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The Dipole Formalism for the Calculation of QCD Jet Cross Sections at Next-to-Leading Order*

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Abstract

In order to make quantitative predictions for jet cross sections in perturbative QCD, it is essential to calculate them to next-to-leading accuracy. This has traditionally been an extremely laborious process. Using a new formalism, imaginatively called the dipole formalism, we are able to construct a completely general algorithm for next-to-leading order calculations of arbitrary jet quantities in arbitrary processes. In this paper we present the basic ideas behind the algorithm and illustrate them with a simple example.

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1 Introduction

Most of the recent progress in the understanding of strong interaction physics at large momentum transfer has been due to the comparison between precise experimental data and very accurate QCD calculations to higher perturbative orders [1].

These higher-order computations have been carried out over a period of about fifteen years because of the difficulties in setting up a general and straightforward calculational procedure. The physical origin of these difficulties is in the necessity of factorizing the long- and short-distance components of the scattering processes and is reflected in the perturbative calculation by the presence of divergences.

In general, when evaluating higher-order QCD cross sections, one has to consider real-emission contributions and virtual-loop corrections and one has to deal with different kind of singularities. The customary *ultraviolet* singularities, present in the virtual contributions, are removed by renormalization. The low-momentum (*soft*) and small-angle (*collinear*) regions instead produce singularities both in the real and in the virtual contributions. In order to handle these divergences, the observable one is interested in has to be properly defined. It has to be a *jet quantity*, that is, a hadronic observable that turns out to be infrared safe and either collinear safe or collinear factorizable: its actual value has to be independent of the number of soft and collinear particles in the final state (see Sect. 4 for a formal definition). In the case of jet quantities, the coherent sum over different (real and virtual) soft and collinear partonic configurations in the final state leads to the cancellation of soft singularities. The left-over collinear singularities are then factorized into the process independent structure and fragmentation functions of partons (parton distributions), leading to predictable scaling violations. As a result, jet cross sections are finite (calculable) at the partonic level order by order in perturbation theory. All the dependence on long-distance physics is either included in the parton distributions or in non-perturbative corrections that are suppressed by inverse powers of the (large) transferred momentum Q that controls the scattering process.

Because of this pattern of singularities, QCD calculations of jet cross sections beyond leading order (LO) are very involved. Owing to the complicated phase space for multi-parton configurations, analytic calculations are in practice impossible for all but the simplest quantities, but the use of numerical methods is far from trivial because soft and collinear singularities present in the intermediate steps have first to be regularized. This is usually done by analytic continuation to a number of space-time dimensions $d = 4 - 2\epsilon$ different from four, which greatly complicates the Lorentz algebra in the evaluation of the matrix elements and prevents a straightforward implementation of numerical integration techniques. Despite these difficulties, efficient computational techniques have been set up, at least to next-to-leading order (NLO), during the last few years.

There are, broadly speaking, two types of algorithm used for NLO calculations: one based on the phase-space *slicing* method and the other based on the *subtraction* method[†]. The main difference between these algorithms and the standard procedures of analytic calculations is that only a minimal part of the full calculation is treated analytically, namely only those contributions giving rise to the singularities. Moreover, for any given process, these contributions are computed in a manner that is independent of the particular

[†]We refer the reader to the Introduction of Ref. [2] for an elementary description of the basic difference between the two methods.

jet observable considered. Once every singular term has been isolated and the cancellation/factorization of divergences achieved, one can perform the remaining part of the calculation in four space-time dimensions. Although, when possible, one still has the freedom of completing the calculation analytically, at this point the use of numerical integration techniques (typically, Monte Carlo methods) is certainly more convenient. First of all, the numerical approach allows one to calculate any number and any type of observable simultaneously by simply histogramming the appropriate quantities, rather than having to make a separate analytic calculation for each observable. Furthermore, using the numerical approach, it is easy to implement different experimental conditions, for example, detector acceptances and experimental cuts. In other words, the phase-space slicing and subtraction algorithms provide the basis for setting up a general-purpose Monte Carlo program for carrying out arbitrary NLO QCD calculations in a given process.

Both the slicing [3] and the subtraction [4] methods were first used in the context of NLO calculations of three-jet cross sections in e^+e^- annihilation. Then they have been applied to other cross sections adapting the method each time to the particular process. Only recently has it become clear that both algorithms are generalizable in a process-independent manner. The key observation is that the singular parts of the QCD matrix elements for real emission can be singled out in a general way by using the factorization properties of soft and collinear radiation [5].

At present, a general slicing algorithm is available for calculating NLO cross sections for *any* number of jets both in lepton [6] and hadron [7] collisions. To our knowledge, fragmentation processes have been considered only in the particular case of direct-photon production [8]. The complete generalization of this method to include fragmentation functions and heavy flavours is still missing.

As for the subtraction algorithm, a general NLO formalism has been set up for computing three-jet observables in e^+e^- annihilation [4,9] and cross sections up to two final-state jets [2,10] in hadron collisions[‡]. Also the treatment of massive partons has been considered in the particular case of heavy-quark correlations in hadron collisions [12].

In this paper, we present the basic idea to set up a *completely general* version of the subtraction algorithm. This generality is obtained by fully exploiting the factorization properties of soft and collinear emission and, thus, deriving new improved factorization formulae, called *dipole factorization formulae*. They allow us to introduce a set of universal counter-terms that can be used for *any* NLO QCD calculation.

For the purpose of illustration, in this short contribution we describe the implementation of our general method to the calculation of jet cross sections in processes with no initial-state hadrons, typically e^+e^- annihilation. Full details of the method and its application to all the other hard-scattering processes will appear elsewhere [13].

We first recall the main features of the subtraction method in Section 2. Then in Section 3 we present our dipole factorization formulae, for the case in which there are no incoming QCD partons. These allow us to calculate cross-sections for an arbitrary number of jets in e^+e^- annihilation, which we do in Section 4. After briefly recapping the resulting formulae in Section 5, we illustrate them with a simple example in Section 6, $e^+e^- \to 3$ jets. Finally in Section 7 we give a summary and outlook.

[‡]An extension of the method for three-jet cross sections has been recently presented [11].

2 The subtraction procedure

Suppose we want to compute a jet cross section σ to NLO, namely

$$\sigma = \sigma^{LO} + \sigma^{NLO} \ . \tag{1}$$

Here the LO cross section σ^{LO} is obtained by integrating the exclusive cross section $d\sigma^B$ in the Born approximation over the phase space for the corresponding jet quantity. Suppose also that this LO calculation involves m partons in the final state. Thus, we write

$$\sigma^{LO} = \int_m d\sigma^B \ , \tag{2}$$

where, in general, all the quantities (QCD matrix elements and phase space) are evaluated in $d = 4 - 2\epsilon$ space-time dimensions. However, by definition, at this LO the phase space integration in Eq. (2) is finite so that the whole calculation can be carried out (analytically or numerically) in four dimensions.

Now we go to NLO. We have to consider the exclusive cross section $d\sigma^R$ with $m+1$ partons in the final-state and the one-loop correction $d\sigma^V$ to the process with m partons in the final state:

$$\sigma^{NLO} \equiv \int d\sigma^{NLO} = \int_{m+1} d\sigma^R + \int_m d\sigma^V \ . \tag{3}$$

The two integrals on the right-hand side of Eq. (3) are separately divergent if $d = 4$, although their sum is finite. Therefore, before any numerical calculation can be attempted, the separate pieces have to be regularized. Using dimensional regularization, the divergences (arising out of the integration) are replaced by double (soft and collinear) poles $1/\epsilon^2$ and single (soft, collinear or ultraviolet) poles $1/\epsilon$. Suppose that one has already carried out the renormalization procedure in $d\sigma^V$ so that all its ultraviolet poles have been removed.

