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Exact Solution of Discrete Two-Dimensional R^2 Gravity

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We exactly solve a special matrix model of dually weighted planar graphs describing pure two-dimensional quantum gravity with an R^2 interaction. It permits us to study the intermediate regimes between the gravitating and flat metric. Flat space is modeled by a regular square lattice, while localised curvature is introduced through lattice defects. No "flattening" phase transition is found with respect to the R^2 coupling: the infrared behaviour of the system is that of pure gravity for any finite R^2 coupling. In the limit of infinite coupling, we are able to extract a scaling function interpolating between pure gravity and a dilute gas of curvature defects on a flat background. We introduce and explain some novel techniques concerning our method of large N character expansions and the calculation of Schur characters on big Young tableaux.
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1. Introduction and overview

Two-dimensional random geometry is now placed at the heart of many models of modern physics, from string theory and two-dimensional quantum gravity, attempting to describe fundamental interactions, to membranes and interface fluctuations in various problems of condensed matter physics. Therefore it is quite important to study in detail the basic universal properties of random geometries, such as their fractal nature, their phase diagram, critical phenomena and correlations of physical quantities of geometric origin. In addition, one might hope to discover entirely new mechanisms for phase transitions.

Over the last fifteen years, considerable progress has been achieved in the understanding of noncritical strings, or 2D quantum gravity in the presence of matter fields with central charge $c \leq 1$. It was based on two different approaches, completing and justifying each other.

The first one, based on the continuous treatment of 2D metric fluctuations by means of quantum Liouville theory, was introduced by Polyakov [3].

The second one, based on a discretisation of the two-dimensional metric in terms of random planar graphs, was first proposed in [4], [5] and [6].

With the help of powerful matrix model techniques this approach allowed for the first time the exact calculation of critical exponents in quantum gravity; both for pure 2D gravity without matter [4],[5], and, subsequently, for various forms of conformal matter with $c \leq 1$ (e.g. $c = -2$ [7] and $c = \frac{1}{2}$ [8]). These results were successfully confirmed and supplemented by the continuous approach in [9], and then in [10].

Following this breakthrough, numerous further developments have led to a rather full understanding of the subject.

Still, we think that some important questions were left unclarified. In this paper, we are addressing one of them: what will happen in the case of pure 2D gravity if we introduce coupling constants favoring the flat configurations among the ensemble of 2D metrics? Can we achieve a transition to a new phase of essentially flat metrics? Or shall we always find – on sufficiently big distances – the familiar behaviour of 2D gravity, whatever these new couplings are? If the last scenario is true, do we have some universality for intermediate scales, when the size of the 2D universe is of the order of the flattening scale?

On first sight, this question appears to have a simple answer, at least from the point of view of the continuous theory. The Euclidean path integral is

$$\mathcal{Z} = \int \mathcal{D}g_{ab} \; e^{-\int d^2z \sqrt{\det g} \; (\mu + \alpha R_g + \frac{1}{\beta_0} R_g^2 + \ldots)}. \tag{1.1}$$

Apart from the cosmological term (controlled by μ) and the (topological) Einstein term proportional to the curvature R_g, it does not seem to be meaningful to put further terms into the action of 2D gravity on dimensional grounds. The simplest term one might want to consider is $\frac{1}{\beta_0}R_g^2$. The bare coupling constant β_0 is however dimensionful and thus should be proportional to the cutoff squared. So it is small and in principle should be dropped, as well as any further higher derivative terms.

On the other hand, no argument has ever been given to exclude the following alternative scenario: increasing the R_g^2 coupling ($\beta_0 \to 0$ in our notation), the theory might entirely restructure *nonperturbatively.* In this case the Liouville approach, which is known to correctly quantise the theory (1.1) in the absence (i.e. $\beta_0 = \infty$) of higher derivative terms, might no longer yield a good description of the theory. Some calculational attempts to address this question within the framework of Liouville theory are inconclusive precisely because they merely perturb by the R^2 term in question (see e.g. [11],[12]).

In this paper, we address and resolve some of these questions in a completely nonperturbative way. We take as a starting point the by now very familiar discrete definition [4],[5],[6] of the path integral (1.1), known to correctly quantise the theory in the absence of higher derivative terms. We then introduce couplings whose naive continuum limit precisely induces the higher terms indicated in (1.1). Technically, this is done by solving an unusual matrix model that generates the dually weighted graphs proposed and studied in our earlier papers [1] and [2].

The paper is organized in the following way:

In section 2, the discretised model of dually weighted graphs is formulated and the main physical results, including the absence of a "flattening" phase transition, the joint scaling function of the R^2 coupling and the cosmological constant and their infrared and ultraviolet asymptotics are presented and discussed in a non-technical fashion. We physically interpret our model as a statistical mechanics system of a "gas" of point-like curvature defects surrounded by flat two-dimensional space. Curvature is quantised, i.e. localised on defects whose deficit angle equals an integer times some fixed value.

In section 3, a review of the character expansion approach to the models of dually weighted graphs (worked out in [1] and [2]) is given.

In section 4, the exact solution of the present model is derived in some detail. The solution is rather complex but nevertheless explicit. Elliptic functions, already crucial in [2], play a central role.

Section 5 demonstrates how to extract the physical results from the involved expressions of section 4.

Section 6 is devoted to conclusions and open questions. We also discuss the important open issue of the universality of our results and point out that fine-tuning the curvature weights – an in principle feasible extension of the present work – might result in new phases of 2D gravity.

Finally, appendix 1. contains the calculation of the Schur characters of $GL(N)$ in the limit of a big Young tableau while some more technical details concerning the exact solution and its verification are given in appendix 2.

2. Definition of the model and physical results

In the particular model used here the two-dimensional random geometry is described by planar graphs (with the topology of the sphere) built from plaquettes ("quadrangulations"). We are summing over all such graphs with the weights $t_2, t_4, t_6, ...,$ corresponding to vertices with 2,4,6,... neighboring plaquettes, respectively. The partition function is

$$\mathcal{Z} = \sum_G \prod_{v_{2q} \in G} {t_{2q}}^{\# v_{2q}}, \tag{2.1}$$

where v_{2q} are the vertices with coordination number $2q$ and $\# v_{2q}$ are the numbers of such vertices in the given graph G. Note that one cannot sum up these graphs by the usual one-matrix model, since we need to control both vertex and face coordination numbers. The discrete, curved manifold is thus described by a graph made from flat squares. The curvature appears exclusively in the vertices and corresponds to the deficit angles around the vertices: $\pi(2-q)$. Thus two neighboring plaquettes contribute positive curvature, four correspond to zero curvature and six and more result in localised negative curvature (see Fig 2.1).

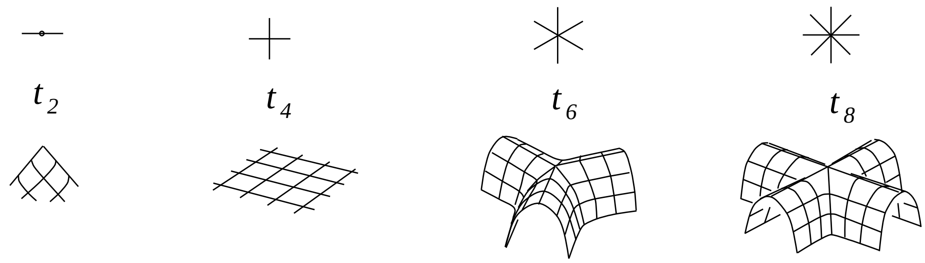

Fig. 2.1 Flat space and curvature defects.

For technical reasons, it proved useful to choose a particular parametrisation of the above weights t_{2q}:

$$t_2 = \sqrt{\lambda}\, t,\ t_4 = \lambda,\ t_6 = \lambda^{\frac{3}{2}}\ \frac{\beta^2}{t},\ ...\ t_{2q} = \lambda^{\frac{q}{2}}\ (\frac{\beta^2}{t})^{(q-2)}. \tag{2.2}$$

With these weights, it is easy to prove, using Euler's theorem, that the partition sum (2.1) becomes

$$\mathcal{Z}(t, \lambda, \beta) = t^4 \sum_G \lambda^A \; \beta^{2(\#v_2 - 4)}, \tag{2.3}$$

where A is the number of plaquettes of the graph G and $\#v_2$ the number of positive curvature defects. Note that the latter are balanced by a gas of negative curvature defects, whose individual probablities are given in (2.2).

We expect this model to describe pure gravity in a sufficiently large interval of β, after tuning the bare cosmological constant λ (controlling the number of plaquettes) to some critical value $\lambda_c(\beta)$. On the other hand, for λ fixed and $\beta = 0$ we entirely suppress curvature defects except for the four positive defects needed to close the regular lattice into a sphere. It is thus clear that β is the precise lattice analog of the bare curvature coupling β_0 in (1.1). The phase $\beta = 0$ of "almost flat" lattices – very different from pure gravity – was discussed in detail in [2].

