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Semi-Classical Blocks and Correlators in Rational and Irrational Conformal Field Theory

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ABSTRACT

The generalized Knizhnik-Zamolodchikov equations of irrational conformal field theory provide a uniform description of rational and irrational conformal field theory. Starting from the known high-level solution of these equations, we first construct the high-level conformal blocks and correlators of all the affine-Sugawara and coset constructions on simple g. Using intuition gained from these cases, we then identify a simple class of irrational processes whose high-level blocks and correlators we are also able to construct.

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1 Introduction

In recent years we have learned that the generic conformal field theory has irrational central charge, even when the theory is unitary. The study of this subject is called irrational conformal field theory (ICFT), which properly includes rational conformal field theory (RCFT) as a small subspace,

$$\text{ICFT} \supset\!\supset \text{RCFT} \tag{1.1}$$

where RCFT is understood here as the affine-Sugawara [1-6] and coset constructions [1,2,7]. A comprehensive review of ICFT is found in Ref.[8].

The foundation of ICFT is affine Lie algebra [9,1] and the general affine-Virasoro construction [10,11],

$$T = L^{ab} \, {}^*_* J_a J_b {}^*_* \tag{1.2}$$

on the currents J_a, $a = 1 \ldots \dim g$ of the general affine algebra. The construction (1.2) is summarized by the Virasoro master equation [10,11] for the inverse inertia tensor L^{ab}, and the system is understood as a conformal spinning top.

The solutions of the master equation show a symmetry hierarchy [12] in ICFT,

$$\text{ICFT} \supset\!\supset H\text{-invariant CFTs} \supset\!\supset \text{Lie } h\text{-invariant CFTs} \supset\!\supset \text{RCFT} \tag{1.3}$$

where the H-invariant CFTs, which are also generically irrational, include all theories with a symmetry H, where H may be a finite group or a Lie group. In this hierarchy, the RCFTs are understood as special cases of exceptionally high symmetry, with ever-increasing symmetry breakdown to the left. The generic ICFT is completely asymmetric.

The central computational tools of the subject are the generalized Knizhnik-Zamolodchikov (KZ) equations of ICFT [13], which provide a unified description of rational and irrational conformal field theory, including powerful new tools for RCFT. In particular, the recent solution of these equations for the general coset correlators [14,15,13] appears to be inaccessible by other methods.

Moreover, the semi-classical or high-level solution of the generalized KZ equations has been known for some time, providing a uniform and apparently simple description of all ICFT $\supset\!\supset$ RCFT on simple g. The high-level solution is deceptively simple, however, because it is expressed in a Lie algebra basis, which is not the block basis in which conformal blocks are conventionally expressed, and it is only in solving the general problem,

- Lie algebra basis $\quad\rightarrow\quad$ block basis

that one confronts the full complexity of the ever-increasing symmetry breakdown of ICFT.

In this paper we begin the study of the known high-level solutions, obtaining the high-level conformal blocks and non-chiral correlators of the simplest and most symmetric cases.

In particular, we will first find closed-form expressions for the high-level conformal blocks and correlators of all the affine-Sugawara and coset constructions. Both results are new.

Using intuition gained in this analysis, we then identify what we believe to be the simplest and most symmetric class of correlators in ICFT, which we call

- the $L(g;H)$-degenerate processes in the H-invariant CFTs.

This is the set of correlators all of whose external states have completely degenerate conformal weights. The set includes all the affine-Sugawara correlators, a highly-symmetric set of coset correlators and a presumably large set of irrational correlators, examples of which are known. For this class of processes, we are also able to find general expressions for the high-level conformal blocks and non-chiral correlators, and we discuss an irrational example with octohedral symmetry in some detail.

2 The High-Level Chiral Correlators of ICFT

Our starting point is the set of high-level four-point chiral correlators of ICFT,

$$Y_L^\alpha(y) = \bar{v}_g^\beta[\mathbb{1} + 2L_\infty^{ab}(\mathcal{T}_a^1\mathcal{T}_b^2 \ln y + \mathcal{T}_a^1\mathcal{T}_b^3 \ln(1-y))]_\beta{}^\alpha + \mathcal{O}(k^{-2}) \tag{2.1a}$$

$$L_\infty^{ab} = \frac{P^{ab}}{2k} \quad , \quad a,b = 1\dots \dim g \tag{2.1b}$$

on simple compact G, where G is the Lie group whose algebra is g, and k is the level of affine g. These correlators were conjectured in Ref.[14], derived in Ref.[15], and were also obtained as solutions of the generalized Knizhnik-Zamolodchikov (KZ) equations of ICFT in Ref.[13].

In what follows, we discuss the notation and concepts involved in the result (2.1).
A. The symmetric matrix L_∞^{ab} in (2.1b) is the high-level form $L \to L_\infty$ of the inverse inertia tensor of any high-level smooth solution of the Virasoro master equation. The matrix P^{ab}, which solves [16,17],

$$P^{ac}\eta_{cd}P^{db} = P^{ab} \tag{2.2}$$

is the high-level projector of the L theory and η_{ab} is the Killing metric of g.

The chiral correlators (2.1) provide a uniform high-level description of the rational and irrational conformal field theories on g, including

$$P_g^{ab} = \eta^{ab} \quad , \quad P_{g/h}^{ab} = P_g^{ab} - P_h^{ab} \tag{2.3}$$

for the affine-Sugawara and coset constructions respectively, where η^{ab} is the inverse Killing metric of g. More generally, the projectors P are closely related to the adjacency matrices of graph theory [18] and generalized graph theory [19] in the partial classification of ICFT. For example, one has [18]

$$P_{ij,kl} = \theta_{ik}(\mathcal{G}_n)\delta_{ij,kl} \quad , \quad 1 \le i < j \le n \quad , \quad 1 \le k < l \le n \tag{2.4}$$

in the graph theory ansatz on $SO(n)$, where $a = (ij)$ is the adjoint index and $\theta(\mathcal{G}_n)$ is the adjacency matrix of any graph $\mathcal{G}_n$ of order n. The level-families classified by the graphs and generalized graphs are generically unitary and irrational on integer levels of the affine algebras.

B. The complex variable y is the anharmonic ratio $y = \frac{z_{12}z_{34}}{z_{14}z_{32}}$.

C. The Greek letters $\alpha, \beta, \ldots$ are composite indices, e.g. $\alpha = (\alpha_1\alpha_2\alpha_3\alpha_4)$, and the correlators may be written schematically as,

$$Y_L^\alpha \sim \langle R_L^{\alpha_1}(\mathcal{T}^1) R_L^{\alpha_2}(\mathcal{T}^2) R_L^{\alpha_3}(\mathcal{T}^3) R_L^{\alpha_4}(\mathcal{T}^4)\rangle \tag{2.5}$$

where $R_L^\alpha(\mathcal{T})$, $\alpha = 1 \ldots \dim \mathcal{T}$ is the broken affine primary field of the L theory corresponding to irreducible matrix representation $\mathcal{T}$ of g. The correlators are written assuming an L-basis [14] for each $\mathcal{T}^i$, where the high-level conformal weight matrix of the broken affine primary field $R_L^{\alpha_i}(\mathcal{T}^i)$ is diagonal,

$$(L_\infty^{ab}\mathcal{T}_a^i\mathcal{T}_b^i)_{\alpha_i}{}^{\beta_i} = \Delta_{\alpha_i}(\mathcal{T}^i)\delta_{\alpha_i}^{\beta_i} \quad , \quad \Delta_{\alpha_i}(\mathcal{T}^i) = \mathcal{O}(k^{-1}) \quad . \tag{2.6}$$

As we will see below, the fact that the broken affine primary conformal weights $\Delta_{\alpha_i}(\mathcal{T}^i)$ are $\mathcal{O}(k^{-1})$ is central for the interpretation of the logarithms in (2.1). Fig.1 shows our conventions for the s and t-channels of the correlators, and the 13 channel is the u-channel.

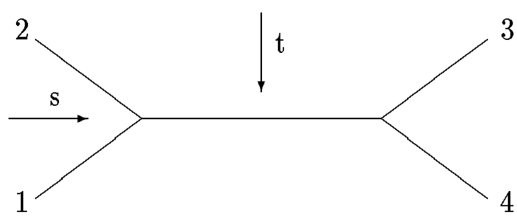

Fig. 1. The correlators.

D. Multiplication of irreps is by tensor product, so that

$$(\mathbb{1})_\alpha^\beta = \delta_\alpha^\beta \equiv \delta_{\alpha_1}^{\beta_1}\delta_{\alpha_2}^{\beta_2}\delta_{\alpha_3}^{\beta_3}\delta_{\alpha_4}^{\beta_4} \tag{2.7a}$$

$$(\mathcal{T}_a^1)_\alpha{}^\beta \equiv (\mathcal{T}_a^1)_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}^{\beta_2}\delta_{\alpha_3}^{\beta_3}\delta_{\alpha_4}^{\beta_4} \tag{2.7b}$$

$$(\mathcal{T}_a^1\mathcal{T}_b^2)_\alpha{}^\beta \equiv (\mathcal{T}_a^1)_{\alpha_1}{}^{\beta_1}(\mathcal{T}_b^2)_{\alpha_2}{}^{\beta_2}\delta_{\alpha_3}^{\beta_3}\delta_{\alpha_4}^{\beta_4} \quad . \tag{2.7c}$$

E. The objects $\bar{v}_g^\alpha$ are arbitrary linear combinations of g-invariant tensors of $\mathcal{T}^1 \otimes \cdots \otimes \mathcal{T}^4$, which satisfy the g-global Ward identity,

$$\bar{v}_g^\beta(\sum_{i=1}^4 \mathcal{T}_a^i)_\beta{}^\alpha = 0 \quad , \quad a = 1 \ldots \dim g \quad . \tag{2.8}$$

F. The matrix irreps $\mathcal{T}_a$ satisfy the hermiticity condition,

$$\mathcal{T}_a^\dagger = \rho_a{}^b \mathcal{T}_b \tag{2.9a}$$

$$(\mathcal{T}_a^\dagger)_\alpha{}^\beta \equiv \eta_{\alpha\rho}\eta^{\beta\sigma}(\mathcal{T}_a)_\sigma{}^{\rho *} \tag{2.9b}$$

where star is complex conjugation and $\eta_{\alpha\beta} = \eta^*_{\alpha\beta}$ is the carrier space metric of irrep $\mathcal{T}$. Moreover, we will consider only unitary theories (integer level of the affine algebra and $L^\dagger(m) = L(-m)$), for which the inverse inertia tensor satisfies

$$L^{ab*} = L^{cd}(\rho^{-1})_c{}^a(\rho^{-1})_d{}^b \tag{2.10}$$

and similarly for L_∞. It follows that all the terms in (2.1) are hermitean, e.g.

$$(2L_\infty^{ab}\mathcal{T}_a^1\mathcal{T}_b^2)^\dagger = 2L_\infty^{ab}\mathcal{T}_a^1\mathcal{T}_b^2 \tag{2.11}$$

with orthonormal, complete sets of eigenvectors and real eigenvalues.

G. The correlators (2.1) are given in the 2-3 symmetric KZ gauge,

$$Y^\alpha(y) = (\prod_{i<j}^4 z_{ij}^{\gamma_{ij}})A^\alpha(z_1, z_2, z_3, z_4) \tag{2.12a}$$

$$\gamma_{12} = \gamma_{13} = 0 \quad , \quad \gamma_{14} = 2\Delta_{\alpha_1} \quad , \quad \gamma_{23} = \Delta_{\alpha_1} + \Delta_{\alpha_2} + \Delta_{\alpha_3} - \Delta_{\alpha_4} \tag{2.12b}$$

$$\gamma_{24} = -\Delta_{\alpha_1} + \Delta_{\alpha_2} - \Delta_{\alpha_3} + \Delta_{\alpha_4} \quad , \quad \gamma_{34} = -\Delta_{\alpha_1} - \Delta_{\alpha_2} + \Delta_{\alpha_3} + \Delta_{\alpha_4} \tag{2.12c}$$

where $A^\alpha(z)$ are the non-invariant chiral four-point correlators.

H. For any conformal field theory in the KZ gauge, the conformal weights $\Delta_{(\mathrm{s})}$, $\Delta_{(\mathrm{u})}$ and $\Delta_{(\mathrm{t})}$ of the s, u and t-channel intermediate states appear in the limiting behaviour,

$$Y^\alpha(y) \sim \begin{cases} y^{\Delta_{(\mathrm{s})} - \Delta_{\alpha_1}(\mathcal{T}^1) - \Delta_{\alpha_2}(\mathcal{T}^2)} & , \ y \to 0 \\ (1-y)^{\Delta_{(\mathrm{u})} - \Delta_{\alpha_1}(\mathcal{T}^1) - \Delta_{\alpha_3}(\mathcal{T}^3)} & , \ y \to 1 \\ \left(\frac{1}{y}\right)^{\Delta_{(\mathrm{t})} + \Delta_{\alpha_1}(\mathcal{T}^1) - \Delta_{\alpha_4}(\mathcal{T}^4)} & , \ y \to \infty \end{cases} . \tag{2.13}$$

Here, we will use these facts in the high-level form

$$y^{\Delta_{(\mathrm{s})} - \Delta_{\alpha_1}(\mathcal{T}^1) - \Delta_{\alpha_2}(\mathcal{T}^2)} = 1 + [\Delta_{(\mathrm{s})} - \Delta_{\alpha_1}(\mathcal{T}^1) - \Delta_{\alpha_2}(\mathcal{T}^2)]\ln y + \mathcal{O}(k^{-2}) \tag{2.14}$$

where we have recalled that the conformal weights of the broken affine primary fields are $\mathcal{O}(k^{-1})$.

I. In Ref.[15], it was shown that the high-level chiral correlators (2.1) have physical singularities in all channels, and that the high-level fusion rules of the broken affine primaries follow the Clebsch-Gordan coefficients of their corresponding matrix irreps.

Symmetry hierarchy in ICFT

The high-level correlators (2.1) provide a uniform description of all IFCT on simple g, which is a bewildering variety of theories [8]. In this paper we make the first attempt to identify simpler, tractable ICFTs among these varieties.

Towards this end, we remind the reader of the symmetry hierarchy [12] in ICFT,

$$\text{ICFT} \supset\supset H\text{-invariant CFTs} \supset\supset \text{Lie } h\text{-invariant CFTs} \supset\supset \text{RCFT} \tag{2.15}$$

which organizes the space of ICFTs on G according to the residual symmetry group $H \subset G$ of the theory. As seen in this hierarchy, the generic ICFT has no residual symmetry group[a], and these generic theories are expected to be the most complex. Consequently, we focus here on the theories with a symmetry, which are also generically irrational.

The set of all ICFTs with a non-trivial symmetry group H (which may be a discrete subgroup of G or a Lie subgroup) is called the set of H-invariant CFTs. Among the H-invariant CFTs, the subspace of theories with a Lie symmetry is called the set of Lie h-invariant CFTs, where $h \subset g$. This subspace includes the affine-Sugawara and coset constructions as a much smaller subspace.

When a theory L is an H-invariant CFT, the correlators (2.1) also satisfy the global H-invariance condition,

$$Y_H \Omega(H) = Y_H \quad , \quad \Omega(H) \in G \quad , \quad \Omega(H)_\alpha{}^\beta = \prod_{i=1}^{4} \Omega(H, \mathcal{T}^i)_{\alpha_i}{}^{\beta_i} \tag{2.16}$$

where $\Omega(H, \mathcal{T}^i)_{\alpha_i}{}^{\beta_i}$ is the subgroup H in matrix irrep $\mathcal{T}^i$. When the theory is a Lie h-invariant CFT, the condition (2.16) reduces to the h-global Ward identity

$$Y_{\text{Lie}\, h} \sum_{i=1}^{4} \mathcal{T}_a^i = 0 \quad , \quad a = 1 \ldots \dim h \tag{2.17}$$

which applies for example in the cases of the affine-Sugawara construction (with $h = g$) and the g/h coset constructions.

For the affine-Sugawara and g/h coset constructions, it is known [4,13] that the resolution of chiral correlators into conformal blocks is a basis change from the Lie algebra basis to the block basis, using the h-invariant tensors defined by (2.17). More generally, one expects that the H-invariant tensors defined by (2.16) will play an analogous role in finding the block bases of the H-invariant CFTs.

3 The Affine-Sugawara Constructions

3.1 The affine-Sugawara blocks

The simplest and most symmetric conformal field theories are the affine-Sugawara constructions [1-6] on G, whose high-level correlators are described by (2.1) with

$$L^{ab}_{g,\infty} = \frac{\eta^{ab}}{2k} \tag{3.1a}$$

[a]In the graph theory ansatz [18] on $SO(n)$, whose high-level projectors are given in (2.4), this corresponds to the fact that the generic graph has no symmetry.

$$Y_g(y)\sum_{i=1}^{4}\mathcal{T}_a^i = 0 \quad , \quad a = 1\ldots\dim g \tag{3.1b}$$

where $P_g^{ab} = \eta^{ab}$ is the inverse Killing metric of g. In this case, the correlators (2.1) are the high-level solutions of the KZ equations [3,4] on simple g.