The general idea of the subtraction method for writing a general-purpose Monte Carlo program is to use the identity

$$d\sigma^{NLO} = \left[d\sigma^R - d\sigma^A \right] + d\sigma^A + d\sigma^V \quad , \tag{4}$$

where $d\sigma^A$ is a proper approximation of $d\sigma^R$ such as to have the same *pointwise* singular behaviour (in d dimensions) as $d\sigma^R$ itself. Thus, $d\sigma^A$ acts as a *local* counterterm for $d\sigma^R$ and, introducing the phase space integration,

$$\sigma^{NLO} = \int_{m+1} \left[d\sigma^R - d\sigma^A \right] + \int_{m+1} d\sigma^A + \int_m d\sigma^V \quad , \tag{5}$$

one can safely perform the limit $\epsilon \to 0$ under the integral sign in the first term on the right-hand side of Eq. (5). Hence, this first term can be integrated numerically in four dimensions.

All the singularities are now associated to the last two terms on the right-hand side of Eq. (5). If one is able to carry out analytically the integration of $d\sigma^A$ over the one-parton subspace leading to the ϵ poles, one can combine these poles with those in $d\sigma^V$, thus cancelling all the divergences, performing the limit $\epsilon \to 0$ and carrying out numerically the remaining integration over the m-parton phase space. The final structure of the calculation is as follows

$$\sigma^{NLO} = \int_{m+1} \left[d\sigma^R - d\sigma^A \right]_{\epsilon=0} + \int_m \left[d\sigma^V + \int_1 d\sigma^A \right]_{\epsilon=0} \quad , \tag{6}$$

and can be easily implemented in a 'partonic Monte Carlo' program, which generates appropriately weighted partonic events with $m+1$ final-state partons and events with m partons.

The key for the subtraction procedure to work is obviously the actual form of $d\sigma^A$. One needs to find an expression for $d\sigma^A$ that fulfils the following properties: *i*) for any given process, $d\sigma^A$ has to be obtained in a way that is independent of the particular jet observable considered; *ii*) it has to exactly match the singular behaviour of $d\sigma^R$ in d dimensions; *iii*) its form has to be particularly convenient for Monte Carlo integration techniques; *iv*) it has to be exactly integrable analytically in d dimensions over the single-parton subspaces leading to soft and collinear divergences.

In Ref. [4], a suitable expression for $d\sigma^A$ for the process $e^+e^- \to 3$ jets was obtained by starting from the explicit expression (in d dimensions) of the corresponding $d\sigma^R$ and by performing extensive partial fractioning of the $3+1$-parton matrix elements, so that each divergent piece could be extracted. This is an extremely laborious and ungeneralizable task, in the sense that having done it for $e^+e^- \to 3$ jets does not help us to do this for, say, $e^+e^- \to 4$ jets or for any other process.

In Ref. [2], the general properties of soft and collinear emission were first used (in the context of the subtraction method) to construct $d\sigma^A$, for one- and two-jet production in hadron collisions, in a way that is independent of the detailed form of the corresponding $d\sigma^R$.

The central proposal of our version of the subtraction method is that one can give a recipe for constructing $d\sigma^A$ that is completely *process independent* (and not simply independent of the jet observable). Starting from our physical knowledge of how the $m+1$-parton matrix elements behave in the soft and collinear limits that produce the divergences, we introduce improved factorization formulae, called dipole formulae (see Sect. 3), which allow us to obtain in a straightforward way (see Sect. 4) a counter-term $d\sigma^A$ satisfying all the properties listed above.

3 Dipole factorization formulae

Notation

In general we use dimensional regularization in $d = 4 - 2\epsilon$ space-time dimensions and consider $d-2$ helicity states for gluons and 2 helicity states for massless quarks. This defines the usual dimensional-regularization scheme. Other dimensional-regularization prescriptions can be used, as well [13].

The dimensional-regularization scale, which appears in the calculation of the matrix elements, is denoted by μ. In the perturbative calculation of physical cross sections, after having combined the renormalized matrix elements, the dependence on μ exactly cancels and is replaced by the dependence on the renormalization scale μ_R. Therefore, in order to avoid a cumbersome notation, we set $\mu = \mu_R$.

Throughout the paper, α_S stands for $\alpha_S(\mu)$, the NLO QCD running coupling evaluated at the renormalization scale μ. The actual value of the QCD coupling $\alpha_S(\mu)$ depends on the renormalization scheme used to subtract the ultraviolet divergences from the (bare) one-loop matrix element (or, equivalently, from $d\sigma^V$ in Eq. (3)).

The d-dimensional phase space, which involves the integration over the momenta $\{p_1, ..., p_m\}$ of m final-state partons, will be denoted as follows

$$\left[\prod_{l=1}^{m} \frac{d^d p_l}{(2\pi)^{d-1}} \, \delta_+(p_l^2) \right] \; (2\pi)^d \, \delta^{(d)}(p_1 + ... + p_m - Q) \equiv d\phi_m(p_1, ..., p_m; Q) \; . \tag{7}$$

In the case of processes without initial-state QCD partons (e^+e^--type processes), the (*tree-level*) matrix element with m QCD partons in the final state has the following general structure (non-QCD partons, namely $\gamma^*, Z^0, W^\pm, \ldots$, carrying a total incoming momentum Q_μ, are always understood)

$$\mathcal{M}_m^{c_1,\ldots,c_m;s_1,\ldots,s_m}(p_1, ..., p_m) \tag{8}$$

where $\{c_1, ..., c_m\}$, $\{s_1, ..., s_m\}$ and $\{p_1, .., p_m\}$ are respectively colour indices ($a = 1, ..., N_c^2 - 1$ different colours for each gluon, $\alpha = 1, .., N_c$ different colours for each quark or antiquark), spin indices ($\mu = 1, ..., d$ for gluons, $s = 1, 2$ for massless fermions) and momenta.

It is useful to introduce a basis $\{|c_1, ..., c_m > \otimes |s_1, ..., s_m >\}$ in colour + helicity space in such a way that

$$\mathcal{M}_m^{c_1,\ldots,c_m;s_1,\ldots,s_m}(p_1, ..., p_m) \equiv \Big(< c_1, ..., c_m| \otimes < s_1, ..., s_m| \Big) |1, ..., m >_m \; . \tag{9}$$