Let us now summarize the main physical results following from the exact solution (for general λ and β) of this model:

1. A long debated question was whether models of the present type undergo a "flattening" phase transition at a finite, non-zero critical value of $\beta = \beta_c$. The weak coupling region $\beta > \beta_c$ would then correspond to the standard phase of pure gravity while a putative novel "smooth" phase of gravity might exist either at $\beta = \beta_c$ or in the entire interval $0 \leq \beta \leq \beta_c$. This would constitute an existence proof of *continuum* 2D R^2 gravity. We find, to the contrary, that *there is no "flattening" phase transition at non-zero* β. For any given β we find the powerlike scaling of standard pure gravity on large scales. This means that no matter how flat the system is on small scales (of the order of $\beta^{-\frac{1}{2}}$), it destabilizes in the infrared into the familiar ensemble of highly fractal "baby-universes".

2. The dependence of the partition sum (2.3) on β and the lattice cosmological constant λ in the vicinity of the flat phase $\beta \sim 0$ and close to $\lambda \sim \lambda_c$ is given by a simple, (presumably) universal scaling function $f(x)$ (defined through $\mathcal{Z}(t, \lambda, \beta) = \frac{4t^4}{15\beta^2} \; f(x)$) reflecting the transition from flat space to pure gravity:

$$f(x) = x^6 - \frac{5}{2}\, x^4 + \frac{15}{8}\, x^2 - \frac{5}{16} - x\, \left(x^2 - 1\right)^{\frac{5}{2}}, \tag{2.4}$$

where the scaling variable x is given, to leading order, by

$$x = \frac{\sqrt{2}}{\pi} \, \frac{1 - \lambda}{\beta}. \tag{2.5}$$

We can distinguish the following features:

(a) There is a degree $\frac{5}{2}$ singularity at $x = 1$, correctly reproducing the universal string susceptibility exponent $\gamma_s = -\frac{1}{2}$ of pure gravity [4],[5]. In view of eq.(2.5), the critical value of the lattice cosmological constant λ is therefore given to leading order by $\lambda_c = 1 - \frac{\pi}{\sqrt{2}}\beta + \mathcal{O}(\beta^2)$. Therefore, in view of (2.3), the characteristic growth of the random surfaces as a function of the lattice area A (= number of plaquettes) is given by

$$\mathcal{Z}(t, A, \beta) \sim \frac{t^4}{\beta^{\frac{9}{2}}} e^{\frac{\pi}{\sqrt{2}} \beta A} A^{-\frac{7}{2}}. \tag{2.6}$$

For any non-zero β we do have exponential growth of the number of surfaces, but one has to go to larger and larger scales (i.e. use more and more plaquettes) to be able to take the continuum limit. If β is exactly zero there is no longer any exponential growth and no pure gravity continuum limit is possible. The prefactor $\beta^{-\frac{9}{2}}$ is found in the exact calculation in section 5; we are not sure whether it is universal.

(b) We further see that taking $\beta \to 0$ *before* the limit $\lambda \to \lambda_c$ corresponds to the limit $x \to \infty$. In this limit one finds $f(x) \sim \frac{5}{128} \frac{1}{x^2} + \mathcal{O}(\frac{1}{x^3})$, that is, the characteristic critical behavior of 2D gravity "silently" disappears and we recover a power series in $\frac{1}{x}$ corresponding, in view of (2.5), to a perturbative expansion in lattice defects β. In this limit the characteristic growth of surfaces as a function of area A is

$$\mathcal{Z}(t, A, \beta) \sim t^4 \; (\; A + \mathcal{O}(\beta^2 A^3) \;). \tag{2.7}$$

The leading order corresponds precisely to the almost flat lattices (with exactly four positive defects) of [2]. The corrections are interpreted as insertions of negative defects, balanced by further positive defects. The typical shape of the surfaces in this limit is a generalisation of the one we found for "almost flat" graphs in [2]: Long, filamentary cylinders growing out from every negative curvature defect.

(c) It is easy to prove that the scaling function (2.4) is the simplest possible function with the limiting properties discussed in (a) and (b).

The above results might be interpreted in terms of a continuum model of quantised curvature defects, in which the localised defects move around like particles in a gas on a flat background space. The deficit angle, $\Delta\theta$, of a defect can take on the values $\Delta\theta = \pi, 0$ and $-\pi$. A positive curvature defect is surrounded by a conical geometry, whereas a negative curvature defect corresponds to a saddle-type insertion (see Fig. 2.1). The higher order negative curvature defects (-2π and higher) would not be expected to play a role in this limit (the entropy from moving two low order defects around would completely dominate that from a single higher order defect). The coupling β can be interpreted as a fugacity controlling the number of defects. The flat space limit $\beta \to 0$ consists of four defects of degree π moving around with respect to one another. Varying the fugacity, β, allows one to smoothly interpolate between flat space, (2.7) (with four defects), and pure gravity (2.6) (with an infinite number of defects).

3. Review of the technique

The appropriate techniques for dealing with models of dually weighted planar graphs were pioneered in two previous publications [1],[2]. Let us recall the key steps of the method and collect the crucial formulas allowing the exact solution of the model. The method is in fact applicable to a situation more general than the one written in eq.(2.1), to be analysed in the present work. The partition function of general dually weighted planar graphs G is defined to be

$$Z(t^*,t) = \sum_G \prod_{v_q^*, v_q \in G} t_q^{*\#v_q^*} \; t_q{}^{\#v_q}, \tag{3.1}$$

where v_q^*, v_q are the vertices with q neighbours on the original and dual graph, respectively, and $\#v_q^*, \#v_q$ are the numbers of such vertices in the given graph G. This expansion is generated by the following matrix model:

$$Z(t^*,t) = \int \mathcal{D}M \; e^{-\frac{N}{2}\mathrm{Tr}\; M^2 \; + \; \mathrm{Tr}\; V_B(MA)}, \tag{3.2}$$

with

$$V_B(MA) = \sum_{k=1}^{\infty} \frac{1}{k} \; \mathrm{Tr} B^k \; (MA)^k. \tag{3.3}$$

The matrices A and B are fixed, external matrices encoding the coupling constants through

$$t_q^* = \frac{1}{q}\frac{1}{N} \; \mathrm{Tr}\; B^q \qquad \text{and} \qquad t_q = \frac{1}{q}\frac{1}{N} \; \mathrm{Tr}\; A^q. \tag{3.4}$$

The model generalises, for $A \neq 1$, the standard one matrix model first solved by Brézin, Itzykson, Parisi and Zuber [14]. It may no longer be successfully treated by changing to eigenvalue variables; a reduction to N variables is nevertheless possible: One expands the potential in terms of the characters on the group:

$$e^{\Sigma_{k=1}^{\infty} \frac{1}{k} \mathrm{Tr} B^k \; \mathrm{Tr}(MA)^k} = \prod_{i,j=1}^{N} \frac{1}{(1 - B_i (MA)_j)} = \frac{1}{N^N} \sum_R \chi_R(B) \; \chi_R(MA). \tag{3.5}$$

Here B_i and $(MA)_j$ are the eigenvalues of the matrices B and MA. The first step involves rewriting the sum over k as a double sum over all the eigenvalues of the matrices B and MA of $-\ln(1 - B_i(MA)_j)$. Exponentiating the log then gives the product in the numerator. The second step uses a group theoretic result to write the product in terms of a sum over characters. The character is defined by the Weyl formula:

$$\chi_{\{h\}}(A) = \frac{\det_{(k,l)}(a_k^{h_l})}{\Delta(a)}, \tag{3.6}$$

where the set of $\{h\}$ are a set of ordered, increasing, non-negative integers, $\Delta(a)$ is the Vandermonde determinant, and the sum over R is the sum over all such sets. The R's label representations of the group $U(N)$ and the sets of integers $\{h\}$ are the usual Young tableau weights defined by $h_i = i - 1 + \#\text{boxes in row } i$ (the index i labels the rows in the Young tableau, $i = 1$ corresponding to the first row). Note that the restriction on the allowed Young tableaux that any row must have at least as many boxes as the row below implies that the $\{h_i\}$ are a set of increasing integers:

$$h_{i+1} > h_i. \tag{3.7}$$

Substituting equation (3.5) into the integral in equation (3.2), we can now do the angular integration using the key identity

$$\int (\mathcal{D}\Omega)_H \; \chi_R(\Omega M \Omega^\dagger A) = d_R^{-1} \; \chi_R(M) \; \chi_R(A), \tag{3.8}$$

where d_R is the dimension of the representation given by $d_R = \Delta(h)/\prod_{i=1}^{N-1} i!)$, and arrive, after performing a Gaussian integral over the eigenvalue degrees of freedom, at the expression

$$Z(t,t^*) = c \sum_{\{h^e,h^o\}} \frac{\prod_i (h_i^e - 1)!! h_i^o!!}{\prod_{i,j}(h_i^e - h_j^o)} \; \chi_{\{h\}}(A) \; \chi_{\{h\}}(B). \tag{3.9}$$

The sum is taken over a subclass of so-called even representations. These are defined as possessing an equal number of even weights h_i^e and odd weights h_i^o (since the mentioned Gaussian integration vanishes if the latter condition is not satisfied). The formula [15] was originally discovered by Itzykson and Di Francesco [15] by summing up "fatgraphs", using purely combinatoric and group theoretic arguments.

