Heterotic/Heterotic Duality in $D = 6, 4$

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Abstract

We consider $E_8 \times E_8$ heterotic compactifications on $K3$ and $K3 \times T^2$. The idea of heterotic/heterotic duality in $D = 6$ has difficulties for generic compactifications since for large dilaton values some gauge groups acquire negative kinetic terms. Recently Duff, Minasian and Witten (DMW) suggested a solution to this problem which only works if the compactification is performed assuming the presence of symmetric gauge embeddings on both E_8's . We consider an alternative in which asymmetric embeddings are possible and the wrong sign of kinetic terms for large dilaton value is a signal of spontaneous symmetry breaking. Upon further toroidal compactification to $D = 4$, we find that the duals in the DMW case correspond to $N = 2$ models in which the β-function of the different group factors verify $\beta_\alpha = 12$ whereas the asymmetric solutions that we propose have $\beta_\alpha = 24$. We check the consistency of these dualities by studying the different large T, S limits of the gauge kinetic function. Dual $N = 1$, $D = 4$ models can also be obtained by the operation of appropriate freely acting twists, as shown in specific examples.

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1 Introduction

In a recent paper [1], Duff, Minasian and Witten (DMW) have provided evidence for the existence of a heterotic/heterotic duality in six dimensions. This duality relates a weakly coupled $E_8 \times E_8$ theory compactified on $K3$ to a strongly coupled $E_8 \times E_8$ theory obtained upon compactification on a differently realized $K3$. Heterotic/heterotic duality was first conjectured in refs. [2, 3, 4, 5] motivated by heterotic/fivebrane duality [6] in $D = 10$. It may also be understood [1] as two alternative (dual) ways of looking at the compactification of the $D = 11$ M-theory on $K3 \times S^1/Z_2$.

In the present note we reexamine the idea of $D = 6$ and $D = 4$ heterotic/heterotic duality concerning different aspects. First, we would like to propose that this duality is not only present in $E_8 \times E_8$ compactifications in which equal instanton numbers are embedded in both factors but is also present in certain asymmetric cases. In these new cases the gauge group is also Higgsed away at generic points in hypermultiplet moduli space but, unlike the DMW case, the dual gauge particles do not have their origin in small instanton effects. Secondly, we examine the $D = 4$ theories obtained in both cases upon further toroidal compactification. We find that the resulting $N = 2$ theories have $\beta_\alpha^{N=2} = 12$ in the DMW case and $\beta_\alpha^{N=2} = 24$ in the new proposed duals for *all* perturbative gauge groups appearing at enhanced symmetry points. We show how the $N = 2$ models obtained in the case of non-symmetric embeddings are related to certain chains of heterotic/type II duals studied in ref. [7]. We also study the large T and/or S limit of the $N = 2$ gauge kinetic functions for those theories and obtain consistency with duality only precisely for the above mentioned values of the β-functions. Finally, we also show how $D = 4$, $N = 1$ duality can be derived upon twisting by appropriate freely-acting symmetries of the $N = 2$ models.

2 $D = 6$ heterotic/heterotic duality

One of the most intuitive hints [4] for the existence of a $D = 6$ heterotic/heterotic duality is the way in which the anomaly eight-form I_8 factorizes into the product of four-forms in $D = 6$. In fact, $I_8 = X_4\tilde{X}_4$, with

$$\begin{aligned} X_4 &= \frac{1}{4(2\pi)^2}\left(trR^2 - v_\alpha trF_\alpha^2\right) \\ \tilde{X}_4 &= \frac{1}{4(2\pi)^2}\left(trR^2 - \tilde{v}_\alpha trF_\alpha^2\right) \quad , \end{aligned} \tag{1}$$

where α runs over the gauge groups in the model. This very symmetric form of I_8 suggests a duality under which one exchanges the tree-level Chern-Simons contribution to the Bianchi identity $dH = \alpha'(2\pi)^2 X_4$ with the one-loop Green-Schwarz corrections to the field equations $d\tilde{H} = \alpha'(2\pi)^2 \tilde{X}_4$. In these expressions v_α is a (positive) tree-level coefficient which is essentially the Kac-Moody level. On the other hand the coefficients $\tilde{v}_\alpha$ are associated to the Green-Schwarz mechanism, they depend on the massless spectrum of the model and they can be positive, negative or zero. The problem with a naive duality exchanging both terms is that a generic $D = 6$ model yields values for $\tilde{v}_\alpha$ that have the wrong sign (see the formulae for $\tilde{v}_\alpha$ below).