Thus $|1, ..., m >_m$ is a vector in colour + helicity space. According to this notation, the matrix element squared (summed over final-state colours and spins) $|\mathcal{M}_m|^2$ can be written as

$$|\mathcal{M}_m|^2 = {}_m\!< 1, ..., m|1, ..., m >_m \; . \tag{10}$$

As for the colour structure§, it is convenient to associate a colour charge $\mathbf{T}_i$ with the emission of a gluon from each parton i. We thus define the square of colour-correlated tree-amplitudes as follows

$$\begin{aligned} |\mathcal{M}_m^{i,k}|^2 &\equiv {}_m\!< 1, ..., m| \, \boldsymbol{T}_i \cdot \boldsymbol{T}_k \, |1, ..., m >_m \\ &= \Big[\mathcal{M}_m^{a_1..b_i...b_k...a_m}(p_1, ..., p_m) \Big]^* \; T^c_{b_i a_i} \, T^c_{b_k a_k} \; \mathcal{M}_m^{a_1..a_i...a_k...a_m}(p_1, ..., p_m) \; , \end{aligned} \tag{11}$$

where $T^a_{cb} \equiv i f_{cab}$ (colour-charge matrix in the adjoint representation) if the emitting particle i is a gluon and $T^a_{\alpha\beta} \equiv t^a_{\alpha\beta}$ (colour-charge matrix in the fundamental representation) if the emitting particle i is a quark (in the case of an emitting antiquark $T^a_{\alpha\beta} \equiv \bar{t}^a_{\alpha\beta} = -t^a_{\beta\alpha}$). It is straightforward to check that the colour-charge algebra is:

$$\boldsymbol{T}_i \cdot \boldsymbol{T}_j = \boldsymbol{T}_j \cdot \boldsymbol{T}_i \quad \text{if} \;\; i \neq j; \qquad \boldsymbol{T}_i^2 = C_i, \tag{12}$$

where C_i is the Casimir operator, that is, $C_i = C_A = N_c$ if i is a gluon and $C_i = C_F = (N_c^2 - 1)/2N_c$ if i is a quark or antiquark.

In this notation, each vector $|1, ..., m >_m$ is a colour singlet, so colour conservation is simply

$$\sum_{i=1}^{m} \boldsymbol{T}_i |1, ..., m >_m = 0. \tag{13}$$

§Within our formalism, there is no need to consider the decomposition of the matrix elements into colour subamplitudes, as in [6,14].

Dipole formulae

The real contribution $d\sigma^R$ to the NLO cross section in Eq. (3) is proportional to the tree-level matrix element $\mathcal{M}_{m+1}$ for producing $m+1$ partons in the final state. The dependence of $|\mathcal{M}_{m+1}|^2$ on the momentum p_j of a final-state parton j is singular in two different phase-space regions: when the momentum p_j vanishes (*soft* region) and/or when it becomes parallel to the momentum p_i of another parton in $\mathcal{M}_{m+1}$ (*collinear* region). This singular behaviour of $|\mathcal{M}_{m+1}|^2$ is well-known [5,15] and universal. Indeed, in the soft and collinear limits, $\mathcal{M}_{m+1}$ is essentially factorizable with respect to $\mathcal{M}_m$, the tree-level amplitude with m partons, and the singular factor only depends on the momenta and quantum numbers of the QCD partons in $\mathcal{M}_m$.

We thus introduce the following dipole factorization formula:

$$|\mathcal{M}_{m+1}(p_1,...,p_{m+1})|^2 = {}_{m+1}\!< 1,...,m+1||1,...,m+1 >_{m+1} = \sum_{k\neq i,j} \mathcal{D}_{ij,k}(p_1,...,p_{m+1}) + \ldots \tag{14}$$

where $\ldots$ stands for terms that are not singular in the limit $p_i \cdot p_j \to 0$ (i.e. when i and j become collinear or when either i or j is soft) and the dipole contribution $\mathcal{D}_{ij,k}$ is given by

$$\mathcal{D}_{ij,k}\,(p_1,...,p_{m+1}) = \frac{-1}{2p_i\cdot p_j} \cdot {}_m\!< 1,..,\widetilde{ij},..,\widetilde{k},..,m+1|\,\frac{\boldsymbol{T}_k\cdot\boldsymbol{T}_{ij}}{\boldsymbol{T}_{ij}^2}\,\boldsymbol{V}_{ij,k}\,|1,..,\widetilde{ij},..,\widetilde{k},..,m+1>_m \;. \tag{15}$$

The m-parton matrix element on the right-hand side of Eq. (15) is obtained from the original $m+1$-parton matrix element by replacing a) the partons i and j with a single parton $\widetilde{ij}$ (which plays the role of *emitter*) and b) the parton k with the parton $\widetilde{k}$ (which plays the role of *spectator*). All of the quantum numbers (colour, flavour) except momenta are assigned as follows. The spectator parton $\widetilde{k}$ has the same quantum numbers as k. The quantum numbers of the emitter parton $\widetilde{ij}$ are obtained according to their conservation in the collinear splitting process $\widetilde{ij} \to i+j$ (i.e. 'anything + gluon' gives 'anything' and 'quark + antiquark' gives 'gluon').

The momenta of the emitter and the spectator are defined as follows

$$\widetilde{p}_{ij}^{\mu} = p_i^{\mu} + p_j^{\mu} - \frac{y_{ij,k}}{1-y_{ij,k}}\,p_k^{\mu} \;\;, \quad \widetilde{p}_k^{\mu} = \frac{1}{1-y_{ij,k}}\,p_k^{\mu} \;\;, \tag{16}$$

where the dimensionless variable $y_{ij,k}$ is given by

$$y_{ij,k} = \frac{p_i p_j}{p_i p_j + p_j p_k + p_k p_i} \;. \tag{17}$$

In the bra-ket on the right-hand side of Eq. (15), $\boldsymbol{T}_{ij}$ and $\boldsymbol{T}_k$ are the colour charges of the emitter and the spectator and $\boldsymbol{V}_{ij,k}$ are matrices in the helicity space of the emitter. These matrices, which depend on $y_{ij,k}$ and on the kinematic variables $\tilde{z}_i$, $\tilde{z}_j$:

$$\tilde{z}_i = \frac{p_i p_k}{p_j p_k + p_i p_k} = \frac{p_i \widetilde{p}_k}{\widetilde{p}_{ij}\widetilde{p}_k} \;, \quad \tilde{z}_j = \frac{p_j p_k}{p_j p_k + p_i p_k} = \frac{p_j \widetilde{p}_k}{\widetilde{p}_{ij}\widetilde{p}_k} = 1 - \tilde{z}_i \;, \tag{18}$$

are universal factors related to the d-dimensional Altarelli-Parisi splitting functions [15]. For fermion + gluon splitting we have (s and s' are the spin indices of the fermion $\widetilde{ij}$ in $< .., \widetilde{ij}, ..|$ and $|.., \widetilde{ij}, .. >$ respectively)

$$
\begin{aligned}
< s|\boldsymbol{V}_{q_i g_j,k}(\tilde{z}_i; y_{ij,k})|s' > &= 8\pi\mu^{2\epsilon}\alpha_{\rm S}\, C_F \left[\frac{2}{1-\tilde{z}_i(1-y_{ij,k})} - (1+\tilde{z}_i) - \epsilon(1-\tilde{z}_i)\right] \delta_{ss'} \\
&\equiv V_{q_i g_j,k}\, \delta_{ss'} \; . \qquad (19)
\end{aligned}
$$