In our first paper [1] we demonstrated how to take the large N limit of this expansion. In this limit the weights $\frac{1}{N}h_i$ condense to give a smooth, stationary distribution $dh\,\rho(h)$, where $\rho(h)$ is a probability density normalized to one. For simplicity we restrict our attention to models in which the matrices A and B are such that traces of all odd powers of A and B are zero. This means that the our random surfaces are made from vertices and faces with even coordination number. As was discussed in [1], this ensures that the support of the density $\rho(h)$ lies entirely on the real axis and thus simplifies the solution of the problem. The matrix A will satisfy this condition if we introduce a $\frac{N}{2} \times \frac{N}{2}$ matrix $\sqrt{a}$ in terms of which A and the character $\chi_{\{h\}}(A)$ are given by

$$A = \begin{bmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{bmatrix} \quad \text{and} \quad \chi_{\{h\}}(A) = \chi_{\{\frac{h^e}{2}\}}(a)\chi_{\{\frac{h^o-1}{2}\}}(a) \; \mathrm{sgn}\Big[\prod_{i,j}(h_i^e - h_j^o)\Big]. \tag{3.10}$$

In order to study the transition from flat to random graphs we choose the potential to be

$$V_A(MA_4) = \sum_{k=1}^{\infty} \frac{1}{2k} \ \mathrm{Tr}[A^{2k}] \ (MA_4)^{2k}. \tag{3.11}$$

Here A_4 is defined to satisfy $\mathrm{Tr}[(A_4)^k] = N\delta_{k,4}$ and A is as defined in (3.10). We are thus studying surfaces made up entirely from squares. A weight $t_{2k} = Tr[A^{2k}]$ is assigned whenever $2k$ squares meet at a vertex (see Fig. 2.1). We are therefore precisely considering the situation defined in eq.(2.1). In order to investigate (3.9) in the large N limit, one attempts to locate the stationary point. This leads to the saddlepoint equation

$$2F(h) + \fint_0^a \ dh' \ \frac{\rho(h')}{h-h'} = -\ln h. \tag{3.12}$$

As has been discussed in detail in [1], this equation actually does not hold on the entire interval $[0, a]$, but only on an interval $[b, a]$ with $0 \le b \le 1 \le a$: Assuming the equation to hold on $[0, a]$ would violate the implicit constraint $\rho(h) \le 1$ following from the restriction (3.7). The density is in fact exactly saturated at its maximum value $\rho(h) = 1$ on the interval $[0, a]$. The solution of (3.12) requires the knowledge of the large N limit of the variation of the characters in eq.(3.10):

$$F(h_k) = 2\frac{\partial}{\partial h_k^e} \ \ln \ \frac{\chi_{\{\frac{h^e}{2}\}}(a)}{\Delta(h^e)}. \tag{3.13}$$

A rather general method for its determination has been one of the main technical achievements of our previous work. In fact, we found the following simple result: Introduce the character function $F(h)$ and the resolvent $H(h)$ as, respectively, the large N limit of (3.13) and

$$H(h) = \int_0^a dh' \ \frac{\rho(h')}{h-h'}. \tag{3.14}$$

We found the weight resolvent $H(h)$ to be very closely related to the standard matrix model eigenvalue resolvent. It provides a direct link between the statistical distribution of Young weights and the correlators of the model:

$$\frac{1}{N} \ \mathrm{Tr} M^{2q} = \frac{\lambda^q}{q} \oint \frac{dh}{2\pi i} \ h^q \ e^{qH(h)}. \tag{3.15}$$

Here the contour encircles the cut of $H(h)$. Further introduce the function $G(h)$ as

$$G(h) = e^{H(h)+F(h)}. \tag{3.16}$$

Its importance stems from the fact that it relates in a simple way the introduced functions $F(h)$,$H(h)$ and the coupling constants t_{2q} [1]:

$$t_{2q} = \frac{1}{q} \oint \frac{dh}{2\pi i} \; G(h)^q. \tag{3.17}$$

Using this equation as well as an alternative representation of the Weyl character (3.6)

$$\chi_{\{h\}}(A) = \det_{(k,l)} \big(P_{h_k+1-l}(\theta)\big) \tag{3.18}$$

as a determinant of Schur polynomials $P_n(\theta)$ defined through

$$e^{\Sigma_{i=1}^{\infty} z^i \theta_i} = \sum_{n=0}^{\infty} z^n P_n(\theta) \quad \text{with} \quad \theta_i = \frac{1}{i}\ \mathrm{Tr}[A^i], \tag{3.19}$$

one derives [2] from the "equation of motion"

$$\frac{\partial}{\partial \theta_q} P_n(\theta) = P_{n-q}(\theta) \quad \text{with} \quad \theta_q = \frac{N}{2q} t_{2q} \tag{3.20}$$

the equation

$$h - 1 = \sum_{q=1}^{Q} \frac{t_{2q}}{G^q} + \sum_{q=1}^{\infty} \frac{2q}{N} \frac{\partial}{\partial t_{2q}} \ln\Big(\chi_{\{\frac{h^e}{2}\}}(a)\Big)\ G^q, \tag{3.21}$$

where the coefficients of the positive powers of G in (3.21) are directly related to the correlators of the matrix model dual to (3.11), i.e. the model with potential $V_{A_4}(\tilde{M}A) = \frac{1}{4}\ (\tilde{M}A)^4$:

$$\frac{2q}{N} \frac{\partial}{\partial t_{2q}} \ln\Big(\chi_{\{\frac{h^e}{2}\}}(a)\Big) = \langle \frac{1}{N} \mathrm{Tr}\ (\tilde{M}A)^{2q} \rangle. \tag{3.22}$$

Here the negative powers of G are fixed by eq.(3.17) while the positive powers are determined through eq.(3.20). If we write formula (3.21) in the form $\hat{h}\ \chi_h(t) = (\sum_{q\geq 0}\ G^{-q}\ t_{2q} + \sum_{q>0}\ G^q\ \frac{2q}{N} \frac{\partial}{\partial t_{2q}})\ \chi_h(t)$ we see that the operator $\hat{h}$ acts on the character like an operator of the derivative of the bosonic field $\frac{d}{dG}\ \phi(G)$ since the commutation relations of $\phi(G)$ for different G's are those of a bosonic field[1]. This fact suggests that the models of dually weighted planar graphs might have a description in terms of integrable hierarchies of differential equations, just as for the (much simpler) case of ordinary matrix models.

We have also assumed for the moment that only a finite number Q of couplings are non-zero (i.e. $t_{2q} = 0$ for $q > Q$). Furthermore, we were able to show in our second work [2] that (3.21) implies the functional equation

$$e^{H(h)} = \frac{(-1)^{(Q-1)} h}{t_Q} \prod_{q=1}^{Q} G_q(h), \tag{3.23}$$

[1] We acknowledge several conversations with I. Kostov concerning this point.

where the $G_q(h)$ are the first Q branches of the multivalued function $G(h)$ defined through (3.21) which map the point $h = \infty$ to $G = 0$. The saddlepoint equation (3.12), together with (3.23), defines a well-posed Riemann-Hilbert problem. It was solved exactly and in explicit detail in [2] for the case $Q = 2$, where the Riemann-Hilbert problem is succinctly written in the form

$$\begin{aligned} 2\not{F}(h) + H(h) &= -\ln(-\frac{h}{t_4}) \\ 2F(h) + \not{H}(h) &= -\ln h, \end{aligned} \tag{3.24}$$

where $\not{H}(h)$ denotes the real part of $H(h)$ on the cut $[b, a]$ and $\not{F}(h)$ denotes the real part of $F(h)$ on a cut $[-\infty, c]$ with $c < b$. This case corresponds, in view of the potential (3.11), to an ensemble of squares being able to meet in groups of four (i.e. flat points) or two (i.e. positive curvature points). We termed the resulting surfaces "almost flat". It turned out that all the introduced functions could be found explicitly in terms of elliptic functions. We also wrote down the Riemann-Hilbert problem for the more complicated case $Q = 3$. Allowing in addition a non-zero negative curvature coupling t_6, it already captures the transition from flat to random surfaces. The explicit equations are

$$\begin{aligned} 2F(h) + \not{H}(h) &= -\ln h \\ F_1(h) + F_2(h) + F_3(h) + 2H(h) &= -\ln(\frac{h}{t_6}). \end{aligned} \tag{3.25}$$

Its exact solution can, in principle, be obtained as well. Some of the steps of the solution of (3.25) as well as for the general case of an arbitrary number of couplings will be presented in appendix 1. Unfortunately it involves already functions more general than elliptic, making the solution much less explicit and transparent. Luckily, however, we are not forced to solve the system (3.25) in order to analyse the problem of the transition from flat to random lattices. Indeed, we merely need to perturb our almost flat lattices by *any* operator containing negative curvature. This physical observation will be used in the next section to achieve the long-sought solution to our problem.

