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Small-Angle Electron–Positron Scattering with a Per Mille Accuracy[1]

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Abstract

The elastic and inelastic high-energy small-angle electron–positron scattering is considered. All radiative corrections to the cross-section with the relative accuracy $\delta\sigma/\sigma = 0.1\%$ are explicitly taken into account. According to the generalized eikonal representation for the elastic amplitude, only diagrams with one exchanged photon are considered. Single photon emission with radiative corrections as well as next-to-leading two-photon and pair production diagrams are evaluated, together with leading three-loop corrections. All contributions have been calculated analytically. We define an experimentally measurable cross-section by integrating the calculated distributions over suitable intervals of angles and energies. To the leading approximation, the results are shown to be described in terms of kernels of electron structure functions. Some numerical results are presented.

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1 Introduction

An accurate verification of the Standard Model is one of the primary aims of LEP [1]. While electroweak radiative corrections to the s-channel annihilation process and to large-angle Bhabha scattering allow a direct extraction of the Standard Model parameters, small angle Bhabha cross-section affects, as an overall normalization condition, all observable cross-sections and represents an equally unavoidable condition towards a precise determination of the Standard Model parameters. The small-angle Bhabha scattering process is used to measure the luminosity of electron–positron colliders. At LEP an experimental accuracy on the luminosity of

$$|\frac{\delta\sigma}{\sigma}| < 0.001 \tag{1}$$

has been reached [2]. However, to obtain the total accuracy, a systematic theoretical error must also be added. This precision calls for an equally accurate theoretical expression for the Bhabha scattering cross-section in order to extract the Standard Model parameters from the observed distributions. An accurate determination of the small-angle Bhabha cross-section and of the luminosity directly affects the determination of absolute cross-sections such as, for example, the determination of the invisible width and of the number of massless neutrino species N_ν [3].

In recent years a considerable attention has been devoted to the Bhabha process [4, 5]. The reached accuracy, is, however, still inadequate [2]. According to these evaluations the theoretical estimates are still incomplete; moreover, they are far from the required theoretical and experimental accuracy [2].

The process that will be considered in this work is that of Bhabha scattering when electrons and positrons are emitted at small angles with respect to the initial electron and positron directions. We have examined the radiative processes inclusively accompanying the main $e^+e^- \to e^+e^-$ reaction at high energies, when both the scattered electron and positron are tagged within the counter aperture.

We assume that the centre-of-mass energies are within the range of the LEP collider $2\epsilon = \sqrt{s} = 90 - 200$ GeV and the scattering angles are within the range $\theta \simeq 10 - 150$ mrad. We assume that the charged-particle detectors have the following polar angle cuts:

$$\theta_1 < \theta_- = \widehat{\boldsymbol{p}_1\boldsymbol{q}_1} \equiv \theta < \theta_3, \qquad \theta_2 < \theta_+ = \widehat{\boldsymbol{p}_2\boldsymbol{q}_2} < \theta_4, \qquad 0.01 \lesssim \theta_i \lesssim 0.1 \text{ rad}\ , \tag{2}$$

where $\boldsymbol{p}_1$, $\boldsymbol{q}_1$ ($\boldsymbol{p}_2$, $\boldsymbol{q}_2$) are the momenta of the initial and of the scattered electron (positron) in the centre-of-mass frame.

In this paper we present the results of our calculations of the electron–positron scattering cross-section with an accuracy of $\mathcal{O}(0.1\%)$. The squared matrix elements of the various exclusive processes inclusively contributing to the $e^+e^- \to e^+e^-$ reaction are integrated in order to define an experimentally measurable cross-section according to suitable restrictions on the angles and energies of the detected particles. The different contributions to the electron and positron distributions, needed for the required accuracy, are presented using analytical expressions.

In order to define the angular range of interest and the implications on the required accuracy, let us first briefly discuss, in a general way, the angle-dependent corrections to the cross-section.

We consider e^+e^- scattering at angles as defined in Eq. (2). Within this region, if one expresses the cross-section by means of a series expansion in terms of angles, the main contribution to the cross-section $d\sigma/d\theta^2$ comes from the diagrams for the scattering amplitudes containing one exchanged photon in the t-channel. These diagrams, as is well known, show a singularity of the type θ^{-4} for $\theta \to 0$, e.g.

$$\frac{d\sigma}{d\theta^2} \sim \theta^{-4} \quad .$$

Let us now estimate the correction of order θ^2 to this contribution. If

$$\frac{d\sigma}{d\theta^2} \sim \theta^{-4}(1 + c_1\theta^2) \quad , \tag{3}$$

then, after integration over θ^2 in the angular range of Eq. (2), we obtain:

$$\int\limits_{\theta^2_{\min}}^{\theta^2_{\max}} \frac{d\sigma}{d\theta^2}\, d\theta^2 \sim \theta^{-2}_{\min}(1 + c_1\theta^2_{\min} \ln \frac{\theta^2_{\max}}{\theta^2_{\min}}). \tag{4}$$

We see that, for $\theta_{\min} = 50$ mrad and $\theta_{\max} = 150$ mrad (we have taken the case where the θ^2 corrections are maximal), the relative contribution of the θ^2 terms is about $5 \times 10^{-3}c_1$. Therefore, the terms of relative order θ^2 must only be kept in the Born cross-section where the coefficient c_1 is not small. In higher orders of the perturbative expansion the coefficient c_1 contains at least one factor α/π and therefore these terms can safely be omitted. This implies that, within our accuracy, only radiative corrections from the scattering-type diagrams contribute. Furthermore only diagrams with one photon exchanged in the t-channel should be taken into account, since, according to the generalized eikonal representation, the large logarithmic terms from the diagrams with the multi photon exchange are cancelled.

Having as a final goal for the experimental cross-section the relative accuracy of Eq. (1), and taking into account that the minimal value of the squared momentum transfer $Q^2 = 2\epsilon^2(1 - \cos\theta)$ in the region defined in Eq. (2) is of the order of 1 GeV^2, we may omit in the following also the terms appearing in the radiative corrections of the type m^2/Q^2, with m equal to the electron (m_e) or the muon (m_μ) mass.

The contents of this paper can be outlined as follows. In Section 2 we discuss the Born cross-section $d\sigma^B$ by taking the Z^0 boson exchange into account and we compute the corrections to it due to the virtual and real soft-photon emission. We define also an *experimentally measurable* cross-section σ_{exp} with the experimental cuts on angles and energies taken into account and we discuss how to obtain it from the differential distributions. We present the results, as discussed above, in the form of an expansion in terms of the scattering angle θ. We introduce the ratio $\Sigma = \sigma_{\text{exp}}/\sigma_0$ by normalizing σ_{exp} with respect to the cross-section $\sigma_0 = 4\pi\alpha^2/\epsilon^2\theta_1^2$. In Section 3, by using a simplified version of the differential cross-section for the small-angle scattering, we discuss the contribution to σ_{exp} from the single bremsstrahlung process. The details of the Sudakov technique we use to calculate the hard-photon emission are given in Appendix A. In Section 4 we find all corrections of $\mathcal{O}(\alpha^2)$ to $\sigma_{\rm exp}$ caused by two virtual and real photons as well as pair emission. In Section 5 we consider the virtual and soft-photon emission accompanyng the single photon bremsstrahlung process. The details of this derivation are given in Appendices B and C. In Section 6 we consider the double hard-photon emission process in both the same-side and opposite-side cases. Details are given in Appendix D. In Section 7 we consider the hard pair production process in both the collinear and semi-collinear kinematical region. The details of this calculation are given in Appendix F. In Appendices D and E are given the expressions for the leading logarithmic approximation in terms of structure functions factorization and the details of the cancellation of the Δ-dependence respectively. In Section 8 the expressions to leading logarithmic $\mathcal{O}(\alpha^3)$ for the e^+e^- and $e^+e^-\gamma$ radiative processes are obtained. In Section 9, finally, estimates of the neglected terms together with numerical results are presented.

A less detailed derivation of these results has been reported elsewhere [6].

2 Born cross-section and one-loop virtual and soft corrections

The Born cross-section for Bhabha scattering within the Standard Model is well known [4]:

$$\frac{\mathrm{d}\sigma^B}{\mathrm{d}\Omega} = \frac{\alpha^2}{8s}\{4B_1 + (1-c)^2 B_2 + (1+c)^2 B_3\}, \tag{5}$$

where

$$\begin{aligned}
B_1 &= (\frac{s}{t})^2 \left|1 + (g_v^2 - g_a^2)\xi\right|^2, \qquad B_2 = \left|1 + (g_v^2 - g_a^2)\chi\right|^2, \\
B_3 &= \frac{1}{2}\left|1 + \frac{s}{t} + (g_v + g_a)^2(\frac{s}{t}\xi + \chi)\right|^2 + \frac{1}{2}\left|1 + \frac{s}{t} + (g_v - g_a)^2(\frac{s}{t}\xi + \chi)\right|^2, \\
\chi &= \frac{\Lambda s}{s - m_z^2 + iM_Z\Gamma_Z}, \qquad \xi = \frac{\Lambda t}{t - M_Z^2}, \\
\Lambda &= \frac{G_F M_Z^2}{2\sqrt{2}\pi\alpha} = (\sin 2\theta_w)^{-2}, \quad g_a = -\frac{1}{2}, \quad g_v = -\frac{1}{2}(1 - 4\sin^2\theta_w), \\
s &= (p_1 + p_2)^2 = 4\varepsilon^2, \qquad t = -Q^2 = (p_1 - q_1)^2 = -\frac{1}{2}\, s\,(1-c), \\
c &= \cos\theta, \qquad \theta = \widehat{\boldsymbol{p}_1\boldsymbol{q}}_1.
\end{aligned}$$

