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Advances in Large N Group Theory and the Solution of Two-Dimensional R^2 Gravity

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Abstract

We review the recent exact solution of a matrix model which interpolates between flat and random lattices. The importance of the results is twofold: Firstly, we have developed a new large N technique capable of treating a class of matrix models previously thought to be unsolvable. Secondly, we are able to make a first precise statement about two-dimensional R^2 gravity. These notes are based on a lecture given at the Cargese summer school 1995. They contain some previously unpublished results.

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The large N expansion of matrix-valued field theories was invented more than twenty years ago by 't Hooft as a way to treat four-dimensional QCD with gauge group $SU(N)$. At $N=\infty$ planar diagrams dominate and it was hoped that this fact would lead either to analytic results or to a reformulation of QCD as a string theory. While the original ideas remain attractive, the program has not yet been successful. It was slowly understood that $N=\infty$ field theories retain much of the complexity of the generic N case.

Some of this complexity remains even in zero and one-dimensional matrix "field theories". In a famous paper[1] by Brézin, Itzykson, Parisi and Zuber it was demonstrated that these so-called matrix models are non-trivial – but still solvable – systems. E.g. the zero dimensional model

$$Z = \int \mathcal{D}M \; e^{-N \; \mathrm{Tr} \; [\frac{1}{2} M^2 \; - \; \Sigma_{q=1}^{\infty} \; t_q M^q]}, \tag{1}$$

where M is a $N \times N$ Hermitian matrix and the t_q's are coupling constants parametrizing a general potential, is solved[1] by changing variables to the eigenvalues of the matrix M and thereby reducing the number of degrees of freedom from N^2 to N. The latter proves possible due to the invariance of the above action and measure under the group $U(N)$. The model's free energy describes at $N=\infty$ a sum over all planar diagrams G of spherical topology, where vertices v_q of order q (there are $\#v_q$ of them in G) are weighted with a factor t_q:

$$\log Z \;\sim\; \mathcal{Z} = \sum_G \prod_{v_q \in G} t_q^{\# v_q} \tag{2}$$

The rather non-trivial combinatorial sum (2) is elegantly calculated by solving (1). At the time[1] the model was considered as a kind of "toy", zero-dimensional QCD (retaining the diagrammatic structure but nothing else) to test a new technique. A few years later, however, it became clear that the results could be used to obtain the solution of a rather interesting physical problem: two-dimensional quantum gravity [2,3,4]. Indeed, it was argued that by tuning the couplings t_q in an appropriate way a continuum limit could be reached at which the planar graphs condense to give a continuum path integral over two-dimensional metrics g_{ab} of spherical topology:

$$\mathcal{Z}_{\mathrm{cont}} = \int \mathcal{D}g_{ab} \; e^{-\int d^2z \sqrt{\det g} \; (\mu + \frac{1}{\alpha} R_g)}. \tag{3}$$

The distance from the critical point in the space of the t_q's turns into a continuum cosmological constant μ controlling the area of the surfaces. We also wrote the Einstein term, which is however known to be a constant in two dimensions. This approach was subsequently worked out and justified by a large number of researchers.

Thus already the simplest matrix model (1) contains non-trivial physics. Various generalisations were solved and shown to give new physical information; e.g. certain simple multi matrix models describe the coupling of $c \leq 1$ conformal matter to 2D gravity (3). Such generalisations usually required the development of new techniques in order to succeed. Turning this around, each technical advance in large N theory usually allows to address a previously inaccessible physical problem. Keeping in mind the final goals – like large N QCD or string theories in physical dimensions – a valuable strategy is, then, to continue to enlarge our tools and methods.

An encouraging example for this strategy was given recently in a series of papers[5,6,7]. We studied a generalisation of (1) consisting in the inclusion of an external matrix field A in the potential:

$$Z = \int \mathcal{D}M \; e^{-N \ \mathrm{Tr}\ [\frac{1}{2}M^2 \ - \ \Sigma_{q=1}^{\infty} \ t_q (MA)^q]}. \tag{4}$$