4. Exact solution of discrete two-dimensional R^2 gravity

In the review of the method of the previous section we recalled that the analytic structure of the multivalued function $G(h)$ (3.16) is in part determined by the weights t_{2q}. Specifically, the structure of the physical sheet and of all the sheets attached to it by the cut of $e^{F(h)}$ is the same as that given by (i) dropping the positive powers of G in (3.21) and

(ii) inverting this truncated equation to obtain G as a function of h. In [2] this observation permitted us to solve the flat space limit. There the equation (3.21) reads:

$$h - 1 = \frac{t_2}{G} + \frac{t_4}{G^2} + \text{ positive powers of } G. \tag{4.1}$$

Dropping the positive powers of G leaves us with a quadratic equation for G. We thus deduced that the physical sheet and all the sheets attached to it by the cut of $e^{F(h)}$ have a quadratic structure. In other words, there is only one sheet attached by the cut of $e^{F(h)}$. This simple structure allowed us to solve the flat space limit directly.

The key observation that allows us to solve the full problem is that we can choose a more general set of weights, including those needed to introduce negative curvature defects, whilst still preserving this simple two sheet structure. Labeling the first two weights as in [2] : t_2, t_4, we now include all the even t_{2q} with $q \geq 3$, assigning them the following weights: $t_{2q} = t_4 \epsilon^{q-2}$. Equation (3.21) can then be written compactly as:

$$h - 1 = \frac{t_2}{G} + \frac{t_4}{G\,(G - \epsilon)} + \text{ positive powers of } G. \tag{4.2}$$

Dropping the positive powers of G and inverting this equation, we see that there are two sheets connected together by a square root cut running between two finite cut points, d and c (see Fig. 4.1). On the physical sheet a further cut (running from b to a), corresponding to $e^{H(h)}$, connects to further sheets.

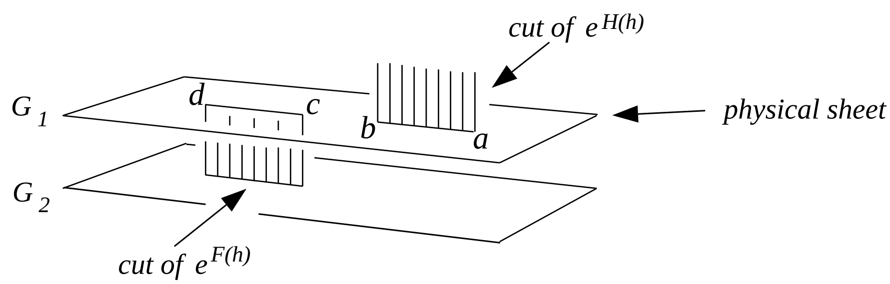

Fig. 4.1 Sheet and cut structure of $G(h)$

To proceed with the solution we relate the function $G(h)$ to the resolvent $H(h)$ by the formula:

$$e^{H(h)} = \frac{h}{\epsilon t_2 - t_4}\; G_1(h)\; G_2(h), \tag{4.3}$$

where $G_1(h)$ and $G_2(h)$ are the physical sheet and the only other sheet attached to it by the cut of $e^{F(h)}$ (see Fig. 4.1). Note that this differs slightly from the form (3.23) derived in [2] since there are now an infinite number of inverse powers of G in (3.21). It can be derived by the methods used in [2].

We now proceed exactly as in [2] . Equation (4.3) can be written as

$$F_1(h) + F_2(h) + H(h) = \ln(\frac{h}{\epsilon t_2 - t_4}), \tag{4.4}$$

where $\ln G_i(h) = F_i(h) + H(h)$. The two sheets $G_1(h)$ and $G_2(h)$ are glued together by the square root cut coming from $F(h)$. The combination $F_1(h) + F_2(h)$, evaluated on the cut of $F(h)$, is twice the constant part of $F(h)$ on the cut (the discontinuous part of $F(h)$ is of opposite sign on $F_1(h)$ and $F_2(h)$ and is therefore canceled). We thus have the two equations

$$\begin{aligned} 2\not{F}(h) + H(h) &= -\ln(\frac{h}{\epsilon t_2 - t_4}) \\ 2F(h) + \not{H}(h) &= -\ln h, \end{aligned} \tag{4.5}$$

the first coming from the large N limit of the character (4.3)and the second, in view of (3.14), being the saddlepoint equation (3.12). These two equations tell us about the behaviour of the function $2F(h) + H(h)$ on the cuts of $F(h)$ and $H(h)$, respectively. We have introduced the notation $\not{F}(h)$ to denote the real part on the cut of $F(h)$, and similarly for $\not{H}(h)$. The principal part integral in (3.12) is thus denoted in (4.5) by $\not{H}(h)$.

Our object is to now reconstruct the analytic function $2F(h)+H(h) = 2\ln G(h)-H(h)$ from its behaviour on its cuts. To do this we first need to understand the complete structure of cuts. We already know the structure of cuts of $H(h)$; it has a logarithmic cut running from $h = 0$ to $h = b$, corresponding to the portion of the density which is saturated at its maximal value of one, and a cut from b to a corresponding to the "excited" part of the density, where the density is less than one. It thus remains for us to understand the cut structure of $\ln G(h)$.

The function $G(h)$ has two cuts on the physical sheet. The first cut, running from b to a, corresponds to the cut of $e^{H(h)}$, the second cut, running from cut point c to cut point d, corresponds to the cut of $e^{F(h)}$ (see Fig. 4.1). To see whether $\ln G(h)$ has any logarithmic cuts, we first notice from (4.2) that $G(h)$ is non zero everywhere in the complex h plane except possibly at infinity. Thus for $\ln G(h)$ the only finite logarithmic cut points are at $h = b$, defined to be the end of the flat part of the density (this corresponds to the end of the cut of $e^{H(h)}$), and possibly the cut point c, defined to be at the end of the cut of $e^{F(h)}$. The only remaining question is whether this logarithmic cut starting at $h = b$ goes off to infinity or terminates at c. For large h we see from (4.2) that $\ln G(h) = \ln\epsilon + \mathcal{O}(\frac{1}{h})$, i.e. there is no logarithmic cut at infinity. We conclude that the cut structure of the function $\ln G(h)$ consists of the cuts corresponding to $e^F(h)$ and $e^{H(h)}$ connected together by a logarithmic cut, whose cut points are b and c (see Fig. 4.1).

We now introduce two functions $\tilde{F}(h)$ and $\tilde{H}(h)$ defined by

$$F(h) = \tilde{F}(h) - \ln\frac{h}{(h-c)} \quad \text{and} \quad H(h) = \tilde{H}(h) + \ln\frac{h}{h-b}. \tag{4.6}$$

They are defined such that they have no logarithmic cuts. The equation (4.5) now reads

$$\begin{aligned} 2\tilde{F}(h) + \tilde{H}(h) &= \ln(t_4 - \epsilon t_2) + \ln\big(-\frac{(h-b)}{(h-c)^2}\big) \\ 2\tilde{F}(h) + \tilde{H}(h) &= \ln\big(\frac{(h-b)}{(h-c)^2}\big). \end{aligned} \tag{4.7}$$

These two equations define the behaviour of $2\tilde{F}(h) + \tilde{H}(h)$ on all of its cuts. By standard methods we now generate the full analytic function $2\tilde{F}(h) + \tilde{H}(h)$. There are four cut points, a and b defining the cut of $\tilde{H}(h)$, and c and d defining the cut of $\tilde{F}(h)$. Their values will be fixed later by boundary conditions. Note that this differs from the flat space limit solved in [2] in that $\tilde{F}(h)$ now has two finite cut points whereas previously it had a semi-infinite cut (i.e. $d \to -\infty$). We generate the full analytic function by performing the contour integral

$$2\tilde{F}(h) + \tilde{H}(h) = r(h)\Bigg[\oint_{C_H} \frac{ds}{2\pi i}\frac{\ln\big(\frac{(s-b)}{(s-c)^2}\big)}{(h-s)\ r(s)} + \oint_{C_F} \frac{ds}{2\pi i}\frac{\ln(t_4 - \epsilon t_2) + \ln\big(-\frac{(s-b)}{(s-c)^2}\big)}{(h-s)\ r(s)}\Bigg], \tag{4.8}$$

where for compactness we have defined

$$r(h) = \sqrt{(h-a)(h-b)(h-c)(h-d)}. \tag{4.9}$$

The contour C_H encircles the cut $[b, a]$, whilst C_F encircles the cut between c and d. They are illustrated in Fig. 4.2(a).

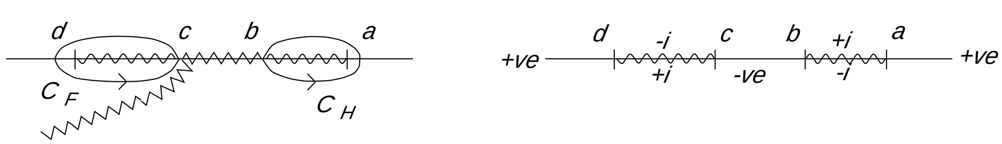

Fig. 4.2 (a)Contours C_H and C_F. (b)Sign convention for square roots of $r(h)$.