Here θ_w is the Weinberg angle. In the small-angle limit ($c = 1 - \theta^2/2 + \theta^4/24 + \ldots$), expanding formula (5) leads to

$$\frac{\mathrm{d}\sigma^B}{\theta\mathrm{d}\theta} = \frac{8\pi\alpha^2}{\varepsilon^2\theta^4}(1 - \frac{\theta^2}{2} + \frac{9}{40}\theta^4 + \delta_{\rm weak}), \tag{6}$$

where $\varepsilon = \sqrt{s}/2$ is the electron or positron initial energy and the weak correction term δ_{weak}, connected with diagrams with Z^0-boson exchange, is given by the expression:

$$\delta_{\text{weak}} = 2g_v^2\xi - \frac{\theta^2}{4}(g_v^2 + g_a^2)Re\,\chi + \frac{\theta^4}{32}(g_v^4 + g_a^4 + 6g_v^2g_a^2)|\chi|^2. \tag{7}$$

From Eq. (7) it can be seen that the contribution c_1^w of the weak correction δ_{weak} into the coefficient c_1 introduced in Eq. (3)

$$c_1^w \lesssim 2g_v^2 + \frac{(g_v^2 + g_a^2)}{4}\frac{M_Z}{\Gamma_Z} + \theta^2_{\max}\frac{(g_v^4 + g_a^4 + 6g_v^2g_a^2)}{32}\frac{M_Z^2}{\Gamma_Z^2} \simeq 1. \tag{8}$$

According to our discussion after Eq. (4) this means that the contribution connected with Z^0-boson exchange diagrams does not exceed 0.3%. We will therefore neglect such diagrams in the calculation of radiative corrections since they could contribute at most with terms $\lesssim 10^{-4}$.

In the pure QED case one-loop radiative corrections to Bhabha cross-section were calculated a long time ago [8]. Taking into account a contribution from soft-photon emission with energy less than a given finite threshold $\Delta\varepsilon$, we have in this case for the cross-section $d\sigma^{(1)}_{QED}$, in the one-loop approximation:

$$\frac{\mathrm{d}\sigma^{(1)}_{QED}}{\mathrm{d}c} = \frac{\mathrm{d}\sigma^{B}_{QED}}{\mathrm{d}c}\,(1 + \delta_{\text{virt}} + \delta_{\text{soft}}), \tag{9}$$

where $\mathrm{d}\sigma^B_{QED}$ is the Born cross-section in the pure QED case (it is equal to $\mathrm{d}\sigma^B$ with $g_a = g_v = 0$) and

$$\begin{aligned}
\delta_{\text{virt}} + \delta_{\text{soft}} &= 2\frac{\alpha}{\pi}\left[2\left(1 - \ln\frac{4\varepsilon^2}{m^2} + 2\ln(\mathrm{ctg}\frac{\theta}{2})\right)\ln\frac{\varepsilon}{\Delta\varepsilon} + \int\limits_{\cos^2(\theta/2)}^{\sin^2(\theta/2)}\frac{\mathrm{d}x}{x}\ln(1-x)\right.\\
&- \left.\frac{23}{9} + \frac{11}{6}\,\ln\frac{4\varepsilon^2}{m^2}\right] + \frac{\alpha}{\pi}\,\frac{1}{(3+c^2)^2}\left[\frac{\pi^2}{3}\,(2c^4 - 3c^3 - 15c)\right.\\
&+ 2\,(2c^4 - 3c^3 + 9c^2 + 3c + 21)\,\ln^2(\sin\frac{\theta}{2})\\
&- 4\,(c^4 + c^2 - 2c)\,\ln^2\cos\frac{\theta}{2} - 4\,(c^3 + 4c^2 + 5c + 6)\,\ln^2(\mathrm{tg}\frac{\theta}{2})\\
&+ \frac{2}{3}\,(11c^3 + 33c^2 + 21c + 111)\,\ln(\sin\frac{\theta}{2}) + 2\,(c^3 - 3c^2 + 7c - 5)\,\ln(\cos\frac{\theta}{2})\\
&+ \left. 2\,(c^3 + 3c^2 + 3c + 9)\,\delta_t - 2\,(c^3 + 3c)(1-c)\,\delta_s\right].
\end{aligned}$$

The value δ_t (δ_s) is defined by contributions to the photon vacuum polarization function $\Pi(t)$ ($\Pi(s)$) as follows:

$$\Pi(t) = \frac{\alpha}{\pi}\left(\delta_t + \frac{1}{3}L - \frac{5}{9}\right) + \frac{1}{4}\,(\frac{\alpha}{\pi})^2L, \tag{10}$$

where

$$L = \ln \frac{Q^2}{m^2}, \qquad Q^2 = -t = 2\varepsilon^2(1-c), \tag{11}$$

and we took into account the leading part of the two-loop contribution in the polarization operator. In the Standard Model, δ_t contains contributions of muons, tau-leptons, W-bosons and hadrons:

$$\delta_t = \delta_t^\mu + \delta_t^\tau + \delta_t^W + \delta_t^H, \qquad \delta_s = \delta_t\ (Q^2 \to -s), \tag{12}$$

the first three contributions are theoretically calculable and can be given as:

$$\begin{aligned}
\delta_t^\mu &= \frac{1}{3}\ln\frac{Q^2}{m_\mu^2} - \frac{5}{9}, \\
\delta_t^\tau &= \frac{1}{2}\, v_\tau\, (1 - \frac{1}{3}v_\tau^2)\, \ln\frac{v_\tau + 1}{v_\tau - 1} + \frac{1}{3}\, v_\tau^2 - \frac{8}{9}, \qquad v_\tau = \sqrt{1 + \frac{4m_\tau^2}{Q^2}}, \\
\delta_t^W &= \frac{1}{4}\, v_W\, (v_W^2 - 4)\, \ln\frac{v_W + 1}{v_W - 1} - \frac{1}{2}\, v_W^2 + \frac{11}{6}, \qquad v_W = \sqrt{1 + \frac{4M_W^2}{Q^2}}.
\end{aligned} \tag{13}$$

The contribution of hadrons cannot be calculated theoretically; instead, it can be given as integration of the experimentally measurable cross-section:

$$\delta_t^H = \frac{Q^2}{4\pi\alpha^2} \int\limits_{4m_\pi^2}^{+\infty} \frac{\sigma^{e^+e^-\to h}\,(x)}{x + Q^2}\, dx. \tag{14}$$

For numerical calculations we will use for $\Pi(t)$ the results of Ref. [9].

In the small scattering angle limit we can present (9) in the following form:

$$\begin{aligned}
\frac{\mathrm{d}\sigma_{QED}^{(1)}}{\mathrm{d}c} &= \frac{\mathrm{d}\sigma_{QED}^{B}}{\mathrm{d}c}\, (1 - \Pi(t))^{-2}\, (1 + \delta), \\
\delta &= 2\frac{\alpha}{\pi}\left[2(1-L)\ln\frac{1}{\Delta} + \frac{3}{2}L - 2\right] + \frac{\alpha}{\pi}\,\theta^2\,\Delta_\theta + \frac{\alpha}{\pi}\,\theta^2\,\ln\Delta, \\
\Delta_\theta &= \frac{3}{16}l^2 + \frac{7}{12}l - \frac{19}{18} + \frac{1}{4}\,(\delta_t - \delta_s), \\
\Delta &= \frac{\Delta\varepsilon}{\varepsilon}, \qquad l = \ln\frac{Q^2}{s} \simeq \ln\,\frac{\theta^2}{4}.
\end{aligned} \tag{15}$$

This representation gives us a possibility to verify explicitly that the terms of relative order θ^2 in the radiative corrections are small. Taking into account that the large contribution proportional to $\ln\Delta$ disappears when we add the cross-section for the hard emission, we can verify once more that such terms can be neglected. Therefore we will omit in higher orders the annihilation diagrams as well as multiple-photon exchange diagrams in the scattering channel. The second simplification is justified by the generalized eikonal representation for small-angle scattering amplitudes. In particular, for the case of elastic processes we have [10]:

$$A(s,t) = A_0(s,t)\, F_1^2(t)\, (1-\Pi(t))^{-1}\, \mathrm{e}^{i\varphi(t)} \left[1 + \mathcal{O}\Big(\frac{\alpha}{\pi}\,\frac{Q^2}{s}\Big)\right], \quad s \gg Q^2 \gg m^2, \tag{16}$$

where $A_0(s,t)$ is the Born amplitude, $F_1(t)$ is the Dirac form factor and $\varphi(t) = -\alpha\, \ln(Q^2/\lambda^2)$ is the Coulomb phase, λ is the *photon mass* auxiliary parameter. The eikonal representation is violated at a three-loop level, but, fortunately, the corresponding contribution to the Bhabha cross-section is small enough ($\sim \alpha^5$) and can be neglected for our purposes. We may consider the eikonal representation as correct within the required accuracy[4].