In fact, independently of any duality hypothesis, the existence of negative values for $\tilde{v}_\alpha$ is in general problematic since, as shown by Sagnotti [8], the exact dilaton dependence of the gauge kinetic terms in $D = 6$ is given (in the Einstein frame) by the expression:

$$L^{D=6}_{gauge} \;=\; -\frac{(2\pi)^3}{8\alpha'}\sqrt{G}\left(v_\alpha e^{-\phi/2} \;+\; \tilde{v}_\alpha e^{\phi/2}\right) trF^2_\alpha \tag{2}$$

where ϕ is the six-dimensional dilaton field. This expression shows that there are only tree-level and one-loop contributions and that the corresponding coefficients are given by v_α, $\tilde{v}_\alpha$. It is obvious that, if any of the $\tilde{v}_\alpha$'s is negative (as happens in the generic case), for some value of the dilaton the theory becomes inconsistent and hence one cannot continuously extrapolate towards strong coupling.

Since the coefficients v_α and $\tilde{v}_\alpha$ play an important role in the discussion below, we will first briefly review the relevant formulae that allow their computation in the $D = 6$, $N = 1$ theory. At the Kac-Moody level $k_\alpha = 1$, the coefficient v_α is given by $v_\alpha = 2, 1, \frac{1}{3}, \frac{1}{6}, \frac{1}{30}$ for $\alpha = SU(N), SO(2N), E_6, E_7, E_8$, respectively [12, 1]. The derivation of the $\tilde{v}_\alpha$ involves constraints on the massless spectrum that follow from the conditions of anomaly cancellation [11, 12]. These conditions include cancellation of the term $tr\, R^4$ that requires $(n_h - n_v) = 244$, where n_h (n_v) is the number of hyper(vector)multiplets.

For groups with an independent fourth-order Casimir, cancellation of $tr\, F^4$ imposes the constraint

$$T_\alpha \;=\; \Sigma_i\; n_i t^i_\alpha \quad , \tag{3}$$

where n_i refers to the number of hypermultiplets in representation R_i. T_α and t^i_α are group-theoretical quantities that appear when decomposing TrF^4_α into factorized and non-factorized terms. In the notation of ref. [12],

$$TrF^4_\alpha = T_\alpha trF^4_\alpha + U_\alpha \left(trF^2_\alpha\right)^2 \qquad ; \qquad tr_{R_i}F^4_\alpha = t^i_\alpha trF^4_\alpha + u^i_\alpha \left(trF^2_\alpha\right)^2 . \tag{4}$$

For example, for $SU(N)$ with $N \geq 4$, $T_{SU(N)} = 2N$ and $t^i_{SU(N)} = 1,\ (N-8)$, $\frac{1}{2}(N^2-17N+54)$, $\frac{1}{6}(N-4)(N^2-23N+96)$ for the fundamental and $2,3,4$-index antisymmetric representations respectively.

In ref. [12], an analysis of the total anomaly led to general expressions for the coefficients $\tilde{v}_\alpha$ at the level $k_\alpha = 1$. We will be mostly interested in the following

$$\begin{aligned}
\tilde{v}_{SU(N)} &= n_{a2} + (N-4)n_{a3} + \frac{1}{2}(N-4)(N-5)n_{a4} - 2 \qquad (N \geq 4) \\
\tilde{v}_{SO(2N)} &= 2^{(N-6)}n_s - 2 \qquad (N \geq 3) \\
\tilde{v}_{SU(2)} &= \frac{n_2 - 16}{6} \quad ; \quad \tilde{v}_{SU(3)} = \frac{n_3 - 18}{6} \\
\tilde{v}_{E_6} &= \frac{n_{27} - 6}{6} \quad ; \quad \tilde{v}_{E_7} = \frac{n_{56} - 4}{6} \quad ; \quad \tilde{v}_{E_8} = -\frac{1}{5} \quad .
\end{aligned} \tag{5}$$

Here n_{aj} refers to the number of j-index antisymmetric $SU(N)$ tensors, n_s to the number of $SO(2N)$ spinorials and the rest of the notation is self-explanatory. Notice that almost always n_i also counts the complex $\overline{R}_i$ representation. The representations appearing in (5) are those allowed by unitarity at $k_\alpha = 1$ and potentially massless. The number of fundamentals, n_f, has been eliminated in the first two groups by virtue of eq. (3).

