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EVIDENCE FOR HETEROTIC/HETEROTIC DUALITY

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ABSTRACT

We re-examine the question of heterotic - heterotic string duality in six dimensions and argue that the $E_8 \times E_8$ heterotic string, compactified on $K3$ with equal instanton numbers in the two E_8's, has a self-duality that inverts the coupling, dualizes the antisymmetric tensor, acts non-trivially on the hypermultiplets, and exchanges gauge fields that can be seen in perturbation theory with gauge fields of a non-perturbative origin. The special role of the symmetric embedding of the anomaly in the two E_8's can be seen from field theory considerations or from an eleven-dimensional point of view. The duality can be deduced by looking in two different ways at eleven-dimensional M-theory compactified on $K3 \times \mathbf{S}^1/\mathbf{Z}_2$.

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1 Introduction

Prior to the recent surge of interest in a duality between heterotic and Type IIA strings [1, 2, 3, 4, 5], it was conjectured (on the basis of $D = 10$ heterotic string/fivebrane duality [6, 7]) that in $D \leq 6$ dimensions there ought to exist a duality between one heterotic string and another [8, 9, 10, 11, 12, 13, 14]. A comparison of the fundamental string solution [15] and the dual solitonic string solution [10, 11] suggests the following $D = 6$ duality dictionary: the dilaton $\widetilde{\Phi}$, the string σ-model metric $\widetilde{G}_{MN}$ and 3-form field strength $\widetilde{H}$ of the dual string are related to those of the fundamental string, Φ, G_{MN} and H by the replacements

$$\begin{aligned}\Phi \rightarrow \widetilde{\Phi} &= -\Phi \\ G_{MN} \rightarrow \widetilde{G}_{MN} &= e^{-\Phi} G_{MN} \\ H \rightarrow \widetilde{H} &= e^{-\Phi} * H\end{aligned} \tag{1}$$

In going from the fundamental string to the dual string, one also interchanges the roles of worldsheet and spacetime loop expansions. Moreover, since the dilaton enters the dual string equations with the opposite sign to the fundamental string, it was argued in [8, 10, 11] that in $D = 6$ the strong coupling regime of the string should correspond to the weak coupling regime of the dual string:

$$\lambda_6 = < e^{\Phi/2} > = 1/\widetilde{\lambda}_6 \tag{2}$$

where λ_6 and $\widetilde{\lambda}_6$ are the fundamental string and dual string coupling constants. Because this duality interchanges worldsheet and spacetime loop expansions – or because it acts by duality on H – the duality exchanges the tree level Chern-Simons contributions to the Bianchi identity

$$\begin{aligned}dH &= \alpha'(2\pi)^2 X_4 \\ X_4 &= \frac{1}{4(2\pi)^2}[\mathrm{tr}R^2 - \Sigma_\alpha v_\alpha \mathrm{tr}{F_\alpha}^2]\end{aligned} \tag{3}$$

with the one-loop Green-Schwarz corrections to the field equations

$$\begin{aligned}d\widetilde{H} &= \alpha'(2\pi)^2 \widetilde{X}_4 \\ \widetilde{X}_4 &= \frac{1}{4(2\pi)^2}[\mathrm{tr}R^2 - \Sigma_\alpha \widetilde{v}_\alpha \mathrm{tr}{F_\alpha}^2]\end{aligned} \tag{4}$$

Here F_α is the field strength of the α^{th} component of the gauge group, tr denotes the trace in the fundamental representation, and $v_\alpha, \widetilde{v}_\alpha$ are constants. (As explained in Appendix A, we may, without loss of generality, choose the string tension measured in the string metric and the dual string tension mesured in the dual string metric to be equal.) In fact, the Green-Schwarz anomaly cancellation mechanism in six dimensions requires that the anomaly eight-form I_8 factorize as a product of four-forms,

$$I_8 = X_4 \widetilde{X}_4, \tag{5}$$

and a six-dimensional string-string duality with the general features summarized above would exchange the two factors.

Until now, there has not been a really convincing example of heterotic-heterotic duality in six dimensions. In [11], it was proposed that the $D = 10$ $SO(32)$ heterotic string compactified to $D = 6$ on $K3$ might be dual to the $D = 10$ $SO(32)$ heterotic fivebrane wrapped around $K3$. However, this candidate for a heterotic/heterotic dual string pair suffered from the following drawbacks:

1) The existence of a fivebrane carrying the requisite $SO(32)$ quantum numbers is still unclear. Even if it exists, its properties are not well-understood.

2) The anomaly eight-form of this model is given by (5) with [19]

$$\begin{aligned} X_4 &= \frac{1}{4(2\pi)^2}[\mathrm{tr}R^2 - \mathrm{tr}{F_{SO(28)}}^2 - 2\mathrm{tr}{F_{SU(2)}}^2] \\ \widetilde{X}_4 &= \frac{1}{4(2\pi)^2}[\mathrm{tr}R^2 + 2\mathrm{tr}{F_{SO(28)}}^2 - 44\mathrm{tr}{F_{SU(2)}}^2], \end{aligned} \tag{6}$$

and one of the gauge coefficients in the second factor enters with the wrong sign.

The structure of this equation actually presents a problem that is independent of any speculation about string-string duality. It was shown by Sagnotti [20] that corrections to the Bianchi identities of the type (3) and to the field equations of the type (4) are entirely consistent with supersymmetry, with no restrictions on the constants v_α and $\widetilde{v}_\alpha$. Moreover, supersymmetry relates these coefficients to the gauge field kinetic energy. In the Einstein metric ${G^c}_{MN} = e^{-\Phi/2}G_{MN}$, the exact dilaton dependence of the kinetic energy of the gauge field $F_{\alpha MN}$, is

$$L_{gauge} = -\frac{(2\pi)^3}{8\alpha'}\sqrt{G^c}\Sigma_\alpha\left(v_\alpha e^{-\Phi/2} + \widetilde{v}_\alpha e^{\Phi/2}\right)\mathrm{tr}F_{\alpha MN}{F_\alpha}^{MN}. \tag{7}$$

Positivity of the kinetic energy for all values of Φ thus implies that v_α and $\widetilde{v}_\alpha$ should both be non-negative, and at least one should be positive. This fails for the $SO(32)$ heterotic string, as we see from the formula for the anomaly eight-form. Some interesting new "phase transition" must occur at the value of Φ at which the $SO(28)$ coupling constant appears to change sign, and at least until this phase transition is understood, its occurrence might well obstruct simple attempts to extrapolate from a string description at large negative Φ to a dual string description at large positive Φ.

In this paper, we shall attempt to remedy these problems as follows.

$1'$) It has recently been recognised that the ten-dimensional $E_8 \times E_8$ heterotic string is related to eleven-dimensional M-theory on $\mathbf{R}^{10} \times \mathbf{S}^1/\mathbf{Z}_2$ [16], just as the ten-dimensional Type IIA string is related to M-theory on $\mathbf{R}^{10} \times \mathbf{S}^1$ [17]. By looking in two different ways at M-theory on $\mathbf{R}^6 \times K3 \times \mathbf{S}^1/\mathbf{Z}_2$, we get a definite framework for deducing string-string duality. This framework shows that the gauge group should be $E_8 \times E_8$, the vacuum gauge bundle should have equal instanton numbers in each E_8 (a situation we will refer to as symmetric embedding [4]), and the duality acts in a non-trivial fashion on the hypermultiplets.

From this eleven-dimensional point of view, one heterotic string comes by wrapping the $D = 11$ membrane around $\mathbf{S}^1/\mathbf{Z}_2$ and the dual heterotic string is obtained by reducing the $D = 11$ fivebrane on $\mathbf{S}^1/\mathbf{Z}_2$ and then wrapping around $K3$. This is quite similar to the eleven-dimensional derivation of heterotic - Type IIA duality, which is recovered if we replace $\mathbf{S}^1/\mathbf{Z}_2$ by $\mathbf{S}^1$ in the above scenario [18].[5]

$2'$) Now let us discuss the anomaly polynomial. Picking a vacuum on $K3$ with equal instanton numbers in each E_8 will break $E_8 \times E_8$ to a subgroup. Generically $E_8 \times E_8$ is completely broken, so there are no questions of whether the gauge contributions to the anomaly eight-form are compatible with duality. But we also want to understand how the duality acts on vacua with non-trivial unbroken gauge groups. For instance, in a vacuum in which the gauge bundle breaks $E_8 \times E_8$ to $E_7 \times E_7$ (a maximal possible unbroken

[4]Symmetric embedding entered naturally in [21] in constructing simple examples of heterotic/Type II duality in four dimensions.

[5]The interpretation of the heterotic string as a wrapping of a fivebrane around $K3$, or around a $K3$ sub-manifold of a Calabi-Yau manifold or Joyce manifold, is presumably the explanation for the ubiquity in string/string duality of $K3$ itself and Calabi-Yau and Joyce manifolds corresponding to fibrations of $K3$.

subgroup of $E_8 \times E_8$) the anomaly eight-form is

$$I_8 = \frac{1}{4(2\pi)^2}[\mathrm{tr}R^2 - \frac{1}{6}\mathrm{tr}F_{E_7}^2 - \frac{1}{6}\mathrm{tr}F_{E_7}^2]\frac{1}{4(2\pi)^2}[\mathrm{tr}R^2] \tag{8}$$

We see that $\widetilde{v}_\alpha = 0$, so (i) there is no wrong sign problem and one can possibly extrapolate to strong coupling without meeting a phase transition, but (ii) since $v_\alpha \neq \widetilde{v}_\alpha$, there is no manifest self-duality. Qualitatively similar results hold (that is, $\widetilde{v}_\alpha = 0$) for any other unbroken subgroup of $E_8 \times E_8$. This qualitative picture depends on having equal instanton numbers in the two E_8's; in any other case, $\widetilde{v}_\alpha < 0$ for some subgroups of $E_8 \times E_8$, and phase transitions of some kind are unavoidable.

Because of (ii), it might appear that duality is impossible, but since as in $1'$) above there is a systematic framework for deducing the duality, we are reluctant to accept this interpretation. We are led therefore to assume that the duality exchanges perturbative gauge fields, that is gauge fields of perturbative origin, with non-perturbative gauge fields. Despite the name, non-perturbative gauge fields, if they appear at all, appear no matter how small the string coupling constant may be; in fact, as the dilaton is part of a tensor multiplet in $K3$ compactification, the unbroken gauge group is independent of the string coupling constant at least if the low energy world can be described by known physics. Non-perturbative gauge fields, that is gauge fields that are not seen in perturbation theory but appear no matter how weak the string coupling constant may be, can therefore only appear at points in moduli space at which perturbation theory breaks down because of a kind of singularity.