We begin by defining the s-channel block basis of g-invariants $v(\mathrm{s},g)^m$ as the solutions of the simultaneous eigenvalue problem and g-global condition

$$(2L_{g,\infty}^{ab}\mathcal{T}_a^1\mathcal{T}_b^2)_\alpha{}^\beta v(\mathrm{s},g)_\beta^m = (\Delta_{(\mathrm{s})}^g(m) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^2))v(\mathrm{s},g)_\alpha^m \tag{3.2a}$$

$$\sum_{i=1}^{4}(\mathcal{T}_a^i)_\alpha{}^\beta v(\mathrm{s},g)_\beta^m = 0 \quad , \quad a = 1\ldots\dim g \quad . \tag{3.2b}$$

The g-global condition (3.2b) is compatible with the eigenvalue problem because the generators $\sum_{i=1}^4\mathcal{T}_a^i$ commute with $L_{g,\infty}^{ab}\mathcal{T}_a^1\mathcal{T}_b^2$. Here $\Delta_g(\mathcal{T}^i)$, $i = 1,2$ are the (unbroken) conformal weights of irrep $\mathcal{T}^i$ under the affine-Sugawara construction,

$$\Delta_{\alpha_i}(\mathcal{T}^i)|_{L=L_g} = \Delta^g(\mathcal{T}^i) = \frac{I(\mathcal{T}^i)}{x+\tilde{h}} \tag{3.3a}$$

$$x = \frac{2k}{\psi_g^2} \tag{3.3b}$$

where ψ_g, $\tilde{h}$, $I(\mathcal{T})$ and x are respectively the highest root and dual Coxeter number of g, the invariant Casimir of irrep $\mathcal{T}$ and the invariant level of the affine algebra. The quantity $\Delta_{(\mathrm{s})}^g(m) = \Delta^g(\mathcal{T}^m)$ is the corresponding conformal weight of an irrep $\mathcal{T}^m$ in $\mathcal{T}^1\otimes\mathcal{T}^2$. The associated dual eigenvalue problem is

$$\bar{v}(\mathrm{s},g)_m^\beta(2L_{g,\infty}^{ab}\mathcal{T}_a^1\mathcal{T}_b^2)_\beta{}^\alpha = \bar{v}(\mathrm{s},g)_m^\alpha(\Delta_{(\mathrm{s})}^g(m) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^2)) \tag{3.4a}$$

$$\bar{v}(\mathrm{s},g)_m^\beta\sum_{i=1}^{4}(\mathcal{T}_a^i)_\beta{}^\alpha = 0 \quad , \quad a = 1\ldots\dim g \tag{3.4b}$$

where $\bar{v}(\mathrm{s},g)_m^\alpha = v(\mathrm{s},g)_\beta^{m*}\eta^{\beta\alpha}$ and $\eta_{\alpha\beta} = \prod_{i=1}^4\eta_{\alpha_i\beta_i}$ is the product of the carrier space metrics.

Because $2L_{g,\infty}^{ab}\mathcal{T}_a^1\mathcal{T}_b^2$ is hermitean we know that the eigenvectors are orthonormal and complete,

$$\bar{v}(\mathrm{s},g)_m v(\mathrm{s},g)^n = \delta_m^n \quad , \quad v(\mathrm{s},g)_\alpha^m\bar{v}(\mathrm{s},g)_m^\beta = (I_g)_\alpha^\beta \tag{3.5}$$

where I_g is the projector onto the g-invariant subspace of $\mathcal{T}^1\otimes\cdots\otimes\mathcal{T}^4$. The relation

$$[L_{g,\infty}^{ab}\mathcal{T}_a^i\mathcal{T}_b^j, I_g] = 0 \quad , \quad 1\le i,j\le 4 \tag{3.6}$$

also holds on the g-invariant subspace defined by (3.2). An explicit solution to the eigenvalue problem and global condition in (3.2) is known [15]

$$\bar{v}(\mathrm{s},g)_m^\alpha = \sum_{\alpha_r\alpha_{\bar{r}}}\bar{w}_\mathrm{s}(r,\xi)^{\alpha_1\alpha_2\alpha_r}\bar{w}_\mathrm{s}(\bar{r},\xi')^{\alpha_3\alpha_4\alpha_{\bar{r}}}\eta_{\alpha_r\alpha_{\bar{r}}} \quad , \quad m = (r,\xi,\xi') \tag{3.7a}$$

$$\bar{w}_\mathrm{s}(r,\xi)^{\beta_1\beta_2\beta_r}(T_a^1+\mathcal{T}_a^2+\mathcal{T}_a^r)_{\beta_1\beta_2\beta_r}{}^{\alpha_1\alpha_2\alpha_r} = 0 \quad , \quad a = 1\ldots\dim g \tag{3.7b}$$

$$\bar{w}_s(r,\xi)^{\beta_1\beta_2\alpha_r}[2L^{ab}_{g,\infty}\mathcal{T}^1_a\mathcal{T}^2_b]_{\beta_1\beta_2}{}^{\alpha_1\alpha_2} = \bar{w}_s(r,\xi)^{\alpha_1\alpha_2\alpha_r}[\Delta^g(\mathcal{T}^r) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^2)] \quad (3.7c)$$

where $\bar{w}_s(r,\xi)^{\alpha_i\alpha_j\alpha_r}$ are the Clebsch-Gordan coefficients of $\mathcal{T}^i \otimes \mathcal{T}^j$ into irrep $\mathcal{T}^r$, ξ labels copies of the same irrep $\mathcal{T}^r$ and $\bar{r}$ is the conjugate representation of r. Using (3.7), it is easy to check that $\Delta^g_{(s)}(m) = \Delta^g(\mathcal{T}^m)$ in (3.2a) is the conformal weight of irrep m under the affine-Sugawara construction.

As an explicit example, one finds for $n\bar{n}\bar{n}n$ correlators on $SU(n)$ that the invariant tensors (3.7) are

$$\bar{v}(\mathrm{s}, SU(n))^\alpha_V = v(\mathrm{s}, SU(n))^V_\alpha = \frac{1}{n}\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} \quad (3.8a)$$

$$\bar{v}(\mathrm{s}, SU(n))^\alpha_A = v(\mathrm{s}, SU(n))^A_\alpha = \frac{1}{\sqrt{n^2-1}}[\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4} - \frac{1}{n}\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}] \quad (3.8b)$$

where V and A are vacuum and adjoint. This is the original example [4] considered by Knizhnik and Zamolodchikov, although our Clebsch basis (3.7), (3.8) is slightly different than theirs.

From (2.1),(3.1) and the completeness relation (3.5), we use eigenvector expansions to define the s-channel conformal blocks $\mathcal{F}^{(\mathrm{s})}_g(y)$ of the affine-Sugawara construction

$$\bar{v}^\alpha_g = \sum_m d(\mathrm{s})^m \bar{v}(\mathrm{s}, g)^\alpha_m \quad (3.9a)$$

$$Y^\alpha_g(y) = \sum_{m,n} d(\mathrm{s})^m \mathcal{F}^{(\mathrm{s})}_g(y)_m{}^n \bar{v}(\mathrm{s}, g)^\alpha_n \quad (3.9b)$$

$$\mathcal{F}^{(\mathrm{s})}_g(y)_m{}^n = \bar{v}(\mathrm{s}, g)_m[\mathbb{1} + 2L^{ab}_{g,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y))]v(\mathrm{s}, g)^n + \mathcal{O}(k^{-2}) \quad (3.9c)$$

as the coefficients of the chiral correlators expanded in the block basis. Here, $d(\mathrm{s})^m$ are a set of undetermined constants.

To study the small y behavior of the s-channel blocks, we rearrange (3.9c) as follows,

$$\mathcal{F}^{(\mathrm{s})}_g(y)_m{}^n = \bar{v}(\mathrm{s}, g)_m[\mathbb{1} + L^{ab}_{g,\infty}\mathcal{T}^1_a\mathcal{T}^2_b \ln y][\mathbb{1} + L^{ab}_{g,\infty}\mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y)]v(\mathrm{s}, g)^n + \mathcal{O}(k^{-2}) \quad (3.10a)$$

$$= [1 + (\Delta^g_{(\mathrm{s})}(m) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^2)) \ln y]$$
$$\times \bar{v}(\mathrm{s}, g)_m[\mathbb{1} + L^{ab}_{g,\infty}\mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y)]v(\mathrm{s}, g)^n + \mathcal{O}(k^{-2}) \quad (3.10b)$$

$$= y^{\Delta^g_{(\mathrm{s})}(m) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^2)}\left[\delta^n_m - c(g)_m{}^n \sum_{p=1}^\infty \frac{y^p}{p}\right] + \mathcal{O}(k^{-2}) \quad (3.10c)$$

$$c(g)_m{}^n = \bar{v}(\mathrm{s}, g)_m L^{ab}_{g,\infty}\mathcal{T}^1_a\mathcal{T}^3_b v(\mathrm{s}, g)^n \quad (3.10d)$$

where we have used the dual eigenvalue problem (3.4a) to obtain (3.10b) and the high-level relation (2.14) to obtain (3.10c). In particular, the eigenvector resolution correctly guarantees that each block has only a single leading singularity,

$$\mathcal{F}^{(\mathrm{s})}_g(y)_m{}^n \underset{y\to 0}{\sim} \delta^n_m y^{\Delta^g_{(\mathrm{s})}(m) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^2)} + \mathcal{O}(k^{-2}) \quad (3.11)$$

followed by integer-spaced secondaries from $\ln(1-y)$. According to eq.(3.11), the leading singularities of the $m=n$ blocks correspond to the s-channel exchange of affine primary states, while the leading singularities of the $m \neq n$ blocks are affine secondaries.

To define block bases for the other channels, we also introduce the u and t-channel g-invariants as solutions to their corresponding eigenvalue problems,

$$2L^{ab}_{g,\infty}\mathcal{T}^1_a\mathcal{T}^3_b v(\mathrm{u},g)^m = (\Delta^g_{(\mathrm{u})}(m) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^3))v(\mathrm{u},g)^m \tag{3.12a}$$

$$2L^{ab}_{g,\infty}\mathcal{T}^2_a\mathcal{T}^3_b v(\mathrm{t},g)^m = (\Delta^g_{(\mathrm{t})}(m) - \Delta^g(\mathcal{T}^2) - \Delta^g(\mathcal{T}^3))v(\mathrm{t},g)^m \tag{3.12b}$$

$$(\sum_{i=1}^4 \mathcal{T}^i_a)v(\mathrm{u},g)^m = (\sum_{i=1}^4 \mathcal{T}^i_a)v(\mathrm{t},g)^m = 0 \quad , \quad a = 1\ldots \dim g \tag{3.12c}$$

$$\bar{v}(\mathrm{u},g)_m v(\mathrm{u},g)^n = \bar{v}(\mathrm{t},g)_m v(\mathrm{t},g)^n = \delta^n_m \tag{3.12d}$$

$$v(\mathrm{u},g)^m \bar{v}(\mathrm{u},g)_m = v(\mathrm{t},g)^m \bar{v}(\mathrm{t},g)_m = I_g \quad . \tag{3.12e}$$

Here $\Delta^g_{(\mathrm{u})}(m) = \Delta^g(\mathcal{T}^m)$ and $\Delta^g_{(\mathrm{t})}(m') = \Delta^g(\mathcal{T}^{m'})$ are the conformal weights under the affine-Sugawara construction of irreps $\mathcal{T}^m$ and $\mathcal{T}^{m'}$ in $\mathcal{T}^1 \otimes \mathcal{T}^3$ and $\mathcal{T}^2 \otimes \mathcal{T}^3$ respectively. Explicit forms of the u and t-channel invariants are obtained formally by a $2 \leftrightarrow 3$ and a $2 \leftrightarrow 4$ interchange respectively in eq.(3.7).

Using the u and t-channel invariants, we define the corresponding u and t-channel blocks $\mathcal{F}^{(\mathrm{u})}_g$ and $\mathcal{F}^{(\mathrm{t})}_g$,

$$Y_g(y) = \sum_{m,n} d(\mathrm{u})^m \mathcal{F}^{(\mathrm{u})}_g(y)_m{}^n \bar{v}(\mathrm{u},g)_n = \sum_{m,n} d(\mathrm{t})^m \mathcal{F}^{(\mathrm{t})}_g(y)_m{}^n \bar{v}(\mathrm{t},g)_n \tag{3.13a}$$

$$\mathcal{F}^{(\mathrm{u})}_g(y)_m{}^n = \bar{v}(\mathrm{u},g)_m[1\!\!1 + 2L^{ab}_{g,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y))]v(\mathrm{u},g)^n + \mathcal{O}(k^{-2}) \tag{3.13b}$$

$$\mathcal{F}^{(\mathrm{t})}_g(y)_m{}^n = \bar{v}(\mathrm{t},g)_m[1\!\!1 + 2L^{ab}_{g,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y))]v(\mathrm{t},g)^n + \mathcal{O}(k^{-2}) \tag{3.13c}$$

in analogy to the s-channel blocks $\mathcal{F}^{(\mathrm{s})}_g$ in eq.(3.9). The limiting behaviour of the u and t-channel blocks,

$$\mathcal{F}^{(\mathrm{u})}_g(y)_m{}^n \underset{y\to 1}{\sim} \delta^n_m (1-y)^{\Delta^g_{(\mathrm{u})}(m) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^3)} + \mathcal{O}(k^{-2}) \tag{3.14a}$$

$$\mathcal{F}^{(\mathrm{t})}_g(y)_m{}^n \underset{y\to\infty}{\sim} (\lambda_g)_m{}^n \left(\frac{1}{y}\right)^{\Delta^g_{(\mathrm{t})}(m) + \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^4)} + \mathcal{O}(k^{-2}) \tag{3.14b}$$

$$(\lambda_g)_m{}^n = (-1)^{\Delta^g_{(\mathrm{t})}(m) + \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^4)} \bar{v}(\mathrm{t},g)_m (-1)^{-2L^{ab}_{g,\infty}\mathcal{T}^1_a\mathcal{T}^2_b} v(\mathrm{t},g)^n \tag{3.14c}$$

is obtained from (3.13) and the corresponding eigenvalue problems. Integer-spaced secondaries are also obtained, as in (3.10), from the cross-channel logarithms. To obtain (3.14b,c) we have also used the g-global Ward identity

$$\bar{v}(\mathrm{t},g)_m[2L^{ab}_{g,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b + \mathcal{T}^2_a\mathcal{T}^3_b + \mathcal{T}^3_a\mathcal{T}^1_b) - \gamma] = 0 \tag{3.15a}$$

$$\gamma = \Delta^g(\mathcal{T}^4) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^2) - \Delta^g(\mathcal{T}^3) \tag{3.15b}$$

to write the t-channel blocks in the alternate form

$$\mathcal{F}_g^{(\mathrm{t})}(y)_m{}^n = \bar{v}(\mathrm{t},g)_m[1 - 2L^{ab}_{g,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln(\frac{1}{y} - 1) + \mathcal{T}^2_a\mathcal{T}^3_b \ln(1-y)) + \gamma\ln(1-y)]v(\mathrm{t},g)^n \tag{3.16}$$
$$+\mathcal{O}(k^{-2}) \quad .$$

Using the inverse of the matrix λ_g, another basis can be found for the t-channel blocks, in which their $y \to \infty$ behaviour is also proportional to Kronecker delta, as for the s and u-channel blocks. In such a basis, however, the crossing relations given below would be more complicated.

In what follows, we introduce a unified notation $\rho =$ s, t, u for the three channels and their corresponding blocks $(\mathcal{F}_g^{(\rho)})_m{}^n$,

$$\bar{v}(\rho,g)_m v(\rho,g)^n = \delta^n_m \quad , \quad v(\rho,g)^m_\alpha \bar{v}(\rho,g)^\beta_m = (I_g)^\beta_\alpha \tag{3.17a}$$

$$Y_g(y) = \sum_{m,n} d(\rho)^m \mathcal{F}_g^{(\rho)}(y)_m{}^n \bar{v}(\rho,g)_n \quad , \quad \rho = \mathrm{s,t,u} \tag{3.17b}$$

$$\mathcal{F}_g^{(\rho)}(y)_m{}^n = \bar{v}(\rho,g)_m[\mathbb{1} + 2L^{ab}_{g,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y))]v(\rho,g)^n + \mathcal{O}(k^{-2}) \tag{3.17c}$$

$$(\mathcal{F}_g^{(\rho)}(y)_m{}^n)^* = \mathcal{F}_g^{(\rho)}(y^*)_n{}^m \tag{3.17d}$$

where the last relation follows by unitarity, that is, hermiticity of the basic matrices in the correlators.

We finally note that the number $B_g(\rho)$ of affine-Sugawara blocks in the ρ-channel,

$$B_g(\rho) = (d_g(\rho))^2 \tag{3.18}$$

is equal to the square of the dimension $d_g(\rho)$ of the g-invariants in that channel.

Crossing relations

Using completeness of the three sets of eigenvectors, one easily finds that the three sets of blocks are related by the crossing relations,

$$(\mathcal{F}_g^{(\rho)})_m{}^n = X_g(\rho\sigma)_m{}^p(\mathcal{F}_g^{(\sigma)})_p{}^q X_g^{-1}(\rho\sigma)_q{}^n \quad , \quad \rho,\sigma = \mathrm{s,t,u} \tag{3.19a}$$

$$X_g(\rho\sigma)_m{}^n = \bar{v}(\rho,g)_m v(\sigma,g)^n + \mathcal{O}(k^{-2}) \tag{3.19b}$$

$$X_g^{-1}(\rho\sigma)_m{}^n = X_g(\sigma\rho)_m{}^n = (X_g(\rho\sigma)_n{}^m)^* \tag{3.19c}$$

where $\rho \neq \sigma$ and $X_g(\rho\sigma)$ is the crossing matrix from channel σ to channel ρ. The last relation (3.19c) says that the crossing matrices $X_g(\rho\sigma)_m{}^n$ are unitary $X_g^\dagger = X_g^{-1}$ for each $\rho \neq \sigma$, and the crossing matrices explicitly satisfy the consistency relations

$$X_g(\rho\sigma)X_g(\sigma\tau)X_g(\tau\rho) = X_g(\rho\tau)X_g(\tau\sigma)X_g(\sigma\rho) = 1 \tag{3.20}$$

which says that we return to the same blocks when we go around an s,u,t loop.

