

# ASYMPTOTIC PROPERTIES OF THE SOLUTIONS OF A DIFFERENTIAL EQUATION APPEARING IN QCD

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## ABSTRACT

We establish the asymptotic behaviour of the ratio  $h'(0)/h(0)$  for  $\lambda \rightarrow \infty$ , where  $h(r)$  is a solution, vanishing at infinity, of the differential equation  $h''(r) = i\lambda\omega(r)h(r)$  on the domain  $0 \leq r < \infty$  and  $\omega(r) = (1 - \sqrt{r}K_1(\sqrt{r}))/r$ . Some results are valid for more general  $\omega$ 's.

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# 1 Introduction

Recently [1], [2] the problem of induced gluon (photon) radiation in a QCD (QED) medium has been studied. The authors have been led to study the solution of a second order differential equation, which, with some change of notations, reads

$$h''(r) = i\lambda\omega(r) h(r) , \quad (1)$$

on the domain  $0 \leq r \leq \infty$ , with  $h(\infty) = 0$ , for large  $\lambda$ , for a particular form of  $\omega$

$$\omega(r) = \frac{1 - \sqrt{r}K_1(\sqrt{r})}{r} , \quad (2)$$

where  $K_1$  is the modified Bessel function of order 1.

Specifically they want to know the asymptotic behaviour of  $h'(0)/h(0)$  for large  $\lambda$ ,  $h(r)$  being a solution of (1) vanishing at infinity. What we want to do here is to obtain this limiting behaviour for the specific  $\omega$  given by Eq. (2) and, at the same time, study a broader class of equations of type (1).

## 2 General Considerations

$\omega(r)$ , as given by Eq. (2) has the integral representation [1]:

$$\omega(r) = \frac{1 - \sqrt{r}K_1(\sqrt{r})}{r} = \int_0^\infty \frac{2 \sin^2\left(\frac{u}{2}\right) du}{(r + u^2)^{3/2}} \quad (3)$$

From (3), one can derive a certain number of properties of  $\omega$ :

$$\omega > 0 \quad \text{and} \quad \lim_{r \rightarrow \infty} \omega = 0 , \quad (4)$$

$$\frac{d\omega}{dr} < 0 , \quad (5)$$

$$\frac{d}{dr} \left| \frac{d\omega}{dr} / \omega \right| < 0 . \quad (6)$$

Equations (4) and (5) are obvious. Inequality (6) can be obtained by writing  $\omega \frac{d^2\omega}{dr^2} - \left(\frac{d\omega}{dr}\right)^2$  as a symmetrized double integral following from (3).

In this section, we shall forget the origin of the problem and assume only properties (4) and (5), and eventually (6), but do not refer to the specific expression (3). We shall write  $\lambda\omega$  as  $W$  and study the general properties of the equation

$$h'' = i Wh . \quad (7)$$

We shall sometimes write  $h$  as

$$h = R \exp i S , \quad (8)$$

$R$  and  $S$  real.

Multiplying (7) by  $h^*$  and taking the real part of both sides we get

$$h^* h'' + 2h^* h' + h^{*''} h = 2h^* h' \quad (9)$$

i.e.,

$$RR'' = R^2 (S')^2 \quad (10)$$

which implies that  $R$  is convex, and since we are interested in the solution vanishing at infinity,  $R$  is decreasing.

There, only the reality of  $W$  has been used. If we take now the imaginary part, we get

$$h^* h'' - h h^{*''} = 2iW|h|^2 ,$$

and integrating from  $r$  to  $\infty$ :

$$R^2 S' = - \int_r^\infty W(r') R^2(r') dr' \quad (11)$$

Since  $W$  is assumed to be positive according to (4), this means that the representation of the solution in the Argand diagram is a spiral turning clockwise and shrinking on the origin as  $r$  goes from 0 to  $\infty$ .

Now we shall try to obtain inequalities relating  $h$  and  $h'$ . First we multiply (7) by  $2h'$  and integrate from  $r$  to  $\infty$ . We get

$$-h'^2(r) = -i W h^2(r) - i \int_r^\infty \frac{dW}{dr'} h^2(r') dr' \quad (12)$$

Since, From (5)  $dW/dr$  is negative, we can apply Weierstrass's mean value theorem [3] in the complex domain:

$$\int_r^\infty \frac{dW}{dr'} h^2(r') dr' = \bar{h}^2 \int_r^\infty \frac{dW}{dr'} dr' = -\bar{h}^2 W(r) , \quad (13)$$

where  $\bar{h}^2$  is contained in the convex hull of  $h^2(r')$ , and  $r'$  runs from  $r$  to  $\infty$ . Since  $|h|$  is decreasing, this leads to the following inequality:

$$|h'(r)|^2 < 2W(r)|h(r)|^2 . \quad (14)$$

Except for a factor 2, this is what one would expect in a semi-classical treatment.