A prototype for non-perturbative gauge fields in the heterotic string are the $SU(2)$ gauge fields that arise for the $SO(32)$ heterotic string when an instanton shrinks to zero size [23]. Such gauge fields have $v_\alpha = 0$; this can be seen either from (a) the form of the anomaly polynomial, as computed in section (4) of [38]; (b) the physical picture of [23] according to which making the heterotic string dilaton smaller causes the $SU(2)$ gauge multiplet to appear "farther down the tube," without changing its physical properties such as the gauge coupling; or (c) the description in [23] in terms of Type I D-branes, where one can explicitly compute the $SU(2)$ gauge coupling and compare to (7).[6] In contrast to non-perturbative gauge fields which have

[6]The vact that $v = 0$ for these non-perturbative gauge fields was in essence also noted by V. Kaplunovsky.

$v_\alpha = 0$, perturbative gauge fields always have $v_\alpha > 0$. In fact, as discussed in Appendix B, v_α is essentially the Kac-Moody level.

Thus, our proposal is that heterotic string-string duality, at least in the case of the symmetric embedding in $E_8 \times E_8$, exchanges perturbative gauge fields of $\widetilde{v}_\alpha = 0$ with non-perturbative gauge fields of $v_\alpha = 0$. An interesting conspiracy of factors makes this perhaps radical-sounding proposal possible. (A) Since non-perturbative gauge fields can only arise at particular loci in hypermultiplet moduli space (where a singularity develops in the $K3$ manifold or its gauge bundle, giving a possible breakdown of perturbation theory as in [23]), the proposal is possible only because with the symmetric embedding in the gauge group, $E_8 \times E_8$ is generically completely broken, and perturbative gauge fields only appear at particular loci in hypermultiplet moduli space. (B) Since the loci in hypermultiplet moduli space which are candidates for nonperturbative gauge fields (because of a singularity in the manifold or the gauge bundle) are different from the loci where symmetry breaking is partly turned off and unbroken perturbative gauge fields appear, the proposal is possible only because the mechanism for string-string duality alluded to in $1'$) gives a duality that acts non-trivially on the hypermultiplets. (C) It is essential that with the symmetric embedding promised in $1'$), one has $\widetilde{v}_\alpha = 0$ for all perturbative gauge fields. If indeed one had $\widetilde{v}_\alpha < 0$ in some case, one would have to face the issue of the phase transition implied by the wrong sign gauge kinetic energy. On the other hand, perturbative gauge fields with $\widetilde{v}_\alpha > 0$ would have to be dual to perturbative gauge fields, leading to a contradiction given that one does not have manifest duality of the perturbative gauge fields.

Once one accepts that after exchanging perturbative and non-perturbative gauge fields, the v_α and the $\widetilde{v}_\alpha$ are equal, the equality of the numerical coefficents appearing in the gauge kinetic terms (7), and in the field equations and Bianchi identities (4), (3) means that there may now be a full-fledged *self-duality* of the $D = 6$ string extending the symmetry of the low energy supergravity and acting on some of the massless fields by (1):

$$\Phi \to -\Phi$$

$$\begin{aligned} G_{MN} &\to e^{-\Phi} G_{MN} \\ H &\to e^{-\Phi} * H \end{aligned} \tag{9}$$

2 The Fundamental String on $K3$ And The Low Energy Supergravity

$K3$ compactification of a chiral string theory in ten dimensions gives a chiral six-dimensional theory. $K3$ is a four-dimensional compact closed simply-connected manifold. It is equipped with a self-dual metric with holonomy group $SU(2)$. It was first considered in a Kaluza-Klein context in [24, 25] where it was used, in particular, as a way of compactifying $D = 11$ supergravity to $D = 7$ and $D = 10$ supergravity to $D = 6$. Our interest here is in $K3$ compactification of the heterotic string. Because of the $SU(2)$ holonomy, half the supersymmetry survives in $K3$ compactification, and hence, starting from $N = 1$ supergravity in $D = 10$, we get $N = 1$ supergravity in $D = 6$ (which has half as many supercharges).

There are four massless $N = 1, D = 6$ supermultiplets to consider:

$$
\begin{array}{ll}
\textit{Supergravity Multiplet} & G_{MN}, \Psi_M^{A+}, B^+{}_{MN} \\
\textit{Tensor Multiplet} & B^-{}_{MN}, \chi^{A-}, \Phi \\
\textit{Hypermultiplet} & \psi^{a-}, \phi^\alpha \\
\textit{Yang-Mills Multiplet} & A_M, \lambda^{A+}
\end{array}
$$

All spinors are symplectic Majorana–Weyl. The two-forms $B^+{}_{MN}$ and $B^-{}_{MN}$ have three-form field strengths that are self-dual and anti-self-dual, respectively. Only with precisely one tensor multiplet added to the supergravity multiplet is there a conventional covariant Lagrangian formulation. In $K3$ compactification, the zero modes of the supergravity multiplet in $D = 10$ give this combination plus 20 massless matter hypermultiplets. The 80 scalars in those multiplets parametrize the coset $SO(20,4)/SO(20) \times SO(4)$ [26, 27, 28], which is the moduli space of conformal field theories on $K3$. No vector multiplets come from the ten-dimensional supergravity multiplet since $K3$ has no isometries and is simply connected.

Six-dimensional vector multiplets and additional hypermultiplets come from reduction on $K3$ of the ten-dimensional gauge group $SO(32)$ or $E_8 \times E_8$, as was first analyzed in [29]. An important constraint comes from the anomaly cancellation equation $dH = (\alpha'/4)\,(\mathrm{tr}R^2 - \sum_\alpha v_\alpha \mathrm{tr}F_\alpha^2)$ which was discussed in the Introduction. A global solution for H exists if and only if the integral over $K3$ of $\mathrm{tr}R^2$ equals that of $\sum_\alpha v_\alpha \mathrm{tr}F_\alpha^2$. This amounts to the statement that the vacuum expectation value of the $SO(32)$ or $E_8 \times E_8$ gauge fields must be a configuration with instanton number 24. In the $E_8 \times E_8$ case, it is the sum of the instanton numbers in the two E_8's that must equal 24. We will be interested mainly in the "symmetric embedding," the case where the instanton number is 12 in each E_8.

With instanton number 12 in each E_8, the generic $E_8 \times E_8$ instanton on $K3$ completely breaks the gauge symmetry. This will be explained in more detail in section (4), together with generalizations. Unbroken gauge symmetry arises if the vacuum gauge bundle takes values in a subgroup G of $E_8 \times E_8$, in which case the unbroken subgroup of $E_8 \times E_8$ is the commutant G of H, that is the subgroup of $E_8 \times E_8$ that commutes with H. The G quantum numbers of the massless hypermultiplets can be determined by decomposing the adjoint representation of $E_8 \times E_8$ under $G \times H$, as in [29].

The Anomaly Polynomial

Now let us explain some statements made in the Introduction about the anomaly polynomials. Let F_i, $i = 1, 2$ be the field strengths of the two E_8's. Let $\mathrm{Tr}{F_i}^2$ be the traces in the adjoint representations of the two E_8's, and let $\mathrm{tr}{F_i}^2 = (1/30)\,\mathrm{Tr}{F_i}^2$. In ten dimensions, the anomaly twelve-form I_{12} factorizes as $I_{12} = X_4 \widetilde{X}_8$, with

$$X_4 = \frac{1}{4(2\pi)^2}[\mathrm{tr}R^2 - \sum_i \mathrm{tr}{F_i}^2] \tag{10}$$

and $\widetilde{X}_8$ the more imposing expression

$$\begin{aligned} \widetilde{X}_8 \sim \frac{3}{4}\left((\mathrm{tr}{F_1}^2)^2 + (\mathrm{tr}{F_2}^2)^2\right) - \frac{1}{4}\left(\mathrm{tr}{F_1}^2 + \mathrm{tr}{F_2}^2\right)^2 \\ - \frac{1}{8}\mathrm{tr}R^2\left(\mathrm{tr}{F_1}^2 + \mathrm{tr}{F_2}^2\right) + \frac{1}{8}\mathrm{tr}R^4 + \frac{1}{32}\left(\mathrm{tr}R^2\right)^2. \end{aligned} \tag{11}$$

The six-dimensional anomaly four-form $\widetilde{X}_4$ is obtained by integrating $\widetilde{X}_8$ over $K3$. Also in writing an anomaly four-form in six dimensions, one understands the six-dimensional field strengths F_1 and F_2 to take values in the unbroken subgroup of the gauge group, that is the part that commutes with the gauge bundle on $K3$. If we let $\langle R^2\rangle$, $\langle {F_1}^2\rangle$, and $\langle {F_2}^2\rangle$ denote the integrals of $\mathrm{tr}R^2$, $\mathrm{tr}{F_1}^2$, and $\mathrm{tr}{F_2}^2$ over $K3$, then

$$\widetilde{X}_4 \sim \mathrm{tr}{F_1}^2\left(\frac{1}{2}\langle {F_1}^2\rangle - \frac{1}{4}\langle {F_2}^2\rangle - \frac{1}{8}\langle R^2\rangle\right)$$

$$+ \,\mathrm{tr}F_2{}^2 \left(\frac{1}{2}\langle F_2{}^2\rangle - \frac{1}{4}\langle F_1{}^2\rangle - \frac{1}{8}\langle R^2\rangle\right) + \mathrm{tr}R^2 \left(\frac{1}{16}\langle R^2\rangle - \frac{1}{8}(\langle F_1{}^2\rangle + \langle F_2{}^2\rangle)\right). \tag{12}$$

The topological condition on the vacuum gauge bundle that was explained above amounts to

$$\langle F_1{}^2\rangle + \langle F_2{}^2\rangle - \langle R^2\rangle = 0. \tag{13}$$

With the use of this equation, one sees that the coefficients $\widetilde{v}_i$ of $\mathrm{tr}F_1{}^2$ and $\mathrm{tr}F_2{}^2$ in $\widetilde{X}_4$ are equal and opposite. So the "wrong sign" problem explained in the Introduction is avoided if and only if the $\widetilde{v}_i$ vanish. (Otherwise, the problem arises in the E_8 that has smaller instanton number.) From the above formulas, the condition for this is that $\langle F_1{}^2\rangle = \langle F_2{}^2\rangle$, that is the two E_8 gauge bundles have equal instanton numbers. So we have recovered the statement in the Introduction that in $E_8 \times E_8$, the sign problem is avoided only for the symmetric embedding, for which the $\widetilde{v}_i$ vanish for all subgroups of $E_8 \times E_8$. A similar analysis for $SO(32)$ recovers the anomaly formula given in the Introduction and in particular shows the occurrence of the sign problem for $SO(32)$.

