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On Finite 4D Quantum Field Theory in Non-Commutative Geometry

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Abstract

The truncated 4-dimensional sphere S^4 and the action of the self-interacting scalar field on it are constructed. The path integral quantization is performed while simultaneously keeping the SO(5) symmetry and the finite number of degrees of freedom. The usual field theory UV-divergencies are manifestly absent.

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1 Introduction

The basic ideas of the non-commutative geometry were developed in [1, 2], and in the form of the matrix geometry in [3, 4]. The applications to physical models were presented in [2, 5], where the non-commutativity was in some sense minimal: the Minkowski space was not extended by some standard Kaluza-Klein manifold describing internal degrees of freedom but just by two discrete points. The algebra of functions on this manifold remains commutative but the complex of the differential forms does not. This led to a new insight on the $SU(2)_L \otimes U(1)_R$ symmetry of the standard model of electro-week interactions. The consideration of gravity was included in [6] Such models, of course, do not lead to UV-regularization, since they do not introduce any modification of the space-time short-distance behaviour.

To achieve the UV-regularization one should introduce a non-commutative deformation of the algebra of functions on a space-time manifold in the Minkowski case, or on the space manifold in the Euclidean version. One of the simplest locally Euclidean manifolds is the sphere S^2. Its non-commutative (fuzzy) deformation was described by [7,8] in the framework of the matrix geometry. More general construction of some non-commutative homogenous spaces was described in [9] using coherent states technique.

The first attempts to construct fields on a truncated sphere were presented in [8,10] within the matrix formulation. Using a more general approach, the fields on truncated S^2 were investigated in detail in [11-13]. In particular, in [11] it was the quantum scalar field on the truncated S^2 and it was explicitely demonstrated that the UV-regularization automatically takes place upon the non-commutative deformation of the algebra of functions.

In this article we extend this approach from the 2-dimensional sphere S^2 to the 4-dimensional one. Since S^4 is not a (co)-adjoint orbit, this extension has some new nontrivial features. We shall introduce only the necessary notions of the noncommutative geometry we need in our approach.

In Sec. 2 we describe briefly the standard (commutative) sphere S^4 as the Hopf fibration $S^7 \rightarrow S^4$ and the scalar self-interacting field on it. The Sec. 3 is devoted to the generalization of the model to the noncommutative truncated sphere S^4 introducing the noncommutative analogue of the Hopf fibration. Then, using Feynman (path) integrals, we perform the quantization of the model in question. Last Sec. 4 contains a brief discussion and concluding remarks.

2 Scalar field on the commutative S^4

Here we describe the standard sphere S^4 in the form which will be suitable for the noncommutative generalization. Our basic tools are the real quaternions

$$\varphi \;=\; \varphi_{(a)} e_a \;\in\; \mathbf{H} \tag{1}$$

with $\varphi_{(a)}$ real and the quaternionic units

$$e_1 \;=\; \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \;,\; e_2 \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;,$$

$$e_3 \;=\; \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \;,\; e_4 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;, \tag{2}$$

satisfying the relations

$$e_i e_j \;=\; -\delta_{ij} - \varepsilon_{ijk} e_k \;,\; e_4 e_i \;=\; e_i e_4 \;. \tag{3}$$

We shall usually write 1 instead of e_4. The coefficient $\varphi_{(0)} = \frac{1}{2}\mathrm{tr}\phi$ is called the real part of the quaternion, and $\varphi_{(i)}$, $i = 1, 2, 3$, are pure quaternionic components. The explicit realization (2) of the quaternionic units allows us to identify the space of quaternions with $\mathbf{C}^2$: any quaternion we represent by 2×2 complex matrix

$$\varphi \;=\; \begin{pmatrix} \varphi_2^* & \varphi_1 \\ -\varphi_1^* & \varphi_2 \end{pmatrix} \;. \tag{4}$$

The quaternionic conjugation $\varphi \to \varphi^*$ defined by

$$e_i \to e_i^* = -e_i \;,\; i = 1, 2, 3 \;,\; e_4 \to e_4^* = e_4 \;,$$

then corresponds to the hermitean conjugation of complex matrices. We shall frequently use both descriptions without an explicit specification. Further, the quaternionic length $|\varphi|$ is defined by

$$|\varphi|^2 \;=\; \varphi^* \varphi \;=\; \varphi_{(a)}^2 \;=\; \det\varphi \;. \tag{5}$$

If $|\varphi| = 1$, φ is called a unit quaternion. The set of unit quaternions is isomorphic to the group $SU(2)$ (and as a topological space to S^3).

