SECOND-ORDER EFFECTS IN BEAM-CONTROL SYSTEMS OF PARTICLE ACCELERATORS

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I. INTRODUCTION

The CERN P.S., like many other proton synchrotrons, incorporates a "beam-control" system[1,2]. This largely eliminates coherent phase-oscillations of the particle bunches, but has, in first approximation, no effect on oscillations of the bunch shape. The possibility of a beam-control phenomenon that might conceivably cause antidamping of bunch-shape oscillations came up during the course of discussions in 1959 among those concerned with the running-in of the CERN Proton Synchrotron, at a time when the beam showed some behaviour that was not fully understood. At the present time the observational evidence is that, if an effect of this type occurs in the C.P.S., it is either negligible or must be of such a sign as to cause damping. The purpose of this note is, therefore, to show that:-

(1) With the type of beam-control system used in the C.P.S., one would expect a significant damping of bunch-shape oscillations.
(2) One cannot exclude, *a priori*, the possibility of a beam-control system causing anti-damping of these oscillations.
(3) It may be possible to devise, or to modify, a beam-control system in such a way that bunch-shape oscillations are damped at an increased rate. This would make it possible to relax certain tolerances or to obtain more reproducible beam behaviour, especially in any synchrotron which uses complicated processes of de-bunching, re-bunching, re-trapping etc.

The algebra that we shall use to investigate such effects, although elementary, is sufficiently complicated to obscure the role of the underlying physical processes, so we give in Section II a qualitative description in terms of phase-plane diagrams In Section III we give an approximate quantitative theory of the intrinsic bunch-shape damping for a beam-control system like that of the C.P.S., and in Section IV this is extended to the effect of possible imperfections or deliberate modifications of such a system.

We have considered this type of phenomenon in relation to the synchrotron oscillations, and for reasons which will appear later we have called it a second-order effect. In fact it may also be considered in relation to the betatron oscillation, and calculated to higher order than the second: these possibilities will be briefly mentioned in Section V.

II. BUNCH-SHAPE OSCILLATIONS

We are not here primarily concerned with oscillations of the bunch of particles as a whole, so let us consider the phase δq, and momentum δp, of a particle with reference to the centre of gravity of the bunch. In linear approximation the equations of motion for these differences will be

$$\begin{aligned} \dot{\delta q} &= -a\ \delta p \\ \dot{\delta p} &= +b\ \delta q \end{aligned} \qquad (1)$$

We disregard the fact that the parameters a and b vary slowly with time during the acceleration cycle. We can choose a suitable scale for p so that $a=b$, so that the trajectories in a δq, δp plane are circles, traced out anticlockwise below transition (a, b positive) and clockwise above transition (a, b negative), with angular frequency equal to that of the synchrotron oscillations

$$\omega_s = \sqrt{ab} \qquad (2)$$

The ideal situation is one in which the occupied region in this plane has circular symmetry and so rotates without visible change: the next degree of complication is introduced by supposing it to be elliptical, as in Fig. 1. Then the bunch signals from a simple pick-up electrode will, as the occupied ellipse rotates, oscillate from "wide" to "high" and back to "wide" at a frequency equal to twice that of the synchrotron oscillations. Such bunch-width oscillations are commonly one of the most obvious features of a pick-up electrode signal.

We now suppose that by some means, perhaps connected with the beam-control system, this bunch-width oscillation modulates the focusing force-constant b in equation (1). One can expect a modulation of b at this frequency, $2\omega_s$, to have considerable consequences, as it converts (1) into a Mathieu equation at its lowest stop band.

The qualitative effect of b-modulation can be illustrated by Fig. 2 and 3. When $|b|$ is larger, the particle trajectories have a taller and narrower form than the circles of Fig. 1; when $|b|$ is less they are wider and flatter. We concentrate attention on two particles e and f, located at the end and on the "flat" of the occupied elliptical region, and follow their behaviour for half a cycle of the synchrotron oscillations.

In Fig. 2 the situation is:–

(a) We suppose b larger than normal while the bunch shape is changing from "wide" to "high", less while it changes from "high" to "wide".

(b) Then the oscillations of particle e increase in amplitude, and in a similar way those of f are damped.

(c) The elliptical occupied region consequently becomes more narrow and elongated: the bunch-width oscillations are antidamped.

