^\star}_A + \widehat{\omega}_A{}^B \wedge \lambda^{i^\star}_B \\
F^\Lambda &\equiv dA^\Lambda + \frac{1}{2}\, g\, f^\Lambda{}_{\Sigma\Gamma}\, A^\Sigma \wedge A^\Gamma + \overline{L}^\Lambda\overline{\psi}_A \wedge \psi_B\epsilon^{AB} + L^\Lambda\overline{\psi}^A \wedge \psi^B\epsilon_{AB}
\end{aligned} \tag{4.7}$$

where $L^\Lambda = e^{\frac{\mathcal{K}}{2}}X^\Lambda$ is the first half (electric) of the symplectic section introduced in equation (2.5). (The second part M_Λ of such symplectic sections would appear in the magnetic field strengths if we did introduce them.)

Finally, in the hypermultiplet sector the covariant derivatives are:

$$\begin{aligned}
\mathcal{U}^{A\alpha} &\equiv \mathcal{U}^{A\alpha}_v\nabla q^v \equiv \mathcal{U}^{A\alpha}_v\left(d\,q^v + g\, A^\Lambda\, k^v_\Lambda(q)\right) \\
\nabla\zeta_\alpha &\equiv d\zeta_\alpha - \frac{1}{4}\omega^{ab}\,\gamma_{ab}\,\zeta_\alpha - \frac{\mathrm{i}}{2}\widehat{Q}\,\zeta_\alpha + \widehat{\Delta}_\alpha{}^\beta\zeta_\beta \\
\nabla\zeta^\alpha &\equiv d\zeta^\alpha - \frac{1}{4}\omega^{ab}\,\gamma_{ab}\,\zeta^\alpha + \frac{\mathrm{i}}{2}\widehat{Q}\,\zeta^\alpha + \widehat{\Delta}^\alpha{}_\beta\zeta^\beta
\end{aligned} \tag{4.8}$$

with

$$\widehat{\Delta}_{\alpha}{}^{\beta} \equiv \widehat{\Delta}^{\gamma\beta}\mathbb{C}_{\gamma\alpha} \quad ; \quad \widehat{\Delta}^{\alpha}{}_{\beta} \equiv \mathbb{C}_{\beta\gamma}\,\widehat{\Delta}^{\alpha\gamma} \tag{4.9}$$

Note that the Kähler weights of all spinor fields are given by the coefficients of $\mathrm{i}\widehat{Q}$ in the definition of their curvatures and covariant derivatives.

Our next task is to write down the $N=2$ space-time lagrangian and the supersymmetry transformation laws of the fields. The method employed for this construction is based on the geometrical approach, review in [29], and a more detailed derivation is given in the appendices A and B of [23]. Actually, one solves the Bianchi identities in $N=2$ superspace and then constructs the rheonomic superspace Lagrangian in such a way that the superspace curvatures and covariant derivatives given by the solution of the Bianchi identities are reproduced by the variational equations of motion derived from the lagrangian. After this procedure is completed the space-time lagrangian is immediately retrieved by restricting the superspace p-forms to space-time. The resulting action can be split in the following way:

$$\begin{aligned} S &= \int \sqrt{-g}\, d^4x \left[\mathcal{L}_k + \mathcal{L}_{4f} + \mathcal{L}'_g\right] \ , \\ \mathcal{L}_k &= \mathcal{L}^{inv}_{kin} + \mathcal{L}_{Pauli} \ , \\ \mathcal{L}_{4f} &= \mathcal{L}^{inv}_{4f} + \mathcal{L}^{non\,inv}_{4f} \ , \\ \mathcal{L}'_g &= \mathcal{L}_{mass} - V(z,\overline{z},q) \ , \end{aligned} \tag{4.10}$$

where $\mathcal{L}^{inv}_{kin}$ consists of the true kinetic terms as well as Pauli-like terms containing the derivatives of the scalar fields. The modifications due to the gauging are contained not only in $\mathcal{L}'_g$ but also in the gauged covariant derivatives in the rest of the lagrangian. We collect the various terms of (4.10) in the table below.

N=2 Supergravity lagrangian

$$\begin{aligned} \mathcal{L}^{inv}_{kin} &= -\frac{1}{2}R + g_{ij^\star}\nabla^\mu z^i \nabla_\mu \overline{z}^{j^\star} + h_{uv}\nabla_\mu q^u \nabla^\mu q^v + \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\left(\overline{\Psi}^A_\mu \gamma_\sigma \rho_{A\nu\lambda} - \overline{\Psi}_{A\mu}\gamma_\sigma \rho^A_{\nu\lambda}\right) \\ &- \frac{\mathrm{i}}{2} g_{ij^\star}\left(\overline{\lambda}^{iA}\gamma^\mu \nabla_\mu \lambda^{j^\star}_A + \overline{\lambda}^{j^\star}_A \gamma^\mu \nabla_\mu \lambda^{iA}\right) - \mathrm{i}\left(\overline{\zeta}^\alpha \gamma^\mu \nabla_\mu \zeta_\alpha + \overline{\zeta}_\alpha \gamma^\mu \nabla_\mu \zeta^\alpha\right) \\ &+ \mathrm{i}\left(\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\mathcal{F}^{-\Sigma\mu\nu} - \mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}\mathcal{F}^{+\Sigma\mu\nu}\right) + \Big\{ - g_{ij^\star}\nabla_\mu \overline{z}^{j^\star}\overline{\Psi}^\mu_A \lambda^{iA} \\ &- 2\mathcal{U}^{A\alpha}_u \nabla_\mu q^u \overline{\Psi}^\mu_A \zeta_\alpha + g_{ij^\star}\nabla_\mu \overline{z}^{j^\star}\overline{\lambda}^{iA}\gamma^{\mu\nu}\Psi_{A\nu} + 2\mathcal{U}^{\alpha A}_u \nabla_\mu q^u \overline{\zeta}_\alpha \gamma^{\mu\nu}\Psi_{A\nu} + \mathrm{h.c.}\Big\} \end{aligned} \tag{4.11}$$