Further Aspects Of The Low Energy Supergravity

As a guide to the kind of dualities one might expect in the string theory, let us look in more detail at the corresponding $N = 1, D = 6$ supergravity theories. We shall follow [20] but with the following modifications: by choosing just one tensor multiplet we may write a covariant Lagrangian as well as covariant field equations; we use the string σ-model metric so as to emphasize the tree-level plus one-loop nature of the Lagrangian; we also write the coupling in terms of the string slope parameter α'; we shall also include *Lorentz* as well as Yang-Mills corrections to the Bianchi indentities and field equations. We shall denote the contribution to the action of L fundamental string loops and $\widetilde{L}$ dual string loops by $I_{L,\widetilde{L}}$. The bosonic part of the action of the takes the form

$$I = I_{00} + I_{01} + I_{10} + ... \tag{14}$$

where

$$I_{00} = \frac{(2\pi)^3}{\alpha'^2} \int d^6x \sqrt{-G} e^{-\Phi} [R_G + G^{MN}\partial_M\Phi\partial_N\Phi - \frac{1}{12} G^{MQ}G^{NR}G^{PS}H_{MNP}H_{QRS}] \tag{15}$$

where $M, N = 0, ..., 5$ are spacetime indices and H is the curl of a 2-form B, where

$$I_{01} = \frac{(2\pi)^3}{8\alpha'} \int d^6x \sqrt{-G} e^{-\Phi} [G^{MP} G^{NQ} \mathrm{tr} R_{MN} R_{PQ} - \Sigma_\alpha v_\alpha G^{MP} G^{NQ} \mathrm{tr} F_{\alpha MN} F_{\alpha PQ}] \tag{16}$$

together with Chern-Simons corrections to H appropriate to (3), and where

$$I_{10} = \frac{(2\pi)^3}{8\alpha'} \int d^6x \sqrt{-G} [G^{MP} G^{NQ} \mathrm{tr} R_{MN} R_{PQ} - \Sigma_\alpha \widetilde{v}_\alpha G^{MP} G^{NQ} \mathrm{tr} F_{\alpha MN} F_{\alpha PQ} - 2\pi \int_{M_6} \Big(\frac{1}{(2\pi)^2 \alpha'} B \widetilde{X}_4 + \frac{1}{3} \omega_3 \widetilde{\omega}_3 \Big) \tag{17}$$

Here ω_3 and $\widetilde{\omega}_3$ obey $2d\omega_3 = X_4$ and $2d\widetilde{\omega}_3 = \widetilde{X}_4$. This last term ensures that H obeys the field equations appropriate to (4). The metric G_{MN} is related to the canonical Einstein metric ${G^c}_{MN}$ by

$$G_{MN} = e^{\Phi/2} {G^c}_{MN} \tag{18}$$

where Φ the $D = 6$ dilaton. There will also be couplings to the hypermultiplets, both charged and neutral, which we shall not attempt to write down. They will belong to some quaternionic manifold, which is probably quite complicated.

The most obvious dual supergravity action is given by a similar expression obtained by replacing each field with its dual counterpart according to the following duality dictionary:

$$\begin{aligned} \widetilde{\Phi} &= -\Phi \\ \widetilde{G}_{MN} &= e^{-\Phi} G_{MN} \\ \widetilde{H} &= e^{-\Phi} * H \\ \widetilde{A}_M &= A_M \end{aligned} \tag{19}$$

where $*$ denotes the Hodge dual. (Since the H equation is conformally invariant, it is not necessary to specify which metric is chosen in forming the dual.) The dual metric $\widetilde{G}_{MN}$ is related to the canonical Einstein metric by

$$\widetilde{G}_{MN} = e^{-\Phi/2} {G^c}_{MN} \tag{20}$$

It is also possible (and will be necessary in our application) to combine the duality just described with a transformation of the hypermultiplets and a permutation of the various possible factors in the gauge group:

$$\alpha \to \pi(\alpha) \tag{21}$$

where $\pi(\alpha)$ is the gauge group into which the duality maps the gauge group α. With or without such a transformation of hypermultiplets, the above dictionary achieves just the right interchange of tree-level and one loop effects required by heterotic/heterotic duality, namely

$$I_{10} \leftrightarrow I_{01} \tag{22}$$

In particular, with $\widetilde{H}$ the field strength of a two-form $\widetilde{B}$, with Chern-Simons corrections appropriate to (4), this duality exchanges the Bianchi identities (3) and field equations (4). When the permutation $\alpha \to \pi(\alpha)$ is taken into account, we have hopefully

$$v_\alpha = \widetilde{v}_{\pi(\alpha)}, \tag{23}$$

reflecting the discrete symmetry (9).

3 Deduction Of String-String Duality From Eleven Dimensions

To *deduce* heterotic-heterotic duality on $K3$, we begin with the eleven-dimensional M-theory on $\mathbf{R}^6 \times K3 \times \mathbf{S}^1/\mathbf{Z}_2$. By looking at this theory in two different ways, we will deduce a duality between heterotic strings. On the one hand, we use the fact that the M-theory on $Y \times \mathbf{S}^1/\mathbf{Z}_2$, for any Y, is equivalent to the $E_8 \times E_8$ heterotic string on Y, with a string coupling constant that is small as the radius of the $\mathbf{S}^1/\mathbf{Z}_2$ shrinks. On the other hand, we use the fact that the M-theory on $Z \times K3$, for any Z, is equivalent to the heterotic string on $Z \times \mathbf{T}^3$, with a string coupling constant that is small when the $K3$ shrinks. The point of starting with $W = \mathbf{R}^6 \times K3 \times \mathbf{S}^1/\mathbf{Z}_2$ is that it can be written as either $Y \times \mathbf{S}^1/\mathbf{Z}_2$, with $Y = \mathbf{R}^6 \times K3$, or as $Z \times K3$, with $Z = \mathbf{R}^6 \times \mathbf{S}^1/\mathbf{Z}_2$.

If we look at W as $Y \times \mathbf{S}^1/\mathbf{Z}_2$, then we deduce that as the $\mathbf{S}^1/\mathbf{Z}_2$ becomes small, the M-theory on W is equivalent to the $E_8 \times E_8$ heterotic string on $Y = \mathbf{R}^6 \times K3$.

A little more subtlety is required if we try to look at W as $Z \times K3$. From this vantage point, it appears that as the $K3$ shrinks, we should get a weakly coupled heterotic string on $Z \times \mathbf{T}^3 = \mathbf{R}^6 \times \mathbf{S}^1/\mathbf{Z}_2 \times \mathbf{T}^3$. This cannot be the right answer, as $\mathbf{R}^6 \times \mathbf{S}^1/\mathbf{Z}_2 \times \mathbf{T}^3$ is unorientable, and the parity-violating heterotic string cannot be formulated on this space. One must note that when one divides $\mathbf{R}^6 \times \mathbf{S}^1 \times K3$ by $\mathbf{Z}_2$ to get the M-theory on $\mathbf{R}^6 \times \mathbf{S}^1/\mathbf{Z}_2 \times K3$, the three-form potential A of the low energy limit of the M-theory is odd under the $\mathbf{Z}_2$. Compactified on $K3$, the three-form gives 22 vector fields that come from the two-dimensional cohomology of $K3$; these are related to the momentum and winding modes and Wilson lines of the heterotic string on $\mathbf{T}^3$. For all the momentum and winding modes of the heterotic string to be odd under the $\mathbf{Z}_2$ means that the $\mathbf{Z}_2$ must act as -1 on the $\mathbf{T}^3$. So when we shrink the $K3$ factor in the M-theory on $\mathbf{R}^6 \times \mathbf{S}^1/\mathbf{Z}_2 \times K3$, we get a heterotic string on not $\mathbf{R}^6 \times \mathbf{S}^1/\mathbf{Z}_2 \times \mathbf{T}^3$ but $\mathbf{R}^6 \times \left(\mathbf{S}^1 \times \mathbf{T}^3\right)/\mathbf{Z}_2$.

Now roughly speaking $\left(\mathbf{S}^1 \times \mathbf{T}^3\right)/\mathbf{Z}_2 = \mathbf{T}^4/\mathbf{Z}_2$ is a $K3$ orbifold, so we have arrived again at a heterotic string on $K3$. Actually, there are a few subtleties hidden here. For one thing, in general we will not really get in this way a $K3$ orbifold. When the M-theory is formulated on $\mathbf{R}^6 \times \mathbf{S}^1/\mathbf{Z}_2 \times K3$, there are propagating E_8 gauge fields on each copy of $\mathbf{R}^6 \times K3$ coming from a fixed point in the $\mathbf{Z}_2$ action on $\mathbf{S}^1$, and a specification of vacuum requires picking a $K3$ instanton on each E_8. A choice of such an instanton represents, at least generically, a departure from a strict orbifold vacuum. Since the vacuum on $\mathbf{R}^6 \times \mathbf{S}^1/\mathbf{Z}_2 \times K3$ was not really an orbifold vacuum, the same will be true by the time we get to a heterotic string on $\mathbf{R}^6 \times \left(\mathbf{S}^1 \times \mathbf{T}^3\right)/\mathbf{Z}_2$.

The fact that M-theory on $\mathbf{R}^6 \times \mathbf{S}^1/\mathbf{Z}_2 \times K3$ turns into a weakly coupled heterotic string in two different limits is a kind of duality between heterotic strings. From the point of view of either one of these limits, the other one is strongly coupled; we will be more precise about this below. An observer studying one of the two limiting heterotic strings sees a strongly coupled limit in which there is a weakly coupled description by a different heterotic string; this is heterotic-heterotic duality.

A few points should still be explained: *(i)* The duality is an electric-magnetic string-string duality in the sense described in the Introduction. *(ii)* The duality acts non-trivially on the hypermultiplets, a fact whose importance was explained in the Introduction. *(iii)* The construction – which obviously requires that the heterotic string gauge group be $E_8 \times E_8$ – works only for the symmetric embedding with equal instanton numbers in the two

E_8's.

The first point is a simple consequence of eleven-dimensional facts. [7] We begin with the fact that the eleven-dimensional M-theory has two-branes and five-branes that are electric-magnetic duals. Consider in general a Kaluza-Klein vacuum in a theory containing a p-brane; for simplicity consider the illustrative case that the vacuum is $Q \times \mathbf{S}^1$ for some Q. A p-brane can be wrapped around $\mathbf{S}^1$, giving a $(p-1)$-brane on Q. Or a p-brane can be "reduced" on $\mathbf{S}^1$, by which we mean simply that one takes the p-brane to be localized at a point on $\mathbf{S}^1$. This gives a p-brane on Q, with the position on $\mathbf{S}^1$ seen an a massless world-volume mode. The two operations of wrapping and reduction are electric-magnetic duals, so that if one starts with dual p-branes and q-branes, the wrapping of one around $\mathbf{S}^1$ and reduction of the other on $\mathbf{S}^1$ gives dual objects on Q.