For quark + antiquark and gluon + gluon splitting we have (μ and ν are the spin indices of the gluon $\widetilde{ij}$ in $< .., \widetilde{ij}, ..|$ and $|.., \widetilde{ij}, .. >$ respectively)

$$
< \mu|\boldsymbol{V}_{q_i \bar{q}_j,k}(\tilde{z}_i)|\nu >= 8\pi\mu^{2\epsilon}\alpha_{\rm S}\, T_R \left[-g^{\mu\nu} - \frac{2}{p_i p_j}\,(\tilde{z}_i p_i^\mu - \tilde{z}_j p_j^\mu)\,(\tilde{z}_i p_i^\nu - \tilde{z}_j p_j^\nu)\right] \equiv V^{\mu\nu}_{q_i\bar{q}_j,k} \; , \qquad (20)
$$

$$
\begin{aligned}
< \mu|\boldsymbol{V}_{g_i g_j,k}(\tilde{z}_i; y_{ij,k})|\nu >= 16\pi\mu^{2\epsilon}\alpha_{\rm S}\, C_A &\left[-g^{\mu\nu}\left(\frac{1}{1-\tilde{z}_i(1-y_{ij,k})}\right.\right. \\
+ \left.\frac{1}{1-\tilde{z}_j(1-y_{ij,k})} - 2\right) + (1-\epsilon)\frac{1}{p_i p_j}&\left.(\tilde{z}_i p_i^\mu - \tilde{z}_j p_j^\mu)\,(\tilde{z}_i p_i^\nu - \tilde{z}_j p_j^\nu)\right] \equiv V^{\mu\nu}_{g_i g_j,k} \; . \qquad (21)
\end{aligned}
$$

The factorization formula in Eq. (14) has a dipole structure with respect to the *colour* and *spin* indices of the factorized partons. As shown in Ref. [13], Eqs. (14,15) coincide with the soft-gluon [5] and Altarelli-Parisi [15] factorization formulae respectively in the soft and collinear limits. However, Eqs. (14,15) are completely well-defined also outside these limiting regions of the phase space. Indeed, in the factorized m-parton matrix element both the emitter $\widetilde{ij}$ and the spectator $\widetilde{k}$ are on-shell ($\widetilde{p}_{ij}^2 = \widetilde{p}_k^2 = 0$) and, performing the replacement $\{i, j, k\} \to \{\widetilde{ij}, \widetilde{k}\}$, momentum conservation is implemented exactly:

$$
p_i^\mu + p_j^\mu + p_k^\mu = \widetilde{p}_{ij}^\mu + \widetilde{p}_k^\mu \quad . \qquad (22)
$$

The importance of these kinematical features is twofold. Firstly, momentum conservation leads to a smooth interpolation between the soft and collinear limits, thus avoiding double counting of overlapping soft and collinear singularities. Secondly, the definition (16) of the dipole momenta allows us to factorize exactly the $m+1$-parton phase space into an m-parton subspace times a single-parton contribution (see Eqs. (31,32)). The first property allows us to construct a counter-term $d\sigma^A$ that produces a pointwise cancellation of the singularities of $d\sigma^R$ as in Eq. (5). The second property makes this counter-term fully integrable analytically over the subspace leading to soft and collinear divergences.

4 The calculation of jet cross sections

The dipole formulae form the basis for our general algorithm for NLO jet calculations, as we describe below. First we define the leading order cross section and give a formal definition of the requirements a jet definition must fulfil. Then we introduce the subtraction term, which cancels all singularities of the real matrix element, and show how it can be integrated in d dimensions to cancel the singularities of the virtual matrix element.

Leading order and jet definition

The Born-level cross section in Eq. (2) has the following expression

$$d\sigma^B = \mathcal{N}_{in} \sum_{\{m\}} d\phi_m(p_1, ..., p_m; Q) \; \frac{1}{S_{\{m\}}} \; |\mathcal{M}_m(p_1, ..., p_m)|^2 \; F_J^{(m)}(p_1, ..., p_m) \;\; , \qquad (23)$$

where $\mathcal{N}_{in}$ includes all the factors that are QCD independent, $\sum_{\{m\}}$ denotes the sum over all the configurations with m partons, $d\phi_m$ is the partonic phase space in Eq. (7), $S_{\{m\}}$ is the Bose symmetry factor for identical partons in the final state and $\mathcal{M}_m$ is the tree-level matrix element.

The function $F_J^{(m)}(p_1, ..., p_m)$ defines the jet observable in terms of the momenta of the m final-state partons. In general, F_J may contain θ-functions (thus, Eq. (23) defines precisely a cross section), δ-functions (Eq. (23) defines a differential cross section), numerical and kinematical factors (Eq. (23) refers to an inclusive observable), or any combination of these. The essential property of $F_J^{(m)}$ is that the jet observable we are interested in has to be infrared and collinear safe. That is, it has to be experimentally (theoretically) defined in such a way that its actual value is independent of the number of soft and collinear hadrons (partons) produced in the final state. In particular, this value has to be the same in a given m-parton configuration and in all $m+1$-parton configurations that are kinematically degenerate with it (i.e. which are obtained from the m-parton configuration by adding a soft parton or replacing a parton with a pair of collinear partons carrying the same total momentum). These properties can be simply restated in a formal way, as follows

$$F_J^{(n+1)}(p_1, .., p_j, .., p_{n+1}) \to F_J^{(n)}(p_1, ..., p_{n+1}) \qquad \text{if} \;\; p_j \to 0 \;\; , \qquad (24)$$

$$F_J^{(n+1)}(p_1, .., p_i, .., p_j, .., p_{n+1}) \to F_J^{(n)}(p_1, .., p_i + p_j, .., p_{n+1}) \qquad \text{if} \;\; p_i \parallel p_j \; , \qquad (25)$$

$$F_J^{(m)}(p_1, ..., p_m) \to 0 \qquad \text{if} \;\; p_i \cdot p_j \to 0 \;\; . \qquad (26)$$

Equations (24) and (25) respectively guarantee that the jet observable is infrared and collinear safe for *any* number n of final-state partons, i.e. to *any* order in QCD perturbation theory. Equation (26) defines the LO cross section, that is, it ensures that the Born-level cross section $d\sigma^B$ in Eq. (23) is well-defined (i.e. finite after integration) in $d = 4$ dimensions.

Next-to-leading order: the subtraction term

The real contribution $d\sigma^R$ to the NLO cross section in Eq. (3) has the same expression as $d\sigma^B$ in Eq. (23), apart from the replacement $m \to m+1$. In particular, the m-parton matrix element $\mathcal{M}_m$ is replaced by $\mathcal{M}_{m+1}$. Therefore an explicit and general form for the local counter-term $d\sigma^A$ in Eq. (4) is provided by the dipole factorization formula (14):

$$\begin{aligned} d\sigma^A \;\; &= \;\; \mathcal{N}_{in} \sum_{\{m+1\}} d\phi_{m+1}(p_1, ..., p_{m+1}; Q) \; \frac{1}{S_{\{m+1\}}} \\ &\cdot \sum_{\substack{\text{pairs} \\ \text{i,j}}} \sum_{k \neq i,j} \mathcal{D}_{ij,k}(p_1, ..., p_{m+1}) \; F_J^{(m)}(p_1, ..\widetilde{p}_{ij}, \widetilde{p}_k, .., p_{m+1}) \;\; . \end{aligned} \qquad (27)$$

Here $\mathcal{D}_{ij,k}(p_1, ..., p_{m+1})$ is the dipole contribution in Eq. (15) and $F_J^{(m)}(p_1, ..\widetilde{p}_{ij}, \widetilde{p}_k, .., p_{m+1})$ is the jet function for the corresponding m-parton state $\{p_1, ..\widetilde{p}_{ij}, \widetilde{p}_k, .., p_{m+1}\}$. Note that this is completely independent of p_i, which is how we are able to integrate $d\sigma^A$ analytically over the phase-space of i without any information about the form of F_J, as we perform below.