On a technical level, this model seemed for a long time unsolvable: The external field destroys the $U(N)$ invariance of the action. Thus none of the usual methods, whose essence consists in reducing the number of degrees of freedom from N^2 to N, appears applicable to (4). We succeeded in nevertheless finding a reduction of the number of degrees of freedom by developing the new method of large N character expansions . This allows us to address new physical questions. Indeed, it is easy to prove that the perturbative expansion in planar graphs of (4) is given by

$$\log Z \ \sim \ \mathcal{Z} = \sum_G \prod_{v_q^*, v_q \in G} t_q^{*\#v_q^*} \ t_q{}^{\#v_q}, \tag{5}$$

where

$$t_q^* = \frac{1}{q} \ \frac{1}{N} \ \mathrm{Tr} \ A^q. \tag{6}$$

The sum is, as in eq.(2), taken over all planar graphs G of spherical topology. But now we have an extra set of coupling constants t_q^* at our disposal: They assign weights to the vertices v_q^* of the *dual* lattice (there are $\#v_q^*$ of them in G). Thus the t_q and t_q^* control the coordination numbers of the vertices and the faces of G, respectively. In particular, if we set $t_q = t_q^* = \delta_{q,4}$ the only surviving graphs in the ensemble $\{G\}$ are regular square lattices[a]. Thus, the model (4),(5) is capable of interpolating between fluctuating random lattices and flat, regular lattices! In the continuum formulation, this property would be achieved by adding higher curvature counterterms to the action of (3):

$$\mathcal{Z}_{\mathrm{cont}} = \int \mathcal{D}g_{ab} \; e^{-\int d^2z \sqrt{\det g} \ (\mu + \frac{1}{\alpha} R_g + \frac{1}{\beta_0} R_g^2 + \ldots)}. \tag{7}$$

Tuning the bare coupling β_0 to zero clearly suppresses any non-flat metric. Our model therefore furnishes a precise invariantly regularized definition of 2D higher curvature gravity. Before discussing the physics of the latter, let us first explain the steps that lead to a solution of the apparently intractable matrix model (4). We will only sketch

[a] Of course there is no square lattice of spherical topology; one needs to also add some positive curvature defects to be able to close the graph into a sphere; see below.

the derivation (and even the results!); a much more detailed discussion can be found in the original papers[5,6,7].

The idea is to expand the potential in (4) in terms of the characters of the product matrix MA. It is clear that this is possible since the potential is a class function on the group (i.e. it only depends on the eigenvalues of the matrix MA). What is less obvious (proven in our first work[5]) is that the expansion coefficients can themselves be written as the characters of an auxiliary external matrix B generating the couplings t_q, just as A generates the couplings t_q^* (see eq.(6)):

$$t_q = \frac{1}{q}\,\frac{1}{N}\ \mathrm{Tr}\ B^q. \tag{8}$$

One then has the expansion

$$e^{N\mathrm{Tr}\ \Sigma_{q=1}^{\infty} t_q (MA)^q} = c \sum_R \chi_R(B)\ \chi_R(MA), \tag{9}$$

with c a numerical constant. The characters are defined by the Weyl formula

$$\chi_{\{h\}}(B) = \frac{\det_{(k,l)}(b_k^{h_l})}{\Delta(b)}, \tag{10}$$

where the b_i are the eigenvalues of the matrix B, $\Delta(b)$ is the Vandermonde determinant of the eigenvalues, the set of $\{h\}$ are a set of ordered, increasing, non-negative integers, and the sum over R is the sum over all such sets. The R's label representations of the group $U(N)$ and the sets of integers $\{h\}$ are the usual Young tableau weights defined by $h_i = i - 1 + \#\text{boxes in row } i$ (the index i labels the rows in the Young tableau, $i = 1$ corresponding to the lowest row). Note that the restriction on the allowed Young tableaux that any row must have at least as many boxes as the row below implies that the $\{h_i\}$ are a set of increasing integers: $h_{i+1} > h_i$. Substituting equation (9) into the integral in equation (4), we can now do the angular integration using the key identity

$$\int (\mathcal{D}\Omega)_H\ \chi_R(\Omega M \Omega^\dagger A) = d_R^{-1}\ \chi_R(M)\ \chi_R(A), \tag{11}$$

where d_R is the dimension of the representation given, up to a constant, by $d_R \sim \Delta(h)$), and arrive, after performing a Gaussian integral over the eigenvalue degrees of freedom, at the expression

$$Z = c \sum_{\{h^e, h^o\}} \frac{\prod_i (h_i^e - 1)!! h_i^o!!}{\prod_{i,j}(h_i^e - h_j^o)}\ \chi_{\{h\}}(A)\ \chi_{\{h\}}(B), \tag{12}$$

where c is an immaterial constant. The sum is taken over a subclass of so-called even representations. These are defined as possessing an equal number of even weights h_i^e

and odd weights h_i^o (since the mentioned Gaussian integration vanishes if the latter condition is not satisfied). The formula (12) was originally discovered by Itzykson and Di Francesco[8] by summing up "fatgraphs", using purely combinatoric and group theoretic arguments. We observe that the matrix model (4) is thus reformulated as a sort of "statistical mechanics model" in Young weight space. The important fact is that there are only N weights h_i; therefore the reduction to N degrees of freedom is achieved.