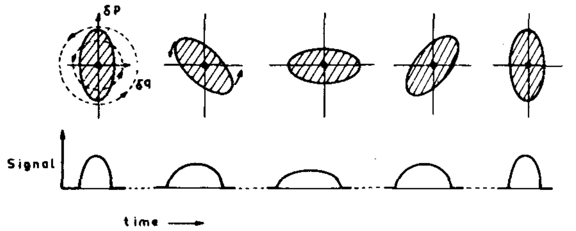

Fig. 1 Bunch-width oscillations. Phase-plane distributions above, corresponding bunch signals below.

The converse situation is shown in Fig. 3:–

(d) We suppose $|b|$ smaller than normal while the bunch shape is changing from "wide" to "high," and conversely

(e) Particle e is damped, f is antidamped.

(f) The elliptical occupied region becomes more nearly circular, so the bunch-width oscillations are reduced. We are assuming that they are the cause of the modulation of $|b|$, so the process tends towards a stable state in which the occupied region is circular in this diagram and nothing changes further.

On the other side of transition the particles and the occupied region rotate in the opposite sense, but the above statements have been formulated in such a way that they remain valid.

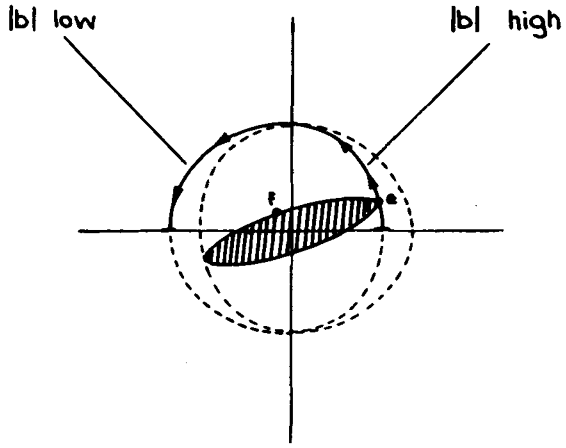

Fig. 2 Bunch-width oscillations antidamped.

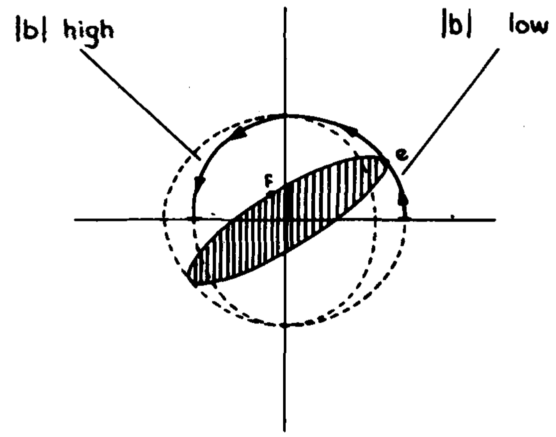

Fig. 3 Bunch-width oscillations damped.

The cases considered are those where the bunch-width oscillations cause a $|b|$ modulation with a phase shift of $\pm 90°$ (at a frequency of $2\omega_s$, so corresponding to $\pm 45°$ on the diagrams), but it is reasonably obvious that any phase shift other than zero or $180°$ will be effective, so a beam-control system in which bunch-width changes do affect $|b|$ is almost certain to show either damping or antidamping of the oscillations. Conditions as to sign and phase can be summarised by writing down:–

If a wider bunch results in a higher $|b|$ with a phase-lead θ,

$$\text{Damping occurs if } 0° < \theta < 180° \tag{3}$$

III. QUANTITATIVE THEORY

The equations of motion for synchrotron oscillations may be taken, for example, from Courant and Snyder[3]; we differ from their notation in using capital omegas for rf and particle revolution frequencies, reserving the small ones for oscillations:

$$\frac{d}{dt}\left(\frac{\Delta E}{\Omega_s}\right) = \frac{eV}{2\pi}(\sin\phi - \sin\phi_0) \tag{4}$$

$$\frac{d\phi}{dt} = \frac{\eta h \Omega_s}{\beta^2}\frac{\Delta E}{E} + \Omega_1 \tag{5}$$

Here η represents the quantity $\gamma_{tr}^{-2} - \gamma^{-2}$, which goes from negative to positive as one crosses transition energy.