Let us now introduce the dimensionless quantity $\Sigma = Q_1^2\ \sigma_{\rm exp}/(4\pi\alpha^2)$, with $Q_1^2 = \varepsilon^2\theta_1^2$, where $\sigma_{\rm exp}$ represents the experimentally observable cross-section:

$$\Sigma = \frac{Q_1^2}{4\pi\alpha^2} \int \mathrm{d}x_1 \int \mathrm{d}x_2\ \Theta(x_1x_2 - x_c) \int \mathrm{d}^2\boldsymbol{q}_1^\perp\ \Theta_1^c \int \mathrm{d}^2\boldsymbol{q}_2^\perp\ \Theta_2^c\ \frac{\mathrm{d}\sigma^{e^+e^-\to e^+(\boldsymbol{q}_2^\perp, x_2)e^-(\boldsymbol{q}_1^\perp, x_1)+X}}{\mathrm{d}x_1\mathrm{d}^2\boldsymbol{q}_1^\perp \mathrm{d}x_2\mathrm{d}^2\boldsymbol{q}_2^\perp}, \tag{17}$$

where $x_{1,2}$, $\boldsymbol{q}_{1,2}^\perp$ are the energy fractions and the transverse components of the momenta of the electron and positron in the final state, sx_c is the experimental cutoff on their invariant mass squared and the functions Θ_i^c do take into account the angular cuts (2):

$$\Theta_1^c = \Theta(\theta_3 - \frac{|\boldsymbol{q}_1^\perp|}{x_1\varepsilon})\ \Theta(\frac{|\boldsymbol{q}_1^\perp|}{x_1\varepsilon} - \theta_1), \qquad \Theta_2^c = \Theta(\theta_4 - \frac{|\boldsymbol{q}_2^\perp|}{x_2\varepsilon})\ \Theta(\frac{|\boldsymbol{q}_2^\perp|}{x_2\varepsilon} - \theta_2). \tag{18}$$

In the case of a symmetrical angular acceptance (we restrict ourselves further to this case only) we have:

$$\theta_2 = \theta_1, \quad \theta_4 = \theta_3, \quad \rho = \frac{\theta_3}{\theta_1} > 1. \tag{19}$$

We will present Σ as the sum of various contributions:

$$\begin{aligned} \Sigma &= \Sigma_0 + \Sigma^\gamma + \Sigma^{2\gamma} + \Sigma^{e^+e^-} + \Sigma^{3\gamma} + \Sigma^{e^+e^-\gamma} \\ &= \Sigma_{00}(1 + \delta_0 + \delta^\gamma + \delta^{2\gamma} + \delta^{e^+e^-} + \delta^{3\gamma} + \delta^{e^+e^-\gamma}), \\ &\qquad\qquad \Sigma_{00} = 1 - \rho^{-2}, \end{aligned} \tag{20}$$

where Σ_0 stands for a modified Born contribution, Σ^γ for a contribution of one-photon emission (real and virtual) and so on. The values of the δ^i as function of x_c are given in table 1 (see

[4]In a recent paper by Fäldt and Osland [8] the authors claimed that the generalized eikonal representation is violated at the two-loop level. We do not agree with their results. In particular, in QED with a massive photon as actually considered in [8], contributions of individual diagrams in the Feynman gauge should have four powers of logarithms at this level, contrary to the Fäldt–Osland three-power result.

in Section 9). Being stimulated by the representation in Eq. (16), we shall slightly modify the perturbation theory, using the full propagator for the t-channel photon, which takes into account the growth of the electric charge at small distances. By integrating Eq. (6) with this convention, we obtain:

$$\Sigma_0 = \theta_1^2 \int\limits_{\theta_1^2}^{\theta_2^2} \frac{\mathrm{d}\theta^2}{\theta^4} (1 - \Pi(t))^{-2} + \Sigma_W + \Sigma_\theta, \tag{21}$$

where Σ_W is the correction due to the weak interaction:

$$\Sigma_W = \theta_1^2 \int\limits_{\theta_1^2}^{\theta_2^2} \frac{\mathrm{d}\theta^2}{\theta^4} \delta_{\text{weak}}\,, \tag{22}$$

and the term Σ_θ comes from the expansion of the Born cross-section in powers of θ^2,

$$\Sigma_\theta = \theta_1^2 \int\limits_{1}^{\rho^2} \frac{\mathrm{d}z}{z} (1 - \Pi(-zQ_1^2))^{-2} \Big(-\frac{1}{2} + z\theta_1^2 \frac{9}{40}\Big). \tag{23}$$

The remaining contributions to Σ in (20) are considered below.

3 Single hard-photon emission

In order to calculate the contribution to Σ due to the hard-photon emission we start from the corresponding differential cross-section written in terms of energy fractions $x_{1,2}$ and transverse components $\boldsymbol{q}_{1,2}^\perp$ of the final particle momenta [12]:

$$\begin{aligned} \frac{\mathrm{d}\sigma_B^{e^+e^-\to e^+e^-\gamma}}{\mathrm{d}x_1 \mathrm{d}^2\boldsymbol{q}_1^\perp \mathrm{d}x_2 \mathrm{d}^2\boldsymbol{q}_2^\perp} &= \frac{2\alpha^3}{\pi^2} \Big\{ \frac{R(x_1; \boldsymbol{q}_1^\perp, \boldsymbol{q}^\perp{}_2)\, \delta(1-x_2)}{(\boldsymbol{q}_2^\perp)^4\, (1 - \Pi(-(\boldsymbol{q}_2^\perp)^2))^2} \\ &+ \frac{R(x_2; \boldsymbol{q}_2^\perp, \boldsymbol{q}_1^\perp)\, \delta(1-x_1)}{(\boldsymbol{q}_1^\perp)^4\, (1 - \Pi(-(\boldsymbol{q}_1^\perp)^2))^2} \Big\} (1 + \mathcal{O}(\theta^2)), \end{aligned} \tag{24}$$

where

$$\begin{aligned} R(x; \boldsymbol{q}_1^\perp, \boldsymbol{q}_2^\perp) &= \frac{1+x^2}{1-x} \Big[\frac{(\boldsymbol{q}_2^\perp)^2 (1-x)^2}{d_1 d_2} - \frac{2m^2(1-x)^2 x}{1+x^2} \,\frac{(d_1 - d_2)^2}{d_1^2 d_2^2} \Big], \\ d_1 &= m^2(1-x)^2 + (\boldsymbol{q}_1^\perp - \boldsymbol{q}_2^\perp)^2, \qquad d_2 = m^2(1-x)^2 + (\boldsymbol{q}_1^\perp - x\boldsymbol{q}_2^\perp)^2, \end{aligned} \tag{25}$$

and we use the full photon propagator for the t-channel photon. Performing a simple azimuthal angle integration of Eq. (24) we obtain for the hard-photon emission the contribution Σ^H:

$$\Sigma^H = \frac{\alpha}{\pi} \int\limits_{x_c}^{1-\Delta} \mathrm{d}x\, \frac{1+x^2}{1-x}\, F(x, D_1, D_3; D_2, D_4), \tag{26}$$

with

$$F=\int\limits_{D_1}^{D_3}\mathrm{d}z_1\int\limits_{D_2}^{D_4}\frac{\mathrm{d}z_2}{z_2}(1-\Pi(-z_2Q_1^2))^{-2}\Big\{\frac{1-x}{z_1-xz_2}(a_1^{-\frac{1}{2}}-xa_2^{-\frac{1}{2}})-\frac{4x\sigma^2}{1+x^2}[a_1^{-\frac{3}{2}}+x^2a_2^{-\frac{3}{2}}]\Big\},\quad(27)$$

where

$$a_1=(z_1-z_2)^2+4z_2\sigma^2,\quad a_2=(z_1-x^2z_2)^2+4x^2z_2\sigma^2,\quad \sigma^2=\frac{m^2}{Q_1^2}(1-x)^2,\qquad(28)$$

and the integration limits in (27) in the symmetrical case are:

$$D_1=x^2,\quad D_2=1,\quad D_3=x^2\rho^2,\quad D_4=\rho^2.\qquad(29)$$

From Eqs. (26)–(29) we have that:

$$\begin{aligned}\Sigma^H &= \frac{\alpha}{\pi}\int\limits_{x_c}^{1-\Delta}\mathrm{d}x\frac{1+x^2}{1-x}\int\limits_{1}^{\rho^2}\frac{\mathrm{d}z}{z^2}(1-\Pi(-zQ_1^2))^{-2}\\ &\times \Big\{[1+\Theta(x^2\rho^2-z)]\;(L-1)+k(x,z)\Big\},\qquad(30)\\ k(x,z) &= \frac{(1-x)^2}{1+x^2}\;[1+\Theta(x^2\rho^2-z)]+\;L_1+\Theta(x^2\rho^2-z)\;L_2+\Theta(z-x^2\rho^2)L_3\,,\end{aligned}$$

where $L=\ln(zQ_1^2/m^2)$ and

$$\begin{aligned}L_1 &= \ln\left|\frac{x^2(z-1)(\rho^2-z)}{(x-z)(x\rho^2-z)}\right|,\qquad L_2=\ln\left|\frac{(z-x^2)(x^2\rho^2-z)}{x^2(x-z)(x\rho^2-z)}\right|,\qquad(31)\\ L_3 &= \ln\left|\frac{(z-x^2)(x\rho^2-z)}{(x-z)(x^2\rho^2-z)}\right|.\end{aligned}$$

It is seen from Eq. (30) that Σ^H contains the auxiliary parameter Δ. This parameter disappears, as it should, in the sum $\Sigma^\gamma=\Sigma^H+\Sigma^{V+S}$, where Σ^{V+S} is the contribution of virtual and soft real photons which can be obtained using Eq. (15):

$$\begin{aligned}\Sigma^\gamma &= \frac{\alpha}{\pi}\int\limits_{1}^{\rho^2}\frac{\mathrm{d}z}{z^2}\int\limits_{x_c}^{1}\mathrm{d}x(1-\Pi(-zQ_1^2))^{-2}\;\Big\{(L-1)P(x)\qquad(32)\\ &\times\;[1+\Theta(x^2\rho^2-z)]+\frac{1+x^2}{1-x}k(x,z)-\delta(1-x)\Big\},\end{aligned}$$

where

$$P(x)=\left(\frac{1+x^2}{1-x}\right)_+=\lim_{\Delta\to0}\left\{\frac{1+x^2}{1-x}\;\theta(1-x-\Delta)+(\frac{3}{2}+2\ln\Delta)\;\delta(1-x)\right\}\qquad(33)$$

is the non-singlet splitting kernel (see Appendix A for details).