The group $Sp(4)$ we identify with the group of 2×2 quaternionic matrices of the form

$$A = \begin{pmatrix} \cos\frac{\theta}{2}\;\alpha & \sin\frac{\theta}{2}\;\gamma\beta^* \\ -\sin\frac{\theta}{2}\;\gamma^*\alpha & \cos\frac{\theta}{2}\;\beta^* \end{pmatrix} \;, \tag{6}$$

where α, β, γ are unit quaternions, and $\theta \in [0, \pi]$ is a real angle.

The Lie algebra $sp(4) = so(5)$ is spanned by 10 antihermitean matrices $\xi_{AB} = -\xi_{BA}$, $A, B = 1, ..., 5$, given as

$$\xi_{a5} \;=\; \begin{pmatrix} 0 & e_a \\ -e_a^* & 0 \end{pmatrix} \;=:\; \xi_a \;,\; \xi_{ab} \;=\; \xi_a \xi_b \;, \tag{7}$$

where $a, b = 1, ..., 4$, $a \neq b$. The matrices ξ_{ab} span the Lie algebra $so(4) = so(3) \oplus so(3)$. Supplementing (7) by 5 matrices

$$\tilde{\xi}_a \;=\; \begin{pmatrix} 0 & e_a \\ e_a^* & 0 \end{pmatrix} \;,\; \tilde{\xi}_5 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;,\; a = 1, ..., 4 \;, \tag{8}$$

we recover the basis of the Lie algebra $su^*(4) = so(5,1)$. It is closely related to the Clifford algebra $C^{4,0}$ with the basis ξ_a, $a = 1, ..., 4$:

$$C^{4,0} = \begin{pmatrix} \xi_1\xi_2\xi_3\xi_4 \\ \xi_a\xi_b\xi_c \;,\; 1 \leq a < b < c \leq 4 \\ \xi_a\xi_b \;,\; 1 \leq a < b \leq 4 \\ \xi_a \;,\; 1 \leq a \leq 4 \\ 1 \end{pmatrix} = \begin{pmatrix} \tilde{\xi}_5 \\ \tilde{\xi}_a \;,\; 1 \leq a \leq 4 \\ \xi_{ab} \;,\; 1 \leq a < b \leq 4 \\ \xi_a \;,\; 1 \leq a \leq 4 \\ 1 \end{pmatrix} \;, \tag{9}$$

where the matrices ξ_a, ξ_{ab} are antihermitean whereas the matrices $\tilde{\xi}_A$, $A = 1, ..., 5$, are hermitean and transform as an $SO(5)$ vector.

The matrices $A \in Sp(4)$ act in a natural way in the space $\mathbf{H}^2$:

$$z \;=\; \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \;\in\; \mathbf{H}^2 \;\rightarrow Az \;=\; \begin{pmatrix} \tilde{a}\varphi + \tilde{b}\chi \\ \tilde{c}\varphi + \tilde{d}\chi \end{pmatrix} \;\in\; \mathbf{H}^2 \;. \tag{10}$$

The sphere S^7 given by the equation

$$z^+ z \;=\; |\varphi|^2 \;+\; |\chi|^2 \;=\; 1 \;, \tag{11}$$

is transitively invariant under this action. Introducing the equivalence relation

$$z \sim z' = z\alpha \ , \ \alpha \ - \ \text{unit quaternion} \ , \tag{12}$$

we recover the sphere S^4 as the Hopf fibration $S^7 \to S^4$. To any equivalence class (13) we assign the $SO(5)$ vector given by the cartesian coordinates in $\mathbf{R}^5$:

$$x_A \ = \ \frac{1}{2}\mathrm{tr}(z^+\tilde{\xi}_A z) \ = \ \frac{1}{2}\mathrm{tr}(z'^+\tilde{\xi}_A z') \ . \tag{13}$$

These are just the cartesian coordinates of the sphere S^4 embedded into $\mathbf{R}^5$ (similar objects were used in [8] within a relativistic context).

As $\mathcal{A}_\infty$ we denote the commutative algebra of analytic functions (polynomials) in the variables x_A, $A = 1, ..., 5$:

$$\Phi(x) \ = \ \sum A_M x^M \ , \ A_M - \text{complex} \ , \tag{14}$$

with the usual point-wise multiplication. Here we used the multiindex notation: $M = (M_1, ..., M_5)$, $x^M = x_1^{M_1} ... x_5^{M_5}$. In $\mathcal{A}_\infty$ we introduce the scalar product

$$(\Phi_1, \Phi_2)_\infty \ = \ I_\infty[\Phi_1^* \Phi_2] \ , \tag{15}$$

where $I_\infty[...]$ denotes the usual $SO(5)$-invariant integral on S^4:

$$I_\infty[...] \ = \ \frac{3}{4\pi^2} \int d^5x \ \delta(x_A^2 - 1) \ [...] \ , \tag{16}$$

where the normalization guarantees that $I_\infty[1] = 1$.