The zigzag line between c and b corresponds to the cut of $\ln\left(\frac{(h-b)}{(h-c)}\right)$, The remaining zigzag line starting at c corresponds to the remaining logarithmic cut, $-\ln(h-c)$. Expanding the contours, catching poles on the way and using the fact that logarithmic cuts have a discontinuity of $\pm i\pi$, we arrive at

$$\begin{aligned} 2F(h)+H(h) = &\ln\frac{t_4-\epsilon t_2}{h} \\ &+r(h)\Big[\int_c^b \frac{ds}{(h-s)\ r(s)} + \int_b^a \frac{ds\,\frac{1}{\pi i}\ln(t_4-\epsilon t_2)}{(h-s)\ r(s)} - \int_{-\infty}^d \frac{ds}{(h-s)\ r(s)}\Big]. \end{aligned} \tag{4.10}$$

Fig. 5(b) clarifies the sign convention for the square root $r(h)$ on the real axis above and below the cuts. Note that, for the cuts of $1/r(h)$ the signs on the cuts are inverted compared to Fig. 5(b), i.e. $+i \leftrightarrow -i$. The integrals in (4.10) are defined to be along the upper side.

To fix the constants a, b, c and d, we expand (4.10) for large h and compare the resulting power series expansion to that obtained from inverting (3.21):

$$2F(h)+H(h) = 2\ln\epsilon + \left(\frac{2t_4}{\epsilon^2}-1\right)\frac{1}{h} \quad + \quad \mathcal{O}\left(\frac{1}{h^2}\right). \tag{4.11}$$

The terms of $\mathcal{O}\left(\frac{1}{h^2}\right)$ depend on the as yet unknown positive powers of G in (4.2). Expanding (4.10) for large h and comparing to (4.11) we find the three boundary conditions:

$$\ln\,(t_4-\epsilon\ t_2) = -\frac{\pi}{K}\ (K'+v), \tag{4.12}$$

$$\epsilon^2 = \frac{4Kq}{i\pi\xi}\ \frac{\theta_1\left(\frac{i\pi v}{K},q\right)}{\theta_1'\left(0,q\right)}, \tag{4.13}$$

$$\frac{2t_4}{\epsilon^2}-1 = -\frac{1}{4}\zeta + \xi\left[\frac{1}{4\ \mathrm{sn}(v,k')}\Upsilon + \Xi\right], \tag{4.14}$$

where we have defined

$$\begin{aligned} \xi =& \sqrt{(a-c)(b-d)}, \quad \zeta = a+b+c+d, \\ \Upsilon =& 3\ \mathrm{cn}(v,k')\ \mathrm{dn}(v,k') - \frac{\mathrm{dn}(v,k')}{\mathrm{cn}(v,k')} - \frac{\mathrm{cn}(v,k')}{\mathrm{dn}(v,k')} \quad \text{and} \\ \Xi =& \frac{\pi}{2K} + E(v,k') + \left(\frac{E}{K}-1\right)\ v. \end{aligned} \tag{4.15}$$

K and K' are the complete elliptic integrals of the first kind with respective moduli k and $k'=\sqrt{1-k^2}$. E is the complete elliptic integral of the second kind with modulus k,

$E(v,k')$ is the incomplete elliptic integral of the second kind with argument v and modulus k' and sn, cn and dn are the Jacobi Elliptic functions. The nome q is defined by

$$q = e^{-\pi\frac{K'}{K}}. \tag{4.16}$$

Finally k and v are defined in terms of the cut points by

$$k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)}, \quad v = \mathrm{sn}^{-1}\big(\sqrt{\frac{a-c}{a-d}}, k'\big). \tag{4.17}$$

The final boundary condition will come from the normalisation of the density.

The density $\rho(h)$ can be obtained from $2F(h)+H(h)$ by using the saddle point equation $2F(h) + \not{H}(h) = -\ln h$ and the fact that the resolvent (3.14) for the Young tableau can be written as $H(h) = \not{H}(h) \mp i\pi\rho(h)$. We thus perform the integrals in (4.10) and, after using the first boundary condition and an identity between elliptic functions[2], we obtain

$$\rho(h) = \frac{u}{K} - \frac{i}{\pi}\ \ln\left[\frac{\theta_4\big(\frac{\pi}{2K}(u-iv),q\big)}{\theta_4\big(\frac{\pi}{2K}(u+iv),q\big)}\right], \tag{4.18}$$

where v is defined in (4.17) and u is defined by

$$u = \mathrm{sn}^{-1}\big(\sqrt{\frac{(a-h)(b-d)}{(a-b)(h-d)}}, k\big). \tag{4.19}$$

Integrating $\rho(h)$ from b to a and equating the answer to $1-b$ to ensure that the density is normalized to 1 (the flat portion from 0 to b gives a contribution b), leads to the final boundary condition:

$$t_4 = \frac{-2Kiq}{\pi^2}\ \frac{\theta_1\big(\frac{i\pi v}{K},q\big)}{\theta_1'\big(0,q\big)}\big[-E + K\big(k'^2\mathrm{sn}^2(v,k') + \frac{\Upsilon\,\Xi}{\mathrm{sn}(v,k')} + 2\,\Xi^2\big)\big]. \tag{4.20}$$

The elliptic modulus k and the parameter v are fixed in terms of the coupling constants t_2, t_4 and ϵ, by equations (4.12) and (4.20) . It is natural to use the parameter β defined by $\beta^2 = \frac{\epsilon t_2}{t_4}$ so that the elliptic parameters k and v are determined through the two parameters t_4 and β. Through (4.17) k and v fix the two independent ratios of differences between cut points. The scale and position of the cut points are then fixed by equations (4.13) and (4.14) .

2 For this and many other relations between Jacobi's elliptic functions and theta functions useful for performing the calculations of this section see e.g. [16],[17].

We now concentrate on the free energy. We choose the definitions for t_2, t_4 and ϵ introduced in section 2:

$$t_2 = \sqrt{\lambda} t, \quad t_4 = \lambda, \quad \epsilon = \sqrt{\lambda}\frac{\beta^2}{t}, \tag{4.21}$$

which implies that

$$t_{2q} = \lambda^{\frac{q}{2}} \left(\frac{\beta^2}{t}\right)^{(q-2)} \quad \text{for} \quad q \geq 2. \tag{4.22}$$

This choice is motivated by several reasons. Firstly we want to have a way of measuring the area of the surfaces. The factors of λ in the coupling constants have thus been chosen in order that the power of λ counts the number of squares A making up the surface (see eq.(2.3)). Each corner of a vertex corresponds to one of the four corners of a square and and so is weighted by $\lambda^{\frac{1}{4}}$. A vertex of $2q$ legs (where $2q$ squares meet) is thus weighted with $\lambda^{\frac{q}{2}}$. The definition of ϵ is such that $\frac{t_2 \epsilon}{t_4} = \beta^2$ as before, so that the elliptic parameters k and v are defined in terms of λ and β. Finally, for surfaces of spherical topology, each negative curvature defect is compensated for by a precise number of positive curvature defects. In particular a negative curvature defect t_{2q} is compensated by $q-2$ positive curvature defects t_2 giving an overall factor of $\beta^{2(q-2)}$. The power of t will thus correspond to the number of excess positive curvature defects. For a surface of spherical topology which takes precisely four extra t_2 defects to close the surface we would therefore expect a factor t^4 (see (2.3)).

We now use a standard formula from matrix models. Denoting the free energy by $\mathcal{Z}(t, \lambda, \beta)$, $Z = e^{-N^2 \mathcal{Z}}$, we have

$$\frac{\partial}{\partial \lambda} \mathcal{Z}(t, \lambda, \beta) = \frac{1}{4\lambda}(\langle \frac{1}{N} \mathrm{Tr} M^2 \rangle - 1). \tag{4.23}$$

This identity corresponds to grabbing hold of one of the propagators of a free energy diagram. From equation (3.15) we see that

$$\langle \frac{1}{N} \mathrm{Tr} M^2 \rangle = \frac{1}{2} + \langle h \rangle. \tag{4.24}$$

where $\langle h \rangle = \int dh\ \rho(h)\ h$. Using the density (4.18) to calculate $\langle h \rangle$ we obtain, after a long calculation,

$$\frac{\partial}{\partial \lambda} \mathcal{Z} = \frac{\xi^2}{4\lambda} \left[\frac{K k'^2 \mathrm{sn}^2}{8\pi\ \mathrm{cn}\ \mathrm{dn}} (1 - k'^2 \mathrm{sn}^4) - \frac{k'^2 \mathrm{sn}^2}{4} - \frac{1}{2\pi} (K\, \Xi - \frac{\pi}{2})(\mathrm{dn}^2 + k'^2 \mathrm{cn}^2) \right.$$
$$\left. + \frac{3\, \Upsilon\, \Phi}{8\pi\ \mathrm{sn}^2} - \frac{\Phi^2}{2\pi^2 \mathrm{sn}^2} - \frac{K\, \Upsilon}{4\pi\ \mathrm{sn}} (1 - \mathrm{dn}^2) + \frac{\Phi\, \Xi}{2\pi\ \mathrm{sn}} + \frac{\Omega\, \Phi}{\pi\ \mathrm{sn}} - \frac{\Omega^2}{2} \right], \tag{4.25}$$

where

$$\Omega = \frac{K}{\pi}(1 - \mathrm{dn}^2 + 2\,\Xi^2) - \Xi \quad \text{and} \quad \Phi = \mathrm{sn}\,E - \Upsilon\big(K\,\Xi - \frac{\pi}{2}\big), \tag{4.26}$$

and for compactness we have denoted $\mathrm{sn}(v,k')$ by sn and similarly for cn and dn. Everything inside the square brackets of this expression is a function of the two elliptic parameters k and v, in other words a function of λ and β. The ξ outside the square bracket can be expressed from (4.13) and (4.21) by

$$\xi = \frac{4t^2Kq}{i\pi\beta^4\lambda}\;\frac{\theta_1\big(\frac{i\pi v}{K},q\big)}{\theta_1'\big(0,q\big)}. \tag{4.27}$$

We thus see that the free energy is written as a function of λ and β times a factor t^4 (see (2.3)). As discussed above, the factor of t^4 corresponds to the four extra t_2 defects needed to close the surface into the topology of a sphere.