4 Radiative corrections to $\mathcal{O}(\alpha^2)$

We consider first virtual two-loop corrections $d\sigma_{VV}^{(2)}$ to the elastic scattering cross-section. Using the representation (16) and the loop expansion for the Dirac form factor F_1

$$F_1 = 1 + \frac{\alpha}{\pi} F_1^{(1)} + (\frac{\alpha}{\pi})^2 F_1^{(2)} \tag{34}$$

one obtains

$$\frac{\mathrm{d}\sigma_{VV}^{(2)}}{\mathrm{d}c} = \frac{\mathrm{d}\sigma_0}{\mathrm{d}c}(\frac{\alpha}{\pi})^2(1-\Pi(t))^{-2}[\,6(F_1^{(1)})^2 + 4F_1^{(2)}]. \tag{35}$$

The one-loop contribution to the form factor is well known:

$$F_1^{(1)} = (L-1)\ln\frac{\lambda}{m} + \frac{3}{4}L - \frac{1}{4}L^2 - 1 + \frac{1}{2}\zeta_2. \tag{36}$$

The two-loop correction can be obtained from the results of Ref. [13]. Let us present it in the form

$$F_1^{(2)} = F_1^{\gamma\gamma} + F_1^{e^+e^-}, \tag{37}$$

where the contribution $F_1^{e^+e^-}$ is related to the vacuum polarization by e^+e^- pairs:

$$\begin{aligned}
F_1^{e^+e^-} &= -\frac{1}{36}L^3 + \frac{19}{72}L^2 - \Big(\frac{265}{216} + \frac{1}{6}\zeta_2\Big)L + \mathcal{O}(1), && (38)\\
F_1^{\gamma\gamma} &= \frac{1}{32}L^4 - \frac{3}{16}L^3 + \Big(\frac{17}{32} - \frac{1}{8}\zeta_2\Big)L^2 + \Big(-\frac{21}{32} - \frac{3}{8}\zeta_2 + \frac{3}{2}\zeta_3\Big)L && (39)\\
&+ \frac{1}{2}(L-1)^2\ln^2\frac{m}{\lambda} + (L-1)\Big[-\frac{1}{4}L^2 + \frac{3}{4}L - 1 + \frac{1}{2}\zeta_2\Big]\ln\frac{\lambda}{m} + \mathcal{O}(1), &&\\
\zeta_2 &= \sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \zeta_3 = \sum_1^\infty \frac{1}{n^3} \approx 1.202\,. &&
\end{aligned}$$

The photon mass λ entering Eqs. (36)–(39) is cancelled in the expression $\mathrm{d}\sigma^{(2)}/\mathrm{d}c$ for the sum of the virtual and soft-photon corrections of the second order $\mathrm{d}\sigma_{VV}^{(2)}/\mathrm{d}c$ (see Eq. (35)), $\mathrm{d}\sigma_{SS}^{(2)}/\mathrm{d}c$ and $\mathrm{d}\sigma_{SV}^{(2)}/\mathrm{d}c$.

The cross-section $\mathrm{d}\sigma_{SS}^{(2)}/\mathrm{d}c$ for the emission of two soft photons, each of energy smaller than $\Delta\varepsilon = \varepsilon\Delta$, is ($\Delta \ll 1$):

$$\mathrm{d}\sigma_{SS}^{(2)} = \mathrm{d}\sigma_0\,(\frac{\alpha}{\pi})^2(1-\Pi(t))^{-2}\,8\Big[(L-1)\ln\frac{m\Delta}{\lambda} + \frac{1}{4}L^2 - \frac{1}{2}\zeta_2\Big]^2, \tag{40}$$

and the virtual correction $\mathrm{d}\sigma_{SV}^{(2)}/\mathrm{d}c$ to the cross-section of the single soft-photon emission is:

$$\mathrm{d}\sigma_{SV}^{(2)} = \mathrm{d}\sigma_0\,(\frac{\alpha}{\pi})^2(1-\Pi(t))^{-2}16F_1^{(1)}\Big[(L-1)\ln\frac{m\Delta}{\lambda} + \frac{1}{4}L^2 - \frac{1}{2}\zeta_2\Big]. \tag{41}$$

The contribution to Σ of this sum, with the exception of the part coming from $F_1^{e^+e^-}$ connected with the vacuum polarization, contains no more than a second power of L. It has the following form:

$$\Sigma^{\gamma\gamma}_{S+V} = \Sigma_{VV} + \Sigma_{VS} + \Sigma_{SS} = (\frac{\alpha}{\pi})^2 \int\limits_1^{\rho^2} \frac{\mathrm{d}z}{z^2}(1-\Pi(-zQ_1^2))^{-2} R^{\gamma\gamma}_{S+V}\,. \tag{42}$$

It is convenient to separate the $R^{\gamma\gamma}_{S+V}$ in the following way:

$$\begin{aligned}
R^{\gamma\gamma}_{S+V} &= r^{\gamma\gamma}_{S+V} + r_{S+V\gamma\gamma} + r^{\gamma}_{S+V\gamma}\,, \\
r^{\gamma\gamma}_{S+V} &= r_{S+V\gamma\gamma} = L^2\Big(2\ln^2\Delta + 3\ln\Delta + \frac{9}{8}\Big) \\
&+ L\Big(-4\ln^2\Delta - 7\ln\Delta + 3\zeta_3 - \frac{3}{2}\zeta_2 - \frac{45}{16}\Big), \\
r^{\gamma}_{S+V\gamma} &= 4[(L-1)\ln\Delta + \frac{3}{4}L - 1]^2.
\end{aligned} \tag{43}$$

The contribution to Σ coming from $F_1^{e^+e^-}$ contains an L^3 term, which is also cancelled when we take into account the soft pair production contribution

$$\begin{aligned}
\mathrm{d}\sigma_S^{e^+e^-} &= (\frac{\alpha}{\pi})^2\mathrm{d}\sigma_0\,(1-\Pi(t))^{-2}R_S^{e^+e^-} = (\frac{\alpha}{\pi})^2\mathrm{d}\sigma_0\,(1-\Pi(t))^{-2}\Big[\frac{1}{9}(L+2\ln\Delta)^3 \\
&- \frac{5}{9}(L+2\ln\Delta)^2 + \Big(\frac{56}{27} - \frac{2}{3}\zeta_2\Big)(L+2\ln\Delta) + \mathcal{O}(1)\Big].
\end{aligned} \tag{44}$$

Thus for the contribution of the virtual and soft e^+ e^- pairs to Σ we have

$$\begin{aligned}
\Sigma^{e^+e^-}_{S+V} &= (\frac{\alpha}{\pi})^2 \int\limits_1^{\rho^2} \frac{\mathrm{d}z}{z^2}(1-\Pi(-zQ_1^2))^{-2} R^{e^+e^-}_{S+V}\,, \\
R^{e^+e^-}_{S+V} &= R_S^{e^+e^-} + 4F_1^{e^+e^-} = L^2\Big(\frac{2}{3}\ln\Delta + \frac{1}{2}\Big) + L\Big(-\frac{17}{6} + \frac{4}{3}\ln^2\Delta \\
&- \frac{20}{9}\ln\Delta - \frac{4}{3}\zeta_2\Big) + \mathcal{O}(1).
\end{aligned} \tag{45}$$

In expressions (43)–(45), $\Delta = \delta\varepsilon/\varepsilon$ is the energy fraction carried by the soft pair, and it is assumed that $2m \ll \delta\varepsilon \ll \varepsilon$. Here we have taken into account only e^+ e^- pair production. An order of magnitude of the pair production radiative correction is less than 0.5%. A rough estimate of the muon pair contribution gives less than 0.05% since $\ln(Q^2/m^2) \sim 3\ln(Q^2/m_\mu^2)$. Contributions of pion and tau lepton pairs give corrections that are still smaller. Therefore, within the 0.1% accuracy, we may omit any pair production contribution except the e^+ e^- one.