We shall use q for $\phi - \phi_0$, and expand the right hand side of (4) as far as the quadratic term:–

$$\sin\phi - \sin\phi_0 = q\cos\phi_0 - \frac{q^2}{2}\sin\phi_0 + \cdots \tag{6}$$

and we take a variable p, for use instead of E/Ω_s, so defined that the trajectories are in linear approximation circles in the q, p plane:–

$$p = +\frac{\Delta E}{\beta}\left(\frac{-2\pi\eta h}{eVE\cos\phi_0}\right)^{1/2}. \tag{7}$$

Then:

$$\begin{aligned} \dot{p} &= aq + b_2 q^2 + \cdots \\ \dot{q} &= -ap + \Omega_1 \end{aligned} \tag{8}$$

$$\text{where } b_2 = -\frac{a}{2}\tan\phi_0. \tag{9}$$

The small-amplitude synchrotron oscillation frequency, ω_s, is given by

$$\omega_s = |a| = \Omega_0(-eV\eta h\cos\phi_0/2\pi E)^{1/2} \tag{10}$$

and a is positive below transition, negative above.

The quantity Ω_1 represents the difference between the actual applied rf frequency and its ideal value. With a beam-control system like that of the C.P.S. it is dependent on $\bar{q}$ and $\bar{p}$, where the bars denote averages over the particles, and we shall later consider cases where it depends on the bunch shape, but in any case it is the same for all individual particles present.

We now consider separately coherent bunch oscillations, i.e., the motion of the centre of gravity of the bunch, $\bar{q}$, $\bar{p}$; and the motion of particles about this centre of gravity. We put

$$\begin{aligned} q &= \bar{q} + \delta q \\ p &= \bar{p} + \delta p \end{aligned} \tag{11}$$

and find equations for the centre of gravity:–

$$\begin{aligned} \dot{\bar{q}} &= -a\bar{p} + \Omega_1 \\ \dot{\bar{p}} &= a\bar{q} + b_2(\bar{q}^2 + \overline{\delta q^2}) \end{aligned} \tag{12}$$

and equations for the differences:–

$$\begin{aligned} \dot{\delta q} &= -a\,\delta p \\ \dot{\delta p} &= (a + 2b_2\bar{q})\delta q + b_2(\delta q^2 - \overline{\delta q^2}) \end{aligned} \tag{13}$$

The important new features of these equations are

(1) We have a forcing term in (12):–

$$b_2\,\overline{\delta q^2}$$

If bunch-width oscillations exist, $\overline{\delta q^2}$ will vary, and this term will produce forced oscillations of the centre of gravity.

(2) Comparing the first bracket in (13) with equation (1), we have

$$b = a + 2b_2\bar{q}$$

So by forcing the centre of gravity of the bunch nearer and further from the peak of the rf wave one can modulate the force constant of the $\dot{\delta p}$, $\dot{\delta q}$ equations.

The quantities δq and δp refer to individual particles. We now want to derive from them two variables that are descriptive of the bunch shape, so that we can replace our diagramatic discussion of Section II by differential equations that will determine the behaviour of bunch-width oscillations. One method is to assume that the outline of the bunch in a q, p plane is nearly circular, and expand the difference in Fourier terms on this circle, so that the first Fourier term represents coherent bunch oscillations, and the second represents the bunch-width oscillations. But since we have considered the coherent oscillations in terms of mean values, it is more satisfactory to consider the bunch-width oscillations in terms of second moments.

We put

$$\left.\begin{aligned} A_2 &= \delta q^2 + \delta p^2 \\ q_2 &= \delta q^2 - \delta p^2 \\ p_2 &= 2\ \delta q\ \delta p \end{aligned}\right\} \qquad (14)$$

One can see that $\overline{q_2}$ is the sort of quantity that we want as a coordinate for bunch-width oscillations: it is positive when the bunch is wide, negative when the bunch is high. And $\overline{p_2}$ is positive or negative according as the occupied ellipse is prominent in the odd or even numbered quadrants.