Observing formulae (5) one immediately realizes the many possibilities for negative $\tilde{v}_\alpha$ that would in turn lead to negative kinetic terms (for large ϕ) in eq.(2). To start with, whenever there is a gauge group without charged hypermultiplets, $\tilde{v}_\alpha$ is negative. Consider for example a $D = 6$ compactification obtained embedding $SU(2)$ bundles with instanton numbers s and s' respectively in each E_8. Modular invariance dictates that $s + s' = 24$. The standard embeding choice ($s = 24$, $s' = 0$) yields a model with gauge group $E_7 \times E_8$, and hypermultiplets transforming like $10(56;1) + 65(1;1)$. In this case $\tilde{v}_{E_7} = 1$ and $\tilde{v}_{E_8} = -1/5$. More generally, a sufficient amount of matter hypermultiplets is required for $\tilde{v}_\alpha > 0$. In the case at hand, for $s, s' \neq 0$, the gauge group is $E_7 \times E_7'$, and $s + s' = 24$ implies $n_{56} + n_{56'} = 8$. Since we need $n_{56} \geq 4$ and $n_{56'} \geq 4$ for non-negative $\tilde{v}_\alpha$'s we are forced to have $n_{56} = n_{56'} = 4$. This corresponds to $s = s' = 12$ and $\tilde{v}_\alpha = 0$ for both E_7's, *i.e.*, the symmetric embedding. It is trivial to check that any non-Abelian subgroup obtained by Higgsing with hypermultiplets also has $\tilde{v}_\alpha = 0$.

The solution proposed by DMW to the wrong sign kinetic term problem is precisely to restrict to compactifications with $\tilde{v}_\alpha = 0$. Since this possibility is obviously not symmetric under the exchange of v_α and $\tilde{v}_\alpha$, it requires that the dual gauge group be generated by non-perturbative (small instanton) effects as suggested in [9] . This hypothesis is consistent with the fact that the known examples of gauge groups generated by these non-perturbative effects verify $v_\alpha = 0$ (unlike the perturbative ones, which obviously have $v_\alpha > 0$). The proposal can be justified [1] by considering this duality as arising from two (dual) ways of looking at the compactification of the $D = 11$ M-theory on $K3 \times S^1/Z_2$. The two dual $D = 6$ theories correspond to $E_8 \times E_8$ heterotic compactifications on $K3$ with symmetric gauge bundles in both E_8's. However, the original and the dual $K3$ are not identical, the dual corresponding to a $K3$ orbifold of type T^4/Z_2.

A generic compactification of the $E_8 \times E_8$ heterotic string on $K3$, with equal $SU(2)$ instanton numbers on both E_8's, gives rise to an $N = 1$, $D = 6$ model with gauge group $E_7 \times E_7$ and hypermultiplets transforming as $4(56, 1)+4(1, 56)$ $+62(1, 1)$. For generic vev's of the hypermultiplets the gauge symmetry is completely broken and one is left with 244 hypermultiplets and no vector multiplets. Thus there is only some gauge group at enhancing points in the moduli space. The idea is that those points in moduli space are different from the ones at which the non-perturbative gauge group may be generated, so that one does not have to deal simultaneously with perturbative and non-perturbative gauge groups, which are supposed to be dual.

Since the dual $K3$ has the structure of (some sort of) T^4/Z_2 orbifold [1], it is interesting to see whether one can construct a Z_2 orbifold with the appropriate characteristics. Let us then consider the twist θ acting on the complexified T^4 coordinates $z_{1,2}$ as $\theta(z_1, z_2) = (-z_1, -z_2)$ and let us look for an embedding on $E_8 \times E_8$ symmetric in both E_8's. It is easy to check that on the $E_8 \times E_8$ lattice there is no modular invariant order-two shift A that is symmetric in both E_8's. Nonetheless, a Z_2 orbifold with symmetric spectrum can be constructed as follows. To begin with, consider just the standard embedding of θ by the gauge lattice shift

$$A = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, 0\right) \times (0, \cdots, 0) \quad . \tag{6}$$

Although this does not look very symmetric, the model is symmetrized if, in addition, we turn on the Wilson line

$$a_1 = \left(\frac{1}{2}, \frac{1}{2}, 0, \cdots, 0\right) \times \left(\frac{1}{2}, \frac{1}{2}, 0, \cdots, 0\right) \quad , \tag{7}$$

along, say, the first compact dimension [1]. The resulting model then has gauge group $(E_7 \times SU(2))^2$, four singlet hypermultiplets in the untwisted sector and hypermultiplets transforming as $4(56, 1; 1, 1) + 4(1, 1; 56, 1)$ $+16(1, 2; 1, 1) + 16(1, 1; 1, 2)$ in

[1] For simple rules to find the massless spectrum in orbifolds with underlying quantized Wilson lines see ref. [10].

the twisted sector. The model is perfectly symmetric and thus constitutes a good candidate for the searched dual T^4/Z_2 orbifold. Notice that upon Higgsing of the two $SU(2)$'s, the particle content of this model corresponds to the particle content of a $K3$ compactification with equal $SU(2)$ instanton numbers embedded in both E_8's, as described above.