We can check that the definition (27) makes the difference $(d\sigma^R - d\sigma^A)$ integrable in $d = 4$ dimensions. Its explicit expression is

$$\begin{aligned} d\sigma^R - d\sigma^A &= \mathcal{N}_{in} \sum_{\{m+1\}} d\phi_{m+1}(p_1, ..., p_{m+1}; Q) \, \frac{1}{S_{\{m+1\}}} \\ &\cdot \Big\{ |\mathcal{M}_{m+1}(p_1, ..., p_{m+1})|^2 \, F_J^{(m+1)}(p_1, ..., p_{m+1}) \\ &- \sum_{\substack{\text{pairs} \\ \text{i,j}}} \sum_{k \neq i,j} \mathcal{D}_{ij,k}(p_1, ..., p_{m+1}) \, F_J^{(m)}(p_1, ..\widetilde{p}_{ij}, \widetilde{p}_k, .., p_{m+1}) \Big\} \; . \end{aligned} \tag{28}$$

Each term in the curly bracket is separately singular in the soft and collinear regions. However, as stated in Sect. 3, in each of these regions both the matrix element $\mathcal{M}_{m+1}$ and the phase space for the $m+1$-parton configuration behave as the corresponding dipole contribution and dipole phase space:

$$|\mathcal{M}_{m+1}(p_1, ..., p_{m+1})|^2 \;\to\; \mathcal{D}_{ij,k}(p_1, ..., p_{m+1}) \; , \tag{29}$$

$$\{p_1, ..p_i, ..p_j, ..p_k, .., p_{m+1}\} \;\to\; \{p_1, ..\widetilde{p}_{ij}, \widetilde{p}_k, .., p_{m+1}\} \; . \tag{30}$$

Thus, because of Eqs. (24) and (25), the singularities of the first term in the curly bracket are cancelled by similar singularities due to the second term. On the other hand, each dipole $\mathcal{D}_{ij,k}$ in Eq. (15) has no other singularities but those due to the m-parton matrix element $|1, .., \widetilde{ij}, .., \widetilde{k}, .., m+1 >_m$. Because of Eq. (26), these singularities are screened (regularized) by the jet function $F_J^{(m)}(p_1, ..\widetilde{p}_{ij}, \widetilde{p}_k, .., p_{m+1})$ in the curly bracket of Eq. (28).

Note that this cancellation mechanism is completely independent of the actual form of the jet defining function but it is essential that $d\sigma^R$ and $d\sigma^A$ are proportional to $F_J^{(m+1)}$ and $F_J^{(m)}$ respectively. Nonetheless, because the terms on the left- and right-hand sides of Eq.(14) depend on the same kinematic variables, $\{p_1, ..., p_{m+1}\}$, both $d\sigma^R$ and $d\sigma^A$ live on the same $m+1$-parton phase space. Thus the numerical integration (in $d = 4$ dimensions) of Eq. (28) via Monte Carlo techniques is straightforward. One simply generates an $m+1$-parton configuration and gives it a positive $(+|\mathcal{M}_{m+1}|^2)$ or negative $(-\mathcal{D}_{ij,k})$ weight. The role of the two different jet functions $F_J^{(m+1)}$ and $F_J^{(m)}$ is that of binning these weighted events into different bins of the jet observable. Any time that the generated configuration approaches a singular region, these two bins coincide and the cancellation of the large positive and negative weights takes place.

Note, also, that the helicity dependence of the splitting kernels $\boldsymbol{V}_{ij,k}$ in Eqs. (20,21) is essential if $d\sigma^A$ is to act as a local counterterm that makes $[d\sigma^R - d\sigma^A]$ integrable in four dimensions. Indeed, the parton azimuthal correlations due to this dependence are not only essential in the most general case when F_J explicitly depends on them, but even when it does not¶.

¶In this case the evaluation of $\int_{m+1} d\sigma^R$ in four dimensions usually involves double angular integrals of the type $\int_{-1}^{+1} d\cos\theta \int_0^{2\pi} d\varphi \;\cos\varphi/(1-\cos\theta)$, where φ is the azimuthal angle. These integrals are mathematically ill-defined. If their numerical integration is attempted, one can obtain any answer whatsoever, depending on the detail of the integration procedure. Performing the integral analytically before going to 4 dimensions, one obtains $\int_{-1}^{+1} d\cos\theta \int_0^{2\pi} d\varphi \;\cos\varphi/(1-\cos\theta)\sin^{-2\epsilon}\theta \sin^{-2\epsilon}\varphi = 0$.

Next-to-leading order: integral of the subtraction term

Having discussed the four-dimensional integrability of $(d\sigma^R - d\sigma^A)$, the only other step we have to consider is the d-dimensional analytical integrability of $d\sigma^A$ over the one-parton subspace leading to soft and collinear divergences. In this respect, our dipole formalism is particularly efficient and simple. The definition (16) of the dipole momenta allows us to *exactly* factorize the phase space of the partons i, j, k into the dipole phase space times a single-parton contribution, as follows

$$d\phi_{m+1}(p_1, .., p_i, p_j, p_k, .., p_{m+1}; Q) = d\phi_m(p_1, .., \widetilde{p}_{ij}, \widetilde{p}_k, .., p_{m+1}; Q) \; [dp_i(\widetilde{p}_{ij}, \widetilde{p}_k)] \;\; , \qquad (31)$$

where

$$[dp_i(\widetilde{p}_{ij}, \widetilde{p}_k)] = \frac{d^d p_i}{(2\pi)^{d-1}} \, \delta_+(p_i^2) \; \Theta(1 - \tilde{z}_i) \, \Theta(1 - y_{ij,k}) \; \frac{(1 - y_{ij,k})^{d-3}}{1 - \tilde{z}_i} \; , \qquad (32)$$

and the kinematic variables $y_{ij,k}$ and $\tilde{z}_i$ are defined in Eqs. (17,18).

Inserting Eq. (31) and the explicit expression (15) for $\mathcal{D}_{ij,k}$ into Eq. (27), we can factorize completely the p_i dependence and carry out the integration over the phase space region (32) with the m parton momenta $\{p_1, .., \widetilde{p}_{ij}, \widetilde{p}_k, .., p_{m+1}\}$ kept fixed. Remarkably, this integration can be exactly performed in closed analytical form in any number of space-time dimensions. As shown in detail in Ref. [13], after integration the spin correlations between $\boldsymbol{V}_{ij,k}$ and $|1, .., \widetilde{ij}, .., \widetilde{k}, .., m+1 >_m$ vanish and only colour correlations survive. The final result for $\int_{m+1} d\sigma^A$ can be written in terms of an m-parton integral of the LO (colour-correlated) matrix element times a factor [13]:

$$\int_{m+1} d\sigma^A = \int_m \left[\int_1 d\sigma^A \right] \;\; = \;\; \int_m \mathcal{N}_{in} \sum_{\{m\}} d\phi_m(p_1, ..., p_m; Q) \, \frac{1}{S_{\{m\}}}$$
$$\cdot \;\; {}_m\!< 1, ..., m| \, \boldsymbol{I}(\epsilon) \, |1, ..., m >_m \;\; F_J^{(m)}(p_1, ..., p_m) \;\; . \qquad (33)$$