The $Sp(4)$ action (10) in the algebra $\mathcal{A}_\infty$ generates $\mathbf{R}^5$ rotations leaving the quantity $x_A^2 = 1$ invariant. The generators of this action (antihermitean with respect to the scalar product given above) are given as

$$\hat{J}_{AB}\Phi \ = \ \frac{1}{2}(\psi^*_\alpha \xi^{\alpha\beta}_{AB}\partial_{\psi^*_\beta} \ + \ \psi_\beta \xi^{\alpha\beta}_{AB}\partial_{\psi_\alpha})\Phi \ . \tag{17}$$

Here $\xi^{\alpha\beta}_{AB}$ are elements of the 4×4 complex matrix assigned to the 2×2 quaternionic matrix ξ_{AB}, and ψ_α, ψ^*_α, $\alpha = 1,...,4$, are complex variables identified with the elements of complex matrices assigned to the quaternions φ and χ in the following way:

$$\psi_1 \ = \ \varphi_1 \ , \ \psi_2 \ = \ \varphi_2 \ , \ \psi_3 \ = \ \chi_1 \ , \ \psi_4 \ = \ \chi_2 \ ,$$

$$\psi^*_1 \ = \ \varphi^*_1 \ , \ \psi^*_2 \ = \ \varphi^*_2 \ , \ \psi^*_3 \ = \ \chi^*_1 \ , \ \psi^*_4 \ = \ \chi^*_2 \ . \tag{18}$$

It follows from (17) that the quantities ψ_α, ψ^*_α, $\alpha = 1,...,4$, transform as S^4 spinors

$$\hat{J}_{AB}\psi_\alpha \ = \ \frac{1}{2}\xi^{\alpha\beta}_{AB}\psi_\beta \ , \ \hat{J}_{AB}\psi^*_\beta \ = \ \frac{1}{2}\xi^{\alpha\beta}_{AB}\psi^*_\alpha \ . \tag{19}$$

Consequently, the quantities x_A, $A = 1,...,5$ given as

$$x_A \ = \ \psi^+\tilde{\xi}_A\psi \ = \ \psi^*_\alpha\tilde{\xi}^{\alpha\beta}_A\psi_\beta \ , \tag{20}$$

where $\tilde{\xi}^{\alpha\beta}_A$ are elements of the complex matrix assigned to $\tilde{\xi}_A$, transforms as a vector in $\mathbf{R}^5$. Moreover, the function $C(x) = x^2_A$ satisfies

$$\hat{J}_{AB}C(x) \ = \ 0 \ , \ A, B = 1,...,5 \ , \tag{21}$$

i.e. C(x) is an invariant function as expected.

The $Sp(4)$ action (17) in the algebra $\mathcal{A}_\infty$ is reducible and we have the following expansion:

$$\mathcal{A}_\infty \ = \ \bigoplus_{p=0}^{\infty} \mathcal{A}^p_\infty \ , \tag{22}$$

where $\mathcal{A}^p_\infty$ is the carrier space of the irreducible representation of the $Sp(4)$ group spanned by the harmonic polynomials Ψ^p_μ of degree p in the variables x_A, $A = 1, ..., 5$. The polynomials Ψ^p_μ are orthonormal with respect to the scalar product (15). The dimension of the space $\mathcal{A}^p_\infty$ is

$$d_p \;=\; \dim \mathcal{A}^p_\infty \;=\; \frac{1}{6}(p+1)(p+2)(2p+3) \;,$$

That means that any field $\Phi \in \mathcal{A}_\infty$ can be expanded as

$$\Phi(x) \;=\; \sum_{p=0}^{\infty} \sum_{\mu=0}^{d_p} a^p_\mu \Psi^p_\mu \;. \tag{23}$$

The topologically trivial real scalar field we identify with fields from $\Phi \in \mathcal{A}_\infty$. The expansion coefficients a^p_μ are real provided that the fuctions Ψ^p_μ are chosen real (if this is not the case the coefficients a^p_μ satisfy some relations that guarantee the reality of the field in question). The space of the real scalar fields we denote as $\mathcal{A}^R_\infty$.

The field action corresponding to the real scalar field Φ is given as

$$S[\Phi] \;=\; I_\infty[\frac{1}{2}(\hat{J}_{AB}\Phi)^2 \;+\; V(\Phi)] \;, \tag{24}$$

where $V(.)$ is a polynomial bounded from below.