We would now like to consider an alternative to the above explicit realization of heterotic/heterotic duality that could occur even if the gauge bundles are embedded differently in both E_8's. Indeed, a negative $\tilde{v}_\alpha$ in eq.(2) is not by itself problematic. It is only problematic if one insists in going to large ϕ. A way to avoid this dangerous limit would be to remain at finite ϕ values (so that the theory is well defined) and Higgs away the gauge groups with negative $\tilde{v}_\alpha$ by giving vev's to the hypermultiplets in the theory. Once all the dangerous gauge groups have been Higgsed away, one can safely take the large ϕ limit and go to strong coupling. This point of view gives a physical interpretation to the large ϕ limit of theories with some negative $\tilde{v}_\alpha$, namely, for large ϕ there is a phase transition in which the corresponding gauge group is spontaneously broken.

It is obvious that this mechanism is not always functional. In particular, there must be sufficient hypermultiplets in the theory to Higgs away completely the wrong sign gauge groups. Consider for example a $E_7 \times E_7'$ theory. Since E_7 has 133 generators one needs a minimum of 3(56)'s to Higgs completely one E_7. We are thus led to the unique choice of an embedding with $s = 14$ and $s' = 10$ that gives rise to hypermultiplets transforming as $5(56,1) + 3(1,56')$+$62(1,1)$. From (5) we find $\tilde{v}_{E_7} = 1/6$ and $\tilde{v}_{E_7'} = -1/6$. However, E_7' can be completely Higgsed away with the $3(1,56')$, leaving behind a $D = 6$ heterotic model with gauge group E_7 and hypermultiplets transforming as $5(56) + 97(1)$ in which

$$\tilde{v}_{E_7} = v_{E_7} = \frac{1}{6} \quad . \tag{8}$$

Notice that not only is $\tilde{v}$ positive but it is also equal to v. Thus in this model explicit heterotic/heterotic duality along the original ideas of refs. [2, 3, 4, 5] seems possible. It is easy to check that any non-Abelian group obtained from this E_7 by Higgsing still verifies the condition $v_\alpha = \tilde{v}_\alpha$. Notice that for generic points in hypermultiplet moduli space the gauge group is again spontaneously broken and, as in the case of symmetric embedding, 244 hypermultiplets remain massless.

This is not the only example in which Higgsing the groups with $\tilde{v}_\alpha < 0$ is feasible. We will now describe a different type of $D = 6$ heterotic compactification in which this procedure works as in the previous example. It is a $D = 6$ model based on the standard Z_6 orbifold with a $E_8\times E_8$ embedding given by $V=\frac{1}{6}(1,1,1,1,-4,0,0,0)\times\frac{1}{6}(1,1,1,1,1,-5,0,0)$. The resulting gauge group is $SU(5)\times SU(4)\times U(1)\times SU(6)\times SU(3)\times SU(2)$ and the massless hypermultiplets are

$$\begin{aligned}
\theta^0 &: (1,\overline{4},-5;1,1,1)+(10,4,1;1,1,1)+(1,1,0;6,3,2)\ +2\,(1,1,0;1,1,1)\\
\theta^1 &: (1,1,\frac{10}{3};1,3,2)+(1,4,-\frac{5}{3};6,1,1)+2\,(1,1,\frac{10}{3};6,1,1)\\
\theta^2 &: 5\,(1,\overline{4},\frac{5}{3};1,\overline{3},1)+4\,(\overline{5},1,-\frac{4}{3};1,\overline{3},1)\\
\theta^3 &: 3\,(1,6,0;1,1,2)+5\,(5,1,-2;1,1,2)\quad ,
\end{aligned}\tag{9}$$

where θ^n indicates to which twisted sector the hypermultiplet belongs. Using (5) we find $\tilde{v}_{SU(5)}=2$, $\tilde{v}_{SU(4)}=4$, $\tilde{v}_{SU(6)}=-2$, $\tilde{v}_{SU(3)}=6$ and $\tilde{v}_{SU(2)}=8$. Only $SU(6)$ has wrong sign but, since for $SU(N)$ one has $v_{SU(N)}=2$, only $SU(5)$ looks promising to produce a self-dual theory. One can check that indeed there are appropriate hypermultiplets to Higgs completely all gauge groups but $SU(5)$. In this way we obtain a $D=6$ $SU(5)$ model with hypermultiplets transforming as $4(10)+22(5)+118(1)$ and $\tilde{v}_{SU(5)}=v_{SU(5)}=2$. As in the previous example, for generic points in hypermultiplet moduli space there is a $D=6$ heterotic theory with no vector multiplets and 244 hypermultiplets.