5 Virtual and soft corrections to the hard-photon emission

By evaluating the corrections arising from the emission of virtual and real soft photons which accompayn a single hard-photon we will consider two cases. The first case corresponds to the emission of the photons by the same fermion. The second one occurs when the hard-photon is emitted by another fermion:

$$\mathrm{d}\sigma\Big|_{H(S+V)} = \mathrm{d}\sigma^{H(S+V)} + \mathrm{d}\sigma_{H(S+V)} + \mathrm{d}\sigma^{H}_{(S+V)} + \mathrm{d}\sigma_{H}^{(S+V)}. \tag{46}$$

In the case when both fermions emit, one finds that:

$$\Sigma^{H}_{(S+V)} + \Sigma_{H}^{(S+V)} = 2\Sigma^{H}(\frac{\alpha}{\pi})\Big[(L-1)\ln\Delta + \frac{3}{4}L - 1\Big], \tag{47}$$

where Σ^H is given in Eq. (30). A more complex expression arises when the radiative corrections are applied to the same fermion line. In this case the cross-section may be expressed in terms of the Compton tensor with a *heavy photon* [14], which describes the process

$$\gamma^*(q) + e^-(p_1) \to e^-(q_1) + \gamma(k) + (\gamma_{\mathrm{soft}}). \tag{48}$$

In the limit of small-angle photon emission we have:

$$\begin{aligned}
\mathrm{d}\sigma^{H(S+V)} &= \frac{\alpha^4 \mathrm{d}x \mathrm{d}^2\boldsymbol{q}_1^{\perp}\mathrm{d}^2\boldsymbol{q}_2^{\perp}}{4x(1-x)(\boldsymbol{q}_2^{\perp})^4\pi^3}[(B_{11}(s_1,t_1) + x^2 B_{11}(t_1,s_1))\rho + T], \qquad (49)\\
T &= T_{11}(s_1,t_1) + x^2 T_{11}(t_1,s_1) + x(T_{12}(s_1,t_1) + T_{12}(t_1,s_1)),\\
\rho &= 2\Big(L - \ln\frac{(\boldsymbol{q}_2^{\perp})^2}{-u_1} - 1\Big)(2\ln\Delta - \ln x) + 3L - \ln^2 x - \frac{9}{2},
\end{aligned}$$

where $\Delta = (\Delta\varepsilon/\varepsilon) \ll 1$, $\Delta\varepsilon$ is the maximal energy of the soft photon, escaping the detectors, and $B_{11}(s_1,t_1) = (-4(\boldsymbol{q}_2^{\perp})^2)/(s_1 t_1) - 8m^2/s_1^2$ is the Born Compton tensor component, and the invariants are: $s_1 = 2q_1 k,\ \ t_1 = -2p_1 k,\ \ u_1 = (p_1 - q_1)^2,\ \ s_1 + t_1 + u_1 = q^2$.

The final result (see Appendix C for details) has the form:

$$\begin{aligned}
\Sigma^{H(S+V)} &= \Sigma_{H(S+V)} = \frac{1}{2}(\frac{\alpha}{\pi})^2 \int\limits_1^{\rho^2}\frac{\mathrm{d}z}{z^2}\int\limits_{x_c}^{1-\Delta}\frac{\mathrm{d}x(1+x^2)}{1-x}\, L\,\Big\{\Big(2\ln\Delta - \ln x + \frac{3}{2}\Big) \qquad (50)\\
&\times\ [(L-1)(1+\Theta) + k(x,z)] + \frac{1}{2}\ln^2 x + (1+\Theta)[-2 + \ln x - 2\ln\Delta]\\
&+\ (1-\Theta)\Big[\frac{1}{2}L\ln x + 2\ln\Delta\ln x - \ln x\ln(1-x)
\end{aligned}$$

$$
\begin{aligned}
&- \quad \ln^2 x - \mathrm{Li}_2(1-x) - \frac{x(1-x)+4x\ln x}{2(1+x^2)}\Big] - \frac{(1-x)^2}{2(1+x^2)}\Big\},\\
\mathrm{Li}_2(x) \quad &\equiv \quad -\int\limits_0^x \frac{\mathrm{d}t}{t}\ln(1-t),
\end{aligned}
$$

where $k(x,z)$ is given in Eq. (30) and $\Theta \equiv \Theta(x^2\rho^2 - z)$.

6 Double hard-photon bremsstrahlung

We now consider the contribution given by the process of emission of two hard photons. We will distinguish two cases: a) the double simultaneous bremsstrahlung in opposite directions along electron and positron momenta, and b) the double bremsstrahlung in the same direction along electron or positron momentum. The differential cross-section in the first case can be obtained by using the factorization property of cross-sections within the impact parameter representation [15]. It takes the following form [12] (see Appendix A):

$$
\frac{\mathrm{d}\sigma^{e^+e^-\to(e^+\gamma)(e^-\gamma)}}{\mathrm{d}x_1\mathrm{d}^2\boldsymbol{q}_1^\perp\mathrm{d}x_2\mathrm{d}^2\boldsymbol{q}_2^\perp} = \frac{\alpha^4}{\pi^3}\int\frac{\mathrm{d}^2\boldsymbol{k}^\perp}{\pi(\boldsymbol{k}^\perp)^4}\,(1-\Pi(-(\boldsymbol{k}^\perp)^2))^{-2}R(x_1;\boldsymbol{q}_1^\perp,\boldsymbol{k}^\perp)R(x_2;\boldsymbol{q}_2^\perp,-\boldsymbol{k}^\perp), \qquad (51)
$$

where $R(x;\boldsymbol{q}^\perp,\boldsymbol{k}^\perp)$ is given by Eq. (25). The calculation of the corresponding contribution Σ_H^H to Σ is analogous to the case of the single hard-photon emission and the result has the form:

$$
\Sigma_H^H = \frac{1}{4}(\frac{\alpha}{\pi})^2\int\limits_0^\infty\frac{\mathrm{d}z}{z^2}(1-\Pi(-zQ_1^2))^{-2}\int\limits_{x_c}^{1-\Delta}\mathrm{d}x_1\int\limits_{x_c/x_1}^{1-\Delta}\mathrm{d}x_2\,\frac{1+x_1^2}{1-x_1}\,\frac{1+x_2^2}{1-x_2}\,\Phi(x_1,z)\Phi(x_2,z), \qquad (52)
$$

where (see Eq. (31)):

$$
\begin{aligned}
\Phi(x,z) \quad &= \quad (L-1)[\Theta(z-1)\Theta(\rho^2-z) + \Theta(z-x^2)\Theta(\rho^2x^2-z)] \qquad (53)\\
&+ \quad L_3[-\Theta(x^2-z) + \Theta(z-x^2\rho^2)] + \Big(L_2 + \frac{(1-x)^2}{1+x^2}\Big)\Theta(z-x^2)\Theta(x^2\rho^2-z)\\
&+ \quad \Big(L_1 + \frac{(1-x)^2}{1+x^2}\Big)\Theta(z-1)\Theta(\rho^2-z)\\
&+ \quad (\Theta(1-z) - \Theta(z-\rho^2))\ln\left|\frac{(z-x)(\rho^2-z)}{(x\rho^2-z)(z-1)}\right|.
\end{aligned}
$$

Let us now turn to the double hard-photon emission in the same direction and the hard e^+ e^- pair production. Here we use the method developed by one of us [16, 17]. We will distinguish the collinear and semi-collinear kinematics of final particles. In the first case all produced particles move in the cones within the polar angles $\theta_i < \theta_0 \ll 1$ centred along the charged-particle momenta (final or initial). In the semi-collinear region only one of the produced particles moves inside those cones, while the other moves outside them. In the totally inclusive cross-section, such a distinction no longer has physical meaning and the dependence on the auxiliary parameter θ_0 disappears.

The contribution of both collinear and semi-collinear regions (we consider for definiteness the emission of both hard photons along the electron, since the contribution of the emission along the positron is the same) has the form (see Appendix B for details):

$$\begin{aligned}\Sigma^{HH} &= \Sigma_{HH} = \frac{1}{4}(\frac{\alpha}{\pi})^2 \int\limits_1^{\rho^2} \frac{\mathrm{d}z}{z^2}(1-\Pi(-zQ_1^2))^{-2} \\ &\times \int\limits_{x_c}^{1-2\Delta} \mathrm{d}x \int\limits_{\Delta}^{1-x-\Delta} \mathrm{d}x_1 \frac{I^{HH}L}{x_1(1-x-x_1)(1-x_1)^2}, \\ I^{HH} &= A;\Theta(x^2\rho^2-z)+B+C\;\Theta((1-x_1)^2\rho^2-z),\end{aligned} \tag{54}$$

where

$$\begin{aligned}A &= \gamma\beta\Big(\frac{L}{2}+\ln\frac{(\rho^2x^2-z)^2}{x^2(\rho^2x(1-x_1)-z)^2}\Big)+(x^2+(1-x_1)^4)\ln\frac{(1-x_1)^2(1-x-x_1)}{xx_1}+\gamma_A, \\ B &= \gamma\beta\left(\frac{L}{2}+\ln\left|\frac{x^2(z-1)(\rho^2-z)(z-x^2)(z-(1-x_1)^2)^2(\rho^2x(1-x_1)-z)^2}{(\rho^2x^2-z)(z-(1-x_1))^2(\rho^2(1-x_1)^2-z)^2(z-x(1-x_1))^2}\right|\right) \\ &+ (x^2+(1-x_1)^4)\ln\frac{(1-x_1)^2x_1}{x(1-x-x_1)}+\delta_B, \\ C &= \gamma\beta\left(L+2\ln\left|\frac{x(\rho^2(1-x_1)^2-z)^2}{(1-x_1)^2(\rho^2x(1-x_1)-z)(\rho^2(1-x_1)-z)}\right|\right) \\ &- 2(1-x_1)\beta-2x(1-x_1)\gamma,\end{aligned}$$

where

$$\begin{aligned}\gamma &= 1+(1-x_1)^2, \qquad \beta=x^2+(1-x_1)^2, \\ \gamma_A &= xx_1(1-x-x_1)-x_1^2(1-x-x_1)^2-2(1-x_1)\beta, \\ \delta_B &= xx_1(1-x-x_1)-x_1^2(1-x-x_1)^2-2x(1-x_1)\gamma.\end{aligned}$$

One may see that the combinations

$$r^{\gamma\gamma}+\Sigma^{H(S+V)}+\Sigma^{HH}, \qquad r_\gamma^\gamma+\Sigma_{S+V}^H+\Sigma_H^{S+V}+\Sigma_H^H \tag{55}$$

with $r^{\gamma\gamma}$ and r_γ^γ normalized (see Eqs. (42,43)) to

$$r^{\gamma\gamma}\to(\frac{\alpha}{\pi})^2\int\limits_1^{\rho^2}\frac{\mathrm{d}z}{z^2}(1-\Pi(-zQ_1^2))^{-2}r_{S+V}^{\gamma\gamma},$$

and

$$r_{\gamma}^{\gamma} \to (\frac{\alpha}{\pi})^2 \int\limits_1^{\rho^2} \frac{\mathrm{d}z}{z^2}(1-\Pi(-zQ_1^2))^{-2} r_{S+V\gamma}^{\gamma},$$

respectively, do not depend on Δ for $\Delta \to 0$ (see Appendix E).