The quantum mean value of some polynomial field functional $F[\Phi]$ is defined as the functional integral over fields from $\Phi \in \mathcal{A}^R_\infty$ by

$$\langle F[\Phi] \rangle = \frac{\int D\Phi e^{-S[\Phi]} F[\Phi]}{\int D\Phi e^{-S[\Phi]}} \;, \tag{25}$$

where $D\Phi = \prod_x d\Phi(x) = \prod_{p,\mu} da^p_\mu$ (eventually, with the reality conditions for a^p_μ included).

Since here $p = 0, 1, ..., \infty$, the formula for the measure is only formal. We shall not discuss the complicated (and not completely solved) problems related to its rigorous definition. As we shall see below, such problems do not appear in the framework of the noncommutative version of the model.

3 Scalar field on the non-commutative S^4

In this section we shall use various unitary irreducible representations of the group $Sp(4)$. Any such representation is characterized by its signature (p, k) with integer $p \geq k \geq 0$ and can be expressed as the Young product

$$(p,k) \ = \ \pi_1^{p-k}\, \pi_2^k \ , \tag{26}$$

of $Sp(4)$ fundamental representations: $\pi_1 = (1, 0)$ - 4 dimensional quaternionic and $\pi_2 = (1, 1)$ - 5 dimensional real (see e.g. [14]). The dimension of the representation (p, k) is

$$d_{pk} \ = \ \frac{1}{6}(p+2)(k+1)(p-k+1)(p+k+2) \ . \tag{27}$$

In the noncommutative (fuzzy) case we replace the commuting parameters (18) by the noncommutative ones. Namely, we shall express the parameters ψ_α, ψ^*_α, $\alpha = 1, ..., 4$ in terms of annihilation and creation operators as

$$\psi_\alpha \ = \ A_\alpha R^{-1/2} \ , \ \psi^*_\alpha \ = \ R^{-1/2} A^*_\alpha \ , \tag{28}$$

where

$$R \ = \ A^*_\alpha A_\alpha \ , \tag{29}$$

so that the condition $\psi^*_\alpha \psi_\alpha = 1$ is satisfied (the operators ψ_α are well defined everywhere except the vacuum; we complete the definition by postulating that they annihilate the vacuum). The operators A_α and A^*_α (* denotes the hermitean conjugation) act in the Fock space $\mathcal{F}$ spanned by the orthonormal vectors $|n\rangle = |n_1, ..., n_4\rangle$ labelled by the occupation numbers n_α, $\alpha = 1, ..., 4$. They satisfy in $\mathcal{F}$ the commutation relations

$$[A_\alpha, A_\beta] \; = \; [A^*_\alpha, A^*_\beta] \; = \; 0 \; , [A_\alpha, A^*_\beta] \; = \; \delta_{\alpha\beta} \; . \tag{30}$$

The operators

$$J_{AB} \; = \; \frac{1}{2} A^*_\alpha \xi^{\alpha\beta}_{AB} A_\beta \; , \; A, B = 1, ..., 5 \; . \tag{31}$$

satisfy in the Fock space $\mathcal{F}$ the $sp(4) = so(5)$ Lie algebra commutation relations. The subspace $\mathcal{F}_N$ with the fixed total occupation number

$$\mathcal{F}_N \; = \; \{|n\rangle \; , \; |n| \; = \; N \; \} \; , \; N \; = \; 0, 1, 2, ... \; . \tag{32}$$

has the dimension

$$d_{N0} \; = \; \begin{pmatrix} N+3 \\ 3 \end{pmatrix} \; . \tag{33}$$

and is the carrier space of $Sp(4)$ unitary irreducible representation $(N, 0)$.

As the $\mathcal{A}_N$ we denote the noncommutative algebra of operators $\mathcal{F}_N \to \mathcal{F}_N$, which can be expressed as polynomials

$$\Phi(x) \; = \; \sum A_M x^M \; , \; A_M - \text{complex} \; , \tag{34}$$

in operators

$$x_A \; = \; \psi^*_\alpha \tilde{\xi}^{\alpha\beta}_A \psi_\beta \; = \; \psi^+ \tilde{\xi}_A \psi \; , \; A = 1, ..., 5 \tag{35}$$

restricted to the space $\mathcal{F}_N$. The operators x_A, $A = 1, ..., 5$, form a vector in $\mathbf{R}^5$.