It is worth remarking that in this type of $D=6$ models in which the unbroken gauge group (at enhanced points) has $\tilde{v}_\alpha=v_\alpha$, one expects the occurrence of heterotic/heterotic duality as a consequence of $D=10$ string/fivebrane duality in the same spirit as originally formulated in refs. [2, 3, 4, 5]. In particular, no small instanton effects seem to be required to obtain duality.

3 $D=4$ heterotic/heterotic duality

Let us now consider the $D=4$ heterotic models obtained upon further compactification of the above $D=6$ heterotic duals on a 2-torus. This case was considered in ref. [5] and briefly mentioned in ref. [1]. The resulting $N=2$ theory has the usual toroidal vector multiplets S,T,U related to the coupling constant and the size and shape of the 2-torus. When the $D=6$ theory is dimensionally reduced to four dimensions, the underlying duality exchanges the roles of S and T [5, 13]. Including mirror symmetry on the torus, one thus expects complete S-T-U symmetry in this type of vacua [5, 1, 15, 16, 17, 18]. Thus, on top of the usual perturbative $SL(2,\mathbb{Z})_T\times SL(2,\mathbb{Z})_U$ dualities, a non-perturbative $SL(2,\mathbb{Z})_S$ S-duality [14] is expected. This $N=2$ model has the toroidal $U(1)^4$ as generic gauge group, and as matter, 244 neutral hypermultiplets (it corresponds to the heterotic construction of model B of ref. [19]). At particular points in moduli space, enhanced gauge groups such as $E_7 \times E_7$ can appear.

A natural question is the following: What is the $D = 4$ equivalent of the $\tilde{v}_\alpha = 0$ or $\tilde{v}_\alpha = v_\alpha$ conditions we had in $D = 6$ in order to have heterotic/heterotic duality? It turns out that the equivalent condition in $D = 4$ can be phrased as a condition on the $N = 2$ β-functions of the gauge groups present at enhanced points in moduli space. Indeed, the $N = 2$, $D = 4$ β-function of a given gauge factor can be written in terms of the corresponding $D = 6$ $\tilde{v}_\alpha$ coefficient.

As an exercise let us consider the case of $SU(N)$ ($N \geq 4$) with the representations that appear at level $k = 1$. With the notation introduced before, the $N = 2$ β-function is given by

$$\beta^{N=2}_{SU(N)} = -2N + n_f + n_{a2}(N-2) + \frac{1}{2}n_{a3}(N-3)(N-2) + \frac{1}{6}n_{a4}(N-4)(N-3)(N-2). \tag{10}$$

Now, the $D = 6$ anomaly factorization condition in eq. (3) implies

$$2N = n_f + n_{a2}(N-8) + \frac{1}{2}n_{a3}(N^2 - 17N + 54) + \frac{1}{6}n_{a4}(N-4)(N^2 - 23N + 96). \tag{11}$$

Combining these two expressions with that for $\tilde{v}_\alpha$ in eq. (5) we arrive at

$$\beta^{N=2}_{SU(N)} = 12 + 6\,\tilde{v}_{SU(N)} \quad . \tag{12}$$

For the other groups in eq. (5) we obtain a similar result. More precisely,

$$\beta^{N=2}_{\alpha} = 12\left(1 + \frac{\tilde{v}_\alpha}{v_\alpha}\right) \quad . \tag{13}$$

Thus, the condition to get heterotic/heterotic duality in $N = 2$, $D = 4$ models reads

$$\begin{aligned} \beta^{N=2}_{\alpha} &= 12 \qquad \text{(symmetric } E_8 \times E_8 \text{ embeddings)} \\ \beta^{N=2}_{\alpha} &= 24 \qquad \text{(non-symmetric } E_8 \times E_8 \text{ embeddings)} \quad . \end{aligned} \tag{14}$$

Notice that in both cases the $N = 2$ models are non-asymptotically free. Notice also that the β-functions of all groups must be equal. In the first case ($\beta_\alpha = 12$), consistently with the DMW hypothesis in $D = 6$, there should be points in moduli space in which new gauge groups of a non-perturbative origin should appear. Those are required to obtain full duality. In the second case ($\beta_\alpha = 24$) this is not expected but explicit duality should be apparent.

