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Dressing Cosets

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Abstract

The account of the Poisson-Lie T-duality is presented for the case when the action of the duality group on a target is not free. At the same time a generalization of the picture is given when the duality group does not even act on σ-model targets but only on their phase spaces. The outcome is a huge class of dualizable targets generically having no local isometries or Poisson-Lie symmetries whatsoever.

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1. The Poisson-Lie (PL) T-duality [1] is the generalization of the traditional non-Abelian T-duality [2]–[6]. It has been demonstrated in [1] and in the series of subsequent papers [7] – [11] that the PL T-duality enjoys most of the structural features of the traditional Abelian T-duality [12] – [17].

The underlying structure of the PL T-duality is the Drinfeld double [18]. The latter is the Lie group which is a sort of twisted product of two its equally dimensional subgroups. These subgroups play the role of the duality and coduality groups in the following sense: The duality group acts on the target of a PL dualizable σ-model and this action is Poisson-Lie symmetric with respect to the coduality group (see [1] for the definition of the PL symmetry). In the dual σ-model the roles of the duality and coduality groups are interchanged.

In the traditional non-Abelian duality the duality group is some Lie group G and coduality group is its Lie coalgebra viewed as the commutative additive group. The Drinfeld double is the cotangent bundle of the group manifold G in this case.

It has been remarked already many times [2, 5, 6] that even in the framework of the traditional non-Abelian duality there is the possibility of a qualitatively new structure which is absent in the Abelian case. It is connected with the fact that a non-Abelian duality group may act with isotropy which means, in other words, that the action is not free. A concrete example of the non-Abelian dual model in the case of the non-free action of the duality group was worked out e.g. in [2, 5, 16, 19] by the standard method of gauging of isometry.

The purpose of this note is to generalize the results of the traditional non-Abelian duality for the not freely acting groups to the general Poisson-Lie case and to find the relevant algebraic structure in terms of the corresponding Drinfeld double. We find that in this case the PL duality relates σ-models on the targets which are respectively cosets of an appropriate (dressing) action of certain residual group on the duality and coduality groups. In general, there is no action of the duality or the coduality group on these cosets, and, consequently, no trace of isometry or Poisson-Lie symmetry of the targets[1]. Still the duality and the coduality group underlie the dynamics of the σ-models in a non-local way.

In what follows, we give the duality invariant description of a Hamiltonian

[1]These 'dressing' cosets σ-models should presumably fit well into the schemes of [20, 21].

dynamical system on the subspace of the loop group of the Drinfeld double and show that this system simultaneously describes the dynamics of the both coset σ-models related by the PL duality. Then we discuss concrete examples of the traditional non-Abelian T-duality and of the 'true' PL duality with both the duality and the coduality groups being non-Abelian.

2. For the description of the Poisson-Lie duality we need the crucial concept of the Drinfeld double which is simply a Lie group D such that its Lie algebra $\mathcal{D}$ can be decomposed into a pair of maximally isotropic subalgebras with respect to a non-degenerate invariant bilinear form on $\mathcal{D}$ [18].

Consider now an n-dimensional linear subspace $\mathcal{E}$ of the $2n$-dimensional Lie algebra $\mathcal{D}$ and its orthogonal complement $\mathcal{E}^\perp$ such that the intersection $\mathcal{E}\cap\mathcal{E}^\perp \equiv \mathcal{F}$ is an isotropic Lie subalgebra of $\mathcal{D}$. Moreover we require that the both subspaces $\mathcal{E}$ and $\mathcal{E}^\perp$ are invariant with respect to the adjoint action of $\mathcal{F}$. We shall show that these data determine a dual pair of σ-models with the targets being the dressing cosets of the groups G and $\tilde{G}$ respectively. These cosets are defined with respect to the dressing action of the group F whose Lie algebra is $\mathcal{F}$. The dressing action of an element $f \in F$ on an element $g \in G$ gives an element $g_1 \in G$ defined as follows

$$fg = g_1\tilde{h}, \quad \tilde{h} \in \tilde{G}. \tag{1}$$

The multiplication in (1) is understood in the sense of the Drinfeld double[2]. By the dressing coset we mean the set of orbits of the dressing action of F on G or $\tilde{G}$.