Comparing Eqs. (33) and (23), we see that the integration of $d\sigma^A$ over the one-parton subspace that produces soft and collinear singularities leads to an expression completely analogous to $d\sigma^B$. One should simply replace the matrix element squared $|\mathcal{M}_m|^2 = {}_m\!< 1, ..., m|1, ..., m >_m$ in $d\sigma^B$ with

$${}_m\!< 1, ..., m| \, \boldsymbol{I}(\epsilon) \, |1, ..., m >_m \; , \qquad (34)$$

where $\boldsymbol{I}(\epsilon)$ is an insertion operator that depends on the colour charges and momenta of the m final-state partons. Its explicit expression is [13]:

$$\boldsymbol{I}(p_1, ..., p_m; \epsilon) = - \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \sum_i \; \frac{1}{\boldsymbol{T}_i^2} \; \mathcal{V}_i(\epsilon) \; \sum_{k \neq i} \boldsymbol{T}_i \cdot \boldsymbol{T}_k \; \left(\frac{4\pi\mu^2}{2 p_i \cdot p_k} \right)^{\epsilon} \;\; , \qquad (35)$$

where the singular factors $\mathcal{V}_i(\epsilon)$ have the following ϵ-expansion[‖]

$$\mathcal{V}_i(\epsilon) = \boldsymbol{T}_i^2 \left(\frac{1}{\epsilon^2} - \frac{\pi^2}{3} \right) + \gamma_i \, \frac{1}{\epsilon} + \gamma_i + K_i + \mathcal{O}(\epsilon) \;\; , \qquad (36)$$

[‖]Their exact expressions in any number $d = 4 - 2\epsilon$ of dimensions are given in [13].

with ($T_R = 1/2$ and N_f is the number of flavours)

$$\gamma_{i=q,\bar{q}} = \tfrac{3}{2}\, C_F \ , \quad \gamma_{i=g} = \tfrac{11}{6}\, C_A - \tfrac{2}{3} T_R N_f \ , \tag{37}$$

$$K_{i=q,\bar{q}} = \left(\tfrac{7}{2} - \tfrac{\pi^2}{6}\right) C_F \ , \quad K_{i=g} = \left(\tfrac{67}{18} - \tfrac{\pi^2}{6}\right) C_A - \tfrac{10}{9} T_R N_f \ . \tag{38}$$

In the calculation of the NLO cross section (6), Eq. (33) has to be combined with the virtual contribution, whose expression in terms of the (renormalized) one-loop matrix element is the following

$$d\sigma^V = \mathcal{N}_{in} \sum_{\{m\}} d\phi_m(p_1, ..., p_m; Q) \, \frac{1}{S_{\{m\}}} \, |\mathcal{M}_m(p_1, ..., p_m)|^2_{(1-loop)} \, F_J^{(m)}(p_1, ..., p_m) \ . \tag{39}$$

As discussed in Ref. [13], the addition of these two contributions correctly produces the cancellation of all the ϵ-poles, thus leading to a finite NLO cross section.

5 Final formulae

The final results of the application of our algorithm to the calculation of jet cross sections with no hadron in the initial state are summarized below.

The full QCD cross section in Eq. (1) contains a LO and a NLO component. Assuming that the LO calculation involves m final-state partons, the LO cross section is given by

$$\sigma^{LO} = \int_m d\sigma^B = \int d\Phi^{(m)} \, |\mathcal{M}_m(p_1, ..., p_m)|^2 \, F_J^{(m)}(p_1, ..., p_m) \ , \tag{40}$$

where $\mathcal{M}_m$ is the tree-level QCD matrix element for producing m partons in the final state and the function $F_J^{(m)}$ defines the particular jet observable we are interested in (see Eqs. (24-26) for the general properties that $F_J^{(m)}$ has to fulfil). The factor $d\Phi^{(m)}$ collects all the relevant phase space factors, i.e. all the remaining terms on the right-hand side of Eq. (23). The whole calculation (phase space integration and evaluation of the matrix element) can be carried out in four space-time dimensions.

According to the subtraction formula, Eq. (6), the NLO cross section is split into two terms with $m+1$-parton and m-parton kinematics, respectively. The contribution with $m+1$-parton kinematics is the following

$$\begin{aligned} \int_{m+1} \left[d\sigma^R - d\sigma^A \right]_{\epsilon=0} \ = \ & \int d\Phi^{(m+1)} \Big\{ \ |\mathcal{M}_{m+1}(p_1, ..., p_{m+1})|^2 \, F_J^{(m+1)}(p_1, ..., p_{m+1}) \\ & - \sum_{\substack{\text{pairs} \\ \text{i,j}}} \sum_{k \neq i,j} \mathcal{D}_{ij,k}(p_1, ..., p_{m+1}) \, F_J^{(m)}(p_1, ..\widetilde{p}_{ij}, \widetilde{p}_k, .., p_{m+1}) \ \Big\} \ , \end{aligned} \tag{41}$$

where the term in the curly bracket is exactly the same as that in Eq. (28): $\mathcal{M}_{m+1}$ is the tree-level matrix element, $\mathcal{D}_{ij,k}$ is the dipole factor in Eq. (15) and $F_J^{(m)}$ is the jet defining function for the corresponding m-parton state (note, again, the difference between the two jet functions $F_J^{(m+1)}$ and $F_J^{(m)}$ in the curly bracket). In spite of their original d-dimensional definition, at this stage the full calculation is carried out in four dimensions.

The NLO contribution with m-parton kinematics is given by

$$\int_m \left[d\sigma^V + \int_1 d\sigma^A \right]_{\epsilon=0} \tag{42}$$
$$= \int d\Phi^{(m)} \left\{ |\mathcal{M}_m(p_1,...,p_m)|^2_{(1-loop)} + {}_m\!< 1,...,m| \, \boldsymbol{I}(\epsilon) \, |1,...,m>_m \right\}_{\epsilon=0} F_J^{(m)}(p_1,...,p_m) \ .$$

The first term in the curly bracket is the one-loop *renormalized* matrix element squared for producing m final-state partons. The second term is obtained by inserting the colour-charge operator of Eq. (35) into the tree-level matrix element for producing m partons as in Eq. (34). These two terms have to be first evaluated in $d = 4 - 2\epsilon$ dimensions. Then one has to carry out their expansion in ϵ-poles (the expansion for the singular factors $\mathcal{V}_i(\epsilon)$ is given in Eq. (36)), cancel analytically (by trivial addition) the poles and perform the limit $\epsilon \to 0$. At this point the phase-space integration is carried out in four space-time dimensions.

6 $e^+e^- \to$ 3 jets

In this Section we consider the simplest non-trivial application of our algorithm, namely the calculation of three-jet observables in e^+e^- annihilation. Thus our formalism can be directly compared with that in Ref. [4].

The LO partonic process to be considered is $e^+e^- \to q(p_1) + \bar{q}(p_2) + g(p_3)$. The corresponding tree-level matrix element is denoted by $\mathcal{M}_3(p_1, p_2, p_3)$. We use customary notation for the kinematic variables: Q^2 is the square of the centre-of-mass energy, $y_{ij} = 2p_i \cdot p_j/Q^2$ and $x_i = 2p_i \cdot Q/Q^2$.