A more subtle bound can be obtained by multiplying (7) by  $h^*$  and integrating from  $r$  to  $\infty$ :

$$-h^* h' = \int_r^\infty |h'(r)|^2 dr' + i \int_r^\infty W(r') |h(r')|^2 dr' . \quad (15)$$

Hence,

$$\left| \int_r^\infty \frac{dW}{dr}(r') h^2(r') dr' \right| < \sup_{r < r' < \infty} \left| \frac{\frac{dW}{dr'}(r')}{W(r')} \right| \times |h| |h'| ,$$

and if property (6) holds:

$$\left| \int_r^\infty \frac{dW}{dr}(r') h^2(r') dr' \right| < \left| \frac{dW}{dr} \right| \times \frac{|h'|^2 + W|h|^2}{2\sqrt{W}} . \quad (16)$$

Inserting in Eq. (12), we get

$$|\rho^2 - i| < \frac{\left| \frac{dW}{dr} \right|}{2|W|^{3/2}} [1 + \rho^2] , \quad (17)$$

with

$$\rho = \frac{h'(r)}{\sqrt{W(r)h(r)}} \quad (18)$$

From (17), we get

$$\left( \rho - \frac{1+i}{\sqrt{2}} \right) \left( \rho + \frac{1+i}{\sqrt{2}} \right) \left| \right| < \frac{\left| \frac{dW}{dr} \right| / |W|^{3/2}}{1 - \frac{1}{2} \left| \frac{dW}{dr} \right| / |W|^{3/2}}$$

provided  $\frac{1}{2} \frac{dW}{dr} / |W|^{3/2} < 1$ , and, noticing that  $\text{Re} \frac{h'}{h} < 0$  since  $|h|$  is decreasing,

$$\left| \rho + \frac{1+i}{\sqrt{2}} \right| < \frac{\sqrt{2} \left| \frac{dW}{dr} \right| / |W|^{3/2}}{1 - \frac{1}{2} \left| \frac{dW}{dr} \right| / |W|^{3/2}} \quad (19)$$

### 3 Asymptotic behaviour of $h'(0)/h(0)$

We return now to our original problem i.e., we take  $W = \lambda\omega$  with  $\omega$  given by Eqs. (2) and (3). What matters is the small  $r$  behaviour of  $\omega$ . For  $r \rightarrow 0$ , we have

$$\omega \simeq \frac{1}{4} \ln \frac{1}{r} . \quad (20)$$

In fact, over the whole range of  $r$ , we have found, numerically, that

$$\omega(r) \simeq \frac{1}{4} \ln \frac{4+r}{r} \quad (21)$$

with an error of at most 10%.

The strategy is to estimate  $h'/h$  for some value of  $r$  sufficiently close to zero, but not zero because of the singularity of  $\omega$ , and to estimate the error made. Without specifying  $r$  yet, and only assuming it is small we have

$$\left| \frac{h'(0)}{\sqrt{W(r)}} + \frac{1+i}{\sqrt{2}} h(0) \right| < \left| \frac{h'(r)}{\sqrt{W(r)}} + \frac{(1+i)}{\sqrt{2}} h(r) \right| + \frac{1}{\sqrt{W(r)}} |h'(r) - h'(0)| + |h(r) - h(0)| . \quad (22)$$

From Eq. (7), we obtain:

$$|h'(r) - h'(0)| = \left| \int_0^r W h dr' \right| < |h(0)| \int_0^r W(r') dr' . \quad (23)$$

Integrating (14) from 0 to  $r$  gives

$$\left| \ln \frac{h(r)}{h(0)} \right| < \int_0^r \sqrt{W(r')} dr' ,$$

and, since,  $|h(r)|$  decreases with  $r$ :

$$|h(r) - h(0)| < |h(0)| \int_0^r \sqrt{W(r')} dr' , \quad (24)$$

and using again the fact that  $W$  decreases, we get, combining (18), (21), (22) and (23)

$$\left| \frac{h'(0)}{\sqrt{W(r)}} + \frac{1+i}{\sqrt{2}} h(0) \right| < |h(0)| \left[ \frac{\sqrt{2} \left| \frac{dW}{dr} \right| / W^{3/2}}{1 - \frac{1}{2} \left| \frac{dW}{dr} \right| / W} + \frac{2}{\sqrt{W}} \int_0^r W(r') dr' \right] , \quad (25)$$

with the restriction  $\left| \frac{dW}{dr} \right| / W < 1$ . The problem is now to optimise  $r$  knowing that  $W \simeq \frac{\lambda}{4} \ln \left( \frac{1}{r} \right)$  for  $r \rightarrow 0$  (notice that the large  $r$  behaviour of  $W$  is completely irrelevant for this problem, as long as  $W \rightarrow 0$ ).

It is not very difficult to see that the qualitative optimum is reached for

$$r = \lambda^{-1/2} (\log \lambda)^{-1} \quad (26)$$

and gives

$$\left| \frac{4}{\sqrt{\lambda \log \lambda}} \frac{h'(0)}{h(0)} + 1 + i \right| < C (\log \lambda)^{-1/2} , \quad (27)$$

which is the final result of the article. This agrees with the result of Ref. [2] (Ref. [1] contained an error).

Whether the right-hand side of (27) gives the order of magnitude of the next term in the asymptotic expansion of  $h'(0)/h(0)$  or not is unclear at the present time. Inequality (16) is somewhat crude, because it disregards the phase changes in the integral in the left-and side. A much more careful analysis is needed. It is difficult but not impossible and is postponed to a future publication.

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# References

- [1] R. Baier, Yu.L. Dokshitzer, S. Peigné and D. Schiff, *Phys.Lett.* **B345** (1995) 277.
- [2] R. Baier, Yu.L. Dokshitzer, A.H. Mueller, S. Peigné and D. Schiff, Preprint in preparation, to be published.
- [3] See, for instance: G. Valiron, *Théorie des Fonctions*, Masson 1948, Paris, page 362.