The most economic description of the common dynamics of the models from the dual pairs is given in terms of the loop group LD of the Drinfeld double. The phase space P is formed by the loops $l(\sigma)$ with the property

$$\partial_\sigma l l^{-1} \in \mathcal{F}^\perp, \tag{2}$$

where $\mathcal{F}^\perp$ denoted the orthogonal complement of $\mathcal{F}$ with respect to the invariant inner product on the double. Note that $\mathcal{F}$ is isotropic, hence $\mathcal{F} \subset \mathcal{F}^\perp$. We also postulate that the loops $l_1(\sigma)$ and $l_2(\sigma)$ such that

$$l_1(\sigma) = l_2(\sigma) l_c, \quad l_c \in D \tag{3}$$

[2]The element g_1 is well defined if f and g are close to unit and for some Drinfeld doubles the definition is entirely correct even globally. For a generic double a special global analysis is required which, however, does not elucidate the main idea of the note and is in fact beyond the scope of it.

by definition describe the same element of the phase space. Note that the right action of D leaves invariant the current component $\partial_\sigma l l^{-1}$.

We define a symplectic two-form Ω on this phase space as the exterior derivative of a polarization one-form α. The latter is most naturally defined in terms of its integral along an arbitrary curve γ in the phase space, parametrized by a parameter τ. From the point of view of the Drinfeld double, this curve is a surface with the topology of cylinder embedded in the double in such a way that constraint (2) is fulfilled. We define

$$\int_\gamma \alpha = \frac{1}{2}\int \langle \partial_\sigma l\ l^{-1}, \partial_\tau l\ l^{-1}\rangle + \frac{1}{12}\int d^{-1}\langle dl\ l^{-1}, [dl\ l^{-1}, dl\ l^{-1}]\rangle. \tag{4}$$

Here $\langle ., . \rangle$ denotes the non-degenerate invariant bilinear form on the Lie algebra $\mathcal{D}$ of the double and in the second term on the r.h.s. we recognize the two-form potential of the WZW three-form on the double. Note that this definition of α is ambiguous due to the ambiguity in the choice of the inverse exterior derivative d^{-1}. However, this ambiguity disappears when the exterior derivative of the one-form α is taken. In other words, the symplectic form Ω is well defined.

We should note that we use the notion of symplectic form somewhat loosely. By this we mean that the symplectic form Ω is closed but it is *not* non-degenerate. From the point of view of the Hamiltonian mechanics this corresponds to the situation occuring in the description of systems with gauge symmetry. The vector fields annihilating Ω (i.e. $\Omega(., v) \equiv 0$) form an algebra under the standard Lie bracket hence they give rise to integrable surfaces (=orbits of the gauge group) in P on which Ω identically vanishes. By factoring the original phase space by these gauge group orbits, we obtain a reduced phase space on which Ω is not only closed but is also non-degenerate. If we define a Hamiltonian on the original phase space which is (gauge) invariant with respect to the action of those vector fields we have a well defined Hamiltonian system on the reduced phase space.

In our concrete situation, we define a Hamiltonian in terms of a certain quadratic form K on $\mathcal{F}^\perp$ such that

$$K(x + x_0) = K(x), \quad x \in \mathcal{F}^\perp, x_0 \in \mathcal{F}. \tag{5}$$

The value of K on some vector $x \in \mathcal{F}^\perp$ is computed as follows: x can be (not uniquely) decomposed as

$$x = x_1 + x_2, \quad x_1 \in \mathcal{E}, \quad x_2 \in \mathcal{E}^\perp \tag{6}$$

and

$$K(x) \equiv \langle x_1, x_1 \rangle - \langle x_2, x_2 \rangle. \tag{7}$$

Note that the value $K(x)$ does not depend on the decomposition (6).