In $\mathcal{A}_N$ we introduce the scalar product

$$(\Phi_1, \Phi_2)_N \;=\; I_N[\Phi_1^* \Phi_2] \;, \tag{36}$$

where $I_N[...]$ is the analog of the $SO(5)$-invariant integral on S^4:

$$I_N[...] \;=\; \frac{1}{d_{N0}} \mathrm{Tr}_N[...] \;. \tag{37}$$

Here $\mathrm{Tr}_N[...]$ denotes the trace in the algebra $\mathcal{A}_N$, and the normalization guarantees that $I_N[1] = 1$.

As a noncommutative analog of (18) we have a commutator action of the $sp(4)$ algebra in $\mathcal{A}_N$:

$$\hat{J}_{AB} \Phi(x) \;=\; [J_{AB}, \Phi(x)] \;, \tag{38}$$

with J_{AB} defined in (31). This is a reducible representation with the following decomposition to $Sp(4)$ irreducible components:

$$(N,0) \otimes (N,0) \;=\; \bigoplus_{p=0}^{N} \bigoplus_{k=0}^{p} (p+k, p-k) \;. \tag{39}$$

This decomposition induces the decomposition of the algebra $\mathcal{A}_N$:

$$\mathcal{A}_N \;=\; \bigoplus_{p=0}^{N} \bigoplus_{k=0}^{p} \; \mathcal{A}_N^{p+k,p-k} \;, \tag{40}$$

where $\mathcal{A}_N^{p'k'}$ is the carrier space of the $Sp(4)$ representation (p', k'). That means that any $\Phi \in \mathcal{A}_N$ can be expanded as

$$\Phi(x) \;=\; \sum_{p=0}^{N} \sum_{k=0}^{p} \sum_{\mu=1}^{d'_{pk}} a_\mu^{p+k,p-k} \Psi_\mu^{p+k,p-k} \;, \tag{41}$$

where $d'_{pk} = d_{p+k,p-k}$ and $\Psi_\mu^{p'k'}$, $\mu = 1, ..., d_{p'k'}$, span the space $\mathcal{A}_N^{p'k'}$.

In the commutative case, the decomposition (22) of the algebra $\mathcal{A}_\infty$ contains only representations $(p,p) = \pi_2^p$ corresponding to terms with $k = 0$ in the decomposition (40).

Note: We would like to stress that it is not essential that the generators x_A, $A = 1, ..., 5$, given in (35) do not close to some Lie algebra (they close to a Lie algebra only after supplementing them by the operators (31)). The following point is important, however: the decomposition (40) of the basic algebra $\mathcal{A}_N$ under symmetry transformation in question (this aspect was less transparent for the truncated sphere S^2, since in this case the generators closed to a Lie algebra, see [11]). The detailed information contained in eq. (40) is necessary for realistic numerical or symbolical calculations.

The topologically trivial configurations of a real scalar field we identify with the subspace

$$\mathcal{A}_N^R \ = \ \bigoplus_{p=0}^{N} \ \mathcal{A}_N^{pp} \ , \tag{42}$$

of symmetric polynomials in x_A, $A = 1, ..., 5$, with real coefficients.

Such fields can be expanded as

$$\Phi(x) \ = \ \sum_{p=0}^{N} \sum_{\mu=1}^{d'_{pk}} a_\mu^p \Psi_\mu^{pp} \ , \tag{43}$$

where the coefficient a_μ^p are real provided that Ψ_μ^{pp} are chosen to be hermitean (if this is not the case the coefficients a_μ^p satisfy some relations that guarantee that the field in question is a hermitean operator in $\mathcal{F}_N$). This guarantees that in the commutative limit $N \to \infty$ we recover from (43) only fields that have the proper form (23).

In the non-commutative case the field action corresponding to the real scalar field Φ is given as

$$S[\Phi] \ = \ I_N[\frac{1}{2}(\hat{J}_{AB}\Phi)^2 \ + \ V(\Phi)] \ , \tag{44}$$

where $V(.)$ is a polynomial bounded from below. Obviously, this action has the following basic properties:

1) it has the full $SO(5)$ symmetry corresponding to S^4 rotations, and

2) it describes a model with a finite number of modes, since in fact, it corresponds to a particular matrix model.

The quantum mean value of some polynomial field functional $F[\Phi]$ is defined as the functional integral

$$\langle F[\Phi]\rangle = \frac{\int D\Phi e^{-S[\Phi]}F[\Phi]}{\int D\Phi e^{-S[\Phi]}} \ . \tag{45}$$

However, here $D\Phi \ = \ \prod_{p,\mu} da^p_\mu$ (eventually with the reality conditions included) is the usual Lebesgue measure, since now the product is finite ($p = 0, 1, ..., N$, and $\mu = 1, ..., d'_{pp}$). The quantum mean values are well defined for any polynomial functional $F[\Phi]$.