The total expression $\Sigma^{2\gamma}$, which describes the contribution to (20) from the two-photon (real and virtual) emission processes is determined by expressions (43), (47), (49), (51) , (53) and (55). Furthermore it does not depend on the auxiliary parameter Δ and has the form:

$$\begin{aligned} \Sigma^{2\gamma} &= \Sigma_{S+V}^{\gamma\gamma} + 2\Sigma^{H(V+S)} + 2\Sigma_{S+V}^{H} + \Sigma_{H}^{H} + 2\Sigma^{HH} \\ &= \Sigma^{\gamma\gamma} + \Sigma_{\gamma}^{\gamma} + (\frac{\alpha}{\pi})^2 \mathcal{L}\phi^{\gamma\gamma}, \qquad \mathcal{L} = \ln\frac{\varepsilon^2\theta_1^2}{m^2}. \end{aligned} \tag{56}$$

The leading contributions $\Sigma^{\gamma\gamma}, \Sigma_{\gamma}^{\gamma}$ have the following forms (see Appendix D):

$$\begin{aligned} \Sigma^{\gamma\gamma} &= \frac{1}{2}(\frac{\alpha}{\pi})^2 \int\limits_1^{\rho^2} \frac{\mathrm{d}z}{z^2} L^2 (1-\Pi(-Q_1^2 z))^{-2} \int\limits_{x_c}^{1} \mathrm{d}x \left\{ \frac{1}{2} P^{(2)}(x) \; [\; \Theta(x^2\rho^2 - z) + 1] \right. \\ &+ \left. \int\limits_x^1 \frac{\mathrm{d}t}{t} P(t)\; P(\frac{x}{t})\; \Theta(t^2\rho^2 - z) \right\}, \end{aligned} \tag{57}$$

$$\begin{aligned} P^{(2)}(x) &= \int\limits_x^1 \frac{\mathrm{d}t}{t} P(t)\; P(\frac{x}{t}) = \lim_{\Delta\to 0} \left\{ \left[\left(2\ln\Delta + \frac{3}{2} \right)^2 - 4\zeta_2 \right] \delta(1-x) \right. \\ &+ \left. 2\left[\frac{1+x^2}{1-x}\left(2\ln(1-x) - \ln x + \frac{3}{2} \right) + \frac{1}{2}(1+x)\ln x - 1 + x \right] \Theta(1-x-\Delta) \right\}, \end{aligned} \tag{58}$$

$$\begin{aligned} \Sigma_{\gamma}^{\gamma} &= \frac{1}{4}(\frac{\alpha}{\pi})^2 \int\limits_0^{\infty} \frac{\mathrm{d}z}{z^2} L^2 (1-\Pi(-Q_1^2 z))^{-2} \int\limits_{x_c}^{1} \mathrm{d}x_1 \int\limits_{x_c/x_1}^{1} \mathrm{d}x_2 P(x_1) P(x_2) \\ &\times [\Theta(z-1)\Theta(\rho^2 - z) + \Theta(z - x_1^2)\Theta(x_1^2\rho^2 - z)] \\ &\times [\Theta(z-1)\Theta(\rho^2 - z) + \Theta(z - x_2^2)\Theta(x_2^2\rho^2 - z)]. \end{aligned} \tag{59}$$

We see that the leading contributions to $\Sigma^{2\gamma}$ may be expressed in terms of kernels for the evolution equation for structure functions.

The function $\phi^{\gamma\gamma}$ in expression Eq. (56) collects the next-to-leading contributions which cannot be obtained by the structure functions method [18]. It has a form that can be obtained from comparison of the results in the leading logarithmic approximation and the logarithmic one given above.

7 Pair production

Pair production process in high-energy $e^+ e^-$ collisions was considerd about 60 years ago (see [12] and references therein). In particular it was found that the total cross-section contains cubic terms in large logarithm L. These terms come from the kinematics when the scattered electron and positron move in narrow (with opening angles $\sim m/\epsilon$) cones and the created pair have the invariant mass of the order of m and moves preferably along either the electron beam direction or the positron one. According to the conditions of the LEP detectors, such a kinematics can be excluded. In the relevant kinematical region a parton-like description could be used giving L^2 and L-enhanced terms.

We accept the LEP 1 conventions whereby an event of the Bhabha process is defined as one in which the angles of the simultaneously registered particles hitting opposite detectors (see Eq. (91)).

The method, developed by one of us (N.P.M.) [16, 17], of calculating the real hard pair production cross-section within logarithmic accuracy consists in separating the contributions of the collinear and semi-collinear kinematical regions. In the first one (CK) we suggest that both electron and positron from the created pair go in the narrow cone around the direction of one of the charged particles [the projectile (scattered) electron $\boldsymbol{p}_1$ ($\boldsymbol{q}_1$) or the projectile (scattered) positron $\boldsymbol{p}_2$ ($\boldsymbol{q}_2$)]:

$$\widehat{\boldsymbol{p}_+\boldsymbol{p}_-} \sim \widehat{\boldsymbol{p}_-\boldsymbol{p}_i} \sim \widehat{\boldsymbol{p}_+\boldsymbol{p}_i} < \theta_0 \ll 1, \qquad \varepsilon\theta_0/m \gg 1, \qquad \boldsymbol{p}_i = \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{q}_1, \boldsymbol{q}_2\,. \tag{60}$$

The contribution of the CK contains terms of order $(\alpha L/\pi)^2$, $(\alpha/\pi)^2 L\ln(\theta_0/\theta)$ and $(\alpha/\pi)^2 L$, where $\theta = \widehat{\boldsymbol{p}_-\boldsymbol{q}_1}$ is the scattering angle. In the semi–collinear region only one of conditions (60) on the angles is fulfilled:

$$\begin{aligned} &\widehat{\boldsymbol{p}_+\boldsymbol{p}_-} < \theta_0, \quad \widehat{\boldsymbol{p}_\pm\boldsymbol{p}_i} > \theta_0\,; \quad \text{or} \quad \widehat{\boldsymbol{p}_-\boldsymbol{p}_i} < \theta_0, \quad \widehat{\boldsymbol{p}_+\boldsymbol{p}_i} > \theta_0\,; \\ &\text{or} \quad \widehat{\boldsymbol{p}_-\boldsymbol{p}_i} > \theta_0, \quad \widehat{\boldsymbol{p}_+\boldsymbol{p}_i} < \theta_0\,. \end{aligned} \tag{61}$$

The contribution of the SCK contains terms of the form:

$$\left(\frac{\alpha}{\pi}\right)^2 L\ln\frac{\theta_0}{\theta}, \qquad \left(\frac{\alpha}{\pi}\right)^2 L. \tag{62}$$

The auxiliary parameter θ_0 drops out in the total sum of the CK and SCK contributions.

All possible mechanisms for pair creation (singlet and non-singlet) as well as the identity of the particles in the final state are taken into account [22]. In the case of small-angle Bhabha scattering only a part of the total 36 tree-type Feynman diagrams are relevant, i.e. the scattering diagrams[5].

[5]We have verified that the interference between the amplitudes describing the production of pairs moving in the electron direction and the positron one cancels. This is known as up–down (interference) cancellation [22].

The sum of the contributions due to virtual pair emission (due to the vacuum polarization insertions in the virtual photon Green's function) and of those due to the real soft pair emission does not contain cubic ($\sim L^3$) terms but depends on the auxiliary parameter $\Delta = \delta\varepsilon/\varepsilon$ ($m_e \ll \delta\varepsilon \ll \varepsilon$, where $\delta\varepsilon$ is the sum of the energies of the soft pair components). The Δ-dependence disappears in the total sum after the contributions due to real hard pair production are added. Before summing one has to integrate the hard pair contributions over the energy fractions of the pair components, as well as over those of the scattered electron and positron:

$$
\begin{aligned}
\Delta &= \frac{\delta\varepsilon}{\varepsilon} < x_1 + x_2, \qquad x_c < x = 1 - x_1 - x_2 < 1 - \Delta, \\
x_1 &= \frac{\varepsilon_+}{\varepsilon}, \qquad x_2 = \frac{\varepsilon_-}{\varepsilon}, \qquad x = \frac{q_1^0}{\varepsilon},
\end{aligned} \tag{63}
$$

where $\varepsilon_\pm$ are the energies of the positron and electron from the created pair. We consider for definiteness the case when the created hard pair moves close to the direction of the initial (or scattered) electron.