$$\begin{aligned} \mathcal{L}_{Pauli} &= \Big\{\mathcal{F}^{-\Lambda}_{\mu\nu}\left(\mathrm{Im}\,\mathcal{N}\right)_{\Lambda\Sigma}[4L^\Sigma \overline{\Psi}^{A\mu}\Psi^{B\nu}\epsilon_{AB} - 4\mathrm{i}\overline{f}^\Sigma_{i^\star}\overline{\lambda}^{i^\star}_A \gamma^\nu \Psi^\mu_B \epsilon^{AB} + \\ &+ \frac{1}{2}\nabla_i f^\Sigma_j \overline{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB}\epsilon_{AB} - L^\Sigma \overline{\zeta}_\alpha \gamma^{\mu\nu}\zeta_\beta \mathbb{C}^{\alpha\beta}] + \mathrm{h.c.}\Big\} \end{aligned} \tag{4.12}$$

$$\mathcal{L}^{inv}_{4f} = \frac{\mathrm{i}}{2}\left(g_{ij^\star}\overline{\lambda}^{iA}\gamma_\sigma \lambda^{j^\star}_B - 2\delta^A_B \overline{\zeta}^\alpha \gamma_\sigma \zeta_\alpha\right)\overline{\Psi}_{A\mu}\gamma_\lambda \Psi^B_\nu \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}$$

$$
\begin{aligned}
&- \frac{1}{6}\left(C_{ijk}\overline{\lambda}^{iA}\gamma^\mu\Psi^B_\mu\,\overline{\lambda}^{jC}\lambda^{kD}\,\epsilon_{AC}\epsilon_{BD} + h.c.\right)\\
&- 2\overline{\Psi}^A_\mu\Psi^B_\nu\,\overline{\Psi}^\mu_A\Psi^\nu_B + 2g_{ij^\star}\,\overline{\lambda}^{iA}\gamma_\mu\Psi^B_\nu\,\overline{\lambda}^{i^\star}_A\gamma^\mu\Psi^\nu_B\\
&+ \frac{1}{4}\left(R_{ij^\star lk^\star} + g_{ik^\star}\,g_{lj^\star} - \frac{3}{2}g_{ij^\star}\,g_{lk^\star}\right)\overline{\lambda}^{iA}\lambda^{lB}\overline{\lambda}^{j^\star}_A\lambda^{k^\star}_B\\
&+ \frac{1}{4}g_{ij^\star}\,\overline{\zeta}^\alpha\gamma_\mu\zeta_\alpha\,\overline{\lambda}^{iA}\gamma^\mu\lambda^{j^\star}_A + \frac{1}{2}\mathcal{R}^\alpha_{\beta ts}\,\mathcal{U}^t_{A\gamma}\,\mathcal{U}^s_{B\delta}\epsilon^{AB}\,C^{\delta\eta}\overline{\zeta}_\alpha\,\zeta_\eta\,\overline{\zeta}^\beta\,\zeta^\gamma\\
&- \left[\frac{\mathrm{i}}{12}\nabla_m\,C_{jkl}\overline{\lambda}^{jA}\lambda^{mB}\overline{\lambda}^{kC}\lambda^{lD}\epsilon_{AC}\epsilon_{BD} + h.c.\right]\\
&+ g_{ij^\star}\,\overline{\Psi}^A_\mu\lambda^{j^\star}_A\,\overline{\Psi}^\mu_B\lambda^{iB} + 2\overline{\Psi}^A_\mu\zeta^\alpha\overline{\Psi}^\mu_A\zeta_\alpha + \left(\epsilon_{AB}\,\mathbb{C}_{\alpha\beta}\,\overline{\Psi}^A_\mu\zeta^\alpha\,\overline{\Psi}^{B|\mu}\zeta^\beta + h.c.\right)
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
\mathcal{L}^{non\,inv}_{4f} = &\Big\{(\mathrm{Im}\,\mathcal{N})_{\Lambda\Sigma}\Big[2L^\Lambda L^\Sigma\left(\overline{\Psi}^A_\mu\Psi^B_\nu\right)^-\left(\overline{\Psi}^C_\mu\Psi^D_\nu\right)^-\epsilon_{AB}\,\epsilon_{CD}\\
&- 8\mathrm{i}\,L^\Lambda\overline{f}^\Sigma_{i^\star}\left(\overline{\Psi}^A_\mu\Psi^B_\nu\right)^-\left(\overline{\lambda}^{i^\star}_A\gamma^\nu\Psi^\mu_B\right)^-\\
&- 2\overline{f}^\Lambda_{i^\star}\overline{f}^\Sigma_{j^\star}\left(\overline{\lambda}^{i^\star}_A\gamma^\nu\Psi^\mu_B\right)^-\left(\overline{\lambda}^{j^\star}_C\gamma_\nu\Psi_{D|\mu}\right)^-\epsilon^{AB}\,\epsilon^{CD}\\
&+ \frac{\mathrm{i}}{2}L^\Lambda\overline{f}^\Sigma_{\ell^\star}\,g^{k\ell^\star}\,C_{ijk}\left(\overline{\Psi}^A_\mu\Psi^B_\nu\right)^-\overline{\lambda}^{iC}\gamma^{\mu\nu}\lambda^{jD}\,\epsilon_{AB}\,\epsilon_{CD}\\
&+ \overline{f}^\Lambda_{m^\star}\overline{f}^\Sigma_{\ell^\star}\,g^{k\ell^\star}\,C_{ijk}\left(\overline{\lambda}^{m^\star}_A\gamma_\nu\Psi_{B\mu}\right)^-\overline{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB}\\
&- L^\Lambda L^\Sigma\left(\overline{\Psi}^A_\mu\Psi^B_\nu\right)^-\overline{\zeta}_\alpha\gamma^{\mu\nu}\zeta_\beta\,\epsilon_{AB}\,\mathbb{C}^{\alpha\beta}\\
&+ \mathrm{i}L^\Lambda\overline{f}^\Sigma_{i^\star}\left(\overline{\lambda}^{i^\star}_A\gamma^\nu\Psi^\mu_B\right)^-\overline{\zeta}_\alpha\gamma_{\mu\nu}\zeta_\beta\,\epsilon^{AB}\mathbb{C}^{\alpha\beta}\\
&- \frac{1}{32}C_{ijk}\,C_{lmn}g^{k\overline{r}}\,g^{n\overline{s}}\,\overline{f}^\Lambda_{\overline{r}}\,\overline{f}^\Sigma_{\overline{s}}\,\overline{\lambda}^{iA}\,\gamma_{\mu\nu}\,\lambda^{jB}\,\overline{\lambda}^{kC}\,\gamma^{\mu\nu}\,\lambda^{lD}\,\epsilon_{AB}\epsilon_{CD}\\
&- \frac{1}{8}L^\Lambda\nabla_i f^\Sigma_j\overline{\zeta}_\alpha\gamma_{\mu\nu}\zeta_\beta\,\overline{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB}\;\epsilon_{AB}\,\mathbb{C}^{\alpha\beta}\\
&+ \frac{1}{8}L^\Lambda\,L^\Sigma\overline{\zeta}_\alpha\gamma_{\mu\nu}\zeta_\beta\,\overline{\zeta}_\gamma\gamma^{\mu\nu}\zeta_\delta\,\mathbb{C}^{\alpha\beta}\mathbb{C}^{\gamma\delta}\Big] + \mathrm{h.c.}\Big\}
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
\mathcal{L}_{mass} = &\; g\Big[2S_{AB}\overline{\Psi}^A_\mu\gamma^{\mu\nu}\Psi^B_\nu + \mathrm{i}g_{ij^\star}W^{iAB}\overline{\lambda}^{j^\star}_A\gamma_\mu\Psi^\mu_B + 2\mathrm{i}N^A_\alpha\overline{\zeta}^\alpha\gamma_\mu\Psi^\mu_A\\
&+ \mathcal{M}^{\alpha\beta}\overline{\zeta}_\alpha\zeta_\beta + \mathcal{M}^\alpha{}_{iB}\overline{\zeta}_\alpha\lambda^{iB} + \mathcal{M}_{iA\ell B}\overline{\lambda}^{iA}\lambda^{\ell B}\Big] + \mathrm{h.c.}
\end{aligned}
\tag{4.15}
$$