One can think of the following consistency check of the proposed ideas. We know the form of the holomorphic $N = 2$ gauge kinetic function f_α for the gauge groups inherited from $E_8 \times E_8$. For a $K3 \times T^2$ compactification of the type discussed here one has [20, 21]

$$f_\alpha \; = \; k_\alpha S_{inv} \; - \; \frac{\beta_\alpha^{N=2}}{4\pi} \log\left(\eta(T)\eta(U)\right)^4 \quad , \tag{15}$$

where η is the Dedekind function and S_{inv} is given by:

$$S_{inv} \equiv S - \frac{1}{2}\partial_T \partial_U h^{(1)}(T,U) - \frac{1}{2\pi}\log\left(J(T) - J(U)\right) + \text{const.} \tag{16}$$

Here $h^{(1)}(T,U)$ is the moduli-dependent one-loop correction to the $N=2$ prepotential $\mathcal{F}$. More explicitly,

$$\mathcal{F} \; = \; -STU + h^{(1)}(T,U) + \cdots \quad , \tag{17}$$

where the ellipsis stands for terms that depend on matter fields, and for simplicity we do not include other moduli fields besides S, T, U. Now, we know that the large-T limit of f_α must reproduce the result in eq. (2). It is not clear that this follows from eqs. (15) and (16). We want to show that indeed this is the case if and only if $\beta_\alpha^{N=2} = 12\left(1 + \frac{\tilde{v}_\alpha}{v_\alpha}\right)$. To this purpose, we need to know the asymptotic large-T limit of S_{inv} and, hence, of $h^{(1)}$.

The one-loop correction $h^{(1)}(T,U)$ was explicitly computed in ref. [23] for the $K3$ representation in terms of a Z_2 orbifold with standard embedding. The authors find

$$h^{(1)}(T,U) = -\frac{1}{(2\pi)^4}\mathcal{L}(T,U) - \frac{c(0)\zeta(3)}{32\pi^4} - \frac{U^3}{12\pi} \qquad \mathrm{Re}\, T > \mathrm{Re}\, U \quad , \tag{18}$$

where

$$\mathcal{L}(T,U) \equiv L_{i_3}\left(e^{-2\pi(T-U)}\right) + \sum_{k,l} c(kl)\, L_{i_3}\left(e^{-2\pi(kT+lU)}\right) \quad . \tag{19}$$

Here $L_{i_3}(x) \equiv \sum_1^\infty x^j/j^3$. For the second Weyl chamber ($\mathrm{Re}\, U > \mathrm{Re}\, T$) the expression for $h^{(1)}(T,U)$ is similar, exchanging $T \leftrightarrow U$. Both expressions are defined up to quadratic terms that have no physical significance. Here the coefficients $c(n)$ are defined by the expansion of the modular form $F(M) \equiv E_6 E_4/\eta^{24} = \sum_{n=-1}^\infty c(n)\, q^n$ with $q \equiv e^{-2\pi M}$, $M = U, T$. In particular $c(0)$ coincides with the difference between the number of vector and hypermultiplets ($n_v - n_h = 4 - 244 = -240$ in these models).

Since our $E_7 \times E_7$ model of the previous section is just the same Z_2 model with standard embedding considered in [23], with the addition of a discrete Wilson line, the moduli dependent part of the prepotential will remain the same. Therefore equations (17), (15) and (18) define the full perturbative prepotential of our model.

From eqs. (18) and (19) one can show that $\partial_T \partial_U h^{(1)}(T,U) \to 0$ for large T. Since $\log J(T) \to 2\pi T$ one then finds for $k_\alpha = 1$

$$\lim_{T\to\infty} f_\alpha \;=\; S \;+ T\left(\frac{\beta_\alpha^{N=2}}{12} \;-\; 1\right) \;=\; S \;+\; \frac{\tilde{v}_\alpha}{v_\alpha}\, T \quad , \tag{20}$$

which is just the $D = 4$ version of formula (2). We thus see that if any of the $\beta_\alpha^{N=2}$ is smaller than 12, the large-T limit gives rise to gauge kinetic terms of the wrong sign. Actually, before Higgsing, the relation with $\tilde{v}_\alpha$ implies that if there is any group with $\beta_\alpha^{N=2} > 12$, there must be another group with $\beta_\alpha^{N=2} < 12$, leading inescapably to negative kinetic terms. More generally, $\beta_\alpha^{N=2} < 12$ is the $D = 4$ equivalent of the wrong sign problem signalled in $D = 6$ by $\tilde{v}_\alpha < 0$.

Notice the different large-T behavior of the two heterotic/heterotic dualities under consideration. In the one proposed by DMW one has $\tilde{v}_\alpha = 0$ and $f \to S$. In the alternative, based on an non-symmetric embedding, one has $f \to S + T$, a $S \leftrightarrow T$ invariant result. We should remark that in this last case, the calculation of [23] does not apply, but the limit in eq.(20) still holds [21].