Now let us apply this wisdom to M-theory on $\mathbf{R}^6 \times \mathbf{S}^1/\mathbf{Z}_2 \times K3$. In eleven dimensions, the M-theory has dual two-branes and five-branes. When the $\mathbf{S}^1/\mathbf{Z}_2$ shrinks, an effective heterotic string in ten dimensions is obtained by wrapping the two-brane around $\mathbf{S}^1/\mathbf{Z}_2$, giving a one-brane which was seen in [16] to have the world-sheet structure of an $E_8 \times E_8$ heterotic string. An effective six-dimensional heterotic string is then obtained by reduction on $K3$. On the other hand, when the $K3$ shrinks, an effective heterotic string is obtained by wrapping the five-brane around $K3$. An effective six-dimensional heterotic string is then obtained by reduction on $\mathbf{S}^1/\mathbf{Z}_2$. Since wrapping the five-brane around $K3$ and reducing it on $\mathbf{S}^1/\mathbf{Z}_2$ is dual to wrapping the membrane around $\mathbf{S}^1/\mathbf{Z}_2$ and reducing it on $K3$, the two effective six-dimensional heterotic strings are electric-magnetic duals to each other in the sense described in the Introduction. This provides an answer to question *(i)* above. To further confirm our understanding, we compute below the six-dimensional string coupling constants of the two heterotic string theories, and show that they are inverses of each other, as expected for a pair of six-dimensional dual strings.

Now we come to question *(ii)*, which is to show that this duality is not the minimal duality suggested by low energy supergravity, but acts non-trivially on the hypermultiplets. In fact, begin with M-theory on $\mathbf{R}^6 \times \mathbf{S}^1/\mathbf{Z}_2 \times K3$, with some modulus for the $K3$ and with a particular choice of E_8 gauge bundles at fixed points. If one shrinks the $\mathbf{S}^1/\mathbf{Z}_2$ one simply gets a heterotic

[7]The following two paragraphs benefited from a discussion with P. Horava.

string on the *same* $K3$, with the same E_8 gauge bundles, that one started with. If instead one shrinks the $K3$, one has an adventure described above involving a non-orbifold vacuum on $(\mathbf{S}^1 \times \mathbf{T}^3)/\mathbf{Z}_2$. This is presumably to be interpreted as a $K3$ (in fact, any $(0,4)$ conformal field theory of the appropriate central charge is believed to describe a $K3$ with a vector bundle), but it certainly does not look like the $K3$ that we started with.[8] We take this to mean that the $K3$ associated with the dual string (obtained by wrapping the five-brane) is *not* the same as the $K3$ we started with in eleven-dimensions, or differently put that the duality acts non-trivially on the hypermultiplets, which are the moduli of $K3$ and the vector bundle. In fact, the action on the hypermultiplets looks rather complicated, and understanding it better would be an important step.

Finally we come to question *(iii)*. As we have presented the eleven-dimensional construction so far, the assignment of instanton numbers to the two E_8 gauge bundles does not seem to matter. But we claim that actually, if examined more closely, the construction works only for the symmetric embedding, that is for equal instanton numbers in the two E_8's. The three-form potential of eleven-dimensional supergravity has a four-form field strength K. M-theory compactifications on $K3$ can be distinguished according to the quantized value of the flux [18]

$$\int_{K3} K = \frac{2\pi m}{T_3}, \qquad m = integer \tag{24}$$

where T_3 is the membrane tension. $K3$ compactification of M-theory has usually been discussed only for $m = 0$. In particular, the statement that when a $K3$ shrinks, one gets a heterotic string with the $K3$ replaced by $\mathbf{T}^3$ holds for $K3$'s with $m = 0$. Intuitively, one would expect $m \neq 0$ to change the behavior that occurs when one tries to shrink a $K3$, because the energy stored in the trapped K field would resist this shrinking.

[8] We have not yet given any explicit argument that the dual heterotic string is an $E_8 \times E_8$ theory with symmetric embedding. That the gauge group of the dual string is $E_8 \times E_8$ rather than $SO(32)$ we infer from the fact that, if one starts with suitable gauge bundles in eleven dimensions, unbroken exceptional gauge groups such as $E_7 \times E_7$ are possible. Moreover, if one starts with the symmetric embedding in eleven dimensions – whose necessity we argue for presently – then there is a symmetry of $\mathbf{S}^1/\mathbf{Z}_2$ that exchanges the two fixed points and the two E_8's. This will carry over in the dual heterotic string theory to a symmetry that exchanges the two E_8's, and the existence of this symmetry indicates that the dual heterotic string has equal instanton numbers in the two E_8's.

One can actually be more precise. The dual heterotic string that arises when one shrinks a $K3$ comes from a five-brane wrapped around the $K3$. But a five-brane cannot wrap around a $K3$ (or any four-manifold) that has $m \neq 0$. The reason for this is that the world-volume spectrum of the five-brane includes a massless two-form with an anti-self-dual three-form field strength T. T does not obey $dT = 0$, but rather (as one can see from equation (3.3) of [40]; another argument is given in [41]) it obeys $dT = K$. The existence of a solution for T means that K must be cohomologically trivial when restricted to the five-brane world-volume; that is, the five-brane cannot wrap around a four-manifold with $m \neq 0$.

The reason that this is relevant is that, as we will argue momentarily, if one compactifies the M-theory on $\mathbf{R}^6 \times \mathbf{S}^1/\mathbf{Z}_2 \times K3$ with instanton number k in one E_8 and $24 - k$ in the other E_8, then the flux of the K field over $K3$ (that is, over any $K3$ obtained by restricting to a generic point in $\mathbf{R}^6 \times \mathbf{S}^1/\mathbf{Z}_2$) is $m = \pm(12 - k)$. (The sign will be explained later.) Therefore, $m = 0$ if and only if $k = 12$, that is, precisely for the symmetric embedding. The eleven-dimensional explanation of heterotic - heterotic duality thus requires $E_8 \times E_8$ with the symmetric embedding.

It remains, then, to explain the relation $m = \pm(12 - k)$. This relation arises upon writing the anomaly cancellation condition

$$dH = \frac{\alpha'}{4} \left(\mathrm{tr} R \wedge R - \mathrm{tr} F_1 \wedge F_1 - \mathrm{tr} F_2 \wedge F_2 \right) \tag{25}$$

in eleven-dimensional terms. The eleven-dimensional version of that equation must involve the five-form dK instead of the four-form dH. This requires incorporating on the right hand side delta-functions supported at the fixed points. If $\mathbf{S}^1$ is parametrized by an angular variable x^{11} such that the $\mathbf{Z}_2$ fixed points are at $x^{11} = 0$ and π, with F_1 supported at the first and F_2 supported at the second, then the eleven-dimensional version of (25) is

$$\begin{gathered} dK = \frac{1}{2\pi T_3} dx^{11} \times \\ \left(\delta(x^{11}) \left(\frac{1}{2} \mathrm{tr} R \wedge R - \mathrm{tr} F_1 \wedge F_1 \right) + \delta(x^{11} - \pi) \left(\frac{1}{2} \mathrm{tr} R \wedge R - \mathrm{tr} F_2 \wedge F_2 \right) \right). \end{gathered} \tag{26}$$

This equation is determined by the following properties: dK vanishes except at fixed points, since (in the absence of five-branes) $dK = 0$ in the eleven-dimensional theory; F_1 and F_2 contribute only at the appropriate values of x^{11}; the two fixed points enter symmetrically; if one integrates over x^{11} and interprets H as the part of the zero mode of K with one index equal to 11, then (26) reduces to (25). Now, let $m(x^{11})$ be the function obtained by integrating $T_3F/2\pi$ over $K3$ at a given value of x^{11}. The $\mathbf{Z}_2$ symmetry implies that $m(-x^{11}) = -m(x^{11})$, and (26) means that $m(x^{11})$ is constant except for jumps at $x^{11} = 0$ or π, the magnitude of the jump being $(2/8\pi^2)\int_{K3}\left(\frac{1}{2}\mathrm{tr}R\wedge R - \mathrm{tr}F_1\wedge F_1\right) = -(2/8\pi^2)\int_{K3}\left(\frac{1}{2}\mathrm{tr}R\wedge R - \mathrm{tr}F_2\wedge F_2\right)$. Hence the constant value of $m(x^{11})$ away from a fixed point, which we earlier called $\pm m$, is

$$\pm(1/8\pi^2)\int_{K3}\left(\frac{1}{2}\mathrm{tr}R\wedge R - \mathrm{tr}F_1\wedge F_1\right).$$

This amounts to the statement that $m = \pm(12-k)$, with k the instanton number in the first E_8, supported at $x^{11} = 0$. This confirms the claim made above and so completes our explanation of why the eleven-dimensional approach to heterotic - heterotic duality requires the symmetric embedding as well as requiring gauge group $E_8\times E_8$.

Anomaly Cancellation By Five-Branes

We cannot resist mentioning an application of these ideas that is somewhat outside our main theme. The equation (26) shows that the curvature $\mathrm{tr}\,R\wedge R$ of $K3$ gives a magnetic source for the K field, that is a contribution to dK supported at fixed points. This is a sort of "anomaly" that must be canceled, since the integral of dK over the compact space $K3\times\mathbf{S}^1/\mathbf{Z}_2$ will inevitably vanish. The conventional string theory way to cancel this anomaly is to use $E_8\times E_8$ instantons on $K3$, using the fact that $-\mathrm{tr}\,F_i\wedge F_i$ also contributes to dK. From this point of view, the "magnetic charge" associated with the $\mathrm{tr}\,R\wedge R$ contribution to dK can be canceled by 24 $E_8\times E_8$ instantons.

There is, however, another standard entity that can contribute to dK; this is the eleven-dimensional five-brane, which is characterized by the fact that dK has a quantized delta-function contribution supported on the five-brane world-volume. This suggests that instead of canceling the total contribution to dK with 24 $E_8\times E_8$ instantons, one could use 23 instantons on the fixed points and one five-brane at some generic point in $K3\times\mathbf{S}^1/\mathbf{Z}_2$. More generally, one could use $24-n$ instantons (distributed as one wishes between the two E_8's) and n five-branes.