$$
\mathrm{V}(z,\overline{z},q) = g^2\Big[\left(g_{ij^\star}k^i_\Lambda k^{j^\star}_\Sigma + 4h_{uv}k^u_\Lambda k^v_\Sigma\right)\overline{L}^\Lambda L^\Sigma + g^{ij^\star}f^\Lambda_i f^\Sigma_{j^\star}\mathcal{P}^x_\Lambda\mathcal{P}^x_\Sigma - 3\overline{L}^\Lambda L^\Sigma\mathcal{P}^x_\Lambda\mathcal{P}^x_\Sigma\Big]
\tag{4.16}
$$

where $\mathcal{F}^{\pm\Lambda}_{\mu\nu} = \frac{1}{2}(\mathcal{F}^\Lambda_{\mu\nu} \pm \frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^\Lambda_{\rho\sigma})$ and $(...)^-$ denotes the self dual part of the fermion bilinears. The mass–matrices are given by:

$$
\begin{aligned}
S_{AB} &= \frac{\mathrm{i}}{2}(\sigma_x)_A{}^C\epsilon_{BC}\mathcal{P}^x_\Lambda L^\Lambda\\
W^{iAB} &= \epsilon^{AB}\,k^i_\Lambda\overline{L}^\Lambda + \mathrm{i}(\sigma_x)_C{}^B\epsilon^{CA}\mathcal{P}^x_\Lambda g^{ij^\star}\overline{f}^\Lambda_{j^\star}\\
N^A_\alpha &= 2\,\mathcal{U}^A_{\alpha u}\,k^u_\Lambda\,\overline{L}^\Lambda\\
\mathcal{M}^{\alpha\beta} &= -\,\mathcal{U}^{\alpha A}_u\,\mathcal{U}^{\beta B}_v\,\varepsilon_{AB}\,\nabla^{[u}k^{v]}_\Lambda\,L^\Lambda\\
\mathcal{M}^\alpha{}_{iB} &= -4\,\mathcal{U}^\alpha_{Bu}\,k^u_\Lambda\,f^\Lambda_i
\end{aligned}
$$

$$\mathcal{M}_{iA|\ell B} = \frac{1}{3}\left(\varepsilon_{AB}\, g_{ij^\star} k^{j^\star}_\Lambda f^\Lambda_\ell + \mathrm{i}(\sigma_x \epsilon^{-1})_{AB}\, \mathcal{P}^x_\Lambda \nabla_\ell f^\Lambda_i\right) \tag{4.17}$$

The coupling constant g in $\mathcal{L}'_g$ is just a symbolic notation to remind that these terms are entirely due to the gauging and vanish in the ungauged theory, where also all gauged covariant derivatives reduce to ordinary ones. Note that in general there is not a single coupling constant, but rather there are as many independent coupling constants as the number of factors in the gauge group. The normalization of the kinetic term for the quaternions depends on the scale λ of the quaternionic manifold, appearing in eq. (2.34), for which we have chosen the value $\lambda = -1$.