In this note, we shall study a dynamical system on the phase space P given by the action

$$S[l(\tau,\sigma)] = \int \alpha - \int H \ d\tau$$

$$= \int d\sigma d\tau \{ \frac{1}{2} \langle \partial_\sigma l \ l^{-1}, \partial_\tau l \ l^{-1} \rangle + \frac{1}{12} d^{-1} \langle dl \ l^{-1}, [dl \ l^{-1}, dl \ l^{-1}] \rangle - \frac{1}{2} K(\partial_\sigma l l^{-1}) \}. \tag{8}$$

It is easy to check that the group action corresponding to the vector fields annihilating the symplectic form Ω is given by the left multiplication of a loop $l(\sigma)$ by an element $f(\sigma)$ from the loop group LF. The Hamiltonian H is invariant with respect to this action.

We conclude that the data P, Ω and H yield a well-defined Hamiltonian system on the reduced phase space $LF \backslash P$. The description of this system in terms of the original phase space P is given by the first order action S which, as it should, indeed possesses the gauge symmetry with respect to the left multiplication of $l(\sigma,\tau)$ by arbitrary $f(\sigma,\tau) \in F$:

$$l(\sigma,\tau) \to f(\sigma,\tau) l(\sigma,\tau). \tag{9}$$

Note that the action S has also a little gauge invariance corresponding to the write multiplication of $l(\sigma,\tau)$ by arbitrary function $l(\tau) \in D$. This small gauge symmetry corresponds to the factorization (3).

3. We show that the Hamiltonian system, defined by P, Ω and H, simultaneously describes dynamics of two σ-models. Their Lagrangians may be obtained directly from the action (8) as follows: Write the field $l(\sigma,\tau)$ in the form

$$l(\sigma,\tau) = g(\sigma,\tau) \tilde{h}(\sigma,\tau). \tag{10}$$

Here $g(\sigma,\tau)$ is an unconstrained element of G and $\tilde{h}(\sigma,\tau)$ is from $\tilde{G}$ in such a way that $l(\sigma,\tau)$ fulfils the constraint (2). Now $\tilde{h}$ can be eliminated from the action (8), yielding the σ-model on the group manifold G with the Lagrangian

$$L = (E + \Pi(g))^{-1} (\partial_+ g g^{-1}, \partial_- g g^{-1}). \tag{11}$$

Here the indices $\pm$ mean the light cone variables on the world sheet and E is a bilinear form on the dual space $\mathcal{G}^*$ of the Lie algebra $\mathcal{G}$ of the group G. The graph of E in $\mathcal{D}$ is precisely the subspace $\mathcal{E}$, i.e.

$$\mathcal{E} = Span\{t + E(t,.), t \in \tilde{\mathcal{G}}\}, \quad \mathcal{E}^\perp = Span\{t - E(.,t), t \in \tilde{\mathcal{G}}\}. \tag{12}$$

$\Pi(g)$ is the bivector field on the group manifold which gives the Poisson-Lie bracket on G (i.e. the multiplication $G \times G \to G$ is the Poisson map) [18, 1, 10].

By choosing the dual ansatz

$$l(\sigma,\tau) = \tilde{g}(\sigma,\tau)h(\sigma,\tau) \tag{13}$$

we arrive at the dual σ-model on the dual group $\tilde{G}$ manifold:

$$\tilde{L} = (E^{-1} + \tilde{\Pi}(\tilde{g}))^{-1}(\partial_+\tilde{g}\tilde{g}^{-1}, \partial_-\tilde{g}\tilde{g}^{-1}). \tag{14}$$

The mutually dual σ-models (11) and (14) appear to live on the targets G and $\tilde{G}$ respectively but, in fact, they do not. The standardly computed symplectic forms on their phase spaces are degenerate. The reason is the gauge symmetry (9) of the original model (8). Therefore the resulting σ-models (11) and (14) possess the same gauge symmetry but now the group F acts from the left not by the standard multiplication as in (9) but by the dressing action (1). Thus the σ-models (11) and (14) live on the targets which are respectively cosets of the dressing action of the group F on G and on $\tilde{G}$.