The expansion (12) can be further generalised. Consider the matrix model

$$Z = \int \mathcal{D}\phi \; e^{-N \; \mathrm{Tr} \; [\frac{1}{2}\phi\phi^+ \; - \; \sum_{k=1}^{\infty} \; g_k (\phi A \phi^+ B)^k]}, \tag{13}$$

where ϕ is a *complex* $N \times N$ matrix. Introducing a third external matrix field C and defining, in analogy with (6),(8),

$$g_k = \frac{1}{k}\frac{1}{N} \; \mathrm{Tr} \; C^k, \tag{14}$$

one finds by the method outlined above the character expansion

$$Z = c \sum_{\{h\}} \frac{\prod_i h_i!}{\Delta(h)} \; \chi_{\{h\}}(A) \; \chi_{\{h\}}(B) \; \chi_{\{h\}}(C), \tag{15}$$

where this time the sum extends over *all* representations. In the special case $g_k = \delta_{k,2}$ the earlier expansion (12) is recovered[b].

The reduction of the number of degrees of freedom is only a *conditio sine qua non.* One next has to take the large N limit of the expansion. The basic idea is the same as for the original model (1) (see[1]), with Young weights replacing eigenvalues: The weights $\frac{1}{N}h_i$ are assumed to freeze into a smooth, stationary distribution $dh \; \rho(h)$, where $\rho(h)$ is a probability density normalized to one. The details, however, turn out to be much more involved. Some simple first examples were worked out in our first paper[5]. An unpleasant feature is that, while the saddlepoint always exists, the support of the density $\rho(h)$ does not necessarily remain on the real axis for completely arbitrary couplings t_q,t_q^*, complicating the general analysis in a significant way. However, if we restrict our attention to models in which the matrices A and B are such that traces of all odd powers of A and B are zero the problem does not arise: $t_{2q+1} = t_{2q+1}^* = 0$. This means that our random surfaces are made from vertices and faces with even coordination numbers. Thus it is easiest to consider surfaces made up from squares (as opposed to, say, triangles): $t_q^* = \delta_{q,4}$. A weight $t_{2q} = \frac{1}{2q}\frac{1}{N}\mathrm{Tr}B^{2q}$ is assigned whenever $2q$ squares meet at a vertex (see Fig. 1).

[b]For this special case the correspondence between the hermitian model (4) and the complex model (13) was already noted previously[8]. For a graphical explanation of this correspondence see our second work[6].

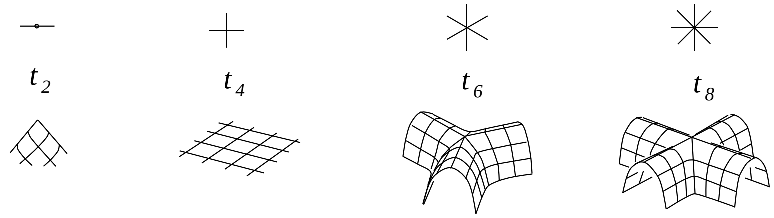

Fig. 1 Flat space and curvature defects.

This model is clearly capable of describing the transition from flat to random graphs. The sum (5) over spherical lattices G_4 built from square plaquettes becomes

$$\mathcal{Z} = \sum_{G_4} \prod_{v_{2q} \in G} {t_{2q}}^{\# v_{2q}}, \tag{16}$$

where v_{2q} are the vertices where $2q$ plaquettes meet and $\# v_{2q}$ are the numbers of such vertices in the given graph G_4. In order to investigate (12) in the large N limit, one attempts to locate the stationary point. This leads to the saddlepoint equation

$$2F(h) + \fint_0^a dh' \; \frac{\rho(h')}{h - h'} = -\ln h. \tag{17}$$

Here $F(h)$ denotes the large N limit $F(h_k) \rightarrow F(h)$ of the variation of the character $\chi_{\{h\}}(B)$ in eq.(12):