Furthermore, using the geometric approach, the form of the supersymmetry transformation laws is also easily deduced from the solution of the Bianchi identities in superspace [23]by interpreting the supersymmetry variations as superspace lie derivatives. One gets

---

Supergravity transformation rules of the Fermi fields

$$\begin{aligned}
\delta\,\Psi_{A\mu} &= \mathcal{D}_\mu\,\epsilon_A - \frac{1}{4}\left(\partial_i\, K\overline{\lambda}^{iB}\epsilon_B - \partial_{i^\star}\, K\overline{\lambda}^{i^\star}_B\epsilon^B\right)\Psi_{A\mu} \\
&\quad -\omega_{Av}^{\ \ B}\,\mathcal{U}^v_{C\alpha}\left(\epsilon^{CD}\,\mathbb{C}^{\alpha\beta}\,\overline{\zeta}_\beta\,\epsilon_D + \overline{\zeta}^\alpha\,\epsilon^C\right)\Psi_{B\mu} \\
&\quad + \left(A_A^{\ \nu B}\eta_{\mu\nu} + A'^{\ \nu B}_A\gamma_{\mu\nu}\right)\epsilon_B \\
&\quad + \left[\mathrm{i}\, g\, S_{AB}\eta_{\mu\nu} + \epsilon_{AB}(T^-_{\mu\nu} + U^+_{\mu\nu})\right]\gamma^\nu\epsilon^B \qquad (4.18) \\
\delta\,\lambda^{iA} &= \frac{1}{4}\left(\partial_j\, K\overline{\lambda}^{jB}\epsilon_B - \partial_{j^\star}\, K\overline{\lambda}^{j^\star}_B\epsilon^B\right)\lambda^{iA} \\
&\quad -\omega^A_{\ \ Bv}\,\mathcal{U}^v_{C\alpha}\left(\epsilon^{CD}\,\mathbb{C}^{\alpha\beta}\,\overline{\zeta}_\beta\,\epsilon_D + \overline{\zeta}^\alpha\,\epsilon^C\right)\lambda^{iB} \\
&\quad - \Gamma^i_{\ jk}\overline{\lambda}^{kB}\epsilon_B\,\lambda^{jA} + \mathrm{i}\left(\nabla_\mu\, z^i - \overline{\lambda}^{iA}\psi_{A\mu}\right)\gamma^\mu\epsilon^A \\
&\quad + G^{-i}_{\mu\nu}\gamma^{\mu\nu}\epsilon_B\epsilon^{AB} + D^{iAB}\epsilon_B \qquad (4.19) \\
\delta\,\zeta_\alpha &= -\Delta_{\alpha v}^{\ \ \beta}\,\mathcal{U}^v_{\gamma A}\left(\epsilon^{AB}\,\mathbb{C}^{\gamma\delta}\,\overline{\zeta}_\delta\,\epsilon_B + \overline{\zeta}^\gamma\,\epsilon^A\right)\zeta_\beta \\
&\quad + \frac{1}{4}\left(\partial_i\, K\overline{\lambda}^{iB}\epsilon_B - \partial_{i^\star}\, K\overline{\lambda}^{i^\star}_B\epsilon^B\right)\zeta_\alpha \\
&\quad + \mathrm{i}\left(\mathcal{U}^{B\beta}_u\,\nabla_\mu\, q^u - \epsilon^{BC}\,\mathbb{C}^{\beta\gamma}\,\overline{\zeta}_\gamma\,\psi_C - \overline{\zeta}^\beta\,\psi^B\right)\gamma^\mu\epsilon^A\epsilon_{AB}\,\mathbb{C}_{\alpha\beta} + g\, N^A_\alpha\,\epsilon_A \qquad (4.20)
\end{aligned}$$

Supergravity transformation rules of the Bose fields

$$\begin{aligned}
\delta\, V^a_\mu &= -\mathrm{i}\,\overline{\Psi}_{A\mu}\,\gamma^a\,\epsilon^A - \mathrm{i}\,\overline{\Psi}^A_\mu\,\gamma^a\,\epsilon_A \qquad (4.21) \\
\delta\, A^\Lambda_\mu &= 2\overline{L}^\Lambda\overline{\psi}_{A\mu}\epsilon_B\epsilon^{AB} + 2L^\Lambda\overline{\psi}^A_\mu\epsilon^B\epsilon_{AB}
\end{aligned}$$

$$+ \quad \left(\mathrm{i}\, f^\Lambda_i\, \overline{\lambda}^{iA} \gamma_\mu \epsilon^B\, \epsilon_{AB} + \mathrm{i}\, \overline{f}^\Lambda_{i^\star}\, \overline{\lambda}^{i^\star}_A \gamma_\mu \epsilon_B\, \epsilon^{AB}\right) \tag{4.22}$$

$$\delta\, z^i = \overline{\lambda}^{iA} \epsilon_A \tag{4.23}$$

$$\delta\, z^{i^\star} = \overline{\lambda}^{i^\star}_A \epsilon^A \tag{4.24}$$

$$\delta\, q^u = \mathcal{U}^u_{\alpha A} \left(\overline{\zeta}^\alpha \epsilon^A + \mathbb{C}^{\alpha\beta} \epsilon^{AB} \overline{\zeta}_\beta \epsilon_B\right) \tag{4.25}$$

where we have:

Supergravity values of the auxiliary fields

$$A_A{}^{\mu B} = -\frac{\mathrm{i}}{4}\, g_{k^\star \ell} \left(\overline{\lambda}^{k^\star}_A \gamma^\mu \lambda^{\ell B} - \delta^B_A\, \overline{\lambda}^{k^\star}_C \gamma^\mu \lambda^{\ell C}\right) \tag{4.26}$$

$$A'_A{}^{\mu B} = \frac{\mathrm{i}}{4}\, g_{k^\star \ell} \left(\overline{\lambda}^{k^\star}_A \gamma^\mu \lambda^{\ell B} - \frac{1}{2}\, \delta^B_A\, \overline{\lambda}^{k^\star}_C \gamma^\mu \lambda^{C\ell}\right) - \frac{\mathrm{i}}{4}\, \delta^B_A\, \overline{\zeta}_\alpha \gamma^\mu \zeta^\alpha \tag{4.27}$$

$$T^-_{\mu\nu} = 2\mathrm{i}\, (\mathrm{Im}\, \mathcal{N})_{\Lambda\Sigma}\, L^\Sigma \left(\widetilde{F}^{\Lambda -}_{\mu\nu} + \frac{1}{8} \nabla_i\, f^\Lambda_j\, \overline{\lambda}^{iA} \gamma_{\mu\nu}\, \lambda^{jB}\, \epsilon_{AB} - \frac{1}{4}\, \mathbb{C}^{\alpha\beta}\, \overline{\zeta}_\alpha \gamma_{\mu\nu}\, \zeta_\beta\, L^\Lambda\right) \tag{4.28}$$

$$T^+_{\mu\nu} = 2\mathrm{i}\, (\mathrm{Im}\, \mathcal{N})_{\Lambda\Sigma}\, \overline{L}^\Sigma \left(\widetilde{F}^{\Lambda +}_{\mu\nu} + \frac{1}{8} \nabla_{i^\star}\, \overline{f}^\Lambda_{j^\star}\, \overline{\lambda}^{i^\star}_A \gamma_{\mu\nu}\, \lambda^{j^\star}_B\, \epsilon^{AB} - \frac{1}{4}\, \mathbb{C}_{\alpha\beta}\, \overline{\zeta}^\alpha \gamma_{\mu\nu}\, \zeta^\beta\, \overline{L}^\Lambda\right) \tag{4.29}$$

$$U^-_{\mu\nu} = -\frac{\mathrm{i}}{4}\, \mathbb{C}^{\alpha\beta}\, \overline{\zeta}_\alpha \gamma_{\mu\nu}\, \zeta_\beta \tag{4.30}$$

$$U^+_{\mu\nu} = -\frac{\mathrm{i}}{4}\, \mathbb{C}_{\alpha\beta}\, \overline{\zeta}^\alpha \gamma_{\mu\nu}\, \zeta^\beta \tag{4.31}$$

$$\begin{aligned} G^{i-}_{\mu\nu} = & -g^{ij^\star} \overline{f}^\Gamma_{j^\star}\, (\mathrm{Im}\, \mathcal{N})_{\Gamma\Lambda} \left(\widetilde{F}^{\Lambda -}_{\mu\nu} + \frac{1}{8} \nabla_k f^\Lambda_\ell \overline{\lambda}^{kA} \gamma_{\mu\nu}\, \lambda^{\ell B}\, \epsilon_{AB}\right. \\ & \left. -\frac{1}{4}\, \mathbb{C}^{\alpha\beta}\, \overline{\zeta}_\alpha \gamma_{\mu\nu}\, \zeta_\beta\, L^\Lambda\right) \end{aligned} \tag{4.32}$$

$$\begin{aligned} G^{i^\star +}_{\mu\nu} = & -g^{i^\star j} f^\Gamma_j\, (\mathrm{Im}\, \mathcal{N})_{\Gamma\Lambda} \left(\widetilde{F}^{\Lambda +}_{\mu\nu} + \frac{1}{8} \nabla_{k^\star} \overline{f}^\Lambda_{\ell^\star} \overline{\lambda}^{k^\star}_A \gamma_{\mu\nu}\, \lambda^{\ell^\star}_B\, \epsilon^{AB}\right. \\ & \left. -\frac{1}{4}\, \mathbb{C}_{\alpha\beta}\, \overline{\zeta}^\alpha \gamma_{\mu\nu}\, \zeta^\beta\, \overline{L}^\Lambda\right) \end{aligned} \tag{4.33}$$

$$D^{iAB} = \frac{\mathrm{i}}{2} g^{ij^\star} C_{j^\star k^\star \ell^\star} \overline{\lambda}^{k^\star}_C\, \lambda^{\ell^\star}_D\, \epsilon^{AC} \epsilon^{BD} + W^{iAB} \tag{4.34}$$

In eqs. (4.28), (4.29), (4.32), (4.33) we have denoted by $\widetilde{F}_{\mu\nu}$ the supercovariant field strength defined by:

$$\widetilde{F}^\Lambda_{\mu\nu} = \mathcal{F}^\Lambda_{\mu\nu} + L^\Lambda \overline{\psi}^A_\mu \psi^B_\nu\, \epsilon_{AB} + \overline{L}^\Lambda \overline{\psi}_{A\mu} \psi_{B\nu} \epsilon^{AB} - \mathrm{i}\, f^\Lambda_i\, \overline{\lambda}^{iA} \gamma_{[\nu} \psi^B_{\mu]}\, \epsilon_{AB} - \mathrm{i}\, \overline{f}^\Lambda_{i^\star}\, \overline{\lambda}^{i^\star}_A \gamma_{[\nu} \psi_{B\mu]}\, \epsilon^{AB}\,. \tag{4.35}$$

Let us make some observation about the structure of the Lagrangian and of the transformation laws.

i) We note that all the terms of the lagrangian are given in terms of purely geometric objects pertaining to the special and quaternionic geometries. Furthermore the Lagrangian does not rely on the existence of a prepotential function $F = F(X)$ and it is valid for any choice of the quaternionic manifold.

ii) The lagrangian is not invariant under symplectic duality transformations. However, in absence of gauging ($g = 0$), if we restrict the lagrangian to configurations where the vectors are on shell, it becomes symplectic invariant[30, 4]. This allows us to fix the terms appearing in $\mathcal{L}_{4f}^{non\,inv}$ in a way independent from supersymmetry arguments.

iii) We note that the field strengths $\mathcal{F}^{\Lambda -}_{\mu\nu}$ originally introduced in the Lagrangian are the free gauge field strengths. The interacting field strengths which are supersymmetry eigenstates are defined as the objects appearing in the transformation laws of the gravitinos and gauginos fields, respectively, namely the bosonic part of $T^{-}_{\mu\nu}$ and $G^{-i}_{\mu\nu}$ defined in eq.s (4.28), (4.32).