4 Topologically nontrivial fields

Let us first discuss the complex scalar field Φ on the commutative sphere $S^4 = S^7/S^3$. In general, the topologically nontrivial field configurations of the field Φ are not globally defined on S^4. However, they can be lifted to a globally defined functions on a principal bundle $S^7 \to S^4$ with $SU(2) = Sp(2)$ being the structural group.

Let us consider the G-bundle $\mathbf{H}^2 \to \mathbf{R}^5$ with $G = SU(2) = Sp(2)$:

(i) The projection from $\mathbf{H}^2$ to $\mathbf{R}^5$ is given by

$$x_A \;=\; \frac{1}{2}\mathrm{tr}(z^+\tilde{\xi}_A z)\ ,\ A = 1, ..., 5\ . \tag{46}$$

(ii) The $SU(2)$ action in $\mathbf{H}^2$ is defined as a right multiplication by the unit quaternion (see (12)):

$$z \in \mathbf{H}^2 \;\to\; z\alpha\ ,\ \alpha \in \mathbf{H}\ ,\ |\alpha| = 1\ . \tag{47}$$

Obviously, x_A, $A = 1, ..., 5$, are invariant under (47).

Any complex field, $\Phi : \mathbf{H}^2 \to \mathbf{C}$, can be expanded as

$$\Phi \;=\; \sum a_{mn}\psi^{*m}\psi^n\ , \tag{48}$$

where $\psi, \psi^* \in \mathbf{C}^4$ are complex parameters assigned in (18) to quaternions (ϕ and χ) enterring z. Above we used the multiindex notation: $\psi^{*m}\psi^n = \psi^{*m_1}...\psi^{*m_4}\psi^{n_1}...\psi^{n_4}$. The topologically nontrivial field configurations on S^4 are obtained by the restriction $z^+z = 1$ which in terms of ψ_μ, ψ^*_μ, $\mu = 1, ..., 4$, can be rewritten as $\psi^*_\mu\psi_\mu = 1$.

These fields are classified according to $SU(2)$ action (46) in the right regular representation:

$$\alpha :\ \Phi(z) \to \Phi(z\alpha)\ . \tag{49}$$

Let us denote the generators of this action by Q_0, $Q_\pm$. As a commuting set of operators we can choose

$$\begin{aligned} Q_0 \;&=\; \frac{1}{2}(\psi^*_\mu\partial_{\psi^*_\mu} \;-\; \psi_\mu\partial_{\psi_\mu})\ , \\ Q \;&=\; \frac{1}{2}(\psi^*_\mu\partial_{\psi^*_\mu} \;+\; \psi_\mu\partial_{\psi_\mu})\ , \end{aligned} \tag{50}$$

the latter directly related to the $SU(2)$ Casimir operator $Q_0^2 + \frac{1}{2}(Q_+Q_- + Q_-Q_+)$. It holds

$$Q_0\psi^{*m}\psi^n \;=\; \frac{1}{2}(|m|-|n|)\psi^{*m}\psi^n \;,$$

$$Q\psi^{*m}\psi^n \;=\; \frac{1}{2}(|m|+|n|)\psi^{*m}\psi^n \;. \tag{51}$$

Consequently, the $\mathcal{A}_\infty$-(bi)modules $\mathcal{H}_q$ formed by the fields of the form

$$\Phi \;=\; \sum a_{mn}\psi^{*m}\psi^n \;,\; q = \frac{1}{2}(|m|-|n|) \;-\; \text{fixed} \;, \tag{52}$$

correspond to the configurations characterized by the topological number $2q \in \mathbf{Z}$, and they classify the topologically nontrivial configurations of the field in question.

In the noncommutative case the parameters ψ_μ, ψ_μ^*, $\mu = 1, ..., 4$, are expressed in terms annihilation and creation operators (see eq. (28)). The fields, of the form (52) with $q = \frac{1}{2}(|m|-|n|)$ fixed, map any space $\mathcal{F}_N$ into the space $\mathcal{F}_M$ with $M = N + q$. Therefore, the topologically nontrivial scalar field configurations we identify with the space $\mathcal{H}_{MN}$ of linear mappings from $\mathcal{F}_N$ to $\mathcal{F}_M$. The space $\mathcal{H}_{MN}$ is a left $\mathcal{A}_N$-modul and a right $\mathcal{A}_M$-modul. Obviously,

$$\mathcal{H}^*_{MN} \;=\; \mathcal{H}_{NM} \;,\; (^*\text{−hermitean conjugation}) \;,$$

$$\mathcal{H}_{LM}\mathcal{H}_{MN} \;=\; \mathcal{H}_{LN} \;,\; \mathcal{H}_{NN} \;=\; \mathcal{A}_N \;. \tag{53}$$