In the special case when $\mathcal{T}^2 \sim \mathcal{T}^3$, the conformal weights exchanged in the u-channel are the same as in the s-channel. In further detail, we have

$$L^{ab}_{g,\infty}(\mathcal{T}^1_a\mathcal{T}^3_b)_\alpha{}^\beta = L^{ab}_{g,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b)_{\alpha'}{}^{\beta'} \tag{3.21}$$

in this case, where $\alpha' = (\alpha_1\alpha_3\alpha_2\alpha_4)$ and similarly for β'. Then we may identify the g-invariants of the u-channel in terms of those of the s-channel

$$v(\mathrm{u},g)^m_\alpha = v(\mathrm{s},g)^m_{\alpha'} \quad , \quad \bar{v}(\mathrm{u},g)^\alpha_m = \bar{v}(\mathrm{s},g)^{\alpha'}_m \tag{3.22}$$

where $m = (r,\xi,\xi')$ is the same irrep $\mathcal{T}^r$ in both channels. It follows from (3.13b), (3.21) and (3.19b) that

$$X_g(\mathrm{su})^{-1} = X_g(\mathrm{su}) \quad , \quad X_g(\mathrm{su})^2 = 1 \tag{3.23a}$$

$$X_g(\mathrm{us})^{-1} = X_g(\mathrm{us}) \quad , \quad X_g(\mathrm{us})^2 = 1 \tag{3.23b}$$

$$\mathcal{F}^{(\mathrm{u})}_g(y)_m{}^n = \mathcal{F}^{(\mathrm{s})}_g(1-y)_m{}^n \quad . \tag{3.23c}$$

Then, using (3.23) in (3.19a), one finds that the affine-Sugawara blocks close under s-u crossing,

$$\mathcal{F}^{(\mathrm{s})}_g(1-y)_m{}^n = X_g(\mathrm{su})_m{}^p\mathcal{F}^{(\mathrm{s})}_g(y)_p{}^q X_g(\mathrm{su})_q{}^n \tag{3.24}$$

as they should in this case.

In the special case when all four representations are the same, one finds that the unitary crossing matrices are also idempotent $X(\rho\sigma)^2 = 1$ and hence $X(\rho\sigma) = X(\sigma\rho)$ for all $\rho \neq \sigma$: then, the Yang-Baxter-like relation,

$$X_g(\rho\sigma)X_g(\sigma\tau)X_g(\tau\rho) = X_g(\tau\rho)X_g(\sigma\tau)X_g(\rho\sigma) = 1 \tag{3.25}$$

follows from the consistency relations (3.20).

3.2 Non-chiral WZW correlators

To construct a set of high-level non-chiral WZW correlators from the affine-Sugawara blocks (3.9c), we take the diagonal construction in the s-channel blocks,

$$Y_g(y^*,y)_\alpha{}^\beta = \sum_{m,n,p} v(\mathrm{s},g)^m_\alpha \mathcal{F}^{(\mathrm{s})}_g(y^*)_m{}^p\mathcal{F}^{(\mathrm{s})}_g(y)_p{}^n\bar{v}(\mathrm{s},g)^\beta_n + \mathcal{O}(k^{-2}) \tag{3.26a}$$

$$= \sum_{m,n,p} (\mathcal{F}^{(\mathrm{s})}_g(y)_p{}^m)^*\mathcal{F}^{(\mathrm{s})}_g(y)_p{}^n\, v(\mathrm{s},g)^m_\alpha\bar{v}(\mathrm{s},g)^\beta_n + \mathcal{O}(k^{-2}) \tag{3.26b}$$

$$\underset{y\to 0}{\sim} \sum_m |y|^{2(\Delta^g_{(\mathrm{s})}(m)-\Delta^g(\mathcal{T}^1)-\Delta^g(\mathcal{T}^2))} v(\mathrm{s},g)^m_\alpha\bar{v}(\mathrm{s},g)^\beta_m + \mathcal{O}(k^{-2}) \tag{3.26c}$$

which shows trivial monodromy around $y = 0$. These correlators satisfy a left and right KZ equation and g-global conditions on the left and right.

To see that these correlators have trivial monodromy around $y=1$ and $y=\infty$, one uses the crossing relations (3.19) of the affine-Sugawara blocks to rewrite the correlator (3.26) in the two alternate forms

$$Y_g(y^*,y)_\alpha{}^\beta = \sum_{m,n,p} (\mathcal{F}_g^{(\mathrm{u})}(y)_p{}^m)^* \mathcal{F}_g^{(\mathrm{u})}(y)_p{}^n \, v(\mathrm{u},g)_\alpha^m \bar{v}(\mathrm{u},g)_n^\beta + \mathcal{O}(k^{-2}) \tag{3.27a}$$

$$= \sum_{m,n,p} (\mathcal{F}_g^{(\mathrm{t})}(y)_p{}^m)^* \mathcal{F}_g^{(\mathrm{t})}(y)_p{}^n \, v(\mathrm{t},g)_\alpha^m \bar{v}(\mathrm{t},g)_n^\beta + \mathcal{O}(k^{-2}) \quad . \tag{3.27b}$$

The forms (3.26) and (3.27) are easily checked against the examples in Ref.[4].

Using completeness and the form (3.9c) of the affine-Sugawara blocks, we also find the summed form of the non-chiral WZW correlators

$$\begin{aligned} Y_g(y^*,y)_\alpha{}^\beta = \Big\{&[\mathbb{1} + 2L^{ab}_{g,\infty}(\mathcal{T}_a^1\mathcal{T}_b^2 \ln y^* + \mathcal{T}_a^1\mathcal{T}_b^3 \ln(1-y^*))]I_g \\ &\times[\mathbb{1} + 2L^{ab}_{g,\infty}(\mathcal{T}_a^1\mathcal{T}_b^2 \ln y + \mathcal{T}_a^1\mathcal{T}_b^3 \ln(1-y))]\Big\}_\alpha{}^\beta + \mathcal{O}(k^{-2}) \end{aligned} \tag{3.28a}$$

$$= \{[\mathbb{1} + 2L^{ab}_{g,\infty}(\mathcal{T}_a^1\mathcal{T}_b^2 \ln |y|^2 + \mathcal{T}_a^1\mathcal{T}_b^3 \ln |1-y|^2)]I_g\}_\alpha{}^\beta + \mathcal{O}(k^{-2}) \tag{3.28b}$$

where I_g is the projector (3.5) onto the G-invariant subspace, and we have used eq.(3.6) to obtain the second form, which explicitly shows two of the trivial monodromies.

Appendix A gives alternate expressions for the g-blocks (3.17) and correlators (3.26) which involve the g-crossing matrices (3.19b).

4 The Coset Constructions

4.1 The coset blocks

The next simplest, and next most symmetric, set of conformal field theories are the g/h coset constructions [1,2,7], whose chiral correlators are defined by (2.1) with

$$L^{ab}_{g/h,\infty} = \frac{P^{ab}_{g/h}}{2k} \quad , \quad P_{g/h} = P_g - P_h \tag{4.1a}$$

$$Y_{g/h}(y) \sum_{i=1}^{4} \mathcal{T}_a^i = 0 \quad , \quad a = 1\ldots \dim h \tag{4.1b}$$

where $h \subset g$. These correlators are the high-level solutions of the general coset equations of Refs.[14,15,13] on simple g, and the results below are the high-level form of the general coset blocks studied in [20,14,15,13].

We begin by reorganizing the high-level coset correlators (2.1) as,

$$\begin{aligned} Y^\alpha_{g/h}(y) = \{&\bar{v}_g[\mathbb{1} + 2L^{ab}_{g,\infty}(\mathcal{T}_a^1\mathcal{T}_b^2 \ln y + \mathcal{T}_a^1\mathcal{T}_b^3 \ln(1-y))] \\ &\times[\mathbb{1} - 2L^{ab}_{h,\infty}(\mathcal{T}_a^1\mathcal{T}_b^2 \ln y + \mathcal{T}_a^1\mathcal{T}_b^3 \ln(1-y))]\}^\alpha + \mathcal{O}(k^{-2}) \end{aligned} \tag{4.2}$$

where we have used (4.1a) and moved the terms of the h theory to the right.

To define the ρ=s, t and u-channel coset blocks, we need the g-invariant tensors $v(\rho,g)^m$, $\bar{v}(\rho,g)_m$ of Section 3, and also the corresponding h-invariant tensors $\psi(\rho,h)$, which satisfy,

$$2L^{ab}_{h,\infty}\mathcal{T}^1_a\mathcal{T}^2_b\psi(\mathrm{s},h)^M = (\Delta^h_{(\mathrm{s})}(M) - \Delta^h_{M_1}(\mathcal{T}^1) - \Delta^h_{M_2}(\mathcal{T}^2))\psi(\mathrm{s},h)^M \tag{4.3a}$$

$$2L^{ab}_{h,\infty}\mathcal{T}^1_a\mathcal{T}^3_b\psi(\mathrm{u},h)^M = (\Delta^h_{(\mathrm{u})}(M) - \Delta^h_{M_1}(\mathcal{T}^1) - \Delta^h_{M_3}(\mathcal{T}^3))\psi(\mathrm{u},h)^M \tag{4.3b}$$

$$2L^{ab}_{h,\infty}\mathcal{T}^2_a\mathcal{T}^3_b\psi(\mathrm{t},h)^M = (\Delta^h_{(\mathrm{t})}(M) - \Delta^h_{M_2}(\mathcal{T}^2) - \Delta^h_{M_3}(\mathcal{T}^3))\psi(\mathrm{t},h)^M \tag{4.3c}$$

$$L^{ab}_{h,\infty}\mathcal{T}^i_a\mathcal{T}^i_b\psi(\rho,h)^M = \Delta^h_{M_i}(\mathcal{T}^i)\psi(\rho,h) \quad , \quad i=1\dots4 \quad , \quad \rho=\mathrm{s,t,u} \tag{4.3d}$$

$$(\sum_{i=1}^4 \mathcal{T}^i_a)\psi(\rho,h)^M = 0 \quad , \quad a=1\dots\dim h \quad , \quad \rho=\mathrm{s,t,u} \tag{4.3e}$$

where $\Delta^h_{(\rho)}(M)$ are the broken conformal weights of h-irreps in the ρ-channel. The eigenvalue problems (4.3a-c) are compatible with the diagonalization of the h conformal weights in (4.3d) because the matrices $L^{ab}_{h,\infty}\mathcal{T}^i_a\mathcal{T}^j_b$ and $L^{ab}_{h,\infty}\mathcal{T}^i_a\mathcal{T}^i_b$ commute. The h-global Ward identities (4.3e) are also compatible with the eigenvalue problems, whose matrices are h-invariant.

These eigenvectors also satisfy completeness and orthonormality,

$$\bar{\psi}(\rho,h)_M\psi(\rho,h)^N = \delta^N_M \quad , \quad \psi(\rho,h)^M\bar{\psi}(\rho,h)_M = I_h \quad , \quad \rho=\mathrm{s,t,u} \tag{4.4a}$$

$$[L^{ab}_{h,\infty}\mathcal{T}^i_a\mathcal{T}^j_b, I_h] = 0 \quad , \quad 1\le i,j\le 4 \tag{4.4b}$$

where I_h is the projection operator onto the h-invariant subspace of $\mathcal{T}^1\otimes\cdots\otimes\mathcal{T}^4$.

As an explicit example, we give the solution for the $U(1)$-invariant s-channel eigenvectors of the coset correlator

$$(\mathcal{T}^1,\mathcal{T}^2,\mathcal{T}^3,\mathcal{T}^4) = (j_1,j_2,j_3,j_4) \quad \text{in} \quad \frac{SU(2)}{U(1)} \quad . \tag{4.5}$$

In this case we need

$$L^{ab}_{U(1),\infty} = \frac{\delta^a_3\delta^b_3}{2k} \quad , \quad \mathcal{T}^i_3 = \sqrt{\psi^2_g}\begin{pmatrix} j_i & & 0 \\ & \ddots & \\ 0 & & -j_i \end{pmatrix} \tag{4.6}$$

where ψ^2_g is the $SU(2)$ root length squared and we have taken the usual magnetic quantum number basis for the matrices, with $\alpha_i = M_i$, $|M_i|\le j_i$. The solution of the eigenvalue problem (4.3a) is then

$$\psi(\mathrm{s},U(1))^M_\alpha = \delta^M_\alpha\,\delta(\sum_{i=1}^4 M_i = 0) \quad , \quad M=(M_1,M_2,M_3,M_4) \tag{4.7a}$$

$$\Delta^{U(1)}_{(\mathrm{s})}(M) = \frac{(M_1+M_2)^2}{x} \quad , \quad \Delta^{U(1)}_{M_i}(\mathcal{T}^i) = \frac{M_i^2}{x} \quad , \quad i=1\dots4 \tag{4.7b}$$

where $x = 2k/\psi_g^2$ is the invariant level of $g = SU(2)$. For more general coset constructions the eigenvectors $\psi(\mathrm{s}, h)$ are squares of products of Clebsch-Gordan coefficients times Clebsch-Gordan coefficients for branching of g irreps into h irreps [21].

Using completeness of $v(g), \bar{v}(g)$ and $\psi(h), \bar{\psi}(h)$, we have [14,15,13]

$$\bar{v}_g^\alpha = \sum_m d(\rho)^m \bar{v}(\rho, g)_m^\alpha \tag{4.8a}$$

$$Y_{g/h}^\alpha(y) = \sum_{m,M} d(\rho)^m \mathcal{C}_{g/h}^{(\rho)}(y)_m{}^M \bar{\psi}(\rho, h)_M^\alpha \tag{4.8b}$$

where $\mathcal{C}_{g/h}^{(\rho)}(y)$ are the coset blocks. Further use of completeness gives the explicit form of the high-level coset blocks

$$\mathcal{C}_{g/h}^{(\rho)}(y)_m{}^M = \mathcal{F}_g^{(\rho)}(y)_m{}^n e(\rho, g/h)_n{}^N (\mathcal{F}_h^{(\rho)}(y)^{-1})_N{}^M \quad , \quad \rho = \mathrm{s}, \mathrm{t}, \mathrm{u} \tag{4.9a}$$

$$\mathcal{F}_h^{(\rho)}(y)_N{}^M = \bar{\psi}(\rho, h)_N [\mathbb{1} + 2L_{h,\infty}^{ab}(\mathcal{T}_a^1 \mathcal{T}_b^2 \ln y + \mathcal{T}_a^1 \mathcal{T}_b^3 \ln(1-y))] \psi(\rho, h)^M + \mathcal{O}(k^{-2}) \tag{4.9b}$$

$$(\mathcal{F}_h^{(\rho)}(y)^{-1})_N{}^M = \bar{\psi}(\rho, h)_N [\mathbb{1} - 2L_{h,\infty}^{ab}(\mathcal{T}_a^1 \mathcal{T}_b^2 \ln y + \mathcal{T}_a^1 \mathcal{T}_b^3 \ln(1-y))] \psi(\rho, h)^M + \mathcal{O}(k^{-2}) \tag{4.9c}$$

$$e(\rho, g/h)_n{}^N = \bar{v}(\rho, g)_n \psi(\rho, h)^N \tag{4.9d}$$

where $\mathcal{F}_g^{(\rho)}$ are the ρ-channel g-blocks (of the affine-Sugawara construction on g) given in eq.(3.17), and $e(\rho, g/h)$ is the embedding matrix of the g-invariants $v(g)$ in the h-invariants $\psi(h)$. The inverse h blocks $\mathcal{F}_h^{-1}$ are the inverse of the h blocks $\mathcal{F}_h$. In Ref.[14], the exact coset blocks were written as $(\mathcal{C}_{g/h})_m{}^M = (\mathcal{F}_g)_m{}^n (\mathcal{F}_h^{-1})_n{}^M$, where $(\mathcal{F}_h^{-1})_n{}^M = e(g/h)_n{}^N (\mathcal{F}_h^{-1})_N{}^M$ in the present notation.