We use (13) to find the equations of motion for the quantities defined by (14), and then average over all particles, neglecting third-moment terms like $\overline{\delta q^2 \delta p}$:–

$$\left.\begin{aligned} \dot{\overline{A_2}} &= 2b_2\bar{q}\overline{p_2} \\ \dot{\overline{q_2}} &= -2a\overline{p_2} - 2b_2\bar{q}\overline{p_2} \\ \dot{\overline{p_2}} &= 2a\overline{q_2} + 2b_2\bar{q}(\overline{A_2}+\overline{q_2}) \end{aligned}\right\} \qquad (15)$$

We linearise these equations by assuming that $\bar{q}$, $\bar{p}$, $\overline{q_2}$, $\overline{p_2}$ are all small, and neglecting quantities of the second order of smallness; we are therefore looking at the case where we have finite amplitudes of synchrotron oscillations, associated with a finite bunch size A, but are working in the limit of small amplitudes of bunch and bunch-shape oscillations:–

$$\begin{aligned} \overline{A_2} &= \text{constant} = A \text{ say.} \\ \dot{\overline{q_2}} &= -2a\overline{p_2} \\ \dot{\overline{p_2}} &= 2a\overline{q_2} + 2Ab_2\bar{q} \end{aligned} \qquad (16)$$

Finally we take as variables for bunch-width oscillations:–

$$\begin{aligned} x &= \overline{q_2}/A \\ y &= \overline{p_2}/A \end{aligned}$$

and their equations of motion are

$$\begin{aligned} \dot{x} &= -2ay \\ \dot{y} &= 2ax + 2b_2\bar{q} \end{aligned} \qquad (17)$$

In the absence of b_2, i.e., taking the linear approximation at the stage of equation (6), these equations indicate bunch-width oscillations of frequency $2\omega_s$, and arbitrary amplitude, depending only on initial conditions.

With the term $2\ b_2\bar{q}$ included, we can expect damping or antidamping if it contains a component of frequency $2\omega_s$ in suitable phase-relationship to x. Let us suppose

$$\bar{q} = Kx \qquad (18)$$

with K in general a complex constant, and look for a solution with the time dependence

$$\exp\ (j\ \omega\ t - \alpha\ t)$$

that is to say, with a (real) frequency ω and a damping-rate of α.

Using $a^2 = \omega_s^2$, one finds

$$\omega + j\ \alpha = 2\omega_s\ (1 + Kb_2/a)^{1/2} \qquad (19)$$

In practice Kb_2/a is small, so bunch-shape oscillations have a frequency

$$\omega \approx 2\omega_s + \omega_s\ \text{Re}\ (Kb_2/a) \qquad (20)$$

$$\approx 2\omega_s \qquad (21)$$

and a damping rate

$$\alpha \approx \omega_s\ \text{Im}\ (Kb_2/a) \qquad (22)$$

The condition, damping if the imaginary part of Kb_2/a is positive, is in agreement with (3).

We have disregarded the possibility of K being frequency-dependent, but it is clear that its value at a frequency of $2\omega_s$ is sufficiently accurate for use in (20) and (22), provided Kb_2/a is indeed small.

To find K, we must go to our equations of motion (12), of the centre of gravity. We have from (14),

$$\overline{\delta q^2} = \tfrac{1}{2}A + \tfrac{1}{2}\overline{q^2} = \tfrac{1}{2}A + \tfrac{1}{2}Ax \qquad (23)$$

and $\bar{q}^2$ we neglect as being of second order, so

$$\begin{aligned} \dot{\bar{q}} &= -a\bar{p} + \Omega_1 \\ \dot{\bar{p}} &= a\bar{q} + \tfrac{1}{2}Ab_2 + \tfrac{1}{2}Ab_2x \end{aligned} \qquad (24)$$

It is now necessary to make some assumptions about the nature of the beam-control system. In the C.P.S., phase-error information is fed into the rf frequency; and radial-error information is used to shift the reference phase from which the phase error is reckoned. So we may put

$$\Omega_1 = -c(q + dK\bar{p}). \tag{25}$$

The term $-c\bar{q}$ represents a phase-error signal fed into the rf system in such a way as to reduce the frequency when the phase error is positive; one aims to make c a positive coefficient. Radial-error information is represented by $d\,\bar{k}\,p$, where k is the conversion factor between $\bar{p}$ and average radial displacement, and $-d$ is the gain with which the radial error shifts the reference phase. For radial stability $c\ d\ k$ should (for low frequencies at least) have the same sign as a, so d is switched from positive to negative as the machine crosses the transition energy. A description of the C.P.S. beam-control system has been given by Schnell[2]; at the present day the system differs from his only in that the radial-error information is fed into the phase loop, not into the amplitude.

We may conveniently put

$$a + c\ d\ k = \rho a \tag{26}$$

ρ is in practice rather large compared with one, and is in a certain sense a measure of the strength of the radial feedback system.