The following is a strong indication that this is correct. Supported on the five-brane world-volume is one tensor multiplet and one hypermultiplet in the sense of $N = 1$ supergravity in $D = 6$. (The hypermultiplet parametrizes the position of the five-brane on $K3$.) Instead, associated with each $E_8 \times E_8$ instanton are precisely 30 hypermultiplets (a number that can be seen in the instanton dimension formula with which we begin the next section). Both the tensor multiplet and the hypermultiplet contribute to the irreducible part of the gravitational anomaly in six dimensions (the part that cannot be canceled by a Green-Schwarz mechanism). From equation (118) of [42], one can see that the contribution to this irreducible anomaly of one tensor multiplet and one hypermultiplet equals that of 30 hypermultiplets, strongly suggesting that the M-theory vacua with $24 - n$ instantons and n-fivebranes (and therefore $n + 1$ tensor multiplets in six dimensions) really do exist. These cannot be related to perturbative heterotic strings, but they might have limits as Type I orientifolds, as in [20]. They are somewhat reminiscent of the M-theory vacua with wandering five-branes found in [41].

Analysis Of The Couplings

We now return to the M-theory on $\mathbf{R}^6 \times \mathbf{S}^1/\mathbf{Z}_2 \times K3$, with the intention of examining somewhat more quantitatively the two limits in which it is related to a heterotic string – the limits in which the $\mathbf{S}^1/\mathbf{Z}_2$ or the $K3$ shrinks to small volume. We want to show that in either such limit, the heterotic string that emerges is *weakly coupled*. In fact, part of the meaning of any claim that the M-theory turns into a string theory in a particular limit should be that the resulting string theory is weakly coupled. We will also, more precisely, show that the coupling constant of the heterotic string obtained by shrinking the $\mathbf{S}^1/\mathbf{Z}_2$ is the inverse of the coupling of the heterotic string obtained by shrinking the $K3$. This is what one would expect given that these strings are electric - magnetic duals. The calculations we will need are quite straightforward given formulas in [2]. In these computations we will not keep track of some absolute constants.

We let R be the radius of $\mathbf{S}^1/\mathbf{Z}_2$, measured with respect to the metric of eleven-dimensional supergravity, and we let V be the volume of the $K3$, likewise measured in eleven-dimensional terms. According to [16, 2], as R goes to zero with large V, one gets a heterotic string with the ten-dimensional string coupling constant being $\lambda_{10} = R^{3/2}$. Also, the string metric differs from the eleven-dimensional metric by a Weyl rescaling, such that the $K3$ volume measured in the string metric is $V_{st} = VR^2$. The six-dimensional string coupling constant λ_6 obeys the standard relation $1/\lambda_6^2 = V_{st}/\lambda_{10}^2$, so we get

$$\lambda_6^2 = \frac{\lambda_{10}^2}{V_{st}} = \frac{R}{V}. \tag{27}$$

This shows that the six-dimensionl string coupling constant is small if $R << V$.

Now we consider the opposite limit in which the $K3$ shrinks. Then one gets in seven dimensions a dual heterotic string $\mathbf{R}^6 \times \mathbf{S}^1/\mathbf{Z}_2$ with (according to formulas in [2]) a seven-dimensional string coupling constant $\widetilde{\lambda}_7 = V^{3/4}$. Also, the string metric of the dual heterotic string theory differs by a Weyl rescaling from the eleven-dimensional metric, such that the radius of the $\mathbf{S}^1/\mathbf{Z}_2$ in the dual string metric is $\widetilde{R}_{st} = V^{1/2}R$. The six-dimensional dual string coupling constant obeys

$$\widetilde{\lambda}_6^2 = \frac{\widetilde{\lambda}_7^2}{R_{st}} = \frac{V}{R}. \tag{28}$$

Putting these formulas together, we get $\lambda_6 = 1/\widetilde{\lambda}_6$, as expected from the fact that the string obtained by wrapping the two-brane around $\mathbf{S}^1/\mathbf{Z}_2$ is dual to the string obtained by wrapping the five-brane around $K3$.

Behavior Under Further Compactification

It is interesting to consider further toroidal compactification to four dimensions, replacing $\mathbf{R}^6$ by $\mathbf{R}^4 \times \mathbf{T}^2$. Starting with a $K3$ vacuum in which the $E_8 \times E_8$ gauge symmetry is completely Higgsed, the toroidal compactification to four dimensions gives an $N = 2$ theory with the usual three vector multiplets S, T and U related to the four-dimensional heterotic string coupling constant and the area and shape of the $\mathbf{T}^2$. When reduced to four dimensions, the six-dimensional string-string duality (9) becomes [14] an operation that exchanges S and T. Since the heterotic string on $\mathbf{T}^2 \times K3$ also has $R \to 1/R$ symmetries that exchange T and U, this is an example with complete $S - T - U$ triality symmetry, as discussed in [5].

Kachru and Vafa [21] made a proposal for a Type II dual of the $E_8 \times E_8$ heterotic string on $\mathbf{T}^2 \times K3$ with this precise vacuum, that is equal instanton numbers in the two E_8's and complete Higgsing. Some evidence for the $S-T$ interchange symmetry has appeared in subsequent study of this example [35, 36, 37].

4 The Duals Of Some Unbroken Gauge Groups

The moduli space $\mathcal{M}_k(E_8)$ of E_8 instantons on $K3$ with instanton number k has a dimension (predicted from the index formula) which is

$$\dim\, \mathcal{M}_k(E_8) = 120k - 992 \tag{29}$$

if k is sufficiently big. We will make frequent use of the special case

$$\dim\, \mathcal{M}_{12}(E_8) = 448. \tag{30}$$

The formula for dim $\mathcal{M}_k$ actually gives the the correct dimension of the moduli space if k is large enough that a $K3$ instanton of instanton number k can completely break the E_8 gauge symmetry. A necessary condition for this to be possible is that the right hand side of (29) must be positive, restricting us to $k \geq 9$. We have checked that complete Higgsing is possible for $k \geq 10$ and do not know if it is possible for $k = 9$.[9] We note that complete Higgsing may be possible for $k = 9$ in conformal field theory even if it does not occur in classical geometry. [10]

The generalization of (29) for an arbitrary simple Lie group G with dual Coxeter number h and dimension dim G is

$$\dim \mathcal{M}_k(G) = 4hk - 4\, \dim\, G, \tag{31}$$

[9] The check for $k = 10$ can be made for instance by starting with instanton number 10 in an $SU(2)$ subgroup of E_8 (a configuration that is possible by standard existence theorems), breaking E_8 to E_7 and giving a low energy spectrum that consists of six **56**'s of E_7. (Note that as the **56** is pseudoreal, it is possible for E_7 to act on 28 hypermultiplets transforming in the **56** of E_7. The spectrum consists of six copies of this.) Sequential Higgsing, turning on the expectation values of successive **56**'s, can then be seen to completely break E_7. For $k = 9$, a similar construction gives five **56**'s of E_7, and sequential Higgsing now leads to a vacuum with an unbroken level one $SU(3)$. Sequential Higgsing does not always give all the possible vacua, as shown in an explicit example in [43], and there may be other branches, but at any rate there is one branch of the $k = 9$ moduli space in which E_8 is generically broken to $SU(3)$.

[10] Six-dimensional supersymmetry permits "phase transitions" in which branches of the moduli space of vacua with different generic unbroken gauge groups meet at a point of enhanced gauge symmetry. For example, according to [43], a branch with generic unbroken $SU(3)$ and a branch with generic complete Higgsing can meet at a point at which the unbroken gauge symmetry is $SU(6)$, with a hypermultiplet spectrum consisting of six **6**'s of $SU(6)$. The necessary enhanced gauge symmetry might well occur in conformal field theory but not in classical geometry.

valid whenever complete Higgsing is possible and in particular whenever k is sufficiently big.

For our problem of E_8 instantons with instanton number 12, complete Higgsing is possible, and the gauge group is generically completely broken. On suitable loci in moduli space, with the property that the instantons fit into a subgroup H of E_8, a subgroup of E_8 is restored – namely the subgroup G that commutes with H, known as the commutant of H. When the vacuum gauge field reaches such a locus, perturbative G gauge fields will appear. According to our discussion in the last section, the un-Higgsing or restoration of G will be dual to the appearance, on some other locus, of a non-perturbative gauge invariance with a gauge group isomorphic to G.

Non-perturbative gauge fields, that is gauge fields that are not seen in conformal field theory but appear no matter how weak the string coupling constant may be, can only arise when the $K3$ or the vacuum gauge bundle develops a singularity, causing string perturbation theory not to be uniformly valid for all states. We would like to find plausible examples of how this works in practice. That is, for suitable groups G, we would like to identify the $K3$ or gauge singularity that generates non-perturbative gauge invariance with gauge group G. We will not be able to do this for all G, but we will find what seem like compelling candidates for some simple cases. Our discussion is necessarily incomplete, and no substitute for actually understanding the map of hypermultiplet moduli space that appeared in the last section.

It seems natural to first consider singularities of the gauge bundle only, keeping the $K3$ smooth. On a smooth $K3$, a singularity of the gauge bundle (keeping the E_8 completely broken) necessarily consists of a certain number of instantons shrinking to points. For example, consider the basic case in which a single instanton shrinks to a point. The effective k of the remaining gauge bundle diminishes by 1, so according to (29) the dimension of the E_8 instanton moduli space drops by 120. However, one is left with four parameters for the position of the point instanton, so actually only $120-4=116$ parameters must be adjusted to make a single instanton shrink.

For the $SO(32)$ heterotic string, a single point instanton gives [23] a non-perturbative $SU(2)$ gauge symmetry. There is no general derivation of this for $E_8 \times E_8$, and it seems doubtful that it is true in general (as illustrated, among other things, by the special role that $k=12$ is about to play). But fortune sometimes favors the brave, and let us ask whether in our particular case, the collapse of an instanton might be dual to an unbroken perturbative

$SU(2)$ gauge symmetry.

E_8 has a maximal subgroup $SU(2)\times E_7$. The $SU(2)$ appearing in such an $SU(2)\times E_7$ is a minimal or (in conformal field theory language) "level one" embedding of $SU(2)$ in E_8, and its commutant is E_7. To get an unbroken level one perturbative $SU(2)$ from one of the E_8's, the vacuum gauge bundle must fit into an E_7 subgroup. As E_7 has $h=18$ and dimension 133, (31) gives $\dim \mathcal{M}_k(E_7)=72k-532$. In particular, $\dim \mathcal{M}_{12}(E_7)=332$. Using also (30) and the fact that $448-332=116$, one must adjust 116 parameters to get a perturbative unbroken $SU(2)$ gauge symmetry. The fact that this is the number of parameters needed to get a point instanton strongly suggests that at $k=12$ the shrinking of an instanton to a point really is dual to the un-Higgsing of an $SU(2)$.