To obtain the limiting behavior of the coset blocks, we first need the corresponding results for the inverse h-blocks,

$$(\mathcal{F}_h^{(\mathrm{s})}(y)^{-1})_N{}^M \underset{y \to 0}{\sim} \delta_N^M y^{-\Delta_{(\mathrm{s})}^h(M) + \Delta_{M_1}^h(\mathcal{T}^1) + \Delta_{M_2}^h(\mathcal{T}^2)} + \mathcal{O}(k^{-2}) \tag{4.10a}$$

$$(\mathcal{F}_h^{(\mathrm{u})}(y)^{-1})_N{}^M \underset{y \to 1}{\sim} \delta_N^M (1-y)^{-\Delta_{(\mathrm{u})}^h(M) + \Delta_{M_1}^h(\mathcal{T}^1) + \Delta_{M_3}^h(\mathcal{T}^3)} + \mathcal{O}(k^{-2}) \tag{4.10b}$$

$$(\mathcal{F}_h^{(\mathrm{t})}(y)^{-1})_N{}^M \underset{y \to \infty}{\sim} (\lambda_h^{-1})_N{}^M \left(\frac{1}{y}\right)^{-\Delta_{(\mathrm{t})}^h(M) - \Delta_{M_1}^h(\mathcal{T}^1) + \Delta_{M_4}^h(\mathcal{T}^4)} + \mathcal{O}(k^{-2}) \tag{4.10c}$$

$$(\lambda_h^{-1})_N{}^M = (-1)^{-\Delta_{(\mathrm{t})}^h(M) - \Delta_{M_1}^h(\mathcal{T}^1) + \Delta_{M_4}^h(\mathcal{T}^4)} \bar{\psi}(\mathrm{t}, h)_N (-1)^{2L_{h,\infty}^{ab} \mathcal{T}_a^1 \mathcal{T}_b^2} \psi(\mathrm{t}, h)^M \quad . \tag{4.10d}$$

The t-channel result in (4.10c,d) was obtained using

$$\begin{aligned} 2L_{h,\infty}^{ab} [\mathcal{T}_a^1 \mathcal{T}_b^2 + \mathcal{T}_a^2 \mathcal{T}_b^3 + \mathcal{T}_a^3 \mathcal{T}_b^1] \psi(\rho, h)^M \\ = L_{h,\infty}^{ab} [\mathcal{T}_a^4 \mathcal{T}_b^4 - \mathcal{T}_a^1 \mathcal{T}_b^1 - \mathcal{T}_a^2 \mathcal{T}_b^2 - \mathcal{T}_a^3 \mathcal{T}_b^3] \psi(\rho, h)^M \end{aligned} \tag{4.11}$$

which follows from (4.3d) and the h-global Ward identity in (4.3e).

Combining (4.10) with the limiting y behavior (3.11), (3.14) of the g-blocks, we then find the limiting behaviour of the coset blocks

$$\mathcal{C}^{(\mathrm{s})}_{g/h}(y)_m{}^M \underset{y\to 0}{\sim} e(\mathrm{s},g/h)_m{}^M y^{\Delta^{g/h}_{(\mathrm{s})}(m,M)-\Delta^{g/h}_{M_1}(\mathcal{T}^1)-\Delta^{g/h}_{M_2}(\mathcal{T}^2)} + \mathcal{O}(k^{-2}) \tag{4.12a}$$

$$\mathcal{C}^{(\mathrm{u})}_{g/h}(y)_m{}^M \underset{y\to 1}{\sim} e(\mathrm{u},g/h)_m{}^M y^{\Delta^{g/h}_{(\mathrm{u})}(m,M)-\Delta^{g/h}_{M_1}(\mathcal{T}^1)-\Delta^{g/h}_{M_3}(\mathcal{T}^3)} + \mathcal{O}(k^{-2}) \tag{4.12b}$$

$$\mathcal{C}^{(\mathrm{t})}_{g/h}(y)_m{}^M \underset{y\to \infty}{\sim} (\lambda_{g/h})_m{}^M y^{\Delta^{g/h}_{(\mathrm{t})}(m,M)+\Delta^{g/h}_{M_1}(\mathcal{T}^1)-\Delta^{g/h}_{M_4}(\mathcal{T}^4)} + \mathcal{O}(k^{-2}) \tag{4.12c}$$

$$(\lambda_{g/h})_m{}^M = (\lambda_g)_m{}^n e(\mathrm{t},g/h)_n{}^N (\lambda_h^{-1})_N{}^M \tag{4.12d}$$

$$\Delta^{g/h}_{(\rho)}(m,M) = \Delta^g_{(\rho)}(m) - \Delta^h_{(\rho)}(M) \tag{4.12e}$$

$$\Delta^{g/h}_{M_i}(\mathcal{T}^i) = \Delta^g(\mathcal{T}^i) - \Delta^h_{M_i}(\mathcal{T}^i) \quad , \quad i = 1\ldots 4 \tag{4.12f}$$

and integer-spaced secondaries for each block as in (3.10). The g/h conformal weights in (4.12e) and (4.12f) are the correct conformal weights of the intermediate and external coset-broken affine primary fields.

We finally note that the number $B_{g/h}(\rho)$ of coset blocks in the ρ-channel,

$$B_{g/h}(\rho) = d_g(\rho)\cdot d_h(\rho) \tag{4.13}$$

is the product of the dimensions $d_g(\rho)$ and $d_h(\rho)$ of the g- and h-invariants in that channel. In fact $d_h(\rho) \geq d_g(\rho)$ because $h \subset g$, so that the inequality

$$B_{g/h}(\rho) \geq B_g(\rho) \tag{4.14}$$

is obtained for comparison of correlators with fixed external g-irreps, where $B_g(\rho)$ in (3.18) is the number of affine-Sugawara blocks in the ρth channel. The result (4.14) is in accord with the intuitive expectation that the number of blocks grows with increased symmetry breaking.

Crossing relations

Following the development of the previous section we find the crossing relations for the embedding matrix and the (inverse) h-blocks,

$$e(\rho,g/h)_m{}^M = X_g(\rho\sigma)_m{}^n e(\sigma,g/h)_n{}^N X_h^{-1}(\rho\sigma)_N{}^M \tag{4.15a}$$

$$(\mathcal{F}^{(\rho)}_h(y)^{-1})_M{}^N = X_h(\rho\sigma)_M{}^P(\mathcal{F}^{(\sigma)}_h(y)^{-1})_P{}^Q X_h^{-1}(\rho\sigma)_Q{}^N \tag{4.15b}$$

where $X_g(\rho\sigma)$ are the g-crossing matrices (3.19b), and $X_h(\rho\sigma)$ are the corresponding h-crossing matrices,

$$X_h(\rho\sigma)_M{}^N = \bar\psi(\rho,h)_M \psi(\sigma,h)^N + \mathcal{O}(k^{-2}) \tag{4.16a}$$

$$X_h^{-1}(\rho\sigma)_M{}^N = X_h(\sigma\rho)_M{}^N = (X_h(\rho\sigma)_N{}^M)^* \tag{4.16b}$$

which are also unitary. Using (3.19a) and (4.15) we obtain the crossing relation of the coset blocks,

$$\mathcal{C}^{(\rho)}_{g/h}(y)_m{}^M = X_g(\rho\sigma)_m{}^n \mathcal{C}^{(\sigma)}_{g/h}(y)_n{}^N X_h^{-1}(\rho\sigma)_N{}^M \tag{4.17}$$

which involve, as expected, the crossing matrices X_g and X_h of g and of h.

The h-crossing matrices satisfy the same consistency relations,

$$X_h(\rho\sigma)X_h(\sigma\tau)X_h(\tau\rho) = X_h(\rho\tau)X_h(\tau\sigma)X_h(\sigma\rho) = 1 \tag{4.18}$$

which were seen for the g-crossing matrices in (3.20).

When the external g-irreps satisfy $\mathcal{T}^2 \sim \mathcal{T}^3$, we find that $X_h(\mathrm{us})^2 = 1$ and $\mathcal{F}_h^{(\mathrm{u})}(y) = \mathcal{F}_h^{(\mathrm{s})}(1-y)$, as for the g-blocks. Together with the corresponding relations for the g-quantities in this case, this implies

$$e(\mathrm{u}, g/h) = e(\mathrm{s}, g/h) \quad , \quad \mathcal{C}^{(\mathrm{u})}_{g/h}(y) = \mathcal{C}^{(\mathrm{s})}_{g/h}(1-y) \tag{4.19}$$

and finally,

$$\mathcal{C}^{(\mathrm{s})}_{g/h}(1-y)_m{}^M = X_g(\mathrm{su})_m{}^n \mathcal{C}^{(\mathrm{s})}_{g/h}(y)_n{}^N X_h(\mathrm{su})_N{}^M \tag{4.20}$$

so that the coset blocks are closed under crossing in this case, as expected.

Fixed external h representations

The crossing relations (4.17) of the coset blocks mix the internal h irreps (M) which arise from different external irreps of h (that is, the h-irreps which arise from the h-decomposition of the g irreps $\mathcal{T}^i$).

To obtain blocks characterized by fixed external irreps of h, we introduce a hermitean projection operator $P_h = P(T^{h1}, T^{h2}, T^{h3}, T^{h4})$ to select any four external h irreps of interest,

$$\psi(\rho,h)_a^{\tilde{M}} \bar{\psi}(\rho,h)_{\tilde{M}}^{\beta} = (P_h)_\alpha^\beta \tag{4.21a}$$

$$P_h \psi(\rho,h)^M = \psi(\rho,h)^{\tilde{M}} \delta_{\tilde{M}}^M \tag{4.21b}$$

$$[L^{ab}_{h,\infty} \mathcal{T}^i_a \mathcal{T}^j_b, P_h] = 0 \quad , \quad 1 \le i,j \le 4 \tag{4.21c}$$

where $\tilde{M}$ runs over the eigenvectors associated to the fixed external set of h irreps. The corresponding set of coset blocks is

$$(\mathcal{C}_{g/h})_m{}^{\tilde{M}} = (\mathcal{F}_g)_m{}^n e(g/h)_n{}^N (\mathcal{F}_h^{-1})_N{}^{\tilde{M}} = (\mathcal{F}_g)_m{}^n e(g/h)_n{}^{\tilde{N}} (\mathcal{F}_h^{-1})_{\tilde{N}}{}^{\tilde{M}} \tag{4.22}$$

and the h-crossing matrices are block diagonal under this decomposition,

$$\begin{aligned} X_h^{-1}(\rho\sigma)_M{}^{\tilde{N}} &= \bar{\psi}(\sigma,h)_M \psi(\rho,h)^{\tilde{N}} = \bar{\psi}(\sigma,h)_M P_h \psi(\rho,h)^{\tilde{N}} \\ &= \delta_M^{\tilde{M}} \bar{\psi}(\sigma,h)_{\tilde{M}} \psi(\rho,h)^{\tilde{N}} = \delta_M^{\tilde{M}} X_h^{-1}(\rho\sigma)_{\tilde{M}}{}^{\tilde{N}} \quad . \end{aligned} \tag{4.23}$$

Then, it follows from (4.22) and (4.23) that

$$\mathcal{C}^{(\rho)}_{g/h}(y)_m{}^{\tilde{M}} = X_g(\rho\sigma)_m{}^n \mathcal{C}^{(\sigma)}_{g/h}(y)_n{}^{\tilde{N}} X_h^{-1}(\rho\sigma)_{\tilde{N}}{}^{\tilde{M}} \tag{4.24}$$

which shows that the selected subset of coset blocks is closed under crossing

The explicit form of these projection operators can be quite complicated in the general case, but there are some simple, highly symmetric cases, where the form is quite simple. As an example, consider the situation when each of the four external g-irreps branches into a single h-irrep, so that the g/h-broken conformal weights of g-irrep $\mathcal{T}^i$ are degenerate,

$$(L^{ab}_{g/h,\infty}\mathcal{T}^i_a\mathcal{T}^i_b)_\alpha{}^\beta = \Delta^{g/h}(\mathcal{T}^i)\delta^\beta_\alpha \quad , \quad i = 1\dots 4 \quad . \tag{4.25}$$

In this case, all the coset-broken components of the g-irrep $\mathcal{T}^i$ are on an equal footing, and one may choose the trivial projector

$$P_h = 1 \quad . \tag{4.26}$$

This is the situation, e.g., in

$$\mathcal{T} = (\mathcal{T}_1, 1) \quad \text{in} \quad \frac{g_{x_1} \times g_{x_2}}{g_{x_1+x_2}} \tag{4.27}$$

examples of which were studied in Ref.[14]. Examples on simple g include

$$\mathcal{T} = n \text{ or } \bar{n} \quad \text{in} \quad \frac{SU(n)_x}{SO(n)_{2x}} = \begin{cases} \frac{SU(3)_x}{SU(2)_{4x}} & , \quad n = 3 \\ \frac{SU(n)_x}{SO(n)_{2x}} & , \quad n \geq 4 \end{cases} \tag{4.28a}$$

$$\mathcal{T} = 2n \quad \text{in} \quad \frac{SO(2n)_x}{SO(n)_x \times SO(n)_x} \tag{4.28b}$$

and the case $n = 3$ of (4.28a) will be considered in detail in Appendix C. In (4.28a) the n of $SU(n)$ is the n of $SO(n) \subset SU(n)$, while in (4.28b) the $2n$ of $SO(2n)$ is the (n, n) of $(SO(n) \times SO(n)) \subset SO(2n)$.

4.2 Non-chiral coset correlators

To construct a set of high-level non-chiral correlators for the coset constructions, we take the s-channel diagonal construction,

$$Y_{g/h}(P_h|y^*, y) = \sum_{m,\tilde{M}} |\mathcal{C}^{(\mathrm{s})}_{g/h}(y)_m{}^{\tilde{M}}|^2 \tag{4.29}$$

which shows trivial monodromy around $y = 0$. To see that (4.29) has trivial monodromy around $y = 1$ and $y = \infty$, one uses the crossing relations (4.24) of the coset blocks to rewrite the coset correlator (4.29) in the two alternate forms,

$$Y_{g/h}(P_h|y^*, y) = \sum_{m,\tilde{M}} |\mathcal{C}^{(\mathrm{u})}_{g/h}(y)_m{}^{\tilde{M}}|^2 = \sum_{m,\tilde{M}} |\mathcal{C}^{(\mathrm{t})}_{g/h}(y)_m{}^{\tilde{M}}|^2 \quad . \tag{4.30}$$

Using completeness and the explicit form (4.9) of the coset blocks, the summed form of these coset correlators is

$$\begin{aligned} Y_{g/h}(P_h|y^*, y) =& \mathrm{Tr}\Big\{[\mathbb{1} + 2L^{ab}_{g/h,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y^* + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y^*))]I_g \\ & \times[\mathbb{1} + 2L^{ab}_{g/h,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y))]P_h\Big\} + \mathcal{O}(k^{-2}) \end{aligned} \tag{4.31a}$$

$$= \mathrm{Tr}[(\mathbb{1} + 2L^{ab}_{g/h,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln|y|^2 + \mathcal{T}^1_a\mathcal{T}^3_b \ln|1-y|^2))I_g P_h] + \mathcal{O}(k^{-2}) \tag{4.31b}$$

where I_g is the projector onto the G-invariant subspace and P_h is the projector onto the desired set of external h representations. To obtain the second form, which explicitly shows two of the trivial monodromies, we used eqs. (3.6) and (4.21c).

5 A Simple Class of Correlators in ICFT

5.1 $L(g;H)$-degenerate states and correlators

In this section, we use the intuition gained in our discussion of the affine-Sugawara and coset constructions above to identify what we believe to be the simplest, most highly symmetric processes in ICFT.

In the first place , we restrict our attention to the ICFTs with a symmetry, that is, to the H-invariant CFTs on g, whose inverse inertia tensors L_H satisfy

$$\omega(H)L_H\,\omega(H)^{-1} = L_H \quad , \quad \omega(H) \in H \tag{5.1}$$

where $H \subset G$ is any subgroup of G, including finite groups and the Lie groups. The matrix $\omega(H)_a{}^b$ is in the adjoint of g. For the H-invariant CFTs, the conformal weight matrix of irrep $\mathcal{T}$ of g and hence the broken conformal weights $\Delta^H_\alpha(\mathcal{T})$ are H-invariant,

$$\Omega(H,\mathcal{T})L^{ab}_H\,\mathcal{T}_a\mathcal{T}_b\;\Omega^{-1}(H,\mathcal{T}) = L^{ab}_H\,\mathcal{T}_a\mathcal{T}_b \quad , \quad \Omega(H,\mathcal{T}) \in H \tag{5.2a}$$

$$\Omega(H,\mathcal{T})_\alpha{}^\beta[\Delta^H_\alpha(\mathcal{T}) - \Delta^H_\beta(\mathcal{T})] = 0 \tag{5.2b}$$

where $\Omega(H,\mathcal{T})_\alpha{}^\beta$ is in irrep $\mathcal{T}$ and we have used (2.6) to obtain (5.2b).

In the H-invariant CFTs, we further restrict ourselves to the most symmetric broken affine primary fields, that is, to the irreps $\mathcal{T}$ of g whose L^{ab}-broken conformal weights $\Delta^H_\alpha(\mathcal{T}) = \Delta^H(\mathcal{T})$, $\alpha = 1\ldots\dim\mathcal{T}$ are completely degenerate,

$$(L^{ab}_H\mathcal{T}_a\mathcal{T}_b)_\alpha{}^\beta = \Delta^H(\mathcal{T})\delta^\beta_\alpha \quad . \tag{5.3}$$

In what follows, such irreps of g are called the $L(g;H)$-degenerate states because, in these cases, the irrep of g decomposes into a unique irrep of H. Finally, we restrict the discussion to the $L(g;H)$-degenerate processes, which are those correlators all of whose external states are $L(g;H)$-degenerate. In this sense, the $L(g;H)$-degenerate processes are the most symmetric correlators in ICFT.

Although they are by no means generic, it is easy to find examples of $L(g;H)$-degenerate states in the H-invariant CFTs. The simplest cases of $L(g;H)$-degenerate states are all the affine primary states of all the affine-Sugawara constructions, which are in fact $L(g;G)$-degenerate.