Then (24) becomes

$$\begin{aligned} \dot{\bar{q}} &= -\rho\ a\ \bar{p} - c\bar{q} \\ \dot{\bar{p}} &= a\ \bar{q} + \tfrac{1}{2}Ab_2 + \tfrac{1}{2}Ab_2 x \end{aligned} \tag{27}$$

The term $\frac{1}{2}Ab_2$ contributes a constant part to the solution, which we shall disregard as it has no effect on the damping [it does however affect the real part of the frequency and invalidate (20)]. If x is oscillatory at some general frequency ω, it contributes

$$\bar{q} = -\tfrac{1}{2}Ab_2\rho a\ (j\omega c - \omega^2 + \rho a^2)^{-1}x \tag{28}$$

We compare with (18), evaluate K at a frequency $2\omega_s$, and obtain the damping rate from (22):—

$$\alpha = -\tfrac{1}{8}A\omega_s \tan^2\phi_0\ \mathrm{Im}\{1 - 4/\rho + 2jc/\rho\omega_s\}^{-1} \tag{29}$$

This can also be written:—

$$\alpha = \tfrac{1}{8}A\omega_s \tan^2\phi_0\ \mathrm{Re}\left\{\frac{\rho\omega_s}{2c}\left[1 - j\,\frac{\omega_s}{2c}\,(\rho - 4)\right]^{-1}\right\} \tag{30}$$

By considering $c = 0$, $\rho = 1$, one may note that there is neither damping nor antidamping in the absence of a beam-control system. There is always some coupling between bunch oscillations and bunch-shape oscillations, but it is not of such a nature as to produce damping or antidamping of the latter until one introduces either a phase-control loop, $c \neq 0$, or a radial servo with Im $\rho \neq 0$, to give the necessary phase-shift mentioned at (3).

The quantities A, ω_s, $\tan^2\phi_0$, are all positive. The requirements of the first-order beam-control system are such that one would aim to make c and ρ predominantly positive real over their useful frequency range, and then (29) predicts damping of the bunch-shape oscillations both before and after transition. We give in Appendix I some further discussion of the behaviour of (29) when c and ρ are complex, together with some rough numerical values for the C.P.S. system. Typical damping rates so calculated are given in Table I, column 3.

TABLE I

Theoretical Damping Rates for Bunch-width Oscillations with the C.P.S. Beam-control System

γ	ω_s(sec^{-1})	α(sec^{-1})
1.055 (injection)	49,000	810
1.5	28,000	110
2	18,000	40
3	8,900	12
4	5,000	4.8
5	2,700	2.1
7.5	1,900	0.9
10	2,200	0.8
15	2,100	0.5
20	1,900	0.4
25	1,700	0.3
30	1,600	0.2

These rates are quite substantial, especially early in the acceleration cycle. The improved performance of the C.P.S. when the beam-control is brought into operation very soon after injection must depend on the fact that these numbers are not antidamping rates, and may be partly due to the fact that they are not negligible. At the time when this rf system was being designed it was realised that questions of bunch-shape could complicate the beam-control process, so it was assumed that it might be necessary to wait for a few tens of synchrotron oscillations before switching in the beam-control[2].

The phase-feedback system of the C.P.S. is based on a measurement of the phase of the peaks, not of the centres of gravity, of the bunch signals. It seems unlikely that this makes much difference to the behaviour of bunch-width oscillations, for these are concerned with a type of bunch deformation which leaves the bunch signal symmetrical about its centre. One would expect differences between the peak and the centre of gravity of this signal to be associated mainly with frequencies of $3\omega_s$, $5\omega_s$, etc., which have little effect on equations like (17).

Of more consequence would be any tendency for the phase-error signal to have a certain bias towards say the front end of the bunch signal: the effects of this will be considered in Section IV.

IV. MODIFICATIONS OR IMPERFECTIONS OF THE BEAM-CONTROL SYSTEM

We now consider cases where the beam-control system is more complicated than the simple situation expressed by (25); let us take instead:

$$\Omega_1 = -c\bar{q} - cdk\bar{p} + \Omega_2 \tag{31}$$

where Ω_2 represents some additional feedback arising from bunch-width oscillations. Provided that all terms that affect the damping rate or frequency of these oscillations are small, it is legitimate to consider them separately and then add up their effects. We therefore drop the b_2 terms from (24), and obtain

$$\begin{aligned} \dot{\bar{q}} &= -\rho a\bar{p} - c\bar{q} + \Omega_2 \\ \dot{\bar{p}} &= a\bar{q} \end{aligned} \tag{32}$$

Then damping rates obtained from (31), (32) will be additional to those of Table I column 3.