$$F(h_k) = 2 \frac{\partial}{\partial h_k^e} \; \ln \; \frac{\chi_{\{\frac{h^e}{2}\}}(\bar{b})}{\Delta(h^e)}. \tag{18}$$

where we have also used that the matrix B will satisfy $\text{Tr} B^{2q+1} = 0$ if we introduce a $\frac{N}{2} \times \frac{N}{2}$ matrix $\bar{b}^{\frac{1}{2}}$ in terms of which B and the character $\chi_{\{h\}}(B)$ are given by

$$B = \begin{bmatrix} \bar{b}^{\frac{1}{2}} & 0 \\ 0 & -\bar{b}^{\frac{1}{2}} \end{bmatrix} \quad \text{and} \quad \chi_{\{h\}}(B) = \chi_{\{\frac{h^e}{2}\}}(\bar{b}) \chi_{\{\frac{h^o - 1}{2}\}}(\bar{b}) \; \text{sgn}[\prod_{i,j} (h_i^e - h_j^o)]. \tag{19}$$

The variation of the character of the matrix A is easily computed directly because of the simple choice $t_q^* = \delta_{q,4}$. As has been discussed in detail before[5], the saddlepoint equation (17) actually does not hold on the entire interval $[0, a]$, but only on an interval $[b, a]$ with $0 \leq b \leq 1 \leq a$: Assuming the equation to hold on $[0, a]$ would violate the implicit constraint $\rho(h) \leq 1$ following from the restriction $h_{i+1} > h_i$. The density is in fact exactly saturated at its maximum value $\rho(h) = 1$ on the interval $[0, a]$. It is useful to introduce in addition the weight resolvent $H(h)$ as follows:

$$H(h) = \int_0^a dh' \; \frac{\rho(h')}{h - h'}. \tag{20}$$

We found the weight resolvent $H(h)$ to be very closely related to the standard matrix model *eigenvalue* resolvent (see[1]). It provides a direct link between the statistical distribution of Young weights and the correlators of the model:

$$\frac{1}{N}\ \mathrm{Tr} M^{2q} = \frac{1}{q} \oint \frac{dh}{2\pi i}\ h^q\ e^{qH(h)}. \tag{21}$$

Here the contour encircles the cut of $e^{H(h)}$.

The solution of (17) evidently requires knowing the function $F(h)$. A rather general method for its determination has been one of the main technical achievements of our work. The method of functionally determining an $N = \infty$ character might prove very useful for other applications. We found the following simple result: Further introduce the function $G(h)$ as

$$G(h) = e^{H(h)+F(h)}, \tag{22}$$

where, again, the contour encircles the cut of $e^{H(h)}$. Its importance stems from the fact that it relates[5] in a simple way the introduced functions $F(h)$,$H(h)$ and the coupling constants t_{2q} :

$$t_{2q} = \frac{1}{q} \oint \frac{dh}{2\pi i}\ G(h)^q. \tag{23}$$

It is then easy to deduce, by changing variables from h to G, the expansion

$$h - 1 = \sum_{q=1}^{Q} \frac{t_{2q}}{G^q} + \sum_{q=1}^{\infty} a_q\ G^q. \tag{24}$$

By considering[6] the alternative representation of the Weyl character as a determinant of Schur polynomials one derives[c] that the coefficients a_q of the positive powers of G in (24) are directly related to correlators of the matrix model dual to (4), i.e. the model with $t_q \leftrightarrow t_q^*$:

$$a_q = \langle \frac{1}{N} \mathrm{Tr}\ (\tilde{M B})^{2q} \rangle. \tag{25}$$

We have also assumed for the moment that only a finite number Q of couplings are non-zero (i.e. $t_{2q} = 0$ for $q > Q$). Furthermore, we were able to show in our second work[6] that (24) implies the functional equation

$$e^{H(h)} = \frac{(-1)^{(Q-1)} h}{t_Q} \prod_{q=1}^{Q} G_q(h), \tag{26}$$

where the $G_q(h)$ are the first Q branches of the multivalued function $G(h)$ defined through (24) which map the point $h = \infty$ to $G = 0$. The resulting picture of the

[c]One also finds the relation $a_q = \frac{2q}{N} \frac{\partial}{\partial t_{2q}} \ln\Big(\chi_{\{\frac{h^e}{2}\}}(\bar{b})\Big)$. As mentioned before[7], rewriting eq.(24) with this expression for a_q suggests a possible relationship to integrable hierarchies of differential equations. This conjecture was developed in several conversations with I. Kostov.

analytic structure of $G(h)$ is as follows: the couplings t_{2q} determine the number of sheets attached to the physical sheet by the cut of $e^{F(h)}$; this number is Q for a finite number of non-zero couplings. In turn, the parameters a_q determine the sheets attached to the physical sheet by the cut of $e^{H(h)}$. This picture is easily verified (and was in fact discovered in this way) for the rather trivial cases where the potential in eq.(4) is at most quadratic. The case $Q = 2$ is shown in Fig. 2.