5 Comments on the scalar potential

A general Ward identity[31] of N-extended supergravity establishes the following formulae for the scalar potential $V(\phi)$ of the theory (in appropriate normalizations for the generic fermionic shifts $\delta\chi^a$)

$$Z_{ab}\delta_A\chi^a\delta^B\overline{\chi}^b - 3\mathcal{M}_{AC}\overline{\mathcal{M}}^{CB} = \delta^A{}_B V(\phi) \quad A,B = 1,\ldots,N \tag{5.36}$$

where $\delta_A\chi^a$ is the extra contribution, due to the gauging, to the spin $\frac{1}{2}$ supersymmetry variations of the scalar vev's, Z_{ab} is the (scalar dependent) kinetic term normalization and $\mathcal{M}_{AC}$ is the (scalar dependent) gravitino mass matrix. Since in the case at hand ($N = 2$) all terms in question are expressed in terms of Killing vectors and prepotentials, contracted with the symplectic sections, we will be able to derive a completely geometrical formula for $V(z,\overline{z},q)$. The relevant terms in the fermionic transformation rules are

$$\begin{aligned}
\delta\psi_{A\mu} &= igS_{AB}\gamma_\mu\epsilon^B \ , \\
\delta\lambda^{iA} &= gW^{iAB}\epsilon_B \ , \\
\delta\zeta_\alpha &= gN^A_\alpha\epsilon_A \ .
\end{aligned} \tag{5.37}$$

In our normalization the previous Ward identity gives

$$V = (g_{ij^\star}k^i_\Lambda k^{j^\star}_\Sigma + 4h_{uv}k^u_\Lambda k^v_\Sigma)\overline{L}^\Lambda L^\Sigma + (U^{\Lambda\Sigma} - 3\overline{L}^\Lambda L^\Sigma)\mathcal{P}^x_\Lambda\mathcal{P}^x_\Sigma \ . \tag{5.38}$$

with $U^{\Lambda\Sigma}$ is defined in (2.17). Above, the first two terms are related to the gauging of isometries of $\mathcal{SK}\otimes\mathcal{Q}$. For an abelian group, the first term is absent. The negative term is the gravitino mass contribution, while the one in $U^{\Lambda\Sigma}$ is the gaugino shift contribution due to the quaternionic prepotential.

Eq. (5.38) can be rewritten in a suggestive form as

$$V = (k_\Lambda, k_\Sigma)\overline{L}^\Lambda L^\Sigma + (U^{\Lambda\Sigma} - 3\overline{L}^\Lambda L^\Sigma)(\mathcal{P}^x_\Lambda\mathcal{P}^x_\Sigma - \mathcal{P}_\Lambda\mathcal{P}_\Sigma) \ , \tag{5.39}$$

where

$$(k_\Lambda, k_\Sigma) = (\, k^i_\Lambda, \quad k^{i^\star}_\Lambda, \quad k^u_\Lambda \,) \begin{pmatrix} 0 & g_{ij^\star} & 0 \\ g_{i^\star j} & 0 & 0 \\ 0 & 0 & 2h_{uv} \end{pmatrix} \begin{pmatrix} k^j_\Sigma \\ k^{j^\star}_\Sigma \\ k^v_\Sigma \end{pmatrix} \tag{5.40}$$

is the scalar product of the Killing vector and we have used eqs. (3.4),(3.16) . $\mathcal{P}^x_\Lambda$ are the quaternionic (triplet) prepotentials and $U^{\Lambda\Sigma}, L^\Lambda$ are special geometry data.

In a theory with only abelian vectors, the potential may still be non-zero due to Fayet-Iliopoulos terms:

$$\mathcal{P}^x_\Lambda = \xi^x_\Lambda \text{ (constant)}; \quad \epsilon^{xyz}\xi^y_\Lambda \xi^z_\Sigma = 0 \ . \tag{5.41}$$

In this case

$$V(z,\overline{z}) = (U^{\Lambda\Sigma} - 3\overline{L}^\Lambda L^\Sigma)\xi^x_\Lambda \xi^x_\Sigma \ . \tag{5.42}$$

Examples with $V(z,\overline{z}) = 0$ but non-vanishing gravitino mass (with $N = 2$ supersymmetry broken to $N = 0$) were given in [27], then generalizing to $N = 2$ the no scale models of $N = 1$ supergravity [34]. These models were obtained by taking a $\xi^x_\Lambda = (\xi_0, 0, 0)$. In this case the expression

$$V = U^{00} - 3\overline{L}^0 L^0 \tag{5.43}$$

reduces to

$$V = (\partial_i K g^{ij^\star} \partial_{j^\star} K - 3)e^K \tag{5.44}$$

which is the $N = 1$ supergravity potential, with solution ($V = 0$) the cubic holomorphic prepotential

$$F(X) = d_{ABC}\frac{X^A X^B X^C}{X^0} \qquad A = 1, \ldots, n \ . \tag{5.45}$$

Another solution is obtained by taking the $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SO(n)}$ coset in the $SO(2,n)$ symmetric parametrization of the symplectic sections $(X^\Lambda, F_\Lambda = \eta_{\Lambda\Sigma} S X^\Sigma \ ; \ X^\Lambda X^\Sigma \eta_{\Lambda\Sigma} = 0 \ , \eta_{\Lambda\Sigma} = (1,1,-1,\ldots,-1))$ where a prepotential F does not exist. In this case

$$U^{\Lambda\Sigma} - 3\overline{L}^\Lambda L^\Sigma = -\frac{1}{i(S-\overline{S})}\eta_{\Lambda\Sigma} \tag{5.46}$$

where we have used the fact that

$$\mathcal{N}_{\Lambda\Sigma} = (S - \overline{S})(\Phi_\Lambda \overline{\Phi}_\Sigma + \overline{\Phi}_\Lambda \Phi_\Sigma) + \overline{S}\eta_{\Lambda\Sigma} \quad , \ \Phi^\Lambda = \frac{X^\Lambda}{(X^\Lambda \overline{X}_\Lambda)^{1/2}} \ . \tag{5.47}$$