Examples of $L(g;h)$-degenerate states in the g/h coset constructions include those mentioned in (4.27) and (4.28). These are RCFT examples in the Lie h-invariant CFTs, and in principle many irrational examples, beyond the coset constructions, can be found among the Lie h-invariant CFTs.

Irrational examples in the much larger set of H-invariant CFTs, beyond the Lie h-invariant CFTs, are already known, including the irrational cases [22]

$$\mathcal{T} = n \text{ or } \bar{n} \quad \text{in} \quad (SU(n)_x)^{\#}_M \tag{5.4a}$$

$$\mathcal{T} = 2n \quad \text{in} \quad (SO(2n)_x)^{\#}_M \tag{5.4b}$$

where H is a finite subgroup of $SO(n) \subset SU(n)$ and $(SO(n) \times SO(n)) \subset SO(2n)$ in (5.4a) and (5.4b) respectively. The case $n = 3$ in (5.4a) will be considered in detail in Section 6.

We should also remark that the $L(g;H)$-degenerate conformal weights of the coset examples in (4.28) and the irrational examples in (5.4) all obey the unified conformal weight formula,

$$\Delta^H_\alpha(\mathcal{T}) = \Delta^H(\mathcal{T}) = \frac{c}{2xn} \tag{5.5}$$

where x is the invariant level of g and c is the central charge, which is rational for the coset constructions and irrational for $SU(n)^{\#}_M$ and $SO(2n)^{\#}_M$. The occurence of
a) $L(g;H)$-degenerate states
b) a unified form of the conformal weights
for these rational and irrational families is not totally surprising, since both families of constructions are contained in the same (maximally-symmetric) ansatz [22] of the Virasoro master equation.

In what follows, we will find uniform formulae for the high-level conformal blocks and correlators of all possible $L(g;H)$-degenerate processes in ICFT.

5.2 Conformal blocks in ICFT

We study only the class of $L(g;H)$-degenerate correlators in the H-invariant CFTs. Fig.2 shows these correlators generically, with one degenerate conformal weight $\Delta^H_i \equiv \Delta^H(\mathcal{T}^i)$, $i = 1 \ldots 4$ for each external state.

Fig. 2. The $L(g;H)$-degenerate correlators.

In this case, we may reorganize the result (2.1) as follows,

$$Y_H^\alpha(y) = \bar{v}_g^\beta \Lambda_H(y)_\beta{}^\alpha + \mathcal{O}(k^{-2}) \tag{5.6a}$$

$$\begin{aligned}\Lambda_H(y) = \mathbb{1} &+ [L^{ab}_{H,\infty}(\mathcal{T}^1_a + \mathcal{T}^2_a)(\mathcal{T}^1_b + \mathcal{T}^2_b) - (\Delta^H_1 + \Delta^H_2)\mathbb{1}]\ln y \\ &+ [L^{ab}_{H,\infty}(\mathcal{T}^1_a + \mathcal{T}^3_a)(\mathcal{T}^1_b + \mathcal{T}^3_b) - (\Delta^H_1 + \Delta^H_3)\mathbb{1}]\ln(1-y)\end{aligned} \tag{5.6b}$$

$$Y_H \Omega(H) = Y_H \quad , \quad \Omega(H) = \prod_{i=1}^{4} \Omega(H, \mathcal{T}^i) \quad . \tag{5.6c}$$

The condition (5.6c), which enforces the H-symmetry of the system, follows from the H-invariance of the relevant matrices

$$[\Lambda_H, \Omega(H)] = 0 \tag{5.7}$$

and the fact that $\bar{v}_g$, being g-invariant, is also invariant under $\Omega(H)$.

To find $\rho =$ s, t and u-channel block bases for the conformal blocks, we introduce the H-invariant eigenvectors $\psi(\rho, H)$ of the ρ-channel,

$$L^{ab}_{H,\infty}(\mathcal{T}^1_a + \mathcal{T}^2_a)(\mathcal{T}^1_b + \mathcal{T}^2_b)\psi(\mathrm{s}, H)^M = \Delta^H_{(\mathrm{s})}(M)\psi(\mathrm{s}, H)^M \tag{5.8a}$$

$$L^{ab}_{H,\infty}(\mathcal{T}^1_a + \mathcal{T}^3_a)(\mathcal{T}^1_b + \mathcal{T}^3_b)\psi(\mathrm{u}, H)^M = \Delta^H_{(\mathrm{u})}(M)\psi(\mathrm{u}, H)^M \tag{5.8b}$$

$$L^{ab}_{H,\infty}(\mathcal{T}^2_a + \mathcal{T}^3_a)(\mathcal{T}^2_b + \mathcal{T}^3_b)\psi(\mathrm{t}, H)^M = \Delta^H_{(\mathrm{t})}(M)\psi(\mathrm{t}, H)^M \tag{5.8c}$$

$$\bar{\psi}(\rho, H)_M \psi(\rho, H)^N = \delta_M^N \quad , \quad \psi(\rho, H)^M_\alpha \bar{\psi}(\rho, H)^\beta_M = (I_H)_\alpha^\beta \tag{5.8d}$$

$$\Omega^{-1}(H)\psi(\rho, H)^M = \psi(\rho, H)^M \quad , \quad \bar{\psi}(\rho, H)_M \Omega(H) = \bar{\psi}(\rho, H)_M \tag{5.8e}$$

where $(I_H)_\alpha^\beta$ is the projector onto the H-invariant subspace of $\mathcal{T}^1 \otimes \cdots \otimes \mathcal{T}^4$. The eigenvalues $\Delta^H_{(\rho)}(M)$ are the L^{ab}-broken conformal weights of the states in the ρ-channel.

We remind the reader that the correlators (5.6) include all the correlators in H-invariant CFTs with $L(g;H)$-degenerate external states. This includes in particular all the correlators of all the affine-Sugawara constructions, in which case the eigenfunctions $\psi(\rho, H)$ may be taken as the g-invariants $v(\rho, g)$ of Section 3, and all the coset correlators whose external states are $L(g;h)$-degenerate, in which case the eigenfunctions $\psi(\rho, H)$ may be identified as the h-invariants $\psi(\rho, h)$ of Section 4.

The $\rho =$ s, t and u-channel conformal blocks $\mathcal{B}_H^{(\rho)}$ are then obtained by inserting completeness sums in (5.6), according to

$$\Lambda_H = \Lambda_H I_H = \Lambda_H \psi(\rho, H)^M \bar{\psi}(\rho, H)_M \quad , \quad \forall\, \rho \quad . \tag{5.9}$$

In this way, we obtain the three expansions,

$$Y_H^\alpha(y) = \sum_{m,M} d(\mathrm{s})^m \mathcal{B}_H^{(\mathrm{s})}(y)_m{}^M \bar{\psi}(\mathrm{s}, H)^\alpha_M \tag{5.10a}$$

$$= \sum_{m,M} d(\mathrm{u})^m \mathcal{B}_H^{(\mathrm{u})}(y)_m{}^M \bar{\psi}(\mathrm{u}, H)^\alpha_M \tag{5.10b}$$

$$= \sum_{m,M} d(\mathrm{t})^m \mathcal{B}_H^{(\mathrm{t})}(y)_m{}^M \bar{\psi}(\mathrm{t},H)_M^\alpha \tag{5.10c}$$

where the ρ-channel blocks $\mathcal{B}_H^{(\rho)}(y)$ are

$$\mathcal{B}_H^{(\rho)}(y)_m{}^M = \bar{v}(\rho,g)_m \Lambda_H(y)\psi(\rho,H)^M + \mathcal{O}(k^{-2}) \quad , \quad \rho = \mathrm{s},\mathrm{t},\mathrm{u} \tag{5.11a}$$

$$= e(\rho,H)_m{}^N \bar{\psi}(\rho,H)_N \Lambda_H(y)\psi(\rho,H)^M + \mathcal{O}(k^{-2}) \tag{5.11b}$$

$$\begin{aligned}\Lambda_H(y) = \mathbb{1} &+ [L^{ab}_{H,\infty}(\mathcal{T}_a^1+\mathcal{T}_a^2)(\mathcal{T}_b^1+\mathcal{T}_b^2) - (\Delta_1^H+\Delta_2^H)\mathbb{1}]\ln y \\ &+ [L^{ab}_{H,\infty}(\mathcal{T}_a^1+\mathcal{T}_a^3)(\mathcal{T}_b^1+\mathcal{T}_b^3) - (\Delta_1^H+\Delta_3^H)\mathbb{1}]\ln(1-y)\end{aligned} \tag{5.11c}$$

$$e(\rho,H)_m{}^M = \bar{v}(\rho,g)_m \psi(\rho,H)^M \quad . \tag{5.11d}$$

Here $e(\rho,H)$ is the embedding matrix of the g-invariants in the H-invariants. These expressions for the high-level conformal blocks of the $L(g;H)$-degenerate correlators in ICFT are among the central results of the paper.

Using (5.8) and (5.11), we find the limiting behavior of the conformal blocks,

$$\mathcal{B}_H^{(\mathrm{s})}(y)_m{}^M \underset{y\to 0}{\sim} e(\mathrm{s},H)_m{}^M y^{\Delta^H_{(\mathrm{s})}(M)-\Delta_1^H-\Delta_2^H} + \mathcal{O}(k^{-2}) \tag{5.12a}$$

$$\mathcal{B}_H^{(\mathrm{u})}(y)_m{}^M \underset{y\to 1}{\sim} e(\mathrm{u},H)_m{}^M (1-y)^{\Delta^H_{(\mathrm{u})}(M)-\Delta_1^H-\Delta_3^H} + \mathcal{O}(k^{-2}) \tag{5.12b}$$

$$\mathcal{B}_H^{(\mathrm{t})}(y)_m{}^M \underset{y\to \infty}{\sim} e(\mathrm{t},H)_m{}^N (\lambda_H)_N{}^M \left(\frac{1}{y}\right)^{\Delta^H_{(\mathrm{t})}(M)+\Delta_1^H-\Delta_4^H} + \mathcal{O}(k^{-2}) \tag{5.12c}$$

$$(\lambda_H)_N{}^M = (-1)^{\Delta^H_{(\mathrm{t})}(M)+\Delta_1^H-\Delta_4^H} \bar{\psi}(\mathrm{t},H)_N (-1)^{-2L^{ab}_{H,\infty}\mathcal{T}_a^1\mathcal{T}_b^2} \psi(\mathrm{t},H)^M \tag{5.12d}$$

and integer-spaced secondaries as in (3.10). These blocks show the correct broken affine-primary fields in each of the three channels.

As an aid to the reader, we note that the t-channel singularities (5.12c) were obtained using the g-global Ward identity on the g-invariants $\bar{v}(\rho,g)_m$,

$$\begin{aligned}&\bar{v}(\rho,g)_m L^{ab}[(\mathcal{T}_a^1+\mathcal{T}_a^2)(\mathcal{T}_b^1+\mathcal{T}_b^2) + (\mathcal{T}_a^2+\mathcal{T}_a^3)(\mathcal{T}_b^2+\mathcal{T}_b^3) + (\mathcal{T}_a^3+\mathcal{T}_a^1)(\mathcal{T}_b^3+\mathcal{T}_b^1)] \\ &= \bar{v}(\rho,g)_m L^{ab}[\mathcal{T}_a^1\mathcal{T}_b^1 + \mathcal{T}_a^2\mathcal{T}_b^2 + \mathcal{T}_a^3\mathcal{T}_b^3 + \mathcal{T}_a^4\mathcal{T}_b^4] \quad .\end{aligned} \tag{5.13}$$

This gives the equivalent form of $\bar{v}(\rho,g)_m\Lambda_H$,

$$\begin{aligned}\bar{v}(\rho,g)_m\Lambda_H(y) = \bar{v}(\rho,g)_m[\mathbb{1} &- (L^{ab}_{H,\infty}(\mathcal{T}_a^1+\mathcal{T}_a^2)(\mathcal{T}_b^1+\mathcal{T}_b^2) - (\Delta_1^H+\Delta_2^H)\mathbb{1})\ln(\frac{1}{y}-1) \\ &- (L^{ab}_{H,\infty}(\mathcal{T}_a^2+\mathcal{T}_a^3)(\mathcal{T}_b^2+\mathcal{T}_b^3) + (\Delta_1^H-\Delta_4^H)\mathbb{1})\ln(1-y)]\end{aligned} \tag{5.14}$$

which gives (5.12c,d) directly.

Number of blocks

We finally note that, for an $L(g;H)$-degenerate process, the number $B_H(\rho)$ of blocks in the ρ-channel

$$B_H(\rho) = d_g(\rho) \cdot d_H(\rho) \tag{5.15}$$

is the product of the dimension $d_g(\rho)$ of g-invariants and the dimension $d_H(\rho)$ of H-invariants in that channel. We know that $d_H(\rho) \geq d_h(\rho) \geq d_g(\rho)$ when H is a finite subgroup of the Lie group generated by $h \subset g$, and hence we obtain the double inequality

$$B_H(\rho) \geq B_{g/h}(\rho) \geq B_g(\rho) \tag{5.16}$$

for comparison of correlators with fixed external g-irreps, where $B_{g/h}(\rho)$ and $B_g(\rho)$ in (4.13) and (3.18) are the number of coset and affine-Sugawara blocks respectively in the ρth channel. This double inequality summarizes the symmetry hierarchy within the $L(g;H)$-degenerate processes, and is in accord with the expectation that the number of blocks increases with increased symmetry breakdown in ICFT.

In Section 6 and Appendix C, we study the $L(g;H)$-degenerate correlator $3\bar{3}\bar{3}3$ under the three constructions,

- the affine-Sugawara construction on $SU(3)$
- the coset construction $SU(3)/SU(2)_{\rm irr}$
- the irrational construction $SU(3)^{\#}_{M}$

to illustrate the double inequality (5.16). As discussed below, the symmetry hierarchy for these three constructions is $SU(3) \supset SU(2)_{\rm irr} \supset O$, where $SU(2)_{\rm irr}$ is the irregular embedding of $SU(2)$ in $SU(3)$ and O is the octohedral group symmetry of the irrational construction.

5.3 Crossing relations

Using the completeness relations (3.17a) and (5.8d) of the g-invariant and H-invariant eigenfunctions respectively, we verify the crossing relations among the blocks,

$$\mathcal{B}_H^{(\rho)}(y)_m{}^M = X_g(\rho\sigma)_m{}^n \mathcal{B}_H^{(\sigma)}(y)_n{}^N X_H^{-1}(\rho\sigma)_N{}^M \tag{5.17a}$$

$$X_H(\rho\sigma)_M{}^N = \bar{\psi}(\rho,H)_M \psi(\sigma,H)^N + \mathcal{O}(k^{-2}) \tag{5.17b}$$

$$X_H^{-1}(\rho\sigma)_M{}^N = X_H(\sigma\rho)_M{}^N = (X_H(\rho\sigma)_N{}^M)^* \tag{5.17c}$$

where $X_g(\rho\sigma)$ is the affine-Sugawara crossing matrix defined in (3.19) and $X_H(\rho\sigma)$ in (5.17b) is another set of unitary crossing matrices, called the H-crossing matrices, from the σ-channel to the ρ-channel.

The H-crossing matrices also satisfy the consistency relations

$$X_H(\rho\sigma)X_H(\sigma\tau)X_H(\tau\rho) = X_H(\rho\tau)X_H(\tau\sigma)X_H(\sigma\rho) = 1 \tag{5.18}$$

in analogy to those found for g and h in (3.20) and (4.18).

When the external g-irreps satisfy $\mathcal{T}^2 \sim \mathcal{T}^3$, we may take

$$\psi(\mathrm{u}, H)_\alpha^M = \psi(\mathrm{s}, H)_{\alpha'}^M \quad , \quad \bar{\psi}(\mathrm{u}, H)_M^\alpha = \bar{\psi}(\mathrm{s}, H)_M^{\alpha'} \tag{5.19}$$

and then one finds that,

$$\Lambda_H(y)_{\alpha'}{}^{\beta'} = \Lambda_H(1-y)_\alpha{}^\beta \tag{5.20a}$$

$$\mathcal{B}_H^{(\mathrm{u})}(y) = \mathcal{B}_H^{(\mathrm{s})}(1-y) \tag{5.20b}$$

$$X_H(\mathrm{su}) = X_H^{-1}(\mathrm{su}) = X_H(\mathrm{us}) \tag{5.20c}$$

where $\alpha' = (\alpha_1\alpha_3\alpha_2\alpha_4)$. It follows that the set of s-channel blocks is closed under crossing

$$\mathcal{B}_H^{(\mathrm{s})}(1-y)_m{}^M = X_g(\mathrm{us})_m{}^n \mathcal{B}_H^{(\mathrm{s})}(y)_n{}^N X_H(\mathrm{us})_N{}^M \tag{5.21}$$

as it should be in this case. Similar relations hold when any two external states are the same.