Fortified by this result, let us consider the possible occurrence of *two* point instantons, first considering the generic case that they are placed at distinct points on $K3$. The number of parameters that must be adjusted to get two point instantons is $2\cdot 116=232$. If one point instanton gives an $SU(2)$ gauge symmetry, then two disjoint point instantons should very plausibly give $SU(2)\times SU(2)$. The dual should involve perturbative un-Higgsing of $SU(2)\times SU(2)$. A level one embedding of $SU(2)\times SU(2)$ in E_8 has commutant $SO(12)$.[11] As $SO(12)$ has $h=10$ and dimension 66, we get $\dim M_{12}(SO(12))=216$. Since $448-216=232$, the expected 232 parameters are needed to restore an $SU(2)\times SU(2)$ subgroup of E_8.

Let us now consider the case of two point instantons at the *same* point. Since one must adjust four more parameters to make the positions of the two point instantons in $K3$ coincide, the number of parameters required is now $2\cdot 116+4=236$. Two coincident point instantons must give a gauge group that contains $SU(2)\times SU(2)$. For the $SO(32)$ heterotic string, two coincident point instantons give gauge group $Sp(2)=SO(5)$; let us make the ansatz that that is true here also. A level one embedding of $SO(5)$ in E_8 has commutant $SO(11)$ (with $SO(5)\times SO(11)\subset SO(16)\subset E_8$). As $SO(11)$ has $h=9$ and dimension 55, one gets $\dim \mathcal{M}_{12}(SO(11))=212$. With $448-212=236$, the expected 236 parameters must be adjusted to see an unbroken $SO(5)$ subgroup of E_8.

[11] Recall that E_8 has a maximal subgroup $SO(16)$, which contains $SO(12)\times SO(4)=SO(12)\times SU(2)\times SU(2)$. Actually, $SO(16)$ admits another inequivalent level one embedding of $SU(2)\times SU(2)$ (the commutant being $SO(8)\times SU(2)\times SU(2)$), but the two embeddings are conjugate in E_8.

Now we move on to consider the case of *three* small instantons. The number of parameters that must be adjusted to create three small instantons at distinct points is $3 \cdot 116 = 348$. (Here we are on shakier grounds, as we will assume that there is a branch of the $k = 9$ moduli space with complete Higgsing, and as explained above we do not know this to be true. This uncertainty will affect many of the observations below.) The generic non-perturbative gauge group for three small instantons should be $SU(2)^3$. A level one embedding of $SU(2)^3$ in E_8 has commutant $SO(8) \times SU(2)$ (with $(SO(8) \times SU(2)) \times SU(2)^3 = SO(8) \times SO(4)^2 \subset SO(16) \subset E_8$). Here there is the new feature that $SO(8) \times SU(2)$ is not simple; we can place k_1 instantons in $SO(8)$ and k_2 in $SU(2)$, with $k_1 + k_2 = 12$. Since $SO(8)$ has $h = 6$ and dimension 28 while $SU(2)$ has $h = 2$ and dimension 3, the dimension of the $SO(8) \times SU(2)$ moduli space is $24k_1 + 8k_2 - 124$. We will make the ansatz of picking k_1 and k_2 to make this as large as possible subject to the constraint that *bona fide* $SU(2)$ instantons on $K3$ of the given k_2 actually exist. That latter constraint forces $k_2 \geq 4$,[12] so to maximize the dimension of the moduli space, we take $(k_1, k_2) = (8, 4)$, and then we find that $\dim \mathcal{M}_{(8,4)}(SO(8) \times SU(2)) = 100$. As $448 - 100 = 348$, one must adjust the expected 348 parameters to restore a level one $SU(2)^3$ subgroup of E_8. In future, when we meet instantons in a group $H = H' \times SU(2)$, we will always set the $SU(2)$ instanton number to be 4, as in the calculation just performed.

One can similarly consider the case of three collapsed instantons that are not at distinct points. For instance, two collapsed instantons at one point and one at a third point should give a non-perturbative gauge group $Sp(2) \times SU(2) = SO(5) \times SU(2)$. To obtain such a configuration, one must adjust $3 \cdot 116 + 4 = 352$ parameters ($3 \cdot 116$ to make three instantons collapse and 4 to make two of the collapsed instantons appear at the same point). The commutant of a level one embedding of $SO(5) \times SU(2)$ in E_8 is $SO(7) \times SU(2)$ (via $SO(7) \times SU(2) \times SO(5) \times SU(2) \subset SO(16) \subset E_8$). Using the fact that $SO(7)$ has $h = 5$ and dimension 21, along with facts already exploited, we get

[12] For $SU(2)$, (31) gives $\dim \mathcal{M}_k(SU(2)) = 8k - 12$. k point instantons would have a $4k$ dimensional moduli space. Honest $SU(2)$ instantons as opposed to collapsed point instantons exist only when $\dim \mathcal{M}_k(SU(2))$ exceeds the dimension of the moduli space of collapsed instantons, that is when $8k - 12 > 4k$ or $k > 3$ The dimension-counting alone does not give a rigorous argument here, but a rigorous argument can be made using the index theorem for the Dirac operator in the instanton field and a vanishing argument.

dim $\mathcal{M}_{(8,4)}(SO(7)\times SU(2)) = 96$. With $448-96 = 352$, one must adjust the expected 352 parameters to restore a perturbative $SO(5)\times SU(2)$ subgroup.

The last example of this kind is the case of three collapsed instantons all at the same point in $K3$. $3\cdot 116+2\cdot 4 = 356$ parameters must be adjusted to achieve this situation. Based on the result of [23] for $SO(32)$, we may guess that the non-perturbative gauge group for three coincident small instantons will be $Sp(3)$. The commutant of a level one embedding of $Sp(3)$ in E_8 is $G_2 \times SU(2)$. (This was found by embedding $Sp(3) \subset SU(6) \subset SO(12) \subset SO(16) \subset E_8$ and then, by hand, determining that the commutant of $Sp(3)$ was $G_2 \times SU(2)$ – a method that also gave the decomposition used later of the E_8 Lie algebra under $Sp(3) \times G_2 \times SU(2)$.) With G_2 having $h = 4$ and dimension 14, one finds that dim $\mathcal{M}_{(8,4)}(G_2\times SU(2)) = 92$; as $448-92 = 356$, one must adjust the expected 356 parameters to restore an $Sp(3)$ subgroup of E_8.

For more than three small instantons, the residual E_8 instanton would have instanton number ≤ 8, so that the right hand side of (29) would be negative. This actually means that the residual instanton cannot completely break the E_8 symmetry, so that perturbative as well as non-perturbative gauge fields will appear. We have not been successful in finding duals of configurations with perturbative as well as non-perturbative gauge fields, and will not discuss this here.

Let us now look in somewhat more detail at the example with three point instantons all at the same point in $K3$, giving a non-perturbative $Sp(3)$ subgroup. $Sp(3)$ has many subgroups that we have seen, such as $Sp(2) \times SU(2)$, $SU(2)^3$, etc. In the non-perturbative description, one sees these subgroups of $Sp(3)$ by perturbing the very exceptional configuration with three coincident point instantons to a less exceptional – but still singular – configuration in which the instantons are not all small or all coincident. In the perturbative description, an $Sp(3)$ which has been restored or un-Higgsed can be broken down to a subgroup by turning back on the ordinary Higgs mechanism.

But when one thinks about doing so, a puzzle presents itself. Apart from subgroups of $Sp(3)$ that we have seen, there are other subgroups of $Sp(3)$, such as $SU(3)$, $U(1)$, $U(1) \times SU(2)$, that do not appear anywhere on the non-perturbative side (in any deformation of the configuration with the three coincident point instantons). What prevents Higgsing of $Sp(3)$ to $SU(3)$ or other unwanted groups?

To answer this question on the perturbative side, we need to know the $Sp(3)$ quantum numbers of the massless hypermultiplets that appear when the $Sp(3)$ is un-Higgsed. The adjoint representation of E_8 decomposes under $G_2 \times SU(2) \times Sp(3)$ as

$$(\mathbf{7}, \mathbf{1}, \mathbf{14}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{14}') \oplus (\mathbf{7}, \mathbf{2}, \mathbf{6}), \tag{32}$$

plus the adjoint of $G_2 \times SU(2) \times Sp(3)$; here $\mathbf{1}$ is a trivial representation, $\mathbf{7}$, $\mathbf{2}$, and $\mathbf{6}$ are the defining representations of G_2, $SU(2)$, and $Sp(3)$, of the indicated dimensions, and $\mathbf{14}$ and $\mathbf{14}'$ are the two fourteen dimensional representations of $Sp(3)$ – the $\mathbf{14}$ is the traceless second rank antisymmetric tensor, and the $\mathbf{14}'$ is a traceless third rank antisymmetric tensor. The decomposition in (32) means that when a level one perturbative $Sp(3)$ is unbroken, the massless hypermultiplets will transform as a certain number of $\mathbf{14}$'s, $\mathbf{14}'$'s, and $\mathbf{6}$'s, determined by index theorems. No other representations of $Sp(3)$ can appear, as they are absent in the adjoint representation of E_8. For instance, the number of $\mathbf{14}$'s will be the index of the Dirac operator on $K3$ with values in the $(\mathbf{7}, \mathbf{1})$ of $G_2 \times SU(2)$, and similarly for the other representations.

Now, the $\mathbf{14}$'s and $\mathbf{6}$'s are the desired representations that can Higgs $Sp(3)$ to the groups that one actually sees on the non-perturbative side, and nothing else. For instance, with $\mathbf{6}$'s alone, one can Higgs only down to $Sp(2)$ or $Sp(1) = SU(2)$ – two of the groups that we actually encountered above. Using also the $\mathbf{14}$ one can get the other desired subgroups of $Sp(3)$. But the $\mathbf{14}'$ would make it possible to Higgs down to unwanted subgroups of $Sp(3)$ such as $SU(3)$ or $U(1)$.

At this point we must recall that there are several loci on which restoration of a level one $Sp(3)$ occurs. They are labeled by the $G_2 \times SU(2)$ instanton numbers (k_1, k_2) with $k_1 + k_2 = 12$. For general (k_1, k_2), the unwanted representation will appear. But we found the situation of three coincident point instantons to be dual to un-Higgsed $Sp(3)$ with $(k_1, k_2) = (8, 4)$. When and only when the instanton number is 4, the Dirac index with values in the $(1, 2)$ of $G_2 \times SU(2)$ vanishes, [13] and the dreaded $\mathbf{14}'$ does not arise.