Consider first the collinear kinematics. There are four different CK regions, when the created pair goes in the direction of the incident (scattered) electron or positron. We will consider only two of them, corresponding to the initial and the final electron directions. For the case of pair emission parallel to the initial electron, it is useful to decompose the particle momenta into longitudinal and transverse components:

$$
\begin{aligned}
p_+ &= x_1 p_1 + p_+^\perp, \qquad p_- = x_2 p_1 + p_-^\perp, \qquad q_1 = x p_1 + q_1^\perp, \\
x &= 1 - x_1 - x_2, \qquad q_2 \approx p_2, \qquad p_+^\perp + p_-^\perp + q_1^\perp = 0,
\end{aligned} \tag{64}
$$

where $p_i^\perp$ are the two-dimensional momenta of the final particles, which are transverse with respect to the initial electron beam direction. It is convenient to introduce dimensionless quantities for the relevant kinematical invariants:

$$
\begin{aligned}
z_i &= \left(\frac{\varepsilon\theta_i}{m}\right)^2, \qquad 0 < z_i < \left(\frac{\varepsilon\theta_0}{m}\right)^2 \gg 1, \\
A &= \frac{(p_+ + p_-)^2}{m^2} = (x_1 x_2)^{-1}[(1-x)^2 + x_1^2 x_2^2 (z_1 + z_2 - 2\sqrt{z_1 z_2}\cos\phi)], \\
A_1 &= \frac{2 p_1 p_-}{m^2} = x_2^{-1}[1 + x_2^2 + x_2^2 z_2], \qquad A_2 = \frac{2 p_1 p_+}{m^2} = x_1^{-1}[1 + x_1^2 + x_1^2 z_1], \\
C &= \frac{(p_1 - p_-)^2}{m^2} = 2 - A_1, \qquad D = \frac{(p_1 - q_1)^2}{m^2} - 1 = A - A_1 - A_2,
\end{aligned} \tag{65}
$$

where ϕ is the azimuthal angle between the $(\boldsymbol{p}_1 \boldsymbol{p}_+^\perp)$ and $(\boldsymbol{p}_1 \boldsymbol{p}_-^\perp)$ planes.

Keeping only the terms from the sum over spin states of the square of the absolute value of the matrix element, which give non-zero contributions to the cross-section in the limit $\theta_0 \to 0$, we find that only 8 from the total 36 Feynman diagrams are essential [22].

The result has the factorized form:

$$\sum_{\text{spins}} |M|^2 \Big|_{p_+,p_-||p_1} = \sum_{\text{spins}} |M_0|^2 \, 2^7\pi^2\alpha^2 \frac{I}{m^4}\,, \qquad (66)$$

where one of the multipliers corresponds to the matrix element in the Born approximation (without pair production):

$$\sum_{\text{spins}} |M_0|^2 = 2^7\pi^2\alpha^2 \left(\frac{s^4+t^4+u^4}{s^2t^2}\right), \qquad (67)$$
$$s = 2p_1p_2, \qquad t = -Q^2x, \qquad u = -s-t,$$

and the quantity I, which stands for the collinear factor, coincides with the expression for I_a obtained in [17]. We write it here in terms of our kinematical variables:

$$\begin{aligned} I &= (1-x_2)^{-2}\left(\frac{A(1-x_2)+Dx_2}{DC}\right)^2 + (1-x)^{-2}\left(\frac{C(1-x)-Dx_2}{AD}\right)^2 \qquad (68)\\ &+ \frac{1}{2xAD}\left[\frac{2(1-x_2)^2-(1-x)^2}{1-x} + \frac{x_1x-x_2}{1-x_2} + 3(x_2-x)\right] \\ &+ \frac{1}{2xCD}\left[\frac{(1-x_2)^2-2(1-x)^2}{1-x_2} + \frac{x-x_1x_2}{1-x} + 3(x_2-x)\right] \\ &+ \frac{x_2(x^2+x_2^2)}{2x(1-x_2)(1-x)AC} + \frac{3x}{D^2} + \frac{2C}{AD^2} + \frac{2A}{CD^2} + \frac{2(1-x_2)}{xA^2D} \\ &- \frac{4C}{xA^2D^2} - \frac{4A}{D^2C^2} + \frac{1}{DC^2}\left[\frac{(x_1-x)(1+x_2)}{x(1-x_2)} - 2\frac{1-x}{x}\right]. \end{aligned}$$

We rewrite the phase volume of the final particles as

$$\begin{aligned} \mathrm{d}\Gamma &= \frac{\mathrm{d}^3q_1\mathrm{d}^3q_2}{(2\pi)^6 2q_1^0 2q_2^0}(2\pi)^4\delta^4(p_1x+p_2-q_1-q_2) \qquad (69)\\ &\times\; m^4 2^{-8}\pi^{-4}x_1x_2\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}z_1\mathrm{d}z_2\frac{\mathrm{d}\phi}{2\pi}\,. \end{aligned}$$

Using the table of integrals given in Appendix F we further integrate over the variables of the created pair. Following a similar procedure in the case when the pair moves in the direction of the scattered electron, integrating the resulting sum over the energy fractions of the pair components, and finally adding the contribution of the two remaining CK regions (when the pair goes in the positron directions), we obtain[6]:

[6]Some misprints, which occur in the expressions for $f(x)$ and $f_1(x)$ in [17, 22], are corrected here.

$$
\begin{aligned}
\mathrm{d}\sigma_{\mathrm{coll}} &= \frac{\alpha^4 \mathrm{d}x}{\pi Q_1^2}\int\limits_1^{\rho^2}\frac{\mathrm{d}z}{z^2}\, L\left\{R_0(x)\left(L+2\ln\frac{\lambda^2}{z}\right)(1+\Theta)\right. && (70)\\
&+ \left. 4R_0(x)\ln x + 2\Theta f(x) + 2f_1(x)\right\}, \qquad \lambda = \frac{\theta_0}{\theta_{\mathrm{min}}},\\
\Theta &\equiv \Theta(x^2\rho^2 - z) = \begin{cases} 1, & x^2\rho^2 > z,\\ 0, & x^2\rho^2 \le z,\end{cases}\\
R_0(x) &= \frac{2}{3}\frac{1+x^2}{1-x} + \frac{(1-x)}{3x}(4+7x+4x^2) + 2(1+x)\ln x,\\
f(x) &= -\frac{107}{9} + \frac{136}{9}x - \frac{2}{3}x^2 - \frac{4}{3x} - \frac{20}{9(1-x)} + \frac{2}{3}[-4x^2 - 5x + 1\\
&+ \frac{4}{x(1-x)}]\ln(1-x) + \frac{1}{3}[8x^2 + 5x - 7 - \frac{13}{1-x}]\ln x - \frac{2}{1-x}\ln^2 x\\
&+ 4(1+x)\ln x\ln(1-x) - \frac{2(3x^2-1)}{1-x}\mathrm{Li}_2(1-x),\\
f_1(x) &= -x\ Ree\ f(\frac{1}{x}) = -\frac{116}{9} + \frac{127}{9}x + \frac{4}{3}x^2 + \frac{2}{3x} - \frac{20}{9(1-x)} + \frac{2}{3}[-4x^2\\
&- 5x + 1 + \frac{4}{x(1-x)}]\ln(1-x) + \frac{1}{3}[8x^2 - 10x - 10 + \frac{5}{1-x}]\ln x\\
&- (1+x)\ln^2 x + 4(1+x)\ln x\ln(1-x) - \frac{2(x^2-3)}{1-x}\mathrm{Li}_2(1-x),\\
\mathrm{Li}_2(x) &\equiv -\int\limits_0^x \frac{\mathrm{d}y}{y}\ln(1-y), \qquad Q_1 = \varepsilon\theta_{\mathrm{min}}, \qquad L = \ln\frac{zQ^2}{m^2},
\end{aligned}
$$

Consider now semi-collinear kinematical regions. We will restrict ourselves again to the case in which the created pair goes close to the electron momentum (initial or final). A similar treatment applies in the CM system in the case in which the pair follows the positron momentum. There are three different semi-collinear regions, which contribute to the cross-section within the required accuracy. The first region includes the events for which the created pair has very small invariant mass:

$$4m^2 \ll (p_+ + p_-)^2 \ll |q^2|,$$

and the pair escapes the narrow cones (defined by θ_0) in both the incident and scattered electron momentum directions. We will refer to this SCK region as $\boldsymbol{p}_+ \parallel \boldsymbol{p}_-$. The reason is the smallness (in comparison with s) of the square of the four-momentum of the virtual photon converting to the pair [22].

The second SCK region includes the events for which the invariant mass of the created positron and the scattered electron is small, $4m^2 \ll (p_+ + q_1)^2 \ll |q^2|$, with the restriction that the positron should escape the narrow cone in the initial electron momentum direction. We refer to it as $\boldsymbol{p}_+ \parallel \boldsymbol{q}_1$ [22].

The third SCK region includes the events in which the created electron goes inside the narrow cone in the initial electron momentum direction, but the created positron does not. We refer to it as $\boldsymbol{p}_- \parallel \boldsymbol{p}_1$ [22].