The identity (5.46) allows one to prove that the tree level potential of an arbitrary heterotic string compactification (including orbifolds with twisted hypermultiplets) is semi-positive definite provided we don't gauge the graviphoton and the gravidilaton vectors (i.e. $\mathcal{P}^x_\Lambda = 0$ for $\Lambda = 0, 1$, $\mathcal{P}^x_\Lambda \neq 0$ for $\Lambda = 2, \ldots, n_V$). On the other hand, it also proves that tree level supergravity breaking may only occurr if $\mathcal{P}^x_\Lambda \neq 0$ for $\Lambda = 0, 1$. This instance is related to models with Scherk-Schwarz mechanism studied in the literature [32, 33].

A vanishing potential can be obtained if $\xi^x_\Lambda = (\xi_\Lambda, 0, 0)$ with

$$\xi_\Lambda \xi_\Sigma \eta^{\Lambda\Sigma} = 0 \ . \tag{5.48}$$

In this case we may also consider the gauge group to be $U(1)^{p+2}\otimes G(n_V-p)$ and introduce $\xi_\Lambda = (\xi_0,\dots,\xi_{p+1},0,\dots,0)$ such that $\xi_\Lambda\xi_\Sigma\eta^{\Lambda\Sigma}=0$ where $\eta^{\Lambda\Sigma}$ is the $SO(2,p)$ Lorentzian metric. The potential is now:

$$V = k^i_\Lambda g_{ij^\star} k^{j^\star}_\Sigma \overline{L}^\Lambda L^\Sigma \quad ; \quad (U^{\Lambda\Sigma} - 3\overline{L}^\Lambda L^\Sigma)\mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma = 0 \tag{5.49}$$

where $k^i_\Lambda L^\Lambda = 0$ for $\Lambda \le p+1$. The gravitino have equal mass

$$\mid m_{3/2} \mid \simeq e^{K/2} \mid \xi_\Lambda X^\Lambda \mid \tag{5.50}$$

with $\xi_\Lambda\xi_\Sigma\eta^{\Lambda\Sigma} = 0$, $\Lambda = 0,\dots,p+1$.

It is amusing to note that the gravitino mass, as a function of the $O(2,p)/O(2)\otimes O(p)$ moduli and of the F-I terms, just coincides with the central charge formula for the level $N_L = 1$ in heterotic string (H-monopoles), if the F-I terms are identified with the $O(2,p)$ lattice electric charges.

Note that, because of the special form of the gauged $\widehat{Q}$, $\widehat{\omega}^x$, we see that whenever $\mathcal{P}_\Lambda \neq 0$ the gravitino is charged with respect to the $U(1)$ factor and whenever $\mathcal{P}^x_\Lambda \neq 0$ the gravitino is charged with respect to the $SU(2)$ factor of the $U(1)\otimes SU(2)$ automorphism group of the supersymmetry algebra. In the case of $U(1)^p$ gauge fields with non-vanishing F-I terms $\xi^x_p = (0,0,\xi_p)$ the gauge field $A^\Lambda_\mu \xi_\Lambda = A_\mu$ gauge a $U(1)$ subgroup of $SU(2)_L$ susy algebra.

Models with breaking of $N=2$ to $N=1$ [19] necessarily require k^u_Λ not to be zero. The minimal model where this happens with $V=0$ is the one based on

$$\mathcal{SK}\otimes\mathcal{Q} = \frac{SU(1,1)}{U(1)} \otimes \frac{SO(4,1)}{SO(4)} \ , \tag{5.51}$$

where a $U(1)\otimes U(1)$ isometry of $\mathcal{Q}$ is gauged. In this case the vanishing of V requires a compensation of the $\delta\lambda,\delta\zeta$ variations with the gravitino contribution

$$4k^u_\Lambda k^v_\Sigma h_{uv} + U^{\Lambda\Sigma}\mathcal{P}^x_\Lambda\mathcal{P}^x_\Sigma = 3\overline{L}^\Lambda L^\Sigma \mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma \ . \tag{5.52}$$

The moduli space of vacua satisfying (5.52) is a four dimensional subspace of (5.51).

One may wonder where are the explicit mass terms for hypermultiplets. In $N=2$ supergravity, since the hypermultiplet mass is a central charge, which is gauged, such term corresponds to the gauging of a $U(1)$ charge. This is best seen if we consider the case where no vector multiplets (and then gauginos) are present. In this case $L^\Lambda = L^0 = 1$ and the potential becomes

$$V = 4h_{uv}k^u k^v - 3\mathcal{P}^x\mathcal{P}^x \tag{5.53}$$

where k^u is the Killing vector of a $U(1)$ symmetry of $\mathcal{Q}$, gauged by the graviphoton and $\mathcal{P}^x$ is the associated prepotential. For $\frac{SO(4,1)}{SO(4)}$ this reproduces the Zachos model [35]. The gauged $U(1)$ in this model is contained in $SU_R(2)$ which commutes with the symmetry $SU_L(2)$ in the decomposition of $SO(4) = SU_L(2)\otimes SU_R(2)$. This model has a local minimum at vanishing hypermultiplet vev at which $U(1)$ is unbroken, and the extrema (at $u=1$) (maxima) which break $U(1)$. The extremal model is when both $n_H = n_V = 0$. Still we may have a pure F-I term

$$V = -3\xi^2 \quad \xi = (\xi,0,0) \tag{5.54}$$

This corresponds to the gauging of a $U(1)\subset SU(2)_L$ and gravitinos have charged coupling. This model corresponds to anti-De Sitter $N=2$ supergravity [36].