Since the solution is somewhat complicated, it is essential to check it carefully. As a first check we investigate two limiting cases. By setting $t_2 = \epsilon = \sqrt{\lambda}$ and $t_4 = \lambda$ (or equivalently $t = \beta = 1$) we correctly recover the solution found in [1] for the model with the external matrices $A = B = \lambda^{\frac{1}{4}}\mathcal{J}$ where $\frac{1}{N}\mathrm{Tr}[\mathcal{J}^q]$ equals one for q even and zero otherwise (see equations (5.8)→(5.11) of [1]). Taking the other limit and setting $\epsilon = 0$ we correctly recover the solution for the flat space limit found in [2]. Both these limits are somewhat subtle to extract since they correspond to singular points in the boundary conditions, and thus have to be derived by a careful taking of limits. We elaborate on this in appendix 2. where further nontrivial checks are also discussed.

5. Extraction of physical results

We are now in a position to derive the physical results announced in section 2. The boundary conditions contain sufficient information to find the critical values of the couplings λ and β corresponding to the continuum limit (the elliptic parameters k and v do not depend on the coupling t which is therefore not fixed by the continuum limit). The most direct way of finding the continuum limit, however, is from the expression for the density (4.18). The first term of the density, $\frac{u}{K}$, is always positive on the interval (b,a). However, for a certain range of values of v, the second term in (4.18) can be negative. The continuum limit corresponds to the point in (q,v) parameter space where the density is no longer positive. Specifically[3] we need to find the point in (q,v) space where the density

[3] This is very similar to ordinary matrix models. There the point in coupling constant space at which the density of *eigenvalues* becomes flat at its end point corresponds to the continuum limit.

becomes exactly flat at the end point a. Taking the derivative w.r.t. h of the density at the end point a and setting it equal to 0 gives

$$1 = \frac{\mathrm{cn}(v,k')}{\mathrm{sn}(v,k')\ \mathrm{dn}(v,k')}\ \Xi, \tag{5.1}$$

where Ξ is defined in (4.15) . This equation defines a line in (q,v) space. Substituting (5.1) into (4.20) , a point on this line can be related to λ by

$$\lambda = -\frac{2Kiq}{\pi^2}\frac{\theta_1(\frac{i\pi v}{K},q)}{\theta_1'(0,q)}\Big[K\frac{\mathrm{dn}^2(v,k')}{\mathrm{cn}^2(v,k')} - E\Big], \tag{5.2}$$

and β by

$$\beta^2 = 1 - \frac{1}{\lambda}e^{-\frac{\pi}{K}(K'+v)}. \tag{5.3}$$

Equations (5.1), (5.2) and (5.3) define the continuum line in the parameter space (β,λ). A plot of this line is shown in Fig. 4.3.

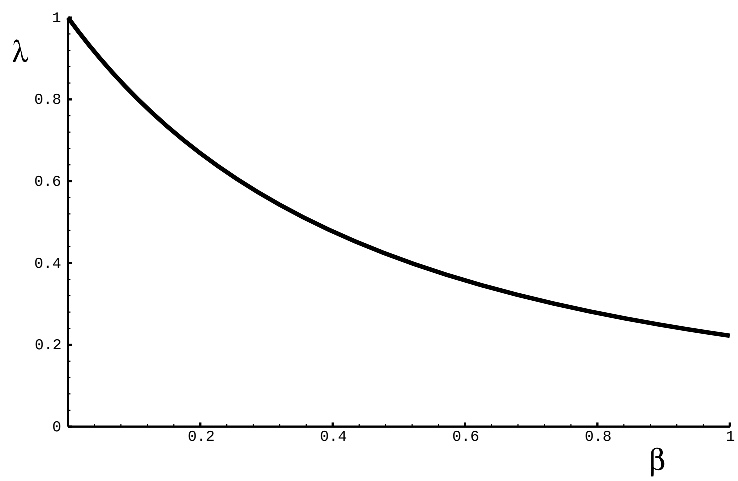

Fig. 4.3 The critical line in the β, λ plane

The point $\beta = 1$ ($\lambda = \frac{2}{9}$) corresponds to the model $A = B = \lambda^{\frac{1}{4}}\mathcal{J}$ mentioned above and solved in [1]. The point $\beta = 0$ ($\lambda = 1$) corresponds to the flat space limit studied in [2]. The continuum line corresponds to a singularity in the mapping between the coupling constants β and λ and the parameters q and v through which physical quantities are expressed. The type of singularity determines the physical regime of the continuum limit. It is easiest to investigate the behaviour of the singularity about the limiting points discussed above. The limit $\beta = 1$ was discussed in [1] , where a standard square root singularity was found, corresponding to the pure gravity regime. We thus concentrate on the limit $\beta \to 0$ and $\lambda \to 1$.

This limit corresponds to $q \to 1$. We can study it most simply by expressing all the dependence on the nome q through the dual parameter q', defined through

$$q' = e^{-\pi \frac{K}{K'}} = e^{-\frac{\pi^2}{\ln q}}. \tag{5.4}$$

We then discard all exponentially suppressed terms, i.e. all powers of q'. This is not the same as setting $q' = 0$ since we keep terms of order $\ln q'$, in particular $K = -\frac{\pi^2}{2\ln q'}(1 + \mathcal{O}(q'))$. For q close to one this is a good approximation[4]. In this limit the boundary conditions fixing v and the nome q are given by

$$\begin{aligned} \frac{1}{1-\beta^2} =& e^{\frac{2zv}{\pi}} \Big[\big(1 + \frac{2v}{\pi}\big) \cos 2v + \frac{1}{\pi}\big(z + \frac{2vz}{\pi} - 1\big) \sin 2v \Big], \\ q =& \big[\lambda(1-\beta^2)\big]^{\left(1+\frac{2v}{\pi}\right)}, \end{aligned} \tag{5.5}$$

where we have defined $z = -\ln(\lambda(1-\beta^2))$. Setting the derivative (w.r.t. v) of the first of the above equations to zero gives the critical line about the limiting point where the surface flattens:

$$\tan v_c = \frac{z}{\pi}. \tag{5.6}$$

This could also have been obtained by discarding all powers of q' in (5.1) . To investigate the type of singularity on the critical line we expand the r.h.s. of the first equation in (5.5) about the critical value of v:

$$\frac{1}{1-\beta^2} = a_0 + a_2(v - v_c)^2 + \mathcal{O}\big((v-v_c)^3\big). \tag{5.7}$$

The coefficient a_0 is given by evaluating (5.5) at v_c. The coefficient a_2 is the second derivative (w.r.t v) of the first equation of (5.5), also evaluated at v_c. It is easy to demonstrate that a_2 cannot be zero for positive, real coupling coefficients. We thus find that the singularity on the critical line is a square root corresponding to pure, two dimensional gravity. A more rigorous study would analyse the singularity of boundary conditions (4.12) and (4.20) at a generic point along the critical line. We have examined this numerically and find a square root singularity for the full length of the critical line. *We can thus conclude that for all finite β we are in the regime of pure, two dimensional gravity.*

[4] Numerically the computation of elliptic functions is performed by writing them as a power series in either q or q', whichever is smaller. The power series converge so rapidly that, even for a generic point, the first few terms are sufficient to give a high degree of accuracy.

We now proceed to extract the physical mechanism of flattening which requires the limit $\beta \to 0$ (corresponding to $v \to 0$). Expanding out (5.5) to lowest order in v, z and β^2 we find

$$v = \frac{\beta}{\sqrt{2}}\big(x - \sqrt{x^2 - 1}\big) \quad \text{and} \quad q = \lambda. \tag{5.8}$$

where we have introduced a scaling parameter x defined by

$$x = 1 + \frac{\sqrt{2}}{\pi}\frac{\mu}{\beta} \quad \text{with} \quad \mu = \lambda_c - \lambda \quad \text{and} \quad \lambda_c = 1 - \frac{\pi\beta}{\sqrt{2}} + \mathcal{O}(\beta^2). \tag{5.9}$$

Note that this is a very natural, dimensionless scaling parameter; μ is the continuum cosmological constant with dimension of inverse area and β controls the number of curvature defects per unit area (and thus also has dimension of inverse area).