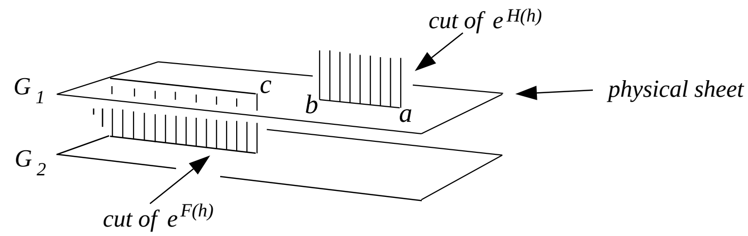

Fig. 2 Sheet and cut structure of $G(h)$

The saddlepoint equation (17), together with (26), defines a well-posed Riemann-Hilbert problem. It was solved exactly and in explicit detail[6] for the case $Q = 2$, where the Riemann-Hilbert problem is succinctly written in the form

$$\begin{aligned} 2F(h) + \not{H}(h) &= \quad -\ln h \\ 2\not{F}(h) + H(h) &= \ln(-\tfrac{t_4}{h}), \end{aligned} \tag{27}$$

The first equation is the saddlepoint equation (17) with $\not{H}(h)$ denoting the real part of $H(h)$ on the cut $[b, a]$ (the righthand cut in Fig. 2). The second equation is (26) with $\not{F}(h)$ denoting the real part of $F(h)$ on a cut $[-\infty, c]$ with $c < b$ (the lefthand cut in Fig. 2). This case corresponds to an ensemble of squares being able to meet in groups of four (i.e. flat points with weight t_4) or two (i.e. positive curvature points with weight t_2) (see Fig. 1). We termed the resulting surfaces “almost flat”. It turned out that all the introduced functions could be found explicitly in terms of elliptic functions. E.g. the density satisfying (17) is given by

$$\rho(h) = \frac{1}{K}\mathrm{sn}^{-1}(\sqrt{\frac{a-h}{a-b}}, k) \quad \text{with} \quad k = \sqrt{\frac{a-b}{a-c}}, \tag{28}$$

where sn^{-1} is an inverse Jacobi elliptic function and K is the complete elliptic integral of the first kind (depending on the modulus k). The expressions for the cutpoints a, b, c as well as the expressions for $H(h), F(h), G(h)$ and the physical correlators a_q (see (25)) were found as well.]

The resulting surfaces are very different from pure gravity. The spherical partition function (16) consists of flat cylinders pinched shut at the two ends, resulting in precisely four defects. Very short, highly twisted cylinders dominate the continuum limit. By exactly calculating the correlators a_q (see eq.(25)) we also analytically solved the combinatorial problem of surfaces with a single negative curvature insertion of arbitrary degree balanced by a gas of positive defects.

To analyse the problem of the transition from flat to random lattices, we merely need to perturb our almost flat lattices by *any* operator containing negative curvature. This physical observation allows us to extend the $Q = 2$ solution in a simple way. Choosing the weights to be

$$t_2 = \sqrt{\lambda}\ t,\ t_4 = \lambda,\ t_6 = \lambda^{\frac{3}{2}}\ \frac{\beta^2}{t},\ ...\ t_{2q} = \lambda^{\frac{q}{2}}\ (\frac{\beta^2}{t})^{(q-2)}, \tag{29}$$

the expansion (24) becomes

$$h - 1 = \frac{t_2}{G} + \frac{t_4}{G\ (G-\epsilon)} + \text{ positive powers of } G, \tag{30}$$

where $\epsilon = \frac{t_4}{t_2}\beta^2$. The derivation of the functional equation (26) is easily modified (note that we now have an infinite number $Q = \infty$ of weights) to give

$$e^{H(h)} = \frac{h}{\epsilon t_2 - t_4}\ G_1(h)\ G_2(h). \tag{31}$$

The essential point is that one keeps the property that only one other sheet $G_2(h)$ is attached to the physical sheet $G_1(h)$ by the cut of $e^{F(h)}$ (see again Fig. 2). The only difference is that the semi-infinite cut $[-\infty, c]$ becomes a finite cut $[d, c]$. This results in modifying the Riemann-Hilbert problem (27) to

$$\begin{aligned} 2F(h) + \not{H}(h) &= \quad -\ln h \\ 2\not{F}(h) + H(h) &= \ln(\tfrac{\epsilon t_2 - t_4}{h}), \end{aligned} \tag{32}$$