The simplest way of obtaining a signal from bunch-width oscillations is to take the bunch signals from a pickup electrode station, perform a peak-rectification process, and then smooth away the rf frequency and its harmonics. Since the bunch is narrow and high when x is negative, the remaining audio-frequency signal will be in antiphase with x; if we amplify it and feed it into the beam-control system in the same way as the phase-error signal, we shall have

$$\Omega_2 = +cgx \tag{33}$$

where g includes any phase shift or sign reversal in this amplifier.

From (32) we then find

$$\bar{q} = g(1 + j\omega/c + \rho a^2/j\omega c)^{-1}x. \tag{34}$$

With (22), (18) and (11) this gives a damping rate

$$\alpha = -\frac{\omega_s}{2}\tan\phi_0 \mathrm{Im}\left\{g\left[1 - \frac{\omega_s}{2c}(\rho - 4)j\right]^{-1}\right\} \tag{35}$$

With the exception of the period immediately after injection, the square bracket is approximately one, and one will obtain the biggest effect by making g imaginary. One should note that $\tan\phi_0$ is not squared in this expression, so that one must switch the sign of g at transition if one wishes to produce damping throughout the acceleration cycle. A possible practical application of such a device would be to increase the damping rate of bunch-shape oscillations from transition upwards: if we switched it in at transition with, for example

$$g \approx -\frac{2}{50}j\cot\phi_0 \approx +0.07j \tag{36}$$

this would damp these oscillations to a ringing time (e^{-1} in amplitude) of 50 radians, or about 30 ms (damping rate $\sim 30\ \mathrm{s}^{-1}$). This might be useful, for example, to reduce the effect of transition tolerances on the effective width of top-energy bunches.

We consider next the possibility that the phase discriminator may have a certain bias towards the front end of the bunch, rather than responding to its true centre of gravity. The electronics of the C.P.S. phase-discriminator is such that this is likely to happen in practice, at least to some small extent. If we take the bunch to be a uniformly filled ellipse in the q, p diagram, its phase-wise half-width is

$$\Delta q = (2A)^{1/2}\left(1 + \frac{x}{2}\right) \tag{37}$$

and its front end is at a phase of

$$\bar{q} - \Delta q \tag{38}$$

Let us suppose that the phase-error signal then produced is

$$\bar{q} - e\Delta q \tag{39}$$

where e is some real coefficient less than one. Disregarding the constant part, this gives

$$\Omega_2 = ec\left(\frac{A}{2}\right)^{1/2}x \tag{40}$$

By comparison with (33), we can obtain the associated damping rate by putting $g = e(A/2)^{1/2}$ into (35)

$$\alpha = -\omega_s e(A/8)^{1/2} \tan\phi_0 \,\mathrm{Im}\left[1 - \frac{\omega_0}{2c}(\rho-4)j\right]^{-1} \quad (41)$$

This again is a quantity which will change sign at transition: if c and ρ are predominantly positive real and ρ is large compared with 4 we shall have antidamping before transition and damping after. With any plausible values of the parameters one finds that this is a relatively small effect.

Clearly there are many other ways in which one can suppose that the bunch-width oscillations affect Ω_1; we shall mention one more. By introducing a small delay into one of the inputs of the synchronous detector, it would be possible to cause the radial-error signal to give extra weight to the back (or front) end of the bunch. This evidently introduces a p_2 or y term into Ω_1, say:—

$$\Omega_1 = -cdk_2 y \quad (42)$$

This gives

$$\alpha = -\frac{a}{2}\tan\phi_0 k_2 \,\mathrm{Re}\left\{d\left[1 - \frac{\omega_s}{2c}(\rho-4)j\right]^{-1}\right\}. \quad (43)$$

V. BETATRON OSCILLATIONS AND HIGHER-ORDER PROCESSES

Coherent transverse betatron oscillations of the beam, due, for example, to an error in the position or direction of injection, can be damped by taking the signal that they produce in a transverse (radial or vertical) pick-up electrode, amplifying this signal at the betatron-oscillation frequency and feeding it in suitable phase relationship into a transverse deflecting electrode at some other point round the ring. (In principle there also exists the possibility that a single electrode structure, suitably tuned and loaded, could perform both the detection and deflection functions.) This is a first-order beam-control process, and will pull the centre of gravity of the occupied region in a betatron-oscillation phase-plane towards the origin.