Expressing the elliptic functions in (4.25) in terms of the dual nome q', dropping all powers of q' and keeping only the leading order contribution in powers of v, we arrive at the free energy about the flattening transition point:

$$\mathcal{Z}(t, \lambda, \beta) = \frac{4t^4}{15\beta^2}\big[x^6 - \frac{5}{2}x^4 + \frac{15}{8}x^2 - \frac{5}{16} - x(x^2 - 1)^{5/2}\big]. \tag{5.10}$$

This is the continuum scaling function corresponding to the flattening transition. For x close to 1, in other words for cosmological constant μ much less than β, there is the standard $\mu^{\frac{5}{2}}$ singularity characteristic of pure gravity. For x very large, corresponding to very small β the free energy has the singularity μ^{-2}. From (5.10) we can also Laplace-transform to the free energy for fixed area

$$\begin{aligned}\mathcal{Z}(t, A, \beta) =& \frac{8t^4}{\pi^2\beta^4 A^3}\left[-\Big(\frac{48}{y^3} + \frac{8}{y}\Big)I_1(y) + \Big(\frac{24}{y^2} + 1\Big)I_0(y)\right] \quad \text{with} \quad y = \frac{\pi}{\sqrt{2}}\beta A, \\ &\sim \begin{cases} t^4\beta^{-\frac{9}{2}}e^{\frac{\pi}{\sqrt{2}}\beta A}A^{-\frac{7}{2}}, & \text{for } A \to \infty \\ t^4 A, & \text{for } A \to 0 \end{cases}\end{aligned} \tag{5.11}$$

where I_0 and I_1 are modified Bessel functions. The limiting behaviours for large and small areas are also shown. For surfaces whose area, A, is large compared to β the exponent $-\frac{7}{2}$ signals that we are in the regime of pure 2D gravity. For surfaces of very small area A there is a linear dependence on the area. The function $\mathcal{Z}(A, \beta, t)$ smoothly interpolates between these two limiting behaviours. *There is no phase transition separating flat space from pure gravity.*

6. Conclusions and open issues

In conclusion, our result strongly suggests that R^2 gravity (1.1) does not seem to exist as a *continuum* theory. The discretisation of the term $\frac{1}{\beta_0}\ R^2$ by the same principle that consistently defines the path integral (1.1) for $\beta_0 = \infty$ (i.e. ordinary 2D gravity) does not lead to any new fixpoints, indicating the *non-perturbative* irrelevance of the dimensionful higher derivative terms of (1.1). Of course one might object that our particular (but generic!) discretisation is too naive. Indeed, it would be gratifying to prove the universality of our result by choosing different weights (2.2) for the curvature terms in our model. In principle, as we hope we have convinced the reader, this can be checked with the help of our formalism. It is not entirely excluded that fine-tuning the weights (2.2) in a subtle way might result in a smooth phase of gravity (for some (non-rigorous, as we believe) arguments in favor of this hypothesis see [18]). However, it is equally evident that any further calculations will be very tedious unless it proves possible to simplify our method in a significant fashion. We should also point out that the fact that the curvature in the present type of discrete models is always necessarily "quantised" in units of π could be an a priori stumbling block for reaching a smooth phase of gravity. Could one find discrete models of gravity that allow for soft, slowly varying curvature fluctuations over many lattice spacings?

The present work presents the first rigorous step towards a full understanding of these issues. It would be very interesting to gain deeper insights by simplifying and extending our novel discrete approach.

One might also attempt to describe a model like ours directly in the continuum. One could start with the conformal metric of a flat surface with localised curvature defects. It can be represented locally as $g_{ab} = \delta_{ab}\ e^{\varphi(z)}$ with

$$\varphi(z) = \sum_{j=1}^{M}\ R_j\ \ln(z - z_j)^2, \tag{6.1}$$

where $R_j = -1, 1$. Symbolically, the partition function might be written as

$$Z(\mu, \beta) = \sum_{M}\ \beta^M\ \int d[z_1, ..., z_M]\ e^{-\mu \int d^2 z\ \sqrt{\det g(z)}}. \tag{6.2}$$

Here we introduced the fugacity of curvature defects β instead of the explicit R^2-term in the action. It serves the same purpose: for $\beta \to 0$ we arrive at the completely flat metric, whereas for $\beta \sim 1$ the system should show the behaviour of pure quantum gravity, at least in the infrared domain. We retained the notation β to denote the parameter playing a role similar to the R^2 coupling in the above discrete model.

This formulation resembles a little bit the two-dimensional Coulomb gas problem. However, the measure of integration $d[z_1, ..., z_M]$ of the positions of the curvature defects is a complicated object: it should take into account the topology of the surface and the existence of zero modes (the action does not depend on some directions in the space of the z_i). It would be very interesting to make this direct continuum formulation more precise.

Another interesting issue is the role of exponential corrections appearing due to the structure of elliptic functions. In fact all physical quantities, such as $\mathcal{Z}(t,\lambda,\beta)$ (see (2.3))), contain exponentially small terms in the limit $\lambda \to \lambda_c$ and $\beta \to 0$, thus leading to an essential singularity at $\beta = 0$, $\lambda = 1$. These terms are precisely the powers of q' that we discarded to arrive at the scaling function (5.10) of section 5. One can obtain the first correction of this type in e.g. the free energy $\bar{f}(\beta)$ per unit area in the thermodynamical limit $\lambda = \lambda_c$:

$$\mathcal{Z}(t,A,\beta) \ \sim \ \frac{t^4}{\beta^{\frac{9}{2}}} \, e^{\bar{f}(\beta)A} \, A^{-\frac{7}{2}} \quad \text{with} \quad \bar{f}(\beta) = \frac{\pi}{\sqrt{2}}\beta\Big[(1+\ldots) + e^{-\frac{\pi\sqrt{2}}{\beta}}(4+\ldots)\Big] \qquad (6.3)$$

where $\bar{f}(\beta) = \lim_{A\to\infty} \frac{1}{A} \ln \mathcal{Z}(t,A,\beta) = \lambda_c$ and the dots denote terms of order β^3 and higher.

These exponential terms are likely to be lattice artifacts. They emerge even in the simplest calculation for the flat closed quadrangulation with four positive curvature defects, where they appear as discrete corrections to the approximation of elliptic sums by integrals close to the continuum limit.

On the other hand, formula (6.3) corresponds to the critical free energy as a function of the curvature fugacity β (i.e. we have already taken the continuum limit). It is possible that the exponential terms might be corrections relevant for the statistical mechanics of random lattices at long distances (of order $\frac{1}{\beta}$) rather than for continuous 2D-gravity.

However, the most interesting and rewarding extension of our methods and ideas would be their generalisation to allow for the coupling of matter to discrete two-dimensional R^2 gravity. It would lead to an entirely novel approach to two-dimensional physics, unifying theories in the absence and presence of two-dimensional quantum gravity.

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Appendix 1. General case: Large N limit of Schur characters

The two equations, (3.25) , in which there are three non-zero couplings, t_2, t_4 and t_6, can be reduced to a single integral equation for the density. The derivation we present below can be generalised in an obvious manner to an arbitrary finite set of non-zero couplings. Other models with different saddlepoint equations can also be treated with only minor modifications.

The initial steps are very similar to the start of section 4, to which we refer the reader for more details. We start from the equation

$$h - 1 = \frac{t_2}{G} + \frac{t_4}{G^2} + \frac{t_6}{G^3} + \text{ positive powers of } G. \tag{1.1}$$

and deduce (by identical reasoning to section 4) that the physical sheet of the function $G(h)$ and all the sheets attached to it by the cut of $e^{F(h)}$ have a cubic structure. We further deduce that the function $\ln G(h)$ has a logarithmic singularity starting on the physical sheet at the point $h = b$, and, by looking at the large h limit of (1.1) (inverting (1.1) to lowest order in $\frac{1}{h}$ gives $\ln G(h) = \frac{1}{3}\ln(\frac{t_6}{h}) + \mathcal{O}(h^{-\frac{1}{3}})$) deduce that the logarithmic cut goes off to infinity. We now study the combination $2\ln G(h) - \tilde{H}(h)$ where $\tilde{H}(h)$ is defined in (4.6) . We find it has the analytic structure shown in Fig. A1

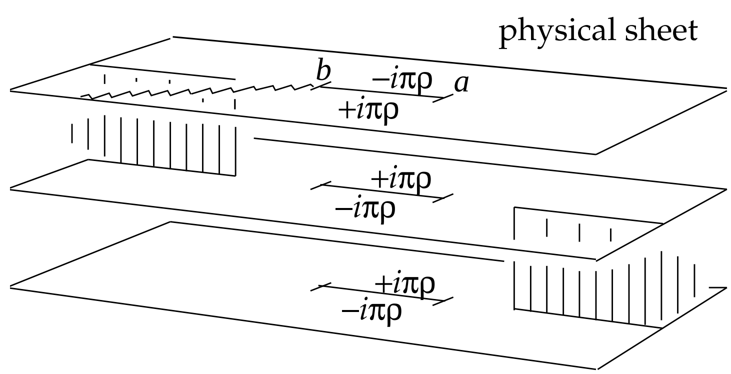

Fig. A1 $2\ln G(h) - \tilde{H}(h)$ in the complex h plane

where the zig-zag line, starting at $h = b$ on the physical sheet, represents the logarithmic cut. It also has a saddlepoint equation (trivially derived from (3.25) and (4.6)) of

$$\Re[2\ln G(h) - \tilde{H}(h)] = -\ln(h - b). \tag{1.2}$$

We now unwind the three sheets of Fig. A1 by parametrising h as a cubic polynomial of a new parameter z

$$h = z^3 - \alpha z + \gamma. \tag{1.3}$$

The coefficients α and γ are such that the cut points of h as a function of z are identical to the cubic cut points of Fig. A1. The three sheets of Fig. A1 (defined in the complex h plane) then unwind to become a single sheet with three cuts (plus a logarithmic cut) in the complex z plane (see Fig. A2).