In every concrete example we may choose convenient gauge slices cutting the orbits of the dressing action and to work out the targets of the σ-models (11) and (14) in terms of some coordinates on the slices. We shall do it explicitly in some cases in order to illustrate the method.

It is interesting to note that there is no natural action of the duality group G on the gauge fixed model (11) . The only exception occurs when F is a subgroup of the group G . In this case the target $F\backslash G$ is the standard coset on which G naturally acts. The isotropy subgroup of this action is precisely F and we recover the standard picture of the traditional non-Abelian duality. But also in this special case, F is not subgroup of $\tilde{G}$ and therefore there is no natural action of $\tilde{G}$ on the dual σ-model target.

The suggested derivation of the gauge-invariant σ-models (11) and (14) from the duality invariant action (8) is technically quite lengthy. It is easier to demonstrate the equivalence of (8), (11) and (14) by a short-cut argument. For concreteness, let us consider the equivalence of the models (8) and (11). The density of the canonical momentum of (11) is a one-form with values in the coalgebra $\mathcal{G}^*$. On the other hand, $\mathcal{G}^*$ is canonically identified with $\tilde{\mathcal{G}}$ by means of the invariant inner product in $\mathcal{D}$. It turns out (see [1, 10] for a detailed argument) that the density of the canonical momentum on an extremal configuration $g(\sigma,\tau)$ can be written as the zero-curvature form $d\tilde{h}\tilde{h}^{-1}$ for some $\tilde{h}(\sigma,\tau) \in \tilde{G}$. Hence for every extremal configuration of the model (11) or, in other words, at every point of the phase space of (11), we may find a configuration $l(\sigma,\tau)$ in the double given by

$$l = g\tilde{h}. \tag{15}$$

This configuration is determined up to the right multiplication by a constant element from $\tilde{G}$.

It is now very easy to check that under the mapping (15) the standard polarization form (pdq) for the σ-model (11) coincides with the polarization form α given in (4) (for a specific choice of d^{-1}). Moreover, the Hamiltonian of (11) also coincides with the Hamiltonian of (8). Thus we conclude that the models (8) and (11) (and in the same way (8) and (14)) are dynamically equivalent. In fact, it is much easier to study the (dressing) gauge invariance of the models (11) and (14) in terms of the standard gauge invariance (9) of the duality invariant action (8), where the invariance of the symplectic form and of the Hamiltonian is manifest.

4. Now there is time for some examples. The simplest one is the sphere with the round metric and invariant 2-form, dualized with respect to $SU(2)$. The double is the cotangent bundle of $SU(2)$, algebra $\mathcal{F}$ is generated by the Pauli matrix σ_3, $\mathcal{E} = Span(\sigma_3, \sigma_+ + iat_+, \sigma_- - ibt_-)$, where t_i is the basis of the coalgebra of $SU(2)$ dual to σ_i and a, b are arbitrary real parameters. The result is the standard round metric on $F\backslash SU(2)$ (=2-sphere) and the standard monopole 2-form as the torsion:

$$\frac{a-b}{2}(d\theta^2 + \sin^2\theta d\phi^2), \quad \frac{a+b}{2}\sin\theta d\theta \wedge d\phi. \tag{16}$$

Now the dressing action of the group F on the coalgebra is simply rotation with respect to the z-axis, the torsion 2-form vanishes and the dual metric on the dressing coset (having the topology of the half-plane) reads

$$\frac{1}{(b-a)\rho}(d\rho+(z-a)dz)(d\rho+(z-b)dz). \qquad (17)$$

Here ρ is one half of the squared distance from the z-axis.