The corresponding second-order process can be effectuated by using a pickup electrode of quadrupole type, which is sensitive to variations of the transverse width of the beam, and feeding its amplified signals into an electric or magnetic quadrupole lens; the important frequency is twice that of the betatron oscillations. A suitable phase relationship will then result in a damping of the oscillations sometimes referred to as "sausaging", typically caused by injection of a mismatched beam.

Analogous processes exist for orders higher than the second; one can consider an occupied region in the phase plane (whether of synchrotron or betatron oscillations) which has n bumps or regions of higher density and n dents or regions of lower denisty round its circumference, compared with the ideal distribution. As this picture rotates there will be an oscillation of the n'th moments of the distribution, at a frequency of n times the fundamental oscillation frequency; and these oscillations can be damped by introducing a modulation of the coefficient of the δq^{n-1} term in the $\delta\dot{p}$ equation of motion, in suitable phase relationship.

For the betatron-oscillation case the physical realisation of an n'th order beam-control process is then relatively obvious: one uses a pickup device and an excited lens, both of $2n$-pole geometry, connected by an amplifier which handles frequencies around $n\omega_\beta$ with a suitable phaseshift. The case of synchrotron oscillations is more complicated: their intrinsic nonlinearity makes the δq^{n-1} coefficient change merely by moving the bunch as a whole, and enables n'th moment bunch-shape oscillations to produce signals on the pickup electrodes without the necessity of a non-linear electrode geometry; consequently any beam-control system can affect bunch-shape oscillations of any order n, and produce damping or antidamping depending on its phase characteristics at the frequency $n\omega_s$.

This may give rise to difficulties in future proton synchrotrons of energy much higher than that of the C.P.S., because they will have a smaller ratio between revolution frequency and synchrotron-oscillation frequency. It becomes difficult to design a first-order beam-control system if the time delay, between obtaining information from the beam and feeding it back, becomes an appreciable fraction of a cycle of the synchrotron oscillation: but it may be more difficult to ensure that the n'th order bunch-shape oscillations are stable, for in their case one must compare this time delay with one n'th of a cycle of synchrotron oscillations.

APPENDIX I

The damping rates for bunch-width oscillations that one would expect with the C.P.S. beam-control system have been calculated from (29) on the following assumptions. For simplicity we have assumed a $\dot{B}$ of 12 kgauss/sec and a $|\sin \phi_0|$ of 0.5 throughout the acceleration cycle. Valves of ω_s have been taken from linear theory, and A from the theoretical adiabatic damping with a full bucket at injection. For the beam-control system, we have assumed the following parameters[2,4]. The phase-feedback coefficient $|c|$ falls off with frequency at 3 *bd* per octave over the whole relevant range and is 1.9×10^6 sec^{-1} at 300 cycles; such a frequency response is unlikely to arise without some phase lag, which we have arbitrarily put at 45°. The radial feedback coefficient d we have assumed to be 0.33 radians per cm at low frequencies, with a fall-off and phase-lag at high frequencies given by a time-constant of 16 μs (3 db down at 10 kc) and a sign-switch at transition. The resulting damping rates are given in Table 1.

It seems unlikely that any other reasonable choice of the cofficients c and d would result in antidamping. From (29) one can obtain a damping condition:—

$$\mathrm{Re}\left(\frac{c}{\rho\omega_s}\right) > \mathrm{Im}\left(\frac{2}{\rho}\right) \qquad (44)$$

For low values of the frequency $2\omega_s$ this is automatically satisfied, for ρ has approximately the same phase lag as c, and c/ω_s is numerically large compared with 2. At high frequencies there is more risk of antidamping, as the phase lag of ρ increases; if, for example, ρ is pure negative imaginary, (44) becomes

$$-\mathrm{Im}\left(\frac{c}{\omega_s}\right) > 2 \qquad (45)$$

so we must then ensure that c should have a sufficient phase lag as well as being sufficiently large compared with $2\omega_s$. There is in practice, however, a limit on the phase lag of ρ; we have from (26):—

$$\rho = 1 + \frac{cdk}{a}$$

and the term cdk/a decreases and ceases to be large compared with one at frequencies where its phase lag is becoming large.

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