We have checked for the special case of the g/h coset constructions that the general high-level blocks (5.11) reduce precisely to the high-level coset blocks computed in (4.9). In further detail, when $L_{g/h} = L_g - L_h$ in Λ, one may factorize

$$\Lambda|_{L=L_{g/h}} = \Lambda_g \Lambda_h^{-1} \tag{5.22a}$$

$$\Lambda_g \equiv \Lambda|_{L=L_g} \quad , \quad \Lambda_h^{-1} \equiv (\Lambda|_{L=L_h})^{-1} \tag{5.22b}$$

and take $\psi(H) = \psi(h)$. Then one follows the steps of Section 4 to see that the blocks,

$$\begin{aligned}
&\mathcal{B}_H^{(\rho)}(y)_m{}^M|_{L=L_{g/h}} \\
&\quad = \bar{v}(\rho, g)_m \Lambda_g(y) v(\rho, g)^n \bar{v}(\rho, g)_n \psi(\rho, h)^N \bar{\psi}(\rho, h)_N \Lambda_h(y)^{-1} \psi(\rho, h)^M \\
&\quad = \mathcal{F}_g^{(\rho)}(y)_m{}^n e(\rho, g/h)_n{}^N (\mathcal{F}_h^{(\rho)}(y)^{-1})_N{}^M \\
&\quad = \mathcal{C}_{g/h}^{(\rho)}(y)_m{}^M
\end{aligned} \tag{5.23}$$

are identical to the coset blocks in (4.9). In the same way, the H-crossing matrices X_H reduce in this case to the h-crossing matrices X_h defined in (4.16).

5.4 Non-chiral correlators in ICFT

For the general $L(g; H)$-degenerate process, we construct a set of high-level non-chiral correlators from the conformal blocks $\mathcal{B}_H^{(\rho)}(y)$ in (5.11) via the diagonal construction,

$$Y_H(y^*, y) = \sum_{m,M} |\mathcal{B}_H^{(\mathrm{s})}(y)_m{}^M|^2 \tag{5.24}$$

which shows trivial monodromy around $y=0$. Using the crossing relations (5.17) we can also rewrite this correlator in terms of u or t-channel blocks

$$Y_H(y^*,y)=\sum_{m,M}|\mathcal{B}_H^{(\mathrm{u})}(y)_m{}^M|^2=\sum_{m,M}|\mathcal{B}_H^{(t)}(y)_m{}^M|^2 \tag{5.25}$$

which show trivial monodromy around $y=0$ and $y=\infty$ respectively.

Using completeness and the explicit form (5.11) of the conformal blocks, we also obtain the summed form of the non-chiral correlators

$$\begin{aligned}Y_H(y^*,y)=&\mathrm{Tr}\{[1\!\!1+2L^{ab}_{H,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b\ln y^*+\mathcal{T}^1_a\mathcal{T}^3_b\ln(1-y^*))]I_g\\&\times[1\!\!1+2L^{ab}_{H,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b\ln y+\mathcal{T}^1_a\mathcal{T}^3_b\ln(1-y))]\}+\mathcal{O}(k^{-2})\end{aligned} \tag{5.26a}$$

$$=\mathrm{Tr}[(1\!\!1+2L^{ab}_{H,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b\ln|y|^2+\mathcal{T}^1_a\mathcal{T}^3_b\ln|1-y|^2))I_g]+\mathcal{O}(k^{-2}) \tag{5.26b}$$

where the last form shows two of the trivial monodromies. One also sees the expected crossing symmetry

$$Y_H(1-y^*,1-y)=Y_H(y^*,y) \tag{5.27}$$

when $\mathcal{T}^2\sim\mathcal{T}^3$. We finally note that the general $L(g;H)$-degenerate correlators (5.26) correctly include the $L(g;h)$-degenerate coset correlators obtained from (4.31) when $P_h=1$.

Using the embedding matrices (5.11d) and the H-crossing matrices (5.17b), Appendix A gives alternate expressions for the blocks and correlators of the $L(g;H)$-degenerate processes in ICFT.

6 Blocks and Correlators in $\mathrm{SU(3)}^{\#}_{\mathrm{M}}$

As an explicit example in irrational conformal field theory, we work out here the high-level conformal blocks and non-chiral correlators for a particular $L(g;H)$-degenerate process in the unitary irrational level-family [22]

$$(SU(3)_x)^{\#}_M \tag{6.1}$$

where x is the invariant level of $SU(3)$. For simplicity below, this construction is often called $SU(3)^{\#}_M$. The construction is included in the larger maximally-symmetric ansatz for all simply-laced g, which was in fact the first set of ICFTs found in the Virasoro master equation. The closely related coset construction $SU(3)_x/SU(2)_{4x}$, which also resides in the maximally-symmetric ansatz, is studied in Appendix C.

The exact forms of the central charge and the conformal weights of the 3 and $\bar{3}$ representations under $(SU(3)_x)^{\#}_M$ are

$$c[(SU(3)_x)^{\#}_M]=\frac{2x}{x+3}\left[2-\frac{x^2-8x+17}{\sqrt{4x^4-28x^3+17x^2+160x-128}}\right] \tag{6.2a}$$

$$\Delta(\mathcal{T}_{(3)}) = \Delta(\mathcal{T}_{(\bar{3})}) = \frac{c}{6x} \tag{6.2b}$$

where the 3-fold degenerate conformal weights in (6.2b) strongly suggest that the 3 and $\bar{3}$ are $L(g;H)$-degenerate representations.

As discussed further in Appendix B, the level-family $(SU(3)_x)^{\#}_M$ has a finite group symmetry

$$H(SU(3)^{\#}_M) = O \subset SU(2)_{\rm irr} \tag{6.3}$$

where O is the octohedral group and $SU(2)_{\rm irr}$ is the irregularly embedded $SU(2) \subset SU(3)$. The degeneracy of the 3 and $\bar{3}$ is due to the octohedral symmetry of the construction, which mixes all three components of each representation. Thus the 3 and $\bar{3}$ are $L(SU(3);O)$-degenerate representations in $(SU(3)_x)^{\#}_M$, as desired.

For the high-level computations in $(SU(3)_x)^{\#}_M$ below, we need only the high-level forms of the inverse inertia tensor (in the Gell-Mann basis) and the degenerate conformal weights,

$$L^{ab}_{O,\infty} = \frac{1}{x\psi_g^2}\theta_a\delta_{ab} \quad , \quad \theta_a = \begin{cases} 1 & a = 1,4,6 \\ 0 & a = 3,8,2,5,7 \end{cases} \tag{6.4a}$$

$$c = 3 + \mathcal{O}(x^{-1}) \tag{6.4b}$$

$$\Delta^O(\mathcal{T}_{(3)}) = \Delta^O(\mathcal{T}_{(\bar{3})}) = \frac{1}{2x} + \mathcal{O}(x^{-2}) \tag{6.4c}$$

which identifies $P^{ab} = \theta_a\delta_{ab}$ as the high-level projector of $SU(3)^{\#}_M$. Moreover, we will consider only the $L(SU(3);O)$-degenerate process $3\bar{3}\bar{3}3$ in $SU(3)^{\#}_M$,

$$\mathcal{T}^1 = \mathcal{T}^4 = \mathcal{T}_{(3)} \quad , \quad \mathcal{T}^2 = \mathcal{T}^3 = \mathcal{T}_{(\bar{3})} \tag{6.5}$$

shown schematically in Fig.3. The matrix irrep of the 3 and $\bar{3}$ in the Gell-Mann basis are given by,

$$\mathcal{T}_{(3)} = \frac{\sqrt{\psi_g^2}}{2}\lambda_a \quad , \quad \mathcal{T}_{(\bar{3})} = \frac{\sqrt{\psi_g^2}}{2}\bar{\lambda}_a \tag{6.6a}$$

$$\bar{\lambda}_a = -\lambda_a^T = \begin{cases} -\lambda_a & a = 3,8,1,4,6 \\ \lambda_a & a = 2,5,7 \end{cases} \tag{6.6b}$$

where λ_a are the Gell-Mann matrices.

Fig. 3. An $L(SU(3);O)$-degenerate correlator in $SU(3)^{\#}_M$.

To compute the high-level blocks in the s-channel, we need to solve the eigenvalue problem (5.8a) for the s-channel O-invariant eigenvectors $\psi(\mathrm{s},O)$, which reads in this case,

$$[-\frac{1}{2x}\sum_{a=1,4,6}\lambda_a^1\lambda_a^2+\frac{1}{x}\mathbb{1}]_\alpha{}^\beta\,\psi(\mathrm{s},O)_\beta^M=\Delta_{(\mathrm{s})}^O(M)\psi(\mathrm{s},O)_\alpha^M \tag{6.7a}$$

$$\prod_{i=1}^4(\omega_l^i)_{\alpha_i}{}^{\beta_i}\psi(\mathrm{s},O)_\beta=\psi(\mathrm{s},O)_\alpha \quad , \quad l=1,2 \tag{6.7b}$$

$$\omega_1=\exp(i\pi\lambda_2/2)=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix} \tag{6.7c}$$

$$\omega_2=\exp(i\pi\lambda_5/2)\exp(i\pi\lambda_7/2)=\begin{pmatrix}0&-1&0\\0&0&1\\-1&0&0\end{pmatrix} \quad . \tag{6.7d}$$

The matrices ω_1 and ω_2 which appear in the O-invariance condition (6.7b) may be taken as the generators of O.

After some algebra, one finds the following orthonormal set of s-channel eigenvectors $\psi(\mathrm{s},O)^M$ and their eigenvalues $\Delta_{(\mathrm{s})}^O(M)$,

$$\psi(\mathrm{s},O)_\alpha^1=\frac{1}{3}\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} \quad , \quad \Delta_{(\mathrm{s})}^O(1)=0 \tag{6.8a}$$

$$\psi(\mathrm{s},O)_\alpha^2=\frac{1}{2\sqrt{3}}[\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}+\delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3}-2\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}\delta_{\alpha_1\alpha_3}] \quad , \quad \Delta_{(\mathrm{s})}^O(2)=\frac{1}{2x} \tag{6.8b}$$

$$\psi(\mathrm{s},O)_\alpha^3=\frac{1}{2\sqrt{3}}[\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}-\delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3}] \quad , \quad \Delta_{(\mathrm{s})}^O(3)=\frac{3}{2x} \tag{6.8c}$$

$$\psi(\mathrm{s},O)_\alpha^4=\frac{1}{3\sqrt{2}}[\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}-3\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}\delta_{\alpha_1\alpha_3}] \quad , \quad \Delta_{(\mathrm{s})}^O(4)=\frac{3}{2x} \tag{6.8d}$$

$$\bar{\psi}(\mathrm{s},O)_M^\alpha=(\psi(\mathrm{s},O)_\beta^M)^*\eta^{\beta\alpha}=\psi(\mathrm{s},O)_\alpha^M \tag{6.8e}$$

where the last relation says that the left and right eigenvectors coincide in this case.

In ICFT, the high-level fusion rules [15] of the broken affine primaries follow the Clebsch-Gordan coefficients of their corresponding matrix irreps, so the s-channel should show the exchange of broken affine primary states corresponding to the vacuum and the adjoint representation,

$$3\otimes\bar{3}=1\oplus 8 \quad . \tag{6.9}$$

Indeed, the first conformal weight in (6.8a) is the conformal weight of the vacuum, and the other three high-level conformal weights in (6.8b-d) are precisely the high-level form of the three degenerate subsets of broken conformal weights of the adjoint (see Appendix B).

Similarly, we can solve for the u and t-channel eigenvectors, which are given by

$$\psi(\mathrm{u},O)^M=\psi(\mathrm{s},O)^M|_{2\leftrightarrow 3} \quad , \quad \Delta_{(\mathrm{u})}^O(M)=\Delta_{(\mathrm{s})}^O(M) \tag{6.10a}$$

$$\psi(\mathrm{t},O)^M = \psi(\mathrm{s},O)^M|_{2\leftrightarrow 4} \quad , \quad \Delta^O_{(\mathrm{t})}(M) = \frac{2}{x} - \Delta^O_{(\mathrm{s})}(M) \tag{6.10b}$$

where $2 \leftrightarrow 3$ and $2 \leftrightarrow 4$ mean respectively $\alpha_2 \leftrightarrow \alpha_3$ and $\alpha_2 \leftrightarrow \alpha_4$ in the explicit expressions of the s-channel eigenvectors (6.8). The result in (6.10a) is in accord with (5.19) since $\mathcal{T}^2 \sim \mathcal{T}^3$, so that the u-channel conformal weights are identical to the ones in the s-channel. The conformal weights found in the t-channel,

$$\Delta^O_{(\mathrm{t})}(M) = (\frac{2}{x}, \frac{3}{2x}, \frac{1}{2x}, \frac{1}{2x}) \tag{6.11}$$

are also in agreement with the L^{ab}-broken conformal weights in the known high-level fusion rule

$$3 \otimes 3 = \bar{3} \oplus 6 \quad . \tag{6.12}$$

In particular, the last value in (6.11) is the completely degenerate conformal weight of the $\bar{3}$ and the first three coincide with the three degenerate subsets (B.11b) of the 6, according to the high-level form (B.13a).

Using eq.(5.17b), the high-level s-u and s-t O-crossing matrices are computed from the eigenvectors as

$$X_O(\mathrm{us})_M{}^N = \psi(\mathrm{u},O)^M \psi(\mathrm{s},O)^N = \frac{1}{6}\begin{pmatrix} 2 & 2\sqrt{3} & 2\sqrt{3} & -2\sqrt{2} \\ 2\sqrt{3} & 3 & -3 & \sqrt{6} \\ 2\sqrt{3} & -3 & 3 & \sqrt{6} \\ -2\sqrt{2} & \sqrt{6} & \sqrt{6} & 4 \end{pmatrix} \tag{6.13a}$$

$$X_O(\mathrm{ts})_M{}^N = \psi(\mathrm{t},O)^M \psi(\mathrm{s},O)^N = \frac{1}{6}\begin{pmatrix} 2 & 2\sqrt{3} & -2\sqrt{3} & -2\sqrt{2} \\ 2\sqrt{3} & 3 & 3 & \sqrt{6} \\ -2\sqrt{3} & 3 & 3 & -\sqrt{6} \\ -2\sqrt{2} & \sqrt{6} & -\sqrt{6} & 4 \end{pmatrix} \tag{6.13b}$$

which are orthogonal and idempotent matrices in this case. The third O-crossing matrix

$$X_O(\mathrm{ut}) = X_O(\mathrm{us}) X_O(\mathrm{ts}) \tag{6.14}$$

follows from the consistency relation (5.18).

For the crossing of the blocks one also needs the high-level affine-Sugawara crossing matrices (3.19b) for $g = SU(3)$. The ρ-channel $SU(3)$-invariant eigenvectors and the corresponding crossing matrices are

$$v(\mathrm{s}, SU(3))^V_\alpha = \frac{1}{3}\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} \tag{6.15a}$$

$$v(\mathrm{s}, SU(3))^A_\alpha = \frac{1}{2\sqrt{2}}[\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4} - \frac{1}{3}\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}] \tag{6.15b}$$

$$v(\mathrm{u}, SU(3)) = v(\mathrm{s}, SU(3))|_{2\leftrightarrow 3} \tag{6.15c}$$

$$v(\mathrm{t}, SU(3))^6_\alpha = \frac{1}{2\sqrt{6}}[\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} + \delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}] \tag{6.15d}$$

$$v(\mathrm{t},SU(3))^{\bar{3}}_{\alpha} = \frac{1}{2\sqrt{3}}[\delta_{\alpha_1\alpha_2}\delta_{a_3\alpha_4} - \delta_{\alpha_1\alpha_3}\delta_{a_2\alpha_4}] \tag{6.15e}$$

$$X_{SU(3)}(\mathrm{us})_m{}^n = v(\mathrm{u},SU(3))^m v(\mathrm{s},SU(3))^n = \frac{1}{3}\begin{pmatrix} 1 & 2\sqrt{2} \\ 2\sqrt{2} & -1 \end{pmatrix} \tag{6.15f}$$

$$X_{SU(3)}(\mathrm{ts})_m{}^n = v(\mathrm{t},SU(3))^m v(\mathrm{s},SU(3))^n = \frac{1}{3}\begin{pmatrix} \sqrt{6} & \sqrt{3} \\ \sqrt{3} & -\sqrt{6} \end{pmatrix} \tag{6.15g}$$

where the labels V, A stand for vacuum and adjoint irrep, and $6, \bar{3}$ for symmetric and antisymmetric irrep. The third g-crossing matrix is given by $X_{SU(3)}(\mathrm{ut}) = X_{SU(3)}(\mathrm{us})$ $X^{-1}_{SU(3)}(\mathrm{ts})$.

Finally, we write down the high-level s-channel conformal blocks (5.11) of $SU(3)^{\#}_{M}$,

$$\mathcal{B}^{(\mathrm{s})}_{O}(y)_m{}^M = e(\mathrm{s},O)_m{}^N[\mathbb{1} + (\Delta^{O}_{(\mathrm{s})} - \frac{1}{x}\mathbb{1})\ln y + (Q^{O}_{(\mathrm{su})} - \frac{1}{x}\mathbb{1})\ln(1-y)]_N{}^M + \mathcal{O}(x^{-2}) \tag{6.16}$$

where

$$e(\mathrm{s},O)_m{}^M = v(\mathrm{s},SU(3))^m \psi(\mathrm{s},O)^M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4}\sqrt{6} & \frac{1}{4}\sqrt{6} & -\frac{1}{2} \end{pmatrix} \tag{6.17a}$$

$$(\Delta^{O}_{(\mathrm{s})})_N{}^M = \Delta^{O}_{(\mathrm{s})}(M)\delta^M_N \tag{6.17b}$$

$$(Q^{O}_{(\mathrm{su})})_N{}^M = \sum_L X_O(\mathrm{us})_N{}^L \Delta^{O}_{(\mathrm{u})}(L) X_O(\mathrm{us})_L{}^M \tag{6.17c}$$

$$\Delta^{O}_{(\mathrm{s})}(M) = \Delta^{O}_{(\mathrm{u})}(M) = (0, \frac{1}{2x}, \frac{3}{2x}, \frac{3}{2x}) \quad . \tag{6.17d}$$

Here we have used the alternate expression (A.9) for the $L(g;H)$-degenerate blocks in Appendix A. The u and t-channel blocks can be computed from the s-channel blocks above using the crossing relation (5.17) and the explicit forms of the crossing matrices $X_{SU(3)}(\mathrm{us}), X_{SU(3)}(\mathrm{ts})$ in (6.15), and $X_O(\mathrm{us}), X_O(\mathrm{ts})$ in (6.13).