So far we have rederived the result of the traditional non-Abelian T-duality [2, 5, 16, 19]. Now we present its generalization, when the cotangent bundle is replaced by $SL(2,C)$ and the coalgebra is replaced by the Borel group B_2 of upper-triangular matrices in $SL(2,C)$ with real entries on the diagonal. The invariant bilinear form on the double is $< a,b >= \frac{1}{\epsilon}Im(tr(ab))$ with an arbitrary real ϵ. $\mathcal{F}$ and $\mathcal{E}$ remain the same, written in terms of the elements of the original basis σ_i and its dual basis t_i (now defined with respect to the inner product on $sl(2,C)$). The metric and the torsion 2-form on the sphere are

$$\frac{1}{\Delta}\frac{a-b}{2}(d\theta^2+\sin^2\theta d\phi^2), \quad \frac{1}{\Delta}(\frac{a+b}{2}+2\epsilon ab\sin^2\frac{\theta}{2})\sin\theta d\theta\wedge d\phi, \qquad (18)$$

$$\Delta=(1+2\epsilon a\sin^2\frac{\theta}{2})(1+2\epsilon b\sin^2\frac{\theta}{2}).$$

The dual torsion 2-form vanishes and the dual metric in appropriate coordinates reads

$$\frac{1}{1+\epsilon z}\frac{1}{(b-a)\rho}(d\rho+(\frac{z+\epsilon z^2/2}{(1+\epsilon z)^2}-a)dz)(d\rho+(\frac{z+\epsilon z^2/2}{(1+\epsilon z)^2}-b)dz). \qquad (19)$$

Note that in the limit $\epsilon\to 0$ our $SL(2,C)$ results (18) and (19) reproduce the traditional non-Abelian duality results (16) and (17). Thus we have obtained a one-parametric deformation of the dual pair of [2, 5, 16, 19].

The data (16) and (18) are defined on the standard coset $F\backslash SU(2)$ where the duality group $SU(2)$ naturally acts. Only the data (17) and (19) are defined on the truly dressing cosets where there is no natural action of the $SU(2)$ coalgebra and the Borel group B_2 on the coset targets (17) and (19) respectively. It is not too difficult to find examples, where the both targets from the dual pair are truly dressing cosets. The corresponding formulas in explicit coordinates are not very illuminating, however.

5. We conclude that there is the natural generalization of the traditional non-Abelian T-duality with a non-freely acting duality group. In the most

general case the duality group does not even act on the σ-model target but in the non-local way on its phase space. Only in the special case when the residual group F is a subgroup of the duality group G the action of G on the σ-model target is local and F is the isotropy group of this action. We should mention that the global aspects of the PL T-duality [11] can be also settled in the case of the dressing cosets. We have shown in [11] that the basic data defining the PL T-duality between D-branes are the $2n$-dimensional Drinfeld double and n-dimensional isotropic subalgebra $\mathcal{A}$ of the Lie algebra of the double. If our algebra $\mathcal{F}$ is also the subalgebra of $\mathcal{A}$ then all results of [11] directly generalize to the dressing cosets.

There remains an important nontrivial open problem: Is a given σ-model a dualizable dressing coset? The nontriviality stems from the fact that even if the answer is in some cases affirmative the duality group does not act on the target and the dressing orbits are in general too 'wild' to be easily recognizable. On the other hand, we find particularly this aspect of our construction promising. The simple data on the double give rise to very non-symmetrically looking σ-models whose targets, metrics and torsions are straightforwardly defined but not easily evaluated explicitly. Needless to say, eventually we hope to establish a connection of the PL T-duality with the mirror symmetry.

Another interesting project consists in considering the subspace $\mathcal{E}$ to be an isotropic subalgebra. The resulting cosets should be topological theories and the PL duality would rotate just the zero modes in a nontrivial way [22]. At the quantum level, a path integral derivation of the dressing cosets should be obtained perhaps by a modification of the derivation due to Tyurin and von Unge [9] or in the way suggested in [10].

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