In the space $\mathcal{H}_{MN}$ we can introduce the scalar product

$$(\Phi_1, \Phi_2)_{MN} \;= \frac{1}{J}\mathrm{Tr}_N(\Phi_1^*\Phi_2) \;=\; \frac{1}{J}\mathrm{Tr}_M(\Phi_2\Phi_1^*) \;=\; (\Phi_2^*, \Phi_1^*)_{NM} \;, \tag{54}$$

where $J = \frac{1}{2}(d_{N0} + d_{M0})$ is a suitable normalization constant.

The space $\mathcal{H}_{MN}$ is the carrier space for the $Sp(4)$ representation $(M,0) \otimes (N,0)$ induced by the generators $\hat{J}_{AB}$ given in eq. (38):

$$\hat{J}_{AB}\Phi \ = \ J_{AB}^{(M)}\Phi \ - \Phi J_{AB}^{(N)}, \tag{55}$$

where $J_{AB}^{(N')}$ denotes the operator defined in (31) restricted to $\mathcal{A}_{N'}$. This is a reducible representation with the following decomposition to $Sp(4)$ irreducible components:

$$\begin{aligned}(M,0) \otimes (N,0) \ &= \ \bigoplus_{p=q}^{\frac{M+N}{2}} \bigoplus_{k=q}^{p} (p+k, p-k) \\ &= \bigoplus_{p=q}^{\frac{M+N}{2}} \bigoplus_{k=q}^{p} \pi_1^{2k} \pi_2^{p-k} \ ,\end{aligned} \tag{56}$$

where $q = \frac{1}{2}|M-N|$. This decomposition induces the decomposition of the space $\mathcal{H}_{MN}$:

$$\mathcal{H}_{MN} \ = \ \bigoplus_{p=q}^{\frac{M+N}{2}} \bigoplus_{k=q}^{p} \mathcal{H}_{MN}^{p+k,p-k} \ , \tag{57}$$

where $\mathcal{H}_{MN}^{p'k'}$ is the carrier space of the $Sp(4)$ representation (p', k'). The number $q = \frac{1}{2}|M-N| \in \mathbf{Z}/2$ classifies the topological configurations of the field in question.

Note: All constructions on the truncated sphere S^4 (and in particular the formulas (56) and (57)) closely resemble those on the truncated sphere S^2 presented in [13]. The field action $S[\Phi, \Phi^*]$ for the field $\Phi \in \mathcal{H}_{MN}$ and the quantum mean values of the field functionals $F[\Phi, \Phi^*]$ can be introduced much in the same way as in the S^2 case. Moreover, we expect that the spinors fields could be introduced on the truncated sphere S^4 analogously as it was done in [12] and [13] in the S^2 case.

5 Concluding Remarks

We have demonstrated above that the interacting scalar field on the non-commutative sphere S^4 represents a quantum system which has the following properties:

1) The model has the full $SO(5)$ space symmetry under the rotations of the sphere S^4. This is exactly the same symmetry as the interacting scalar field on the standard sphere S^4 possesses.

2) The field has only a finite number of modes. Then the number of degrees of freedom is finite and this leads to the non-perturbative UV-regularization, i.e. all quantum mean values of polynomial field functionals are well defined and finite.

In our approach the UV cut-off in the number of modes is supplemented with a highly non-trivial vertex modification due to nontrivial products of fields. Our UV-regularization is non-perturbative and is completely determined by the algebra $\mathcal{A}_N$. It is originated by the short-distance structure of the space, and does not depend on the field action of the model in question.

Moreover, it can be shown that the Schwinger functions

$$S_n(F) = \langle F_n[\Phi] \rangle \ , \tag{58}$$

where $F_n[\Phi] = \sum \alpha^{p_1 \cdots p_n}_{\mu_1 \cdots \mu_n} (\Psi^{p_1}_{\mu_1}, \Phi)_N \ldots (\Psi^{p_n}_{\mu_n}, \Phi)_N$ satisfy the Osterwalder-Schrader axioms:

(OS1) *Hermiticity*

$$S_n^*(F) = S_n(\Theta F) \ ,$$

where ΘF is the involution defined by $\Theta F_n[\Phi] = (F_n[\Phi])^*$.

(OS2) *Covariance*

$$S_n(F) = S_n(\mathcal{R}F) \ ,$$

where $\mathcal{R}F$ is a mapping of functionals induced by $SO(5)$ rotations.