At NLO, two different real-emission subprocesses contribute: a) $e^+e^- \to q(p_1)+\bar{q}(p_2)+g(p_3)+g(p_4)$; b) $e^+e^- \to q(p_1)+\bar{q}(p_2)+q(p_3)+\bar{q}(p_4)$. In addition one has to compute the one-loop correction to the LO process.

The calculation of the subtracted cross section (41) for the subprocess a) involves the evaluation of the following dipole contributions: $\mathcal{D}_{13,2}, \mathcal{D}_{13,4}, \mathcal{D}_{23,1}, \mathcal{D}_{23,4}, \mathcal{D}_{34,1}, \mathcal{D}_{34,2}$. The associated colour algebra can be easily performed in closed form because the several colour projections of the three-parton matrix element completely factorize**. Thus we do not need to compute any colour-correlated tree amplitudes and we find

$$\begin{aligned}
\mathcal{D}_{13,2}(p_1,p_2,p_3,p_4) &= \frac{1}{2p_1p_3}\left(1 - \frac{C_A}{2C_F}\right) V_{q_1g_3,2} \, |\mathcal{M}_3(\widetilde{p}_{13}, \widetilde{p}_2, p_4)|^2 \ , \\
\mathcal{D}_{13,4}(p_1,p_2,p_3,p_4) &= \frac{1}{2p_1p_3}\frac{C_A}{2C_F} V_{q_1g_3,4} \, |\mathcal{M}_3(\widetilde{p}_{13}, p_2, \widetilde{p}_4)|^2 \ , \\
\mathcal{D}_{34,1}(p_1,p_2,p_3,p_4) &= \frac{1}{2p_3p_4}\frac{1}{2} V^{\mu\nu}_{g_3g_4,1} \, \mathcal{T}_{\mu\nu}(\widetilde{p}_1, p_2, \widetilde{p}_{34}) \ .
\end{aligned} \tag{43}$$

The dipoles contributions $\mathcal{D}_{23,1}, \mathcal{D}_{23,4}, \mathcal{D}_{34,2}$ are obtained respectively from $\mathcal{D}_{13,2}, \mathcal{D}_{13,4}, \mathcal{D}_{34,1}$ by the replacement $p_1 \leftrightarrow p_2$.

**If the LO matrix element involves two or three partons, the colour algebra can always be carried out in closed (factorized) form.

In the case of the subprocess b) we have to consider the following dipole contributions: $\mathcal{D}_{12,3}, \mathcal{D}_{12,4}, \mathcal{D}_{14,2}, \mathcal{D}_{14,3}, \mathcal{D}_{23,1}, \mathcal{D}_{23,4}, \mathcal{D}_{34,1}, \mathcal{D}_{34,2}$. Performing the colour algebra we get

$$\mathcal{D}_{34,1}(p_1,p_2,p_3,p_4) = \frac{1}{2p_3p_4}\frac{1}{2}\, V^{\mu\nu}_{q_3\bar{q}_4,1}\mathcal{T}_{\mu\nu}(\widetilde{p}_1,p_2,\widetilde{p}_{34}) \;, \tag{44}$$

and all the other dipoles are obtained by the corresponding permutation of the parton momenta.

The splitting functions $V_{ij,k}$ of Eqs. (43,44) are given in Eqs. (19-21). The tensor $\mathcal{T}_{\mu\nu}$ is the squared amplitude for the LO process $e^+e^- \to q\bar{q}g$ not summed over the gluon polarizations (μ and ν are the gluon spin indices and $-g^{\mu\nu}\mathcal{T}_{\mu\nu} = |\mathcal{M}_3|^2$). This can be easily calculated. In the case of jet observables averaged over the directions of the incoming leptons (un-oriented events) we find (in $d=4$ dimensions)

$$\mathcal{T}^{\mu\nu}(p_1,p_2,p_3) = -\frac{1}{x_1^2+x_2^2}\;|\mathcal{M}_3(p_1,p_2,p_3)|^2\; T^{\mu\nu} \;, \tag{45}$$

where

$$\begin{aligned} T^{\mu\nu} \;=\;& +2\frac{p_1^\mu p_2^\nu}{Q^2} + 2\frac{p_2^\mu p_1^\nu}{Q^2} - 2\frac{1-x_1}{1-x_2}\frac{p_1^\mu p_1^\nu}{Q^2} - 2\frac{1-x_2}{1-x_1}\frac{p_2^\mu p_2^\nu}{Q^2} \\ & -\frac{1-x_1-x_2+x_2^2}{1-x_2}\left[\frac{p_1^\mu p_3^\nu}{Q^2}+\frac{p_3^\mu p_1^\nu}{Q^2}\right] - \frac{1-x_2-x_1+x_1^2}{1-x_1}\left[\frac{p_2^\mu p_3^\nu}{Q^2}+\frac{p_3^\mu p_2^\nu}{Q^2}\right] \\ & +\left(1+\tfrac{1}{2}x_1^2+\tfrac{1}{2}x_2^2-x_1-x_2\right) g^{\mu\nu} \;. \end{aligned} \tag{46}$$

To complete the NLO calculation we also need the virtual cross section. In the case of un-oriented events, we take the one-loop matrix element in the $\overline{\rm MS}$ renormalization scheme from Ref. [4] (we use slightly different notation):

$$\begin{aligned} |\mathcal{M}_3(p_1,p_2,p_3)|^2_{(1-loop)} =& \;|\mathcal{M}_3(p_1,p_2,p_3)|^2\,\frac{\alpha_S}{2\pi}\frac{1}{\Gamma(1-\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon} \\ \cdot\Big\{ &-\frac{1}{\epsilon^2}\left[(2C_F-C_A)y_{12}^{-\epsilon}+C_A\left(y_{13}^{-\epsilon}+y_{23}^{-\epsilon}\right)\right] - \frac{1}{\epsilon}\left(3C_F+\frac{11}{6}C_A-\frac{2}{3}T_RN_f\right) \\ &+\frac{\pi^2}{2}(2C_F+C_A)-8C_F\Big\} + \frac{\alpha_S}{2\pi}\left[F(y_{12},y_{13},y_{23})+\mathcal{O}(\epsilon)\right] \;, \end{aligned} \tag{47}$$

where $F(y_{12},y_{13},y_{23})$ is defined in Eq. (2.21) of Ref. [4].

The explicit evaluation of the insertion operator $\boldsymbol{I}(\epsilon)$ in Eqs. (34,35) gives:

$$\begin{aligned} {}_3\!< 1,2,3|\;\boldsymbol{I}(\epsilon)\;|1,2,3>_3 =& \;|\mathcal{M}_3(p_1,p_2,p_3)|^2\,\frac{\alpha_S}{2\pi}\frac{1}{\Gamma(1-\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon} \\ \Big\{ &\frac{1}{\epsilon^2}\left[(2C_F-C_A)y_{12}^{-\epsilon}+C_A\left(y_{13}^{-\epsilon}+y_{23}^{-\epsilon}\right)\right] + \frac{1}{\epsilon}\left(2\gamma_q+\gamma_g\right) \\ &-\gamma_q\frac{1}{C_F}\left[(2C_F-C_A)\ln y_{12}+\frac{1}{2}C_A\ln(y_{13}y_{23})\right] - \frac{1}{2}\gamma_g\ln(y_{13}y_{23}) \\ &-\frac{\pi^2}{3}(2C_F+C_A)+2(\gamma_q+K_q)+\gamma_g+K_g+\mathcal{O}(\epsilon)\Big\} \;. \end{aligned} \tag{48}$$