[13] A short-cut to deduce this directly from (31) without having to go back to the general index theorem is as follows. From (31), dim $\mathcal{M}_4(SU(3)) = 20$ but also dim $\mathcal{M}_4(SU(2) \times U(1)) = 20$ (in the latter case one understands that the instantons are all in the $SU(2)$, and that $h = 0$ for $U(1)$). As these numbers are equal, for instanton number four an

Computing the other indices, one finds that the hypermultiplet spectrum consists of one **14** and thirty-two **6**'s. (For unclear reasons, but surely not coincidentally, this is the spectrum that appears in the ADHM construction of instantons with instanton number three with gauge group $SO(32)$ on $\mathbf{R}^4$.) With this spectrum, the perturbative $Sp(3)$ can be Higgsed down precisely to those subgroups that we found on the non-perturbative side.

Inclusion Of K3 Singularities

To go farther, since we have exhausted the list of singularities of a completely Higgsed gauge bundle on a generic $K3$, a natural step is to look for non-perturbative gauge fields whose origin involves a $K3$ singularity. This is a potentially vast subject (unless one has a systematic point of view), with many possibilities to consider. We will point out a few candidates.

Since we found the dual to an un-Higgsed $Sp(3)$, let us look for the dual to a relatively small group containing $Sp(3)$, namely $SU(6)$. The commutant of a level one $SU(6)$ is $SU(3) \times SU(2)$ (E_8 has a maximal subgroup $SU(6) \times SU(3) \times SU(2)$). The $SU(3) \times SU(2)$ instanton moduli space with instanton numbers $(8,4)$ has dimension 84 (using the fact that $SU(3)$ has $h = 3$ and dimension eight); as $448 - 84 = 364$, 364 parameters must be adjusted to restore a perturbative $SU(6)$ gauge symmetry. We should try to interpret this number in terms of a singular $K3$, since we have exhausted what can be done with non-singular $K3$'s. The simplest $K3$ singularity is an A_1 singularity; in classical geometry one must adjust three parameters to make such a singularity, but to make the conformal field theory singular, one must adjust also a theta angle [39], making four. Since one must adjust 120 parameters to make an instanton collapse and fix its position, we are tempted to write $364 = 3 \cdot 120 + 4$ and to propose that the dual of a perturbative $SU(6)$ is a $K3$ with an A_1 singularity at which there are also three point instantons.

To check out this idea further, we note that in general, to get the A_1 singularity with k point instantons sitting on top of it, one should expect to have to adjust

$$w_k = 120k + 4 \tag{33}$$

$SU(3)$ instanton on $K3$ automatically takes values in an $SU(2) \times U(1)$ subgroup, so that there are no zero modes in the **2** of $SU(2)$ by which one could deform to an irreducible $SU(3)$ instanton.

parameters. We have already considered the $k = 3$ case. Let us look at $k = 2$. Here we have two coincident point instantons at the singularity. The gauge group is at least $Sp(2)$, which we would get with two coincident instantons without the A_1 singularity. A relatively small group containing $Sp(2)$ is $SU(4)$; let us ask if this is the right group for two point instantons on top of an A_1 singularity. In fact, the commutant of $SU(4)$ is $SO(10)$ (think of the chain $SU(4) \times SO(10) = SO(6) \times SO(10) \subset SO(16) \subset E_8$). $SO(10)$ has $h = 8$ and dimension 45, and dim $\mathcal{M}_{12}(SO(10)) = 204$. Thus $448 - 204 = 244$ parameters must be adjusted to restore a perturbative $SU(4)$ symmetry, in agreement with the $k = 2$ case of (33).

One might ask about the $k = 1$ case of (33). We were not able to find a group whose restoration involves adjusting 124 parameters. However, note that the results so far can be expressed by stating that an A_1 singularity that captures k point instantons gives an $SU(2k)$ gauge group. If we suppose that that holds also for $k = 1$, then the A_1 singularity with one point instanton gives an $SU(2)$ gauge group, which is the same gauge group we obtained for one point instanton without the A_1 singularity. So we postulate that with only one point instanton, the coincidence with an A_1 singularity leads to no enhancement of the gauge group. In essence, for each k, the k coincident point instantons give $SU(2k)$ or $Sp(k)$ depending on whether or not they lie at an A_1 singularity, but the fact that $SU(2) = Sp(1)$ means that for $k = 1$ the A_1 singularity gives no enhanced gauge symmetry.

If we feel bold, we can observe that since in classical geometry an A_1 singularity is the same as a $\mathbf{Z}_2$ orbifold singularity, it is possible for an A_1 singularity to capture a *half-integral* number of point instantons. Why not then try to use (33) for half-integral k? This turns out to work perfectly, though we find this somewhat puzzling for a reason stated below. Since k point instantons on the A_1 singularity gave an $SU(2k)$ gauge group at least for $k = 2, 3$, we compare the $k = 5/2$ and $k = 3/2$ cases to $SU(5)$ and $SU(3)$, respectively. The commutant of $SU(5)$ is $SU(5)$ (E_8 has a maximal subgroup $SU(5) \times SU(5)$). Since $SU(5)$ has $h = 5$ and dimension 24, $\dim \mathcal{M}_{12}(SU(5)) = 144$; as $448 - 144 = 304$, 304 parameters must be adjusted to restore a perturbative $SU(5)$, in agreement with the $k = 5/2$ case of (33). Likewise, $SU(3)$ has commutant E_6, which has $h = 12$ and dimension 78, so dim $\mathcal{M}_{12}(E_6) = 264$. With $448 - 264 = 184$, one must adjust 184 parameters to restore an $SU(3)$ gauge symmetry, in agreement with the $k = 3/2$ case of (33). Now, however, we must confess to what is unsettling about these "successes" for $SU(3)$ and $SU(5)$: in classical geometry (33) is not valid for half-integral k (there is a correction involving an eta-invariant), and we do not know why this simple formula seems to work in string theory.

We have encountered many but not all subgroups of $SU(6)$ in this discussion. Just as in the discussion of $Sp(3)$, we should ask on the perturbative side to what subgroups the $SU(6)$ can be Higgsed. For this we need the fact that under $SU(3) \times SU(2) \times SU(6)$ the adjoint of E_8 decomposes as $(\mathbf{3}, \mathbf{2}, \overline{\mathbf{6}}) \oplus (\overline{\mathbf{3}}, \mathbf{2}, \mathbf{6}) \oplus (\mathbf{3}, \mathbf{1}, \mathbf{15}) \oplus (\overline{\mathbf{3}}, \mathbf{1}, \overline{\mathbf{15}}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{20})$ (plus the adjoint). The $SU(6)$ content will consist of a certain number of $\mathbf{20}$'s as well as $\mathbf{6}$'s and $\mathbf{15}$'s and their complex conjugates. The $\mathbf{20}$'s are unwanted as they would enable Higgsing to subgroups of $SU(6)$ that would not have an interpretation in terms of a perturbation of an A_1 singularity with three point instantons. Happily, because the instanton number in the $SU(2)$ is four, the Dirac index with values in the $(\mathbf{1}, \mathbf{2})$ of $SU(3) \times SU(2)$ vanishes, the same lucky fact we used in the $Sp(3)$ discussion, so again the unwanted representation does not occur.

The actual spectrum thus consists only of $\mathbf{6}$'s, $\mathbf{15}$'s, and their conjugates. With these representations, one can break $SU(6)$ only to groups that have natural interpretations using the above ideas. ($SU(5)$ and $SU(3)$ do occur – they can be reached by Higgsing with $\mathbf{6}$'s – so the $k = 5/2$ and $k = 3/2$ cases of (33) are needed.) For instance, with the $\mathbf{15}$ one can Higgs down to $SU(4) \times SU(2)$, which corresponds to two point instantons at an A_1 singularity and one somewhere else. The commutant of $SU(4) \times SU(2)$ is again $SU(4) \times SU(2)$. As $\dim \mathcal{M}_{(8,4)}(SU(4) \times SU(2)) = 88$, one must adjust $448 - 88 = 360$ parameters to see a perturbative $SU(4) \times SU(2)$ gauge symmetry. We write $360 = 244 + 116$ and propose that in the dual interpretation, one adjusts 244 parameters to get two point instantons at an A_1 singularity and 116 to get one point instanton somewhere else. Similarly, by using also a $\mathbf{6}$, one can Higgs $SU(6)$ to $SU(3) \times SU(2)$, whose commutant in E_8 is another $SU(6)$. As $\dim \mathcal{M}_{12}(SU(6)) = 148$, one must adjust $448 - 148 = 300$ parameters to see a perturbative $SU(3) \times SU(2)$. On the non-perturbative side, we write $300 = 184 + 116$ and interpret this as 184 parameters to get an A_1 singularity that absorbs $3/2$ point instantons, and 116 to get a point instanton somewhere else.

One more example of a similar kind, though not obtained by Higgsing of this particular $SU(6)$, is to look for an unbroken level one $SU(3) \times SU(3)$ subgroup of E_8. The commutant of $SU(3) \times SU(3)$ is another $SU(3) \times SU(3)$

(E_8 has a subgroup $SU(3)^4$). As dim $\mathcal{M}_{k_1,k_2}(SU(3)\times SU(3)) = 80$ (for any k_1, k_2 with $k_1 + k_2 = 12$), one must adjust $448 - 80 = 368$ parameters to get an unbroken perturbative $SU(3)\times SU(3)$. Writing $368 = 164 + 164$, we propose that the dual of this consists of two disjoint A_1 singularities each of which has captured 3/2 point instantons.

We should note that similar examples with codimension very close to 448, like the $k = 7/2$ case of (33), do not seem to work. We suspect that this is because in these cases complete Higgsing to the expected group is not possible.

5 Higher Loops

In this section, we will compare the loop expansion of the fundamental string to that of the dual string. The loop expansion of the fundamental string for any given physical observable is an expansion of that observable in powers of e^{Φ}, valid as as aymptotic expansion near $\Phi = -\infty$. The expansion takes the general form $\sum_{n\geq n_0} b_n\, e^{n\Phi}$ where n_0 depends on what observable is considered. (In the Einstein metric, the exponents may not be integers.) The perturbation expansion of the dual string is an analogous asymptotic expansion in powers of $e^{-\Phi}$, valid near $\Phi = +\infty$. The general form is $\sum_{m\geq m_0} c_m e^{-m\Phi}$. We would like to make a term-wise comparison of these expansions, but in general such a term-wise comparison of asymptotic expansions of a function about two different points is not valid. One situation in which such a termwise comparison *is* valid is the case that each series has only finitely many terms; thus given an equality $\sum_{n=n_0}^{n_1} b_n e^{n\Phi} = \sum_{m=m_0}^{m_1} c_m e^{-m\Phi}$ between finite sums, the exponents and coefficients must be equal.