It may still be explicitly solved in terms of elliptic functions; e.g. the density (28) is generalised to

$$\rho(h) = \frac{u}{K} - \frac{i}{\pi}\ \ln\left[\frac{\theta_4(\frac{\pi}{2K}(u-iv), q)}{\theta_4(\frac{\pi}{2K}(u+iv), q)}\right] \tag{33}$$

with

$$q = e^{-\pi\frac{K'}{K}}, \quad \text{and} \quad k = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}, \tag{34}$$

where K and K' are the complete elliptic integrals of the first kind with respective moduli k and $k' = \sqrt{1-k^2}$ and u and v are given by

$$u = \text{sn}^{-1}(\sqrt{\frac{(a-h)(b-d)}{(a-b)(h-d)}}, k) \quad \text{and} \quad v = \text{sn}^{-1}(\sqrt{\frac{a-c}{a-d}}, k'). \tag{35}$$

Again, the cutpoints a, b, c, d can be explicitly obtained as functions of the couplings λ, β, t.

Let us also mention that the weights (29) can be further generalised while keeping the quadratic (two-sheeted) structure of the function $G(h)$. The idea is to shift away the simple pole at $G = 0$ in the expansion (30), i.e.

$$h - 1 = \frac{c_1}{G - \epsilon_1} + \frac{c_2}{G - \epsilon_2} + \text{ positive powers of } G. \tag{36}$$

Here c_1 and c_2 are two constants given by $c_1 = \frac{t_2 \epsilon_2 - t_4}{\epsilon_2 - \epsilon_1}$ and $c_2 = \frac{t_4 - t_2 \epsilon_1}{\epsilon_2 - \epsilon_1}$. It leads to the Riemann-Hilbert problem

$$\begin{aligned} 2F(h) + \not{H}(h) &= -\ln h \\ 2\not{F}(h) + H(h) &= \ln\left(\tfrac{(\epsilon_1+\epsilon_2)t_2 - t_4}{h} + \epsilon_1 \epsilon_2 \tfrac{h-1}{h}\right), \end{aligned} \tag{37}$$

The weights generalizing (29) by one extra parameter can be found explicitly by expanding (36) in inverse powers of G. We have not worked out this further explicitly solvable case in detail. Analysing it would furnish an interesting universality check of our result.

Having at hand the explicit solution of the model for the weights (29), we are in a position to analyse the problem of discrete 2D R^2 gravity. With these weights it is easy to prove, using Euler's theorem, that the partition sum (16) becomes

$$\mathcal{Z}(t, \lambda, \beta) = t^4 \sum_{G_4} \lambda^A \; \beta^{2(\#v_2 - 4)}, \tag{38}$$

where A is the number of plaquettes of the graph G and $\#v_2$ the number of positive curvature defects. Note that the latter are balanced by a gas of negative curvature defects, whose individual probabilities are given in (29).

We expect this model to describe pure gravity in a sufficiently large interval of β, after tuning the bare cosmological constant λ (controlling the number of plaquettes) to some critical value $\lambda_c(\beta)$. On the other hand, for λ fixed and $\beta = 0$ we entirely suppress curvature defects except for the four positive defects needed to close the regular lattice into a sphere. It is thus clear that β is the precise lattice analog of the bare curvature coupling β_0 in the continuum path integral (7). The phase $\beta = 0$ of "almost flat" lattices – very different from pure gravity – studied in detail in the second paper[6] was discussed above.

Let us now summarize the main physical results following from the exact solution (for general λ and β) of this model:

1. A long debated question was whether models of the present type undergo a "flattening" phase transition at a finite, non-zero critical value of $\beta = \beta_c$. The weak coupling region $\beta > \beta_c$ would then correspond to the standard phase of pure gravity while a putative novel "smooth" phase of gravity might exist either at $\beta = \beta_c$ or in the entire interval $0 \le \beta \le \beta_c$. This would constitute an existence proof of *continuum*

2D R^2 gravity. We found analysing the exact solution, to the contrary, that *there is no "flattening" phase transition at non-zero* β. For any given β we find the powerlike scaling of standard pure gravity on large scales. This means that no matter how flat the system is on small scales (of the order of $\beta^{-\frac{1}{2}}$), it destabilizes in the infrared into the familiar ensemble of highly fractal "baby-universes".