Heterotic/heterotic duality tells us that there should be a heterotic dual model with the roles of S and T exchanged. In particular, the dual kinetic function $\tilde{f}_\alpha$ is obtained by making the replacement $T \leftrightarrow S$ in eq.(15) . Thus

$$\tilde{f}_\alpha \;=\; k_\alpha T_{inv} \;-\; \frac{\beta_\alpha^{N=2}}{4\pi} \log(\eta(S)\eta(U))^4 \quad . \tag{21}$$

We find for $k_\alpha = 1$

$$\lim_{S\to\infty} \tilde{f}_\alpha \;=\; T \;+ S\left(\frac{\beta_\alpha^{N=2}}{12} \;-\; 1\right) \;=\; T \;+\; \frac{\tilde{v}_\alpha}{v_\alpha}\, S \quad , \tag{22}$$

which again shows the special roles of $\beta_\alpha^{N=2} = 12, 24$. We can have non-perturbative gauge groups only for $\beta_\alpha^{N=2} = 12$ whereas only for $\beta_\alpha^{N=2} = 24$ can we have the standard perturbative limit.

There is an interesting connection between the class of models with $\beta_\alpha^{N=2} = 24, 12$ discussed here and $D = 4$ heterotic/type II duality. Indeed, the models discussed in this section are just a particular class of $E_8 \times E_8$ compactifications on $K3 \times T^2$ with appropriate gauge bundles. The particularity of both types of heterotic/heterotic duals is that the gauge symmetry coming from $E_8 \times E_8$ can be Higgsed away completely. As remarked in ref. [7], any heterotic model of this kind necessarily yields 244 hypermultiplets due to anomaly constraints. In addition there are four $U(1)$'s coming from three vector multiplets (which contain S,T and U) plus the graviphoton. This particle content was denoted as $(244, 4)$ in ref. [19].

In ref. [19], a candidate type II dual for the theory with $(244, 4)$ particle content was proposed as a compactification on a Calabi-Yau hypersurface in $\mathbb{P}(1,1,2,8,12)$, which is a $K3$ fibration. The heterotic dual was constructed as a $K3 \times T^2$ compactification with symmetric embedding, as discussed in section one. On the other hand, we proposed a different heterotic construction for this dual in ref. [7]. This was based on the Z_6 compactification described in the previous section. The interest of such a construction is that we were able to identify a chain of heterotic/type II duals continuously connected by Higgsing to the $(244, 4)$ model. The same chain can be derived from the $E_7 \times E_7$ model with instanton numbers $s = 14$ and $s' = 10$. Sequential Higgsing [7] produces models with particle content $(139, 7)$, $(162, 6)$, $(191, 5)$ and finally $(244, 4)$. The remarkable point is that for all these models, that have $v_\alpha = \tilde{v}_\alpha$ at points of enhanced gauge symmetry, we were able to identify *type II* dual candidates as compactifications on certain $K3$ fibrations [16]. Thus, at least for the last four elements of the chain, there are three dual descriptions, namely, a type II compactification and two dual heterotic descriptions. It would be certainly interesting to study more deeply the different connections among the various dualities appearing in the above models.

4 $N = 1, D = 4$ heterotic/heterotic duality

To obtain $N = 1$, $D = 4$ heterotic models, one can perform a $K3 \times T^2$ compactifications along with some twist that preserves just one supersymmetry. If the twist acts freely on $K3 \times T^2$ it is reasonable to expect that the S-T-U permutation invariance symmetry should remain and that the strongly coupled limit of the resulting $N = 1$ model should be given by the weakly coupled limit of an analogous $N = 1$ model with the roles of S and T exchanged. Since we want a free action, the natural choice is to mod out by a Z_2 twist ω that acts on $K3$ as the (freely-acting) Enriques involution and on T^2 by a reflection $T^2 \to -T^2$. In this way there will be one unbroken supersymmetry. Modular invariance requires that the twist should be accompanied by some action on the gauge group.

Although this is a general procedure that should in principle work for any $K3 \times T^2$, let us apply it to our explicit symmetric construction of section 1, in which we consider T^4/Z_2 instead of $K3$. In this case, the action of θ, with embedding by the shift A in (6) and the Wilson line in (7), is supplemented by the action of ω with some gauge embedding. This action need not be symmetric in both E_8s as long as the twist ω is freely-acting.

In practice we then consider an $N=1$, $Z_2 \times Z_2$ orbifold with action on the three complex coordinates of T^6 given by

$$\begin{aligned}
\theta\,(z_1, z_2, z_3) &= (-z_1, -z_2, z_3) \\
\omega\,(z_1, z_2, z_3) &= (z_1, -z_2, -z_3) \;+\; \left(\frac{1}{2}, \frac{1}{2}, 0\right) \quad .
\end{aligned} \tag{23}$$

The twist θ gives the original $K3 \times T^2$ whereas ω corresponds to the simultaneous action of the Enriques involution and a reflection of z_3 [24]. Again, modular invariance requires that ω come along with some action in the gauge degrees of freedom. The shift

$$B = \left(0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0\right) \times \left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right) \tag{24}$$

can be shown to verify all modular invariance constraints.