An important reason for such a series to have only finitely many terms is that supersymmetry may allow only finitely many terms in the expansion of a given observable in powers of $e^{\pm\Phi}$. For instance, in this paper we have exploited the fact that low energy gauge field kinetic energy has a Φ-dependence with only two possible terms. In what follows, we will work out the consequences of a term-wise comparison of the two perturbation expansions in any situation in which such a comparison is valid, including but not limited to the case in which supersymmetry allows only finitely many terms.

The fundamental string involves two kinds of loop expansion: quantum $D = 6$ string loops (L) with loop expansion parameter $\alpha' e^{\Phi}$ and classical 2-

dimensional σ-model loops with loop expansion parameter α'. Following [8], let us consider the purely gravitational contribution to the resulting effective action, using the string σ-model metric:

$$\mathcal{L}_{L,L+m} = a_{L,L+m} \frac{(2\pi)^3}{\alpha'^2} \sqrt{-G} e^{-\Phi} \left(\alpha' e^{\Phi} \right)^L \alpha'^m R^n, \tag{34}$$

where R^n is symbolic for a scalar contribution of n Riemann tensors each of dimension two. One could also include covariant derivatives of R (or other interactions with a known transformation law under duality), but for our purposes (34) will be sufficient. The $a_{L,L+m}$ are numerical coefficients, not involving π. The subscripts L and $L+m$ have been chosen in anticipation of the relation $L+m=\widetilde{L}$, with $\widetilde{L}$ the number of loops in the dual theory. Since

$$[\mathcal{L}_{L,L+m}] = 6, \qquad [\alpha'] = -2, \tag{35}$$

we have, on dimensional grounds,

$$m = n - 1 - 2L. \tag{36}$$

Likewise, the theory of the dual string involves quantum $D=6$ dual string loops ($\widetilde{L}$) with loop expansion parameter $\alpha' e^{-\Phi}$ and classical 2-dimensional σ-model loops with loop expansion parameter α'. The corresponding Lagrangian using the dual σ-model metric is

$$\widetilde{\mathcal{L}}_{\widetilde{L}+\widetilde{m},\widetilde{L}} = \widetilde{a}_{\widetilde{L}+\widetilde{m},L} \frac{(2\pi)^3}{\alpha'^2} \sqrt{-\widetilde{G}} e^{\Phi} \left(\alpha' e^{-\Phi} \right)^{\widetilde{L}} \alpha'^{\widetilde{m}} \widetilde{R}^n. \tag{37}$$

Again, on dimensional grounds,

$$\widetilde{m} = n - 1 - 2\widetilde{L}. \tag{38}$$

Our fundamental assumption is that $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are related by duality which implies, in particular, that the purely gravitational contributions should be identical when written in the same variables. So from (18) and (20) and transforming to the canonical metric, but dropping the c superscript, we find:

$$\mathcal{L}_{L,L+m} = \frac{(2\pi)^3}{\alpha'^2} a_{L,L+m} \alpha'^{L+m} e^{-m\Phi/2} \sqrt{-G} R^n, \tag{39}$$

$$\widetilde{\mathcal{L}}_{\widetilde{L}+\widetilde{m},\widetilde{L}} = \frac{(2\pi)^3}{\alpha'^2}\widetilde{a}_{\widetilde{L}+\widetilde{m},\widetilde{L}}\alpha'^{\widetilde{L}+\widetilde{m}}e^{\widetilde{m}\Phi/2}\sqrt{-G}R^n \tag{40}$$

where we have also dropped the dilaton derivative terms. We find that $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ do coincide provided

$$m + \widetilde{m} = 0, \tag{41}$$

i.e from (36) and (38), provided

$$m = \widetilde{L} - L = -\widetilde{m}, \tag{42}$$

$$n = L + \widetilde{L} + 1, \tag{43}$$

with

$$a_{L,\widetilde{L}} = \widetilde{a}_{L,\widetilde{L}}. \tag{44}$$

Hence the total Lagrangian can be elegantly written

$$\mathcal{L} = \sum_{L,\widetilde{L}} \mathcal{L}_{L,\widetilde{L}}.$$

$$\mathcal{L}_{L,\widetilde{L}} = a_{L,\widetilde{L}}(2\pi)^3\alpha'^{L+\widetilde{L}-2}e^{(L-\widetilde{L})\Phi/2}\sqrt{-G}R^{L+\widetilde{L}+1}. \tag{45}$$

Thus we see that under heterotic string/string duality, the worldsheet loop expansion of one string corresponds to the spacetime loop expansion of the other. Moreover, (45) gives rise to an infinite number of non-renormalization theorems. (As explained at the beginning of this section, these theorems hold if it is known *a priori* that each perturbation expansion has only finitely many terms, and perhaps under wider but presently unknown hypotheses.) The first of these is the absence of a cosmological term $\sqrt{-G}R^0$. The second states that $\sqrt{-G}R^1$ appears only at $(L = 0, \widetilde{L} = 0)$ and hence the tree level action does not get renormalized. The third states that $\sqrt{-G}R^2$ appears only at $(L = 0, \widetilde{L} = 1)$ and $(L = 1, \widetilde{L} = 0)$, and so on. Since F has the same dimension as R, similar restrictions will apply to the pure Yang-Mills and mixed gravity-Yang Mills Lagrangians. The $(L = 0, \widetilde{L} = 0)$, $(L = 0, \widetilde{L} = 1)$ and $(L = 1, \widetilde{L} = 0)$ terms correspond respectively to the I_{00}, I_{01} and I_{10} of the previous section. Self-duality, given by (9), imposes the further constraint

$$a_{L,\widetilde{L}} = a_{\widetilde{L},L} \tag{46}$$

and means that the spacetime and worldsheet loop expansions are in fact identical.

In the situation in which the term-by-term comparison of the two expansions is justified by a supersymmetry argument showing that each expansion has only finitely many terms, it would be interesting to compare the restrictions on exponents following from supersymmetry with those that follow from duality. In principle, duality might give a restriction more severe or less severe than the one that follows from duality, but in the few familiar cases the two restrictions agree.

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A The String Tension And The Dual String Tension

We will here explore from an eleven-dimensional point of view what is entailed in setting the fundamental and dual string tensions equal. Since the string is obtained by wrapping a membrane of tension T_3 around S^1/Z_2 with radius R and the dual string is obtained by wrapping a fivebrane of tension $\widetilde{T}_6$ around $K3$ of volume V, the string tension $\hat{T}$ and dual string tension $\hat{\widetilde{T}}$, measured in the $D = 11$ metric are given by

$$\begin{aligned} \hat{T} &= RT_3 \\ \hat{\widetilde{T}} &= V\widetilde{T}_6 \end{aligned} \tag{47}$$

Let us denote length scales measured in the $D = 11$ metric $\hat{G}$, $D = 6$ string metric G and $D = 6$ dual string metric $\widetilde{G}$ as $\hat{L}$, L and $\widetilde{L}$, respectively. Since

$$\begin{aligned} \hat{G} &= R^{-1}G \\ \hat{G} &= V^{-1}\widetilde{G} \end{aligned} \tag{48}$$

they are related by

$$
\begin{aligned}
L^2 &= R\hat{L}^2 \\
\widetilde{L}^2 &= V\hat{L}^2
\end{aligned}
\tag{49}
$$

Since the string tension measured in the string metric, T, and the dual string tension measured in the dual string metric, $\widetilde{T}$, both have dimensions $(length)^{-2}$, they are given by

$$
\begin{aligned}
T &= R^{-1}\hat{T} = T_3 \\
\widetilde{T} &= V^{-1}\hat{\widetilde{T}} = \widetilde{T}_6
\end{aligned}
\tag{50}
$$

and are therefore both independent of R and V. In fact, since [18]

$$
\widetilde{T}_6 \sim {T_3}^2 \tag{51}
$$

we may, without loss of generality, choose units for which T and $\widetilde{T}$ are equal.

B A Note On The v's and $\tilde{v}$'s

An important feature of the heterotic/heterotic duality conjecture concerns the numerical coefficients v_α appearing in X_4 and $\tilde{v}_\alpha$ appearing in $\widetilde{X}_4$. Here we wish to make a remark on the precise normalization of these coefficients. In string perturbation theory each v is given by [33]

$$
v\mathrm{tr}F^2 = \frac{n}{h}\mathrm{Tr}F^2 \tag{52}
$$

where n is the level of the Kac-Moody algebra, h is the dual Coxeter number, tr denotes the trace in the fundamental representation and Tr is the trace in the adjoint representation. For example,

$$
\begin{array}{lcllcl}
h_{SU(N)} & = & N & \mathrm{Tr}{F_{SU(N)}}^2 & = & 2N\mathrm{tr}{F_{SU(N)}}^2 \\
h_{SO(N)} & = & N-2 & \mathrm{Tr}{F_{SO(N)}}^2 & = & (N-2)\mathrm{tr}{F_{SO(N)}}^2 \\
h_{Sp(N)} & = & N+1 & \mathrm{Tr}{F_{Sp(N)}}^2 & = & (2N+2)\mathrm{tr}{F_{Sp(N)}}^2 \\
h_{G_2} & = & 4 & \mathrm{Tr}{F_{G_2}}^2 & = & 4\mathrm{tr}{F_{G_2}}^2 \\
h_{F_4} & = & 9 & \mathrm{Tr}{F_{F_4}}^2 & = & 3\mathrm{tr}{F_{F_4}}^2 \\
h_{E_6} & = & 12 & \mathrm{Tr}{F_{E_6}}^2 & = & 4\mathrm{tr}{F_{E_6}}^2 \\
h_{E_7} & = & 18 & \mathrm{Tr}{F_{E_7}}^2 & = & 3\mathrm{tr}{F_{E_7}}^2 \\
h_{E_8} & = & 30 & \mathrm{Tr}{F_{E_8}}^2 & = & 30\mathrm{tr}{F_{E_8}}^2
\end{array}
\tag{53}
$$

The values of h were used in section (4).

So, for $G = E_7,\ E_6,\ SO(10),\ SU(5)$ one finds $v_G = n/6,\ n/3,\ n,\ 2n$, respectively. In the case of $SO(32)$ with an $k = 24$ embedding, one finds from (6) that $(n_{SO(28)} = 1, n_{SU(2)} = 1)$ but $(\widetilde{n}_{SO(28)} = \mp 2, \widetilde{n}_{SU(2)} = \pm 22)$ and hence we encounter the "wrong-sign problem"[14]. A similar problem arises for $E_8 \times E_8$ except in the case of symmetric embedding where both $k_i = 12$. For generic embeddings one finds that $n_i = 1$ but

$$\widetilde{n}_i = \frac{1}{2}(k_i - 12) \tag{54}$$

Since one is limited to $k_1 + k_2 = 24$, one factor will always have the wrong sign except when for the $k = 12$ case discussed in this paper, for which the $\widetilde{n}_i$ both vanish.

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