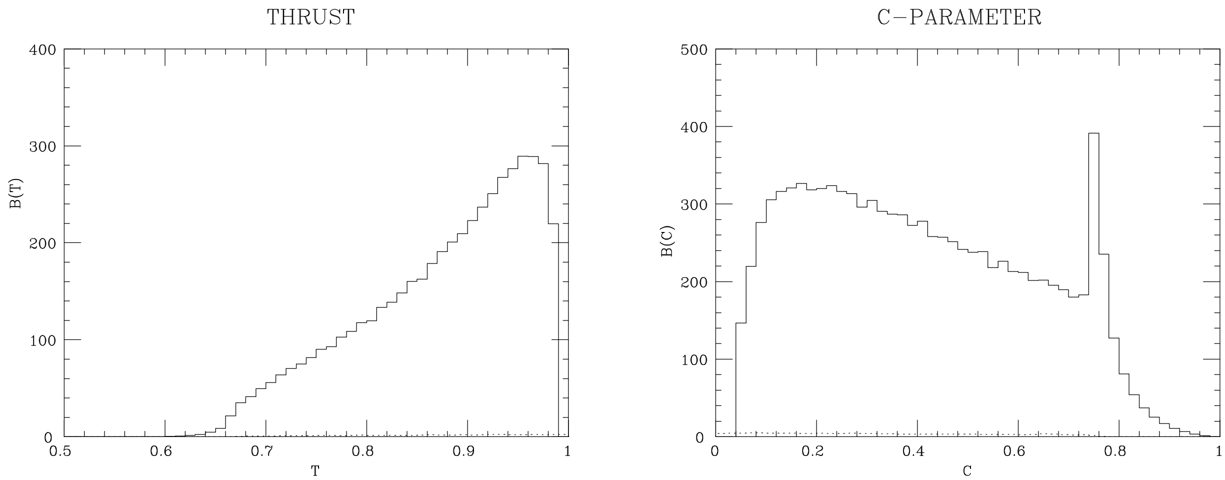

Figure 1: Coefficient of $(\alpha_S/2\pi)^2$ for the thrust and C-parameter distributions. The dotted histograms show the size of the statistical errors.

Combining the one-loop matrix element (47) with the result (48) according to Eq. (42) and using the explicit expressions (37,38) for γ_i and K_i, all the pole terms cancel. Note that as well as the pole terms, the closely related π^2 and $\ln^2$ terms cancel:

$$|\mathcal{M}_3(p_1,p_2,p_3)|^2_{(1-loop)} + {}_3\!< 1,2,3|\, \boldsymbol{I}(\epsilon)\, |1,2,3>_3 = |\mathcal{M}_3(p_1,p_2,p_3)|^2 \cdot \frac{\alpha_S}{2\pi}\left[-\frac{3}{2}(2C_F - C_A)\ln y_{12} - \frac{1}{3}(5C_A - T_R N_f)\ln(y_{13}y_{23}) + 2C_F + \frac{50}{9}C_A - \frac{16}{9}T_R N_f\right] + \frac{\alpha_S}{2\pi}\left[F(y_{12},y_{13},y_{23}) + \mathcal{O}(\epsilon)\right] \; . \tag{49}$$

We have implemented these results as a working Monte Carlo program[††] and the results are in good agreement with Ref. [9] for all distributions shown there. As an example we show the NLO coefficients for the thrust and C-parameter distributions in Fig. 1. We find that in general, the numerical convergence is similar to the program of Ref. [9], except close to the two-jet region in which ours becomes progressively better. More details of the Monte Carlo program and the generalization to oriented three-jet events will be presented elsewhere.

7 Summary and outlook

In this letter we have presented the basic idea to set up a completely general algorithm for calculating jet cross sections in NLO QCD. By general we mean that the algorithm applies to *any* jet observable in a given scattering process as well as to *any* hard-scattering process. The algorithm overcomes all the analytical difficulties related to the treatment of soft and collinear divergences in the perturbative expansion. The output of the algorithm is given in terms of effective matrix elements (the contributions in the curly bracket of Eqs. (41,42)) with built-in cancellation of soft and collinear singularities. These effective matrix elements can be numerically or analytically (whenever possible) integrated over the available phase space to compute the actual value of the NLO cross section. If the numerical approach is

[††]The program can be obtained from `http://surya11.cern.ch/users/seymour/nlo/`.

chosen, Monte Carlo integration techniques can be easily implemented to provide a general-purpose Monte Carlo program for carrying out NLO QCD calculations in any given process.

Starting from the very general idea of the subtraction method, we have discussed how one can automatically construct a pointwise and integrable counter-term $d\sigma^A$ for the real contribution $d\sigma^R$ to the NLO cross section. This counter-term is independent of the details of the process under study and can be handled once and for all to define the effective matrix elements mentioned above. We have shown how this procedure works in practice for processes with no hadron in the initial state (typically, e^+e^- annihilation) and any number of jets in the final state. In this case, the final result is given by the master formulae in Eqs. (41,42). We have provided explicit expressions for both the universal dipole factors $\mathcal{D}_{ij,k}$ and $\boldsymbol{I}$. Having these factors at our disposal, the only other ingredients necessary for the full NLO calculation, are the following

- a set of independent colour projections[‡‡] of the matrix element squared at the Born level, summed over parton polarizations, in d dimensions;
- the one-loop matrix element in d dimensions;
- an additional projection of the Born level matrix element over the helicity of each external gluon in four dimensions;
- the tree-level NLO matrix element in four dimensions.

These few ingredients are sufficient for writing, in a straightforward way, a general-purpose NLO Monte Carlo algorithm. Note in particular that there is no need to extract a proper counter-term $d\sigma^A$ starting from a cumbersome expression for $d\sigma^R$ in d dimensions. The NLO matrix element contributing to $d\sigma^R$ can be evaluated directly in four space-time dimensions thus leading to an extreme simplification of the Lorentz algebra.

The key point of our method for constructing the counter-term $d\sigma^A$ is the dipole formalism described in Sect. 3. Starting from the universal behaviour of the QCD matrix elements in the soft and collinear regions, we have introduced improved factorization formulae based on a dipole structure with respect to the momenta, colours and helicities of the QCD partons. Our dipole formulae correctly match the well-known singularities of the QCD scattering amplitudes in the soft and collinear limits. Moreover, these limits are approached smoothly, thus avoiding double counting of overlapping soft and collinear divergences. This smooth transition is possible because our dipole formalism is explicitly Lorentz covariant and the dipole formulae exactly fulfil momentum conservation.

In the present paper, these main features of the dipole formulae have been used in the context of NLO computations of jet cross sections with no initial-state hadron. For lepton-hadron and hadron-hadron collisions, perturbative QCD calculations face additional difficulties related to the factorization of initial-state collinear singularities. Likewise, to calculate cross sections for processes in which a final state hadron is identified, the equivalent final-state collinear singularities must be dealt with. In a companion paper [13], we show that the dipole formalism overcomes these difficulties in a simple and general manner. Thus we can explicitly provide a set of universal counter-terms or, equivalently, effective (non-singular) matrix elements that can be straightforwardly used for *any* NLO QCD calculation.

[‡‡] Actually, if the total number of QCD partons involved in the LO matrix element is less than or equal to three, one simply needs its incoherent sum over the colours (see Sect. 6).

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