In agreement with (5.15), the number of blocks for this $L(SU(3);O)$-invariant process is

$$B_O(\rho) = 2 \cdot 4 = 8 \quad . \tag{6.18}$$

Because of the increasing symmetry breakdown,

$$O \subset SU(2)_{\mathrm{irr}} \subset SU(3) \tag{6.19}$$

the number (6.18) is larger than the number of blocks

$$B_{SU(3)}(\rho) = 2 \cdot 2 = 4 \quad , \quad B_{SU(3)/SU(2)} = 2 \cdot 3 = 6 \tag{6.20}$$

for the same correlator under the affine-Sugawara construction and the closely related coset construction studied in Appendix C. Taken together, (6.18) and (6.20) are an illustration of the double inequality (5.16).

Using eqs.(A.13), (A.14) we also find the following expression for the high-level non-chiral correlators of $SU(3)^{\#}_M$,

$$Y_O(y^*,y)=\sum_{M,N} E(\mathrm{s},O)_M{}^N[\mathbb{1}+(\Delta^O_{(\mathrm{s})}-\frac{1}{x}\mathbb{1})\ln|y|^2+(\Delta^O_{(\mathrm{s})}-\frac{1}{x}\mathbb{1})\ln|1-y|^2]_N{}^M+\mathcal{O}(x^{-2}) \tag{6.21a}$$

$$E(\mathrm{s},O)_M{}^N=\sum_m (e(\mathrm{s},O)_m{}^M)^* e(\mathrm{s},O)_m{}^N=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{3}{8} & \frac{3}{8} & -\frac{1}{8}\sqrt{6} \\ 0 & \frac{3}{8} & \frac{3}{8} & -\frac{1}{8}\sqrt{6} \\ 0 & -\frac{1}{8}\sqrt{6} & -\frac{1}{8}\sqrt{6} & \frac{1}{4} \end{pmatrix} \tag{6.21b}$$

where we have used $X_O(\mathrm{us})E(\mathrm{s},O)X_O(\mathrm{us}) = E(\mathrm{s},O)$ and the diagonal s-channel conformal weight matrix $\Delta^O_{(\mathrm{s})}$ is given in (6.17b). This result explicitly shows the crossing symmetry (5.27), as it should since $\mathcal{T}^2 \sim \mathcal{T}^3$ in this case.

We finally remark that the high-level blocks and correlators of the K-conjugate theory

$$SU(3)/SU(3)^{\#}_M \quad , \quad \tilde{L}=L_{SU(3)}-L \tag{6.22}$$

can be easily obtained from the results above, by substituting everywhere the K-conjugate conformal weights $\tilde{\Delta}(\mathcal{T})=\Delta^g(\mathcal{T})-\Delta(\mathcal{T})$ for the conformal weights $\Delta(\mathcal{T})$. Moreover, the results above can easily be extended to the $L(g;H)$-degenerate correlators $n\bar{n}\bar{n}n$ in the larger family of ICFTs called $SU(n)^{\#}_M$ [22]; in this case, the number of H-invariant tensors stays the same, with closely analogous forms for all the more general results.

7 Conclusions

The generalized KZ equations of ICFT provide a uniform description of the chiral correlators of rational and irrational conformal field theory, and the solution of these equations is known at high level on simple g. The apparent simplicity of this result is deceptive, however, because the solution describes a vast variety of generically irrational conformal field theories ranging from the most symmetric (the RCFTs) to totally asymmetric (the generic ICFT).

In this paper, we have begun the resolution of the high-level chiral correlators into high-level conformal blocks and non-chiral correlators, beginning with the simplest and most symmetric classes.

In particular, we began by working out the high-level blocks and correlators of all the

- affine-Sugawara constructions on simple g
- coset constructions on simple g.

Both results are new, and the results for the cosets are apparently inaccessible by other methods.

Based on this analysis, we then identified what we believe to be the simplest and most symmetric class of correlators in ICFT. These are the

- $L(g;H)$-degenerate processes in H-invariant CFTs on simple g

which are those correlators whose external states have entirely degenerate conformal weights $\Delta_\alpha = \Delta$. This class of correlators includes all the affine-Sugawara correlators, a highly-symmetric subset of coset correlators and a presumably large set of irrational correlators, examples of which are known.

For this simple class of correlators we were able to find the general expression for the high-level blocks and non-chiral correlators, and we worked out an irrational example with octohedral symmetry on $SU(3)$.

Further information is needed to go beyond the leading order of the $L(g;H)$-degenerate processes in ICFT. The central question here is whether the number of conformal blocks remains finite, as we found in the semi-classical limit, or increases with the order of k^{-1}, as expected generically in ICFT. At finite values of the level, one will also need to consider the roles of the affine cutoff [4,13] and fixed-point resolution [23].

The more immediate open direction is to find the high-level conformal blocks of irrational correlators beyond the set of $L(g;H)$-degenerate processes. Here one also expects an ever-increasing number of blocks as one confronts the progressively larger symmetry breakdown of ICFT, signalled by the L^{ab}-broken conformal weights Δ_α.

In this direction, we remind the reader of the known singularities of the invariant flat connections W which govern the exact (finite level) correlators of ICFT. For example, it is known that [8]

$$W(\tilde{u},u)_\alpha{}^\beta \underset{\tilde{u},u\to 0}{=} \left(\frac{u}{\tilde{u}}\right)^{\Delta_{\alpha_1}(\mathcal{T}^1)+\Delta_{\alpha_2}(\mathcal{T}^2)-\Delta_{\beta_1}(\mathcal{T}^1)-\Delta_{\beta_2}(\mathcal{T}^2)} \frac{(2L^{ab}\mathcal{T}_a^1\mathcal{T}_b^2)_\alpha{}^\beta}{u} \tag{7.1a}$$

$$= \begin{cases} \frac{(2L_\infty^{ab}\mathcal{T}_a^1\mathcal{T}_b^2)_\alpha{}^\beta}{u} + \mathcal{O}(k^{-2}) & \text{(high } k\text{)} \quad (7.1\text{b}) \\ \frac{(2L^{ab}\mathcal{T}_a^1\mathcal{T}_b^2)_\alpha{}^\beta}{u} & (L(g;H)\text{-degenerate}) \quad (7.1\text{c}) \end{cases}$$

where u and $\tilde{u}$ are the variables of the theory and its K-conjugate theory respectively. The result (7.1a) shows the apparently non-Fuchsian α,β dependent shielding factor, which is hidden in the high-level limit (7.1b), and which simplifies to unity at all levels, shown in (7.1c), for the $L(g;H)$-degenerate processes. We believe that this phenomenon underlies the simplicity of the class of $L(g;H)$-degenerate processes in ICFT, and it may be necessary to consider this factor in the physical interpretation of the high-level logarithmic singularities of correlators beyond the simple class we have considered here.

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Appendix A: Alternate expressions for blocks and correlators

In this appendix, we use the relevant crossing matrices to give alternate expressions for the conformal blocks and correlators of any set of external states in the affine-Sugawara constructions (see Section 3) and of any $L(g;H)$-degenerate process in the more general H-invariant CFTs (see Section 5).

Affine-Sugawara constructions

We begin with the ρ-channel affine-Sugawara blocks in (3.17),

$$\mathcal{F}_g^{(\rho)}(y)_m{}^n = \bar{v}(\rho,g)_m[\mathbb{1} + 2L^{ab}_{g,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y))]v(\rho,g)^n + \mathcal{O}(k^{-2}) \quad . \tag{A.1}$$

Using the definitions (3.2), (3.12) of the g-invariant ρ-channel eigenvectors $v(\rho,g)$ and the g-crossing matrices X_g in (3.19b), we have the g-crossing relations,

$$\bar{v}(\rho,g)_m = X_g(\rho\sigma)_m{}^n\bar{v}(\sigma,g)_n \quad , \quad v(\rho,g)^m = v(\sigma,g)^n X_g(\sigma\rho)_n{}^m \quad . \tag{A.2}$$

Using these relations, we obtain the alternate form of the affine-Sugawara blocks,

$$\begin{aligned}\mathcal{F}_g^{(\rho)}(y)_m{}^n = [\mathbb{1} &+ (Q^g_{(\rho\mathrm{s})} - (\Delta^g(\mathcal{T}^1)+\Delta^g(\mathcal{T}^2))\mathbb{1})\ln y \\ &+ (Q^g_{(\rho\mathrm{u})} - (\Delta^g(\mathcal{T}^1)+\Delta^g(\mathcal{T}^3))\mathbb{1})\ln(1-y)]_m{}^n + \mathcal{O}(k^{-2})\end{aligned} \tag{A.3}$$

where

$$(Q^g_{(\rho\sigma)})_m{}^n = \begin{cases} (\Delta^g_{(\rho)})_m{}^n = \Delta^g_{(\rho)}(m)\delta^n_m & , \quad \rho=\sigma \\ \sum_l X_g(\rho\sigma)_m{}^l\Delta^g_{(\sigma)}(l)X_g(\sigma\rho)_l{}^n & , \quad \rho\neq\sigma \end{cases} \quad . \tag{A.4}$$

and $\Delta^g_{(\rho)}$ are the ρ-channel affine-Sugawara conformal weights.

In this form, one immediately sees the correct s and u-channel singularities as given in (3.11) and (3.14a) respectively. To obtain a form which shows the t-channel singularities

(3.14b) more explicitly, one again uses the g-global Ward identity (3.15) on $\bar{v}_g$ to rewrite the t-channel blocks as

$$\begin{aligned}\mathcal{F}_g^{(\mathrm{t})}(y)_m{}^n = [1\!\!1 - (Q^g_{(\mathrm{ts})} - (\Delta^g(\mathcal{T}^1)+\Delta^g(\mathcal{T}^2))1\!\!1)\ln(\frac{1}{y}-1) \\ - (Q^g_{(\mathrm{tt})} + (\Delta^g(\mathcal{T}^1)-\Delta^g(\mathcal{T}^4))1\!\!1)\ln(1-y)]_m{}^n + \mathcal{O}(k^{-2})\end{aligned} \tag{A.5}$$

where $Q^g_{(\mathrm{t}\sigma)}$, $\sigma = \mathrm{s},\mathrm{t}$ is given in (A.4). The equivalence of the two forms (A.3) and (A.5) of the t-channel blocks can be verified directly, using the ρ=t form of the conformal weight sum rule,

$$\Delta^g_{(\rho)} + X_g(\rho\sigma)\Delta^g_{(\sigma)}X_g(\sigma\rho) + X_g(\rho\tau)\Delta^g_{(\tau)}X_g(\tau\rho) = \sum_{i=1}^4 \Delta^g(\mathcal{T}^i) \quad , \quad \rho\neq\sigma\neq\tau\neq\rho \tag{A.6}$$

which is itself a direct consequence of the g-global Ward identity (3.15).

Substitution of the alternate forms (A.3) of the affine-Sugawara blocks in the expression (3.26) for the affine-Sugawara correlators then gives the corresponding alternate form for the non-chiral correlators,

$$\begin{aligned}Y_g(y^*,y)_\alpha{}^\beta = &\sum_{m,n}[1\!\!1 + (Q^g_{(\rho\mathrm{s})} - (\Delta^g(\mathcal{T}^1)+\Delta^g(\mathcal{T}^2))1\!\!1)\ln|y|^2 \\ &+ (Q^g_{(\rho\mathrm{u})} - (\Delta^g(\mathcal{T}^1)+\Delta^g(\mathcal{T}^3))1\!\!1)\ln|1-y|^2]_m{}^n\, v(\rho,g)_\alpha^m \bar{v}(\rho,g)_n^\beta + \mathcal{O}(k^{-2})\end{aligned} \tag{A.7}$$

which explicitly shows trivial monodromy around $y=0$ and 1.

$L(g;H)$-degenerate processes

Following the development for the affine-Sugawara constructions above, we may find similar alternate forms for the blocks and correlators of the general $L(g;H)$-degenerate process.

Using the definitions (5.8) of the H-invariant eigenvectors $\psi(\rho,H)$ and the H-crossing matrices X_H in (5.17), we have the H-crossing relations,

$$\bar{\psi}(\rho,H)_M = X_H(\rho\sigma)_M{}^N\bar{\psi}(\sigma,H)_N \quad , \quad \psi(\rho,H)^M = \psi(\sigma,H)^N X_H(\sigma\rho)_N{}^M \quad . \tag{A.8}$$

Using these relations in the form (5.11b), we obtain the following alternate form of the $L(g;H)$-degenerate blocks,

$$\begin{aligned}\mathcal{B}_H^{(\rho)}(y)_m{}^M = e(\rho,H)_m{}^N[1\!\!1 + (Q^H_{(\rho\mathrm{s})} - (\Delta_1^H+\Delta_2^H)1\!\!1)\ln y \\ + (Q^H_{(\rho\mathrm{u})} - (\Delta_1^H+\Delta_3^H)1\!\!1)\ln(1-y)]_N{}^M + \mathcal{O}(k^{-2})\end{aligned} \tag{A.9}$$

where $e(\rho,H)$ are the ρ-channel embedding matrices (5.11d) and

$$(Q^H_{(\rho\sigma)})_M{}^N = \begin{cases} (\Delta^H_{(\rho)})_M{}^N = \Delta^H_{(\rho)}(M)\delta_M^N & , \quad \rho=\sigma \\ \sum_L X_H(\rho\sigma)_M{}^L\Delta^H_{(\sigma)}(L)X_H(\sigma\rho)_L{}^N & , \quad \rho\neq\sigma \end{cases} \quad . \tag{A.10}$$

with $\Delta^H_{(\rho)}$ the ρ-channel conformal weights. These results include all the correlators of the affine-Sugawara constructions and all the $L(g;h)$-degenerate processes of the g/h coset constructions.

These forms explicitly show the correct s and u-channel singularities (5.12a) and (5.12b). To obtain a form which shows the t-channel singularities (5.12c,d) more explicitly, one uses again the g-global Ward identity (5.13) to rewrite the t-channel blocks as

$$\begin{aligned}\mathcal{B}_H^{(\mathrm{t})}(y)_m{}^n = e(\mathrm{t},H)_m{}^N[\mathbb{1} - (Q^H_{(\mathrm{ts})} - (\Delta_1^H + \Delta_2^H)\mathbb{1})\ln(\frac{1}{y}-1) \\ - (Q^H_{(\mathrm{tt})} + (\Delta_1^H - \Delta_4^H)\mathbb{1})\ln(1-y)]_N{}^M + \mathcal{O}(k^{-2})\end{aligned} \tag{A.11}$$

where $Q^H_{(\mathrm{t}\sigma)}$, $\sigma = \mathrm{s},\mathrm{t}$ is given in (A.10). This equivalent form of the t-channel blocks follows directly from the ρ =t form of the conformal weight sum rule in $L(g;H)$-degenerate processes,

$$\begin{aligned}e(\rho,H)[\Delta^H_{(\rho)}+X_H(\rho\sigma)\Delta^H_{(\sigma)}X_H(\sigma\rho) + X_H(\rho\tau)\Delta^H_{(\tau)}X_H(\tau\rho)] \\ = e(\rho,H)\sum_{i=1}^{4}\Delta_i^H \qquad , \qquad \rho\neq\sigma\neq\tau\neq\rho\end{aligned} \tag{A.12}$$

which is itself a direct consequence of the g-global Ward identity (5.13).

Finally, we give the corresponding alternate form of the non-chiral correlators (5.24), using the expression (A.9) for the blocks,

$$\begin{aligned}Y(y^*,y) = \sum_{M,N} E(\rho,H)_M{}^N[\mathbb{1} + (Q^H_{(\rho\mathrm{s})} - (\Delta_1^H + \Delta_2^H)\mathbb{1})\ln|y|^2 \\ + (Q^H_{(\rho\mathrm{u})} - (\Delta_1^H + \Delta_3^H)\mathbb{1})\ln|1-y|^2]_N{}^M + \mathcal{O}(k^{-2})\end{aligned} \tag{A.13}$$

where

$$E(\rho,H)_M{}^N = \sum_m (e(\rho,H)_m{}^M)^* e(\rho,H)_m{}^N \quad . \tag{A.14}$$

This form of the correlator explicitly shows the two trivial monodromies around $y=0$ and 1.

Appendix B: The level-families $\mathbf{SU(3)^\#_M}$ and $\mathbf{SU(3)/SU(2)_{irr}}$

In this appendix we review [22,16] various results for the unitary irrational level-family

$$(SU(3)_x)^\#_M \tag{B.1}$$

and the closely-related level-family of the coset construction

$$\frac{SU(3)}{SU(2)_{\mathrm{irr}}} = \frac{SU(3)_x}{SU(2)_{4x}} \tag{B.2}$$

both of which occur in the maximally-symmetric ansatz on $SU(3)$. $SU(2)_{\rm irr}$ denotes the irregularly embedded $SU(2)$ subgroup of $SU(3)$ generated by $J_{2,5,7}$. The results given here are used in Section 6 and Appendix C.