2. The dependence of the partition sum (38) on β and the lattice cosmological constant λ in the vicinity of the flat phase $\beta \sim 0$ and close to $\lambda \sim \lambda_c$ is given by a simple, (presumably) universal scaling function $f(x)$ (defined through $\mathcal{Z}(t, \lambda, \beta) = \frac{4t^4}{15\beta^2} f(x)$) reflecting the transition from flat space to pure gravity:

$$f(x) = x^6 - \frac{5}{2} x^4 + \frac{15}{8} x^2 - \frac{5}{16} - x (x^2 - 1)^{\frac{5}{2}}, \tag{39}$$

where the scaling variable x is given, to leading order, by

$$x = \frac{\sqrt{2}}{\pi} \frac{1 - \lambda}{\beta}. \tag{40}$$

We can distinguish the following features:

(a) There is a degree $\frac{5}{2}$ singularity at $x = 1$, correctly reproducing the universal string susceptibility exponent $\gamma_s = -\frac{1}{2}$ of pure gravity[2,3]. In view of eq.(40), the critical value of the lattice cosmological constant λ is therefore given to leading order by $\lambda_c = 1 - \frac{\pi}{\sqrt{2}}\beta + \mathcal{O}(\beta^2)$. Therefore (see (38)), the characteristic growth of the random surfaces as a function of the lattice area A (= number of plaquettes) is given by

$$\mathcal{Z}(t, A, \beta) \sim \frac{t^4}{\beta^{\frac{9}{2}}} e^{\frac{\pi}{\sqrt{2}} \beta A} A^{-\frac{7}{2}}. \tag{41}$$

For any non-zero β we do have exponential growth of the number of surfaces, but one has to go to larger and larger scales (i.e. use more and more plaquettes) to be able to take the continuum limit. If β is exactly zero there is no longer any exponential growth and no pure gravity continuum limit is possible. The prefactor $\beta^{-\frac{9}{2}}$ is found in the exact calculation in section 5; we are not sure whether it is universal.

(b) We further see that taking $\beta \rightarrow 0$ *before* the limit $\lambda \rightarrow \lambda_c$ corresponds to the limit $x \rightarrow \infty$. In this limit one finds $f(x) \sim \frac{5}{128} \frac{1}{x^2} + \mathcal{O}(\frac{1}{x^3})$, that is, the characteristic critical behavior of 2D gravity "silently" disappears and we recover a power series in $\frac{1}{x}$ corresponding, in view of (40), to a perturbative expansion in lattice defects β. In this limit the characteristic growth of surfaces as a function of area A is

$$\mathcal{Z}(t, A, \beta) \sim t^4 (A + \mathcal{O}(\beta^2 A^3)). \tag{42}$$

The leading order corresponds precisely to the almost flat lattices (with exactly four positive defects) studied in our second work[6]. The corrections are interpreted as insertions of negative defects, balanced by further positive defects. The typical shape of the surfaces in this limit is a generalisation of the one we found for "almost flat" graphs: Long, filamentary cylinders growing out from every negative curvature defect.

(c) It is easy to prove that the scaling function (39) is the simplest possible function with the limiting properties discussed in (a) and (b).

The above results might be interpreted in terms of a continuum model of quantized curvature defects, in which the localised defects move around like particles in a gas on a flat background space. The deficit angle, $\Delta\theta$, of a defect can take on the values $\Delta\theta = \pi, 0$ and $-\pi$. A positive curvature defect is surrounded by a conical geometry, whereas a negative curvature defect corresponds to a saddle-type insertion (see Fig. 1). The higher order negative curvature defects (-2π and higher) would not be expected to play a role in this limit (the entropy from moving two low order defects around would completely dominate that from a single higher order defect). The coupling β can be interpreted as a fugacity controlling the number of defects. The flat space limit $\beta \to 0$ consists of four defects of degree π moving around with respect to one another. Varying the fugacity, β, allows one to smoothly interpolate between flat space, (42) (with four defects), and pure gravity (41) (with an infinite number of defects).