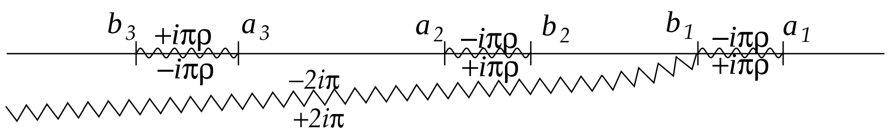

Fig. A2 $2 \ln G(h(z)) - \tilde{H}(h(z))$ in the complex z plane

Defining

$$\mathcal{F}(z) = 2 \ln G(h(z)) - \tilde{H}(h(z)) - 2 \ln\left(\frac{t_6^{\frac{1}{3}}}{z - b_1}\right), \tag{1.4}$$

we see that $\mathcal{F}(z)$ is an analytic function in the complex z plane with three cuts corresponding to the images of the cut of the density. Furthermore, we see from the large h limits of $\ln G(h)$ and $\tilde{H}(h)$, given by $\ln G(h) = \frac{1}{3} \ln(\frac{t_6}{h}) + \mathcal{O}(h^{-\frac{1}{3}})$ and $\tilde{H}(h) = \mathcal{O}(\frac{1}{h})$, that the function $\mathcal{F}(z)$ behaves as $\mathcal{O}(\frac{1}{z})$ for large z. The function $\mathcal{F}(z)$ can thus be generated entirely from the discontinuities of its cuts:

$$\mathcal{F}(z) = \int_{b_1}^{a_1} ds \frac{\rho(h(s))}{z - s} - \int_{b_2}^{a_2} ds \frac{\rho(h(s))}{z - s} - \int_{b_3}^{a_3} ds \frac{\rho(h(s))}{z - s}. \tag{1.5}$$

We now use the identity $(z_1 - z_1')(z_1 - z_2')(z_1 - z_3') = h - h' = (z_1 - z_1')(z_2 - z_1')(z_3 - z_1')$, where the z_i are the three roots of (1.3) and the z_i' are the three roots of (1.3) with h set to h'. Along with equation (1.4) , this leads to a saddle point equation in the complex z plane:

$$-\!\!\!\!\!\!\int_{b_1}^{a_1} ds\, \rho(h(s)) \Big[\frac{1}{z - s} - \frac{1}{z_2(z) - s} - \frac{1}{z_3(z) - s}\Big] = \ln\Big[\frac{(z - b_1)}{(z - b_2)(z - b_3)}\Big] - \frac{2}{3} \ln t_6, \tag{1.6}$$

where z, z_2 and z_3 are the three roots of (1.3) . z_2 and z_3 are thus functions of the first root z. With the parametrisation of (1.3) they would be given by $z_{2(3)}(z) = \frac{z}{2} \pm \sqrt{\alpha - \frac{3z^2}{4}}$.

The generalisation to more than three weights is obvious. However, the functions ρ that satisfy such an equation, even for the case of just three weights, involve functions more general than elliptic and probably more general even than hyperelliptic.

As a final comment, notice that by unwinding the analytic structure of the function $\ln G(h)$ using the mapping (1.3) as in the discussion above we arrive at a closed expression for the function $\ln G(h) = F(h) + H(h)$:

$$\ln G(h) = \int_{b_1}^{a_1} \frac{ds\, \rho(h(s))}{z - s} + \ln\left(\frac{t_6^{\frac{1}{3}}}{z - b_1}\right), \tag{1.7}$$

where z is related to h by (1.3). Given a representation, specified in the large N limit by the density $\rho(h)$ (which then defines $H(h)$ through (3.14)), we thus have a closed expression for the logarithmic derivative of the character, $F(h)$, (3.13). The coupling t_6, is incorporated directly into the expression (1.7), the two remaining couplings, t_2 and t_4, determine, through (3.21), the coefficients α and γ of the mapping (1.3). It is again trivial to generalize this result to more than three weights.

Appendix 2. Verification of the solution

2.1. Limit $\beta \to 1$

This corresponds to the model with $A = B = \lambda^{\frac{1}{4}} \mathcal{J}$ studied in [1] . Setting $t_2 = \epsilon = \sqrt{\lambda}$ and $t_4 = \lambda$ (or equivalently $t = \beta = 1$) the first boundary condition, (4.12), implies immediately that $K' \to \infty \Rightarrow k = 0 \Rightarrow c = d$. The values of E, K, $E(v, k')$, $\mathrm{sn}(v, k')$, Ξ and Υ are then given by

$$E = K = \frac{\pi}{2}, \quad E(v, k') = \mathrm{sn}(v, k') = 1, \quad \Xi = 2 \quad \text{and} \quad \Upsilon = -\sqrt{\frac{a-c}{a-b}} - \sqrt{\frac{a-b}{a-c}},. \tag{2.1}$$

where we have used the definition of v in terms of the cut points, (4.17), along with standard identities $\mathrm{cn}^2 = 1 - \mathrm{sn}^2$ and $\mathrm{dn}^2 = 1 - k'^2 \mathrm{sn}^2$ to find Υ. After using (4.13) to eliminate the theta functions in (4.20), and also using (4.14), we find that $c = d = 0$. In other words the cut of F(h) disappears. We also find the first condition fixing a and b:

$$\sigma - 3\varphi - 1 = 0 \quad \text{where} \quad \sigma = \left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^2 \quad \text{and} \quad \varphi = \left(\frac{\sqrt{a} - \sqrt{b}}{2}\right)^2. \tag{2.2}$$

The limit of the third boundary condition, (4.13), is more subtle. We shift the argument of the theta function using the identity: $\theta_1(z,q) = qe^{-2iz}\theta_1(\frac{i\pi K'}{K} - z, q)$ and carefully take the limit $\frac{\pi}{2K}(K'-v) \to \sinh^{-1}\sqrt{\frac{b}{a-b}}$ to find the final boundary condition

$$3\lambda^2\sigma^3 - \sigma + 1 = 0. \tag{2.3}$$

These were the conditions found in [1] leading to $\lambda_c = \frac{2}{9}$. The density (4.18) can likewise be reduced down to that found in [1].

2.2. Limit $\beta \to 0$ i.e. $\epsilon \to 0$

This limit recovers the flatspace solution found in [2]. In this limit $d \to -\infty$ and the density (4.18) trivially reduces to that found in [2]. To see that the boundary conditions give the correct limit and also to check perturbatively that the next few powers in β correctly correspond to negative curvature insertions we expanded the results in powers of v. Expansions of elliptic functions can be found in most mathematical tables. A comprehensive list is found in [16]. Below we give the expansion formula for the theta function which is less common

$$\frac{\theta_1\left(\frac{i\pi v}{K}, q\right)}{\theta_1'\left(0, q\right)} = \frac{i\pi}{K}v - \frac{2i\pi}{3K}\left(1 + k'2 - 3\frac{E}{K}\right)v^3 + \mathcal{O}(v^5). \tag{2.4}$$

Expanding out (4.12) and (4.20) leads to

$$v = \frac{K}{2\pi}\beta^2 + \mathcal{O}(\beta^4) \qquad \text{and} \qquad q = t_4 - \frac{t_4\beta^2}{2} + \mathcal{O}(\beta^4). \tag{2.5}$$

For $\beta = 0$ we recover the first boundary condition of [2] . Using the identities $(a-c) = \frac{\text{sn dn}}{\text{cn}}\xi$ and $\epsilon = \frac{\beta^2 t_4}{t_2}$ along with (2.5) and (4.13) and working to lowest order we find the second boundary condition of [2]:

$$\frac{t_2}{\sqrt{t_4}} = \frac{\pi}{K}\sqrt{a-c}. \tag{2.6}$$

A careful expansion of (4.14) then leads to the final boundary condition found in [2]:

$$a = 1 + \frac{t_2^2}{\pi^2 t_4}(K^2 - EK). \tag{2.7}$$

A final check of the solution can be performed by expanding out the expression for the free energy (4.25) in powers of v and hence in powers of β. To lowest order we correctly recovered the free energy found in [2]. Expanding out to the first non-zero order in β we retrieved the order β^2 contribution. The result matched exactly with the calculation of a single negative curvature insertion of t_6, calculated in [2].

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