(OS3) *Reflection positivity*

$$\sum_{n,m\in\mathcal{I}} S_{n+m}(\Theta F_n \otimes F_m) \geq 0 \ .$$

(OS4) *Symmetry*

$$S_n(F) = S_n(\pi F) \ ,$$

where πF is a functional obtained from F by arbitrary permutation of indices of $\alpha^{p_1\ldots p_n}_{\mu_1\ldots\mu_n}$.

Note: We do not include the last Osterwalder-Schrader axiom - the cluster property, since the compact manifold requires a special treatment (however, it can be recovered in the limit when the radius of the sphere grows to infinity, but these considerations go beyond the presented scheme). Qualitatively, the properties of the Schwinger functions are the same as those valid for the truncated sphere S^2, see [11]. We would like to stress that the properties of standard Schwinger functions not included above (e.g. support, or singularity and growth, specification) are essential again in the commutative limit $N \rightarrow \infty$.

The usual divergencies will appear only in the commutative limit $N \rightarrow \infty$. It would be very interesting to isolate the large N behaviour nonperturbatively. By this we mean the Wilson-like approach in which the renormalization group flow in the space of Lagrangeans is studied. In this context a connection may be found with similar recent works [15].

Combining the results of this paper with those of [11-13] we obtain a set of UV-regularized Euclidean quantum field models on S^2 and S^4:

a) the scalar field on the truncated S^2 which is super-renormalizable,

b) the Neveu-Schwarz model on the truncated S^2 which is renormalizable,

c) the scalar field on the truncated S^4 with Φ^4 interaction which is renormalizable too.

Analogous models formulated on standard Euclidean planes ($\mathbf{R}^2$ or $\mathbf{R}^4$ instead of spheres) served as important examples for the proof of the existence of quantum fields in continuum Euclidean spaces in the framework of Wilson approach (see [16, 17] for the super-renormalizable case, and [18, 19] for renormalizable one).

We have an alternative approach: the regularization procedure is non-perturbative and preserves all space symmetries of the models in question. The UV-regularization in our scheme can be interpreted as a direct consequence of the short-distance structure induced by the non-commutative geometry of the underlying space. This can lead to a better understanding of the origin and properties of divergencies in quantum field theory.

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References

[1] A. Connes, Publ. IHES *62* (1986) 257

[2] A. Connes, Non-Commutative Geometry (Academic Press, London, 1994)

[3] M. Dubois-Violette, C. R. Acad. Sci. Paris *307*, Ser. I (1988) 403

[4] M. Dubois-Violette, R. Kerner and J. Madore, Journ. Math. Phys. *31* (1990) 316

[5] R. Coquereaux, G. Esposito-Farese and G. Vaillant, Nucl. Phys. *B353* (1991) 689

[6] A. H. Chamsedine, G. Felder and J. Fröhlich, Commun. Math. Phys. 155 (1993) 205

[7] J. Madore, J. Math. Phys. *32* (1991) 332

[8] J. Madore, Non-Commutative Geometry and Applications (Cambridge University Press 1995)

[9] H. Grosse and P. Prešnajder, Lett. Math. Phys. *33* (1995) 171

[10] H. Grosse and J. Madore, Phys. Lett. *B283* (1992) 218

[11] H. Grosse, C. Klimčík and P. Prešnajder, *Towards finite quantum field theory in non-commutative geometry* - preprint CERN-TH/95-138, UWThPh-19-1995, hep-th /9505175

[12] H. Grosse, C. Klimčík and P. Prešnajder, *Field theory on a supersymmetric lattice* - preprint CERN-TH/95-195, UWThPh-20-1995, hep-th/9507074

[13] H. Grosse, C. Klimčík and P. Prešnajder, *Topologically nontrivial field configurations in noncommutative geometry* - preprint CERN-TH/95-264, UWThPh-32-1995, hep-th/9510083

[14] D. P. Zhelobenko and A. I. Stern, Group representations, (Nauka, Moscow 1983) - in Russian

[15] A. Kempf, *On Quantum Field Theory with Non-Zero Minimal Uncertainties in Positions and Momenta*, hep-th/9602085 and references therein

[16] B. Simon, The $P(\Phi)_2$ Euclidean (quantum) field theory, (Princeton Univ. Press 1974)

[17] J. Glimm and A. Jaffe, Quantum physics. A functional intergal point of view (Springer-Verlag, New York 1981)

[18] K. Gawędzki and A. Kupiainen, Commun. Math. Phys. *99* (1985) 197

[19] K. Gawędzki and A. Kupiainen, Commun. Math. Phys. *102* (1985) 1