In the (Cartesian) Gell-Mann basis (6.6), the maximally-symmetric construction $(SU(3)_x)^{\#}_M$ has the form [22]

$$L^{ab} = \frac{1}{\psi_g^2}\ell_a\delta^{ab} \quad , \quad \ell_a = \begin{cases} \ell_c & a=3,8 \\ \ell_h & a=2,5,7 \\ \ell_r & a=1,4,6 \end{cases} \tag{B.3a}$$

$$T = \frac{1}{\psi_g^2} {}^*_*[\ell_c(J_3^2+J_8^2)+\ell_h(J_2^2+J_5^2+J_7^2)+\ell_r(J_1^2+J_4^2+J_6^2)]{}^*_* \tag{B.3b}$$

$$c = x(2\ell_c + 3\ell_h + 3\ell_r) \tag{B.3c}$$

where T is the stress tensor, ψ_g is the highest root of $SU(3)$, x is the affine level and c is the central charge. The exact form of c is given in (6.2a), but we refer to [22] for the exact[b] forms of the coefficients $\ell_{c,h,r}$. The construction above includes the coset construction $SU(3)/SU(2)_{\rm irr}$ as a special case when the further symmetry relation

$$\ell_c = \ell_r \equiv \ell_{g/h} \tag{B.4}$$

is obeyed.

$SU(3)^{\#}_M$ is an H-invariant CFT with symmetry group [16]

$$H(SU(3)^{\#}_M) = O = \text{ octohedral group} \subset SU(2)_{\rm irr} \tag{B.5}$$

where O is the octohedral group (rotational symmetry group of the cube, with order 24) and $SU(2)_{\rm irr}$ is the irregular embedding of $SU(2)$ in $SU(3)$. The octohedral group includes the elements

$$\Omega_{(2)} = \exp(i\frac{\pi}{\sqrt{\psi_g^2}}J_2(0)) \quad , \quad \Omega_{(5)} = \exp(i\frac{\pi}{\sqrt{\psi_g^2}}J_5(0)) \quad , \quad \Omega_{(7)} = \exp(i\frac{\pi}{\sqrt{\psi_g^2}}J_7(0)) \tag{B.6}$$

where $J_a(0)$ are the zero modes of the currents J_a, and in particular we may take the two elements ω_1 and ω_2,

$$\omega_1 = \Omega_{(2)} \quad , \quad \omega_2 = \Omega_{(5)}\Omega_{(7)} \tag{B.7}$$

which satisfy

$$\omega_1^4 = 1 \quad , \quad \omega_2^3 = 1 \tag{B.8a}$$

$$\omega_1\omega_2^2\omega_1 = \omega_2 \quad , \quad \omega_1\omega_2\omega_1 = \omega_2\omega_1^2\omega_2 \tag{B.8b}$$

as the generators of the octohedral group.

The coset construction $SU(3)/SU(2)_{\rm irr}$ has the larger Lie group symmetry

$$H(SU(3)/SU(2)_{\rm irr}) = SU(2)_{\rm irr} \tag{B.9}$$

[b]The relation to the notation of Ref.[22] is $\ell_c = 3\lambda$, $\ell_h = (L_- - L_+)/2$ and $\ell_r = (L_- + L_+)/2$.

because of the symmetry relation (B.4).

The 3 and $\bar{3}$ are $L(g;H)$-degenerate irreps of $SU(3)^{\#}_{M}$ and $SU(3)/SU(2)_{\rm irr}$ with completely degenerate conformal weights,

$$\Delta(\mathcal{T}_{(3)}) = \Delta(\mathcal{T}_{(\bar{3})}) = \frac{c}{6x} \qquad (3) \tag{B.10}$$

where the number in parentheses denotes the degeneracy.

For $(SU(3)_x)^{\#}_{M}$ one also finds the L^{ab}-broken conformal weights of the 8 (adjoint) and 6 (symmetric),

$$\Delta(\mathcal{T}_{(8)}) = \begin{cases} \frac{3}{2}(\ell_h + \ell_r) & (2) \\ \ell_c + \frac{1}{2}(\ell_h + 3\ell_r) & (3) \\ \ell_c + \frac{1}{2}(3\ell_h + \ell_r) & (3) \end{cases} \tag{B.11a}$$

$$\Delta(\mathcal{T}_{(6)}) = \Delta(\mathcal{T}_{(\bar{6})}) = \begin{cases} \frac{2}{3}(2\ell_c + 3\ell_r) & (1) \\ \frac{4}{3}\ell_c + \frac{1}{2}(3\ell_h + \ell_r) & (2) \\ \frac{1}{3}\ell_c + \frac{3}{2}(\ell_h + \ell_r) & (3) \end{cases} \quad . \tag{B.11b}$$

These forms show that the 8 and the 6 each split into three subsets of degenerate weights, in agreement with the block analysis of Section 6.

For $SU(3)_x/SU(2)_{4x}$ the splitting is reduced to two subsets,

$$\Delta_{g/h}(\mathcal{T}_{(8)}) = \begin{cases} \frac{3}{2}(\ell_h + \ell_{g/h}) & (5) \\ \frac{1}{2}(\ell_h + 5\ell_{g/h}) & (3) \end{cases} \tag{B.12a}$$

$$\Delta_{g/h}(\mathcal{T}_{(6)}) = \Delta_{g/h}(\mathcal{T}_{(\bar{6})}) = \begin{cases} \frac{10}{3}\ell_{g/h} & (1) \\ \frac{3}{2}\ell_h + \frac{11}{6}\ell_{g/h} & (5) \end{cases} \tag{B.12b}$$

according to (B.4) and (B.11). These forms are in agreement with the coset block analysis of Appendix C.

For the computations of Section 6 and Appendix C, we need the high-level forms of the two constructions,

$$(SU(3)_x)^{\#}_{M} \quad : \ell_r = \frac{1}{x} + \mathcal{O}(x^{-2}) \quad , \quad \ell_c = \ell_h = \mathcal{O}(x^{-2}) \quad , \quad c = 3 + \mathcal{O}(x^{-1}) \tag{B.13a}$$

$$\frac{SU(3)_x}{SU(2)_{4x}} \quad : \ell_{g/h} = \frac{1}{x} + \mathcal{O}(x^{-2}) \quad , \quad \ell_h = \mathcal{O}(x^{-2}) \quad , \quad c = 5 + \mathcal{O}(x^{-1}) \tag{B.13b}$$

which can be used with (B.10), (B.11) and (B.12) to obtain the high-level forms of all the quantities discussed in this appendix.

Appendix C: Blocks and correlators in $\mathbf{SU(3)/SU(2)_{irr}}$

As an explicit example in rational conformal field theory, we work out in this appendix the high-level conformal blocks and correlators of a particular $L(g;h)$-degenerate process in the level-family of the coset construction

$$\frac{g}{h} = \frac{SU(3)_x}{SU(2)_{4x}} = \frac{SU(3)}{SU(2)_{\rm irr}} \tag{C.1}$$

which is included in the family of coset examples (4.28a).

This level-family has the Lie symmetry $SU(2)_{\rm irr}$, and the 3 and $\bar{3}$ representations are $L(SU(3);SU(2)_{\rm irr})$-degenerate.

For the high-level computations in $SU(3)/SU(2)_{\rm irr}$ below, we need the high-level form of the inverse inertia tensor (in the Gell-Mann basis) and the degenerate conformal weights,

$$L^{ab}_{g/h,\infty} = \frac{1}{x\psi_g^2}\theta_a\delta_{ab} \quad , \quad \theta_a = \begin{cases} 1 & a = 3,8,1,4,6 \\ 0 & a = 2,5,7 \end{cases} \tag{C.2a}$$

$$\Delta^{g/h}(\mathcal{T}_{(3)}) = \Delta^{g/h}(\mathcal{T}_{(\bar{3})}) = \frac{5}{6x} + \mathcal{O}(x^{-2}) \tag{C.2b}$$

and we will consider here the same process, that is $3\bar{3}\bar{3}3$, which we studied for $SU(3)^{\#}_{M}$ in Section 6.

To compute the high-level blocks in the s-channel, we need to determine the s-channel eigenvectors $\psi(\mathrm{s},SU(2))$ from the eigenvalue problem (5.8), which reads in this case,

$$[-\frac{1}{2x}\sum_{a=\substack{3,8\\1,4,6}} \lambda^1_a\lambda^2_a + \frac{5}{3x}\mathbb{1}]_\alpha{}^\beta\ \psi(\mathrm{s},SU(2))^M_\beta = \Delta^{g/h}_{(\mathrm{s})}(M)\psi(\mathrm{s},SU(2))^M_\alpha \tag{C.3a}$$

$$\sum_{i=1}^{4}(\lambda^i_a)_\alpha{}^\beta\psi(\mathrm{s},SU(2))_\beta = 0 \quad , \quad a = 2,5,7 \quad . \tag{C.3b}$$

Here we have used the properties (6.6b) of the Gell-Mann matrices, and the global condition (C.3b) enforces the $SU(2)_{\rm irr}$-invariance of the coset construction.

After some algebra, the following orthonormal set of s-channel eigenvectors is found

$$\psi(\mathrm{s},SU(2))^1_\alpha = \frac{1}{3}\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} \quad , \quad \Delta^{g/h}_{(\mathrm{s})}(1) = 0 \tag{C.4a}$$

$$\psi(\mathrm{s},SU(2))^2_\alpha = \frac{1}{2\sqrt{5}}[\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4} + \delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3} - \frac{2}{3}\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}] \quad , \quad \Delta^{g/h}_{(\mathrm{s})}(2) = \frac{3}{2x} \tag{C.4b}$$

$$\psi(\mathrm{s},SU(2))^3_\alpha = \frac{1}{2\sqrt{3}}[\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4} - \delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3}] \quad , \quad \Delta^{g/h}_{(\mathrm{s})}(3) = \frac{5}{2x} \tag{C.4c}$$

$$\bar{\psi}(\mathrm{s},SU(2))^\alpha_M = (\psi(\mathrm{s},SU(2))^M_\beta)^*\eta^{\beta\alpha} = \psi(\mathrm{s},SU(2))^M_\alpha \tag{C.4d}$$

where the last relation says that the left and right eigenvectors coincide.

The results (C.4) are in agreement with the high-level fusion rule (6.9); in particular, the conformal weight in (C.4a) corresponds to the vacuum, while the remaining two weights in (C.4b,c) are the high-level form of the two degenerate subsets of the coset-broken conformal weights of the adjoint (see eqs.(B.12a) and (B.13b)).

Similarly, we can solve for the u and t-channel eigenvectors, which are given by

$$\psi(\mathrm{u},SU(2))^M=\psi(\mathrm{s},SU(2))^M|_{2\leftrightarrow 3}\quad,\quad \Delta^{g/h}_{(\mathrm{u})}(M)=\Delta^{g/h}_{(\mathrm{s})}(M) \tag{C.5a}$$

$$\psi(\mathrm{t},SU(2))^M=\psi(\mathrm{s},SU(2))^M|_{2\leftrightarrow 4}\quad,\quad \Delta^{g/h}_{(\mathrm{t})}(M)=\frac{10}{3x}-\Delta^{g/h}_{(\mathrm{s})}(M) \tag{C.5b}$$

where $2\leftrightarrow 3$ and $2\leftrightarrow 4$ mean $\alpha_2\leftrightarrow\alpha_3$ and $\alpha_2\leftrightarrow\alpha_4$ in the explicit expressions (C.4) for the s-channel eigenvectors. Since $\mathcal{T}^2\sim\mathcal{T}^3$, the result in (C.5a) is a special case of (5.19) and the u-channel conformal weights are identical to those in the s-channel. The conformal weights of the t-channel,

$$\Delta^{g/h}_{(\mathrm{t})}(M)=(\frac{11}{6x},\frac{10}{3x},\frac{5}{6x}) \tag{C.6}$$

are also in agreement with the coset-broken conformal weights of the high-level fusion rule (6.12). In particular, the last value is the completely degenerate conformal weight of the $\bar{3}$ and the first two coincide with the two degenerate subsets (B.12b) of the 6, according to the high-level form (B.13b).

Using (5.17b), the $SU(2)$-crossing matrices are computed from the eigenvectors as

$$X_{SU(2)}(\mathrm{us})_M{}^N=\psi(\mathrm{u},SU(2))^M\psi(\mathrm{s},SU(2))^N=\frac{1}{6}\begin{pmatrix}2&2\sqrt{5}&2\sqrt{3}\\2\sqrt{5}&1&-\sqrt{15}\\2\sqrt{3}&-\sqrt{15}&3\end{pmatrix} \tag{C.7a}$$

$$X_{SU(2)}(\mathrm{ts})_M{}^N=\psi(\mathrm{t},SU(2))^M\psi(\mathrm{s},SU(2))^N=\frac{1}{6}\begin{pmatrix}2&2\sqrt{5}&-2\sqrt{3}\\2\sqrt{5}&1&\sqrt{15}\\-2\sqrt{3}&\sqrt{15}&3\end{pmatrix} \tag{C.7b}$$

which are orthogonal and idempotent matrices in this case. The third crossing matrix $X_{SU(2)}(\mathrm{ut})=X_{SU(2)}(\mathrm{us})X_{SU(2)}(\mathrm{ts})$ follows from the consistency relations (5.18).

Finally, we use the $SU(3)$ eigenvectors (6.15) and the alternate expression (A.9) for the general $L(g;H)$-degenerate blocks to write down the s-channel coset blocks of $SU(3)/SU(2)_{\mathrm{irr}}$,

$$\mathcal{C}^{(\mathrm{s})}_{g/h}(y)_m{}^M=e(\mathrm{s},g/h)_m{}^N[\mathbb{1}+(\Delta^{g/h}_{(\mathrm{s})}-\frac{5}{3x}\mathbb{1})\ln(y)+(Q^{g/h}_{(\mathrm{su})}-\frac{5}{3x}\mathbb{1})\ln(1-y))]_N{}^M+\mathcal{O}(x^{-2}) \tag{C.8}$$

where

$$e(\mathrm{s},g/h)_m{}^M=v(\mathrm{s},SU(3))^m\psi(\mathrm{s},SU(2))^M=\begin{pmatrix}1&0&0\\0&\frac{1}{4}\sqrt{10}&\frac{1}{4}\sqrt{6}\end{pmatrix} \tag{C.9a}$$

$$(\Delta^{g/h}_{(s)})_N{}^M=\Delta^{g/h}_{(s)}(M)\delta_N^M \tag{C.9b}$$

$$(Q^{g/h}_{(\mathrm{su})})_N{}^M=\sum_L X_{SU(2)}(\mathrm{us})_N{}^L\Delta^{g/h}_{(\mathrm{u})}(L)X_{SU(2)}(\mathrm{us})_L{}^M \tag{C.9c}$$

$$\Delta^{g/h}_{(\mathrm{s})}(M)=\Delta^{g/h}_{(\mathrm{u})}(M)=(0,\frac{3}{2x},\frac{5}{2x})\quad. \tag{C.9d}$$

The u and t-channel blocks can be computed from the s-channel blocks above using the crossing relation (5.17a) and the explicit forms of the crossing matrices $X_{SU(3)}(\mathrm{us})$, $X_{SU(3)}(\mathrm{ts})$ in (6.15) and $X_{SU(2)}(\mathrm{us}), X_{SU(2)}(\mathrm{ts})$ in (C.7).

In accord with (4.13), the number of blocks in this process is

$$B_{SU(3)/SU(2)}(\rho) = 2 \cdot 3 = 6 \tag{C.10}$$

while the same process under the affine-Sugawara construction on $SU(3)$ and the irrational construction $SU(3)^{\#}_{M}$ showed 4 and 8 blocks respectively. This is in accord with the double inequality (5.16) and the increasing symmetry breakdown $O \subset SU(2)_{\mathrm{irr}} \subset SU(3)$ of the three constructions.

Using eqs.(A.13), (A.14), we also find the following expression for the high-level non-chiral correlators of $SU(3)/SU(2)_{\mathrm{irr}}$,

$$Y_{g/h}(y^*, y) = \sum_{M,N} E(\mathrm{s}, g/h)_M{}^N [\mathbb{1} + (\Delta^{g/h}_{(\mathrm{s})} - \frac{5}{3x}\mathbb{1}) \ln |y|^2 + (\Delta^{g/h}_{(\mathrm{s})} - \frac{5}{3x}\mathbb{1}) \ln |1-y|^2]_N{}^M + \mathcal{O}(x^{-2}) \tag{C.11a}$$

$$E(\mathrm{s}, g/h)_M{}^N = \sum_m (e(\mathrm{s}, g/h)_m{}^M)^* e(\mathrm{s}, g/h)_m{}^N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{5}{8} & \frac{1}{8}\sqrt{15} \\ 0 & \frac{1}{8}\sqrt{15} & \frac{3}{8} \end{pmatrix} \tag{C.11b}$$

where we have used $X_{SU(2)}(\mathrm{us}) E(\mathrm{s}, g/h) X_{SU(2)}(\mathrm{us}) = E(\mathrm{s}, g/h)$ and where the diagonal s-channel conformal weight matrix $\Delta^{g/h}_{(\mathrm{s})}$ is given in (C.9b). This result explicitly shows the crossing symmetry (5.27), as it should since $\mathcal{T}^2 \sim \mathcal{T}^3$ in this case.

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