Acknowledgements

A.C. and S. F. would like to thank the Institute for Theoretical Physics at U. C. Santa Barbara for its kind hospitality and where part of this work was completed.

References

[1] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19; Nucl. Phys. B431 (1994) 484.

[2] A. Klemm, W. Lerche, S. Theisen and S. Yankielovicz, Phys. Lett. B344 (1995) 169; P. Argyres and A. Faraggi, Phys. Rev. Lett. 74 (1995) 3931.

[3] A. Ceresole, R. D'Auria and S. Ferrara, Phys. Lett. 339B (1994) 71, hep-th/9408036.

[4] A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, Nucl. Phys. B444 (1995) 92, hep-th/9502072.

[5] C. M. Hull and P.K. Townsend, Nucl. Phys. B438 (1995) 109, hep-th/9410167.

[6] E. Witten, Nucl. Phys. B443 (1995) 85, hep-th/9503124.

[7] M. J. Duff, Nucl. Phys. B442 (1995) 47; for a recent review see also "Electric/Magnetic Duality and its Stringy Origins", hep-th/9509106.

[8] S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, Phys. Lett. B361 (1995) 59, hep-th/9505162.

[9] S. Kachru and C. Vafa, Nucl. Phys. B450 (1995) 69, hep-th/9505105.

[10] M. Billó, A. Ceresole, R. D'Auria, S. Ferrara, P. Frè, T. Regge, P. Soriani and A. Van Proeyen, "A Search for non-perturbative Dualities of Local N=2 Yang-Mills Theories from Calabi-Yau Threefolds", preprint hep-th/9506075, to appear on Class. and Quantum Grav.

[11] A. Klemm, W. Lerche and P. Mayr, Phys. Lett. B357 (1995) 313, hep-th/9506112.

[12] J. Bagger and E. Witten Nucl. Phys. B222 (1983) 1.

[13] N. J. Hitchin, A. Karlhede, U. Lindstrom and M. Rocek, "HyperKähler Metrics and Supersymmetry", Commun. math. Phys. 108 (1987) 535.

[14] K. Galicki, Comm. Math. Phys. 108 (1987) 117.

[15] R. D'Auria, S. Ferrara and P. Fré, Nucl. Phys. B359 (1991) 705.

[16] B. de Wit, P. G. Lauwers and A. Van Proeyen Nucl. Phys. B255 (1985) 569.

[17] L. Castellani, R. D'Auria and S. Ferrara, Phys. Lett. 241B (1990) 57; Class. Quantum Grav. 7 (1990) 1767.

[18] E. Cremmer, S. Ferrara, L. Girardello and A. Van Proeyen, Nucl. Phys. B212 (1983) 413.

[19] S. Ferrara, L. Girardello and M. Porrati, "Minimal Higgs Branch for the Breaking of Half of the Supersymmetry in N=2 Supergravity", preprint CERN-TH/95-268, hep-th/9510074.

[20] B. Greene, D. Morrison and A. Strominger, Nucl. Phys. B451 (1995) 109, hep-th/9504145.

[21] J. Polchinski and A. Strominger, "New vacua for Type II String Theory", preprint hep-th/9510227.

[22] I. Antoniadis, H. Partouche and T. R. Taylor, "Spontaneous Breaking of N=2 Global Supersymmetry", preprint hep-th/9512006.

[23] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and T. Magri, "N=2 Supergravity and N=2 Yang-Mills Theory on fully General Scalar Manifolds", in preparation.

[24] A. Ceresole, R. D'Auria and S. Ferrara, "The Symplectic Structure of N=2 Supergravity and its Central Extension", Proceedings of the ICTP Trieste Conference on "S-Duality and Mirror Symmetry", Miramare, Trieste, Italy, June 1995, hep-th/9509160.

[25] B. de Wit, P.G. Lauwers, R. Philippe, Su S.Q. and A. Van Proeyen, Phys. Lett. 134B (1984) 37; S. J. Gates, Nucl. Phys. B238 (1984) 349; B. de Wit and A. Van Proeyen, Nucl. Phys. B245 (1984) 89.

[26] J. P. Derendinger, S. Ferrara and A. Masiero, Phys. Lett. 143B (1984) 133.

[27] E. Cremmer, C. Kounnas, A. Van Proeyen, J.P. Derendinger, S. Ferrara, B. De Wit and L. Girardello, Nucl. Phys. B250 (1985) 385.

[28] B. de Wit and H. Nicolai, Nucl. Phys. B208 (1982) 323.

[29] L. Castellani, R. D'Auria and P. Fré, "Supergravity and Superstring Theory: a Geometric Perspective", World Scientific (1991) , Volumes 1, 2, 3.

[30] S. Ferrara, J. Scherk and B. Zumino, Nucl. Phys. B121 (1977) 393.

[31] S. Ferrara and L. Maiani, Proc. V Silarg Symp., World Scientific (1986); S. Cecotti, L. Girardello and M. Porrati, Nucl. Phys. B268 (1986) 295; A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and E. Maina, Nucl. Phys. B268 (1986) 317.

[32] S. Ferrara, C. Kounnas, M. Porrati and F. Zwirner, Nucl. Phys. B318 (1989) 75.

[33] I. Antoniadis, Phys. Lett. B246 (1990) 377.

[34] E. Cremmer, C. Kounnas, S. Ferrara and D. Nanopoulos, Phys. Lett. 133B (1983) 61.

[35] C. Zachos Phys. Lett. 76B (1978) 329.

[36] D.Z. Freedman, A. Das Nucl. Phys. B120 (1977) 221