With the above ingredients we then obtain an $N=1$ model with gauge group $E_6 \times U(1)^2 \times SU(8) \times U(1)$. The massless spectrum includes the untwisted moduli $T_3 \equiv T$ and $U_3 \equiv U$. There are no charged multiplets in the untwisted subsectors associated to the first two complex planes. In the third-plane untwisted subsector we find $(27 + \overline{27}; 1)$ $+(1; 70)$ $+ 4(1; 1)$ under $E_6 \times SU(8)$. In the θ-sector we find $4(27 + \overline{27}; 1)$$+4(1; 28 + \overline{28})$$+72(1; 1)$. There are no massless particles in the sectors twisted by ω and $\theta\omega$, since ω acts freely.

The specific $N=1$ example outlined above was derived from an $N=2$ theory with symmetric embedding. However, the method employed in its construction can in principle be applied to obtain $N=1$ examples from $N=2$ models with non-symmetric embedding and hypermultiplet content allowing for complete Higgsing.

As stated before, when ω acts freely, the resulting $N=1$ model is expected to have a strongly coupled limit whose dynamics can be described by a similar but weakly coupled $N=1$ model in which S and T are exchanged. As in the $N=2$ case, a natural simple check of this idea can be performed by comparing the gauge kinetic functions of the original and the dual theory. At first sight this does not look very promising, since the unbroken gauge groups in the $N=1$ models have in general different $N=1$ β-functions ($b^{N=1}_{E_6} = -6$ and $b^{N=1}_{SU(8)} = 10$ in the above orbifold example). But let us proceed with the argument, this objection notwithstanding.

In $N=1$ models the gauge kinetic function $f^{N=1}_\alpha$ is also equal to $k_\alpha S$ at tree level and has only one-loop corrections. However, as shown in ref. [20], the one-loop term only receives (moduli-dependent) contributions from sectors of the orbifold with extended $N=2$ supersymmetry. Moreover, the contribution of these $N=2$ subsectors is precisely of the form given in eq.(15), with the coefficient of the log term proportional to the $N=2$ β-function of the $N=2$ subsectors of the theory. Since

ω acts freely, the only $N = 2$ subsector is the underlying initial $N = 2$, $T^4/Z_2 \times T^2$ orbifold. Thus, when ω is freely-acting we have the result

$$f_\alpha^{N=1} = f_\alpha^{N=2} \;\; ; \;\; \tilde{f}_\alpha^{N=1} = \tilde{f}_\alpha^{N=2} \quad . \tag{25}$$

Hence, the $N = 1$ gauge kinetic function is consistent with duality and S-T-U permutation symmetry. Non-perturbative information about the Kähler potential of this class of models should in principle be extractable by an appropriate truncation of the results for the prepotential of the underlying $N = 2$ theory.

Several comments concerning this class of $N = 1$ models are in order. First, notice that, although the underlying $N = 2$ parent model was asymptotically non-free, the $N = 1$ gauge groups can be asymptotically free and have unequal β-functions. This fact, could allow us to perform a truly non-perturbative analysis of the gaugino condensation process in $N = 1$ models and its possible relation to the breakdown of supersymmetry.

Secondly, this class of $N = 1$ models is a relatively constrained class. Indeed, the underlying $N = 2$ model is essentially uniquely determined as the model with 4 vector multiplets and 244 hypermultiplets. Of course, this model, which just has $U(1)^4$ gauge invariance at generic points in moduli space, can get enhanced gauge symmetries as large as $(E_7 \times SU(2))^2$ at some points. Nevertheless, the $N = 1$ class of models obtained by twisting is not so much constrained since there is a variety of freely acting twists that can be effected (in particular different gauge embeddings are possible). Thirdly, note that these models, like their $D = 6$ parent, have no unbroken gauge group left for generic values of the chiral field scalars. They look quite similar to the $N = 1$ examples built in refs. [24, 25].

There is another potentially interesting way to derive $N = 1$ heterotic/heterotic dual pairs. As we said, in $D = 6$ we have a duality relating $E_8 \times E_8$ heterotic compactifications on different realizations of $K3$, say $K3$ and $K3'$, with symmetric gauge embeddings. In analogy with a similar situation in heterotic/type II duality [26], it is reasonable to conjecture that if we compactify the $E_8 \times E_8$ heterotic down to $D = 4$ on a Calabi-Yau manifold that happens to be *a $K3$-fibration over* $\mathbb{P}_1$ [16, 25], the result should be dual to a compactification on another Calabi-Yau manifold corresponding to *a $K3'$-fibration over* $\mathbb{P}_1$. It would be quite interesting to find explicit examples of this kind of conjectured dual pairs.

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