One might also attempt to develop this picture directly in the continuum. One could start with the conformal metric of a flat surface with localised curvature defects. It can be represented locally as $g_{ab} = \delta_{ab}\; e^{\varphi(z)}$ with

$$\varphi(z) = \sum_{j=1}^{M} R_j \;\ln(z - z_j)^2, \tag{43}$$

where $R_j = -1, 1$. Symbolically, the partition function might be written as

$$Z(\mu, \underline{\;}) = \sum_{M} \beta^M \int d[z_1, ..., z_M]\; e^{-\mu \int d^2z\; \sqrt{\det g(z)}}. \tag{44}$$

Here we introduced the fugacity of curvature defects β instead of the explicit R^2-term in the action. It serves the same purpose: for $\beta \to 0$ we arrive at the completely flat metric, whereas for $\beta \sim 1$ the system should show the behaviour of pure quantum gravity, at least in the infrared domain. We retained the notation β to denote the parameter playing a role similar to the R^2 coupling in the above discrete model.

This formulation resembles a little bit the two-dimensional Coulomb gas problem. However, the measure of integration $d[z_1, ..., z_M]$ of the positions of the curvature defects is a complicated object: it should take into account the topology of the surface and the existence of zero modes (the action does not depend on some directions in the space of the z_i). It would be very interesting to make this direct continuum formulation more precise.

Another interesting issue is the role of exponential corrections appearing due to the structure of elliptic functions. In fact all physical quantities, such as $\mathcal{Z}(t, \lambda, \beta)$ (see eq.(38)), contain exponentially small terms in the limit $\lambda \to \lambda_c$ and $\beta \to 0$, thus leading to an essential singularity at $\beta = 0$, $\lambda = 1$. One can obtain the first correction of this type in e.g. the free energy $\bar{f}(\beta)$ per unit area in the thermodynamical limit $\lambda = \lambda_c$:

$$\mathcal{Z}(t, A, \beta) \;\sim\; \frac{t^4}{\beta^{\frac{9}{2}}}\; e^{\bar{f}(\beta)A}\; A^{-\frac{7}{2}} \quad \text{with} \quad \bar{f}(\beta) = \frac{\pi}{\sqrt{2}}\beta\Big[(1+\ldots) + e^{-\frac{\pi\sqrt{2}}{\beta}}(4+\ldots)\Big] \tag{45}$$

where $\bar{f}(\beta) = \lim_{A\to\infty}\frac{1}{A}\ln\mathcal{Z}(t,A,\beta) = \lambda_c$ and the dots denote terms of order β^3 and higher.

These exponential terms are likely to be lattice artifacts. They emerge even in the simplest calculation for the flat closed quadrangulation with four positive curvature defects, where they appear as discrete corrections to the approximation of elliptic sums by integrals close to the continuum limit.

On the other hand, formula (45) corresponds to the critical free energy as a function of the curvature fugacity β (i.e. we have already taken the continuum limit). It is possible that the exponential terms might be corrections relevant for the statistical mechanics of random lattices at long distances (of order $\frac{1}{\beta}$) rather than for continuous 2D-gravity.

In conclusion, we have tried to emphasize two points in this presentation:

1. *Technical advances in large N group theory are possible.* Apparently hopelessly difficult problems like the model (4) become tractable when one changes the technique.

Our approach raises a number of interesting questions. First of all, the method we have employed seems rather indirect and is definitely very involved. Is there a hidden simplicity we have missed so far? Are there alternative ways to obtain our results? Secondly, one should ask whether the method could be adapted to other theories describing new interesting physics; in particular gauge theories and strings in physical dimensions.

2. *Each such advance leads to the capability to address new physics questions. The present advance enables us to treat the problem of two-dimensional higher curvature gravity.*

We have presented for the first time an exactly solvable model interpolating between flat space and 2D quantum gravity. This addresses a long-standing open problem. We would like to stress, however, that we have by no means demonstrated yet the *universality* of our result. Could it be that finetuning the weights t_q leads to new phases of gravity? Here we certainly do not mean to rederive the usual multicritical phases of the one-matrix model (1). We rather envision tuning the weights in a subtle way so as to reach a phase of *smooth* 2D gravity. Certainly the analytic complexity of the result could be a hint that much more waits to be discovered. (Compare e.g. the densities (28),(33) of the model (4) under study with the rather trivial algebraic densities obtained in the absence of the external field.) Before addressing this issue, we repeat, one first has to simplify the method enough to allow for a deeper insight into the analytic structure of the solution for general couplings t_q.

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