The differential cross-section takes the following form:

$$\begin{aligned} \mathrm{d}\sigma &= \frac{\alpha^4}{8\pi^4 s^2}\frac{|M|^2}{q^4}\frac{\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x}{x_1x_2x}\mathrm{d}^2\boldsymbol{p}_+^\perp\mathrm{d}^2\boldsymbol{p}_-^\perp\mathrm{d}^2\boldsymbol{q}_1^\perp\mathrm{d}^2\boldsymbol{q}_2^\perp\delta(1-x_1-x_2-x) \\ &\times \ \delta^{(2)}(\boldsymbol{p}_+^\perp+\boldsymbol{p}_-^\perp+\boldsymbol{q}_1^\perp+\boldsymbol{q}_2^\perp)\,, \end{aligned} \tag{71}$$

where x_1 (x_2), x and $\boldsymbol{p}_+^\perp$ ($\boldsymbol{p}_-^\perp$), $\boldsymbol{q}_1^\perp$ are the energy fractions and the perpendicular momenta of the created positron (electron) and the scattered electron (positron) respectively; $s=(p_1+p_2)^2$ and $q^2=-Q^2=(p_2-q_2)^2=-\varepsilon^2\theta^2$ are the centre-of-mass energy squared and the momentum transferred squared; the matrix element squared $|M|^2$ takes different forms according to the different SCK regions.

For the differential cross-section in the $\boldsymbol{p}_+ \parallel \boldsymbol{p}_-$ region we have (see, for details, [20]):

$$\begin{aligned} \mathrm{d}\sigma_{\boldsymbol{p}_+\parallel\boldsymbol{p}_-} &= \frac{\alpha^4}{\pi}L\ \mathrm{d}x\ \mathrm{d}x_2\frac{\mathrm{d}(\boldsymbol{q}_2^\perp)^2}{(\boldsymbol{q}_2^\perp)^2}\frac{\mathrm{d}(\boldsymbol{q}_1^\perp)^2}{(\boldsymbol{q}_1^\perp+\boldsymbol{q}_2^\perp)^2} \\ &\times \ \frac{\mathrm{d}\phi}{2\pi}\frac{1}{(\boldsymbol{q}_1^\perp+x\boldsymbol{q}_2^\perp)^2}\left[(1-x_1)^2+(1-x_2)^2-\frac{4xx_1x_2}{(1-x)^2}\right], \end{aligned} \tag{72}$$

where ϕ is the angle between the two-dimensional vectors $\boldsymbol{q}_1^\perp$ and $\boldsymbol{q}_2^\perp$, $\boldsymbol{q}_{1,2}^\perp$ are the transverse momentum components of the final electrons, $x_{1,2}$ are their energy fractions ($x=1-x_1-x_2$). At this stage it is necessary to use the restrictions on the two-dimensional momenta $\boldsymbol{q}_1^\perp$ and $\boldsymbol{q}_2^\perp$. They appear when the contribution of the CK region (which in this case represents the narrow cones with opening angle θ_0 in the momentum directions of both incident and scattered electrons) is excluded:

$$\left|\frac{\boldsymbol{p}_+^\perp}{\varepsilon_+}\right| > \theta_0, \qquad \left|\boldsymbol{r}^\perp\right| = \left|\frac{\boldsymbol{p}_+^\perp}{\varepsilon_+}-\frac{\boldsymbol{q}_1^\perp}{\varepsilon_2}\right| > \theta_0\,, \tag{73}$$

where ε_+ and ε_2 are the energies of the created positron and the scattered electron respectively. In order to exclude $p_+^\perp$ from the above equation we use the conservation of the perpendicular momentum, in this case:

$$\boldsymbol{q}_1^\perp+\boldsymbol{q}_2^\perp+\frac{1-x}{x_1}\boldsymbol{p}_+^\perp=0.$$

In the semi–collinear region $\boldsymbol{p}_+ \parallel \boldsymbol{q}_1$ we obtain:

$$\begin{aligned} \mathrm{d}\sigma_{\boldsymbol{p}_+\parallel\boldsymbol{q}_1} &= \frac{\alpha^4}{\pi}L\ \mathrm{d}x\ \mathrm{d}x_2\ \frac{\mathrm{d}(\boldsymbol{q}_2^\perp)^2}{(\boldsymbol{q}_2^\perp)^2}\frac{\mathrm{d}(\boldsymbol{q}_1^\perp)^2}{(\boldsymbol{q}_1^\perp)^2} \\ &\times \ \frac{\mathrm{d}\phi}{2\pi}\frac{1}{(\boldsymbol{q}_1^\perp+x\boldsymbol{q}_2^\perp)^2}\frac{x^2}{(1-x_2)^2}\left[(1-x)^2+(1-x_1)^2-\frac{4xx_1x_2}{(1-x_2)^2}\right], \end{aligned} \tag{74}$$

with the restrictions

$$\left|\frac{\boldsymbol{p}_-^\perp}{\varepsilon_-} - \frac{\boldsymbol{q}_1^\perp}{\varepsilon_2}\right| > \theta_0, \qquad \boldsymbol{p}_-^\perp + \boldsymbol{q}_2^\perp + \frac{1-x_2}{x}\boldsymbol{q}_1^\perp = 0. \tag{75}$$

Finally for the $\boldsymbol{p}_- \parallel \boldsymbol{p}_1$ semi–collinear region we get:

$$\begin{aligned} \mathrm{d}\sigma_{\boldsymbol{p}_-\parallel\boldsymbol{p}_1} &= \frac{\alpha}{4\pi}L\ \mathrm{d}x\ \mathrm{d}x_2\ \frac{\mathrm{d}(\boldsymbol{q}_2^\perp)^2}{(\boldsymbol{q}_2^\perp)^2}\frac{\mathrm{d}(\boldsymbol{q}_1^\perp)^2}{(\boldsymbol{q}_1^\perp)^2} \\ &\times \frac{\mathrm{d}\phi}{2\pi}\frac{1}{(\boldsymbol{q}_1^\perp+\boldsymbol{q}_2^\perp)^2}\biggl[\frac{(1-x)^2+(1-x_1)^2}{(1-x_2)^2} - \frac{4xx_1x_2}{(1-x_2)^4}\biggr]. \end{aligned} \tag{76}$$

The restriction due to the exclusion of the collinear region when the created pair moves inside a narrow cone in the direction of the initial electron has the form

$$\frac{|\boldsymbol{p}_+^\perp|}{\varepsilon_1} > \theta_0, \qquad \boldsymbol{p}_+^\perp + \boldsymbol{q}_1^\perp + \boldsymbol{q}_2^\perp = 0. \tag{77}$$

In order to obtain the finite expression for the cross-section we have to add $\mathrm{d}\sigma_{\boldsymbol{p}_+\parallel\boldsymbol{p}_-} + \mathrm{d}\sigma_{\boldsymbol{p}_+\parallel\boldsymbol{q}_1} + \mathrm{d}\sigma_{\boldsymbol{p}_-\parallel\boldsymbol{p}_1}$ to the contribution of the collinear kinematics region (70) and those due to the production of virtual and soft pairs. Taking into account the leading and next-to-leading terms we can write the full hard pair contribution including also the pair emission along the positron direction, after the integration over x_2 as

$$\begin{aligned} \sigma_{\mathrm{hard}} &= 2\,\frac{\alpha^4}{\pi Q_1^2}\int\limits_1^{\rho^2}\frac{\mathrm{d}z}{z^2}\int\limits_{x_c}^{1-\Delta}\mathrm{d}x\Bigl\{L^2(1+\Theta)R(x) + \mathcal{L}[\Theta F_1(x) + F_2(x)]\Bigr\}, \\ F_1(x) &= d(x) + C_1(x), \qquad F_2(x) = d(x) + C_2(x), \\ d(x) &= \frac{1}{1-x}\Bigl(\frac{8}{3}\ln(1-x) - \frac{20}{9}\Bigr), \\ C_1(x) &= -\frac{113}{9} + \frac{142}{9}x - \frac{2}{3}x^2 - \frac{4}{3x} - \frac{4}{3}(1+x)\ln(1-x) \\ &+ \frac{2}{3}\frac{1+x^2}{1-x}\Bigl[\ln\frac{(x^2\rho^2-z)^2}{(x\rho^2-z)^2} - 3\mathrm{Li}_2(1-x)\Bigr] + (8x^2+3x-9-\frac{8}{x} \\ &- \frac{7}{1-x})\ln x + \frac{2(5x^2-6)}{1-x}\ln^2 x + \beta(x)\ln\frac{(x^2\rho^2-z)^2}{\rho^4}, \\ C_2(x) &= -\frac{122}{9} + \frac{133}{9}x + \frac{4}{3}x^2 + \frac{2}{3x} - \frac{4}{3}(1+x)\ln(1-x) \\ &+ \frac{2}{3}\frac{1+x^2}{1-x}\Bigl[\ln\left|\frac{(z-x^2)(\rho^2-z)(z-1)}{(x^2\rho^2-z)(z-x)^2}\right| + 3\mathrm{Li}_2(1-x)\Bigr] \\ &+ \frac{1}{3}(-8x^2 - 32x - 20 + \frac{13}{1-x} + \frac{8}{x})\ln x + 3(1+x)\ln^2 x \end{aligned} \tag{78}$$

$$
\begin{aligned}
&+\quad \beta(x)\ln\left|\frac{(z-x^2)(\rho^2-z)(z-1)}{x^2\rho^2-z}\right|, \qquad \beta = 2R(x)-\frac{2}{3}\frac{1+x^2}{1-x}, \\
R(x) &= \frac{1}{3}\frac{1+x^2}{1-x}+\frac{1-x}{6x}(4+7x+4x^2)+(1+x)\ln x. \qquad (79)
\end{aligned}
$$

Eq. (78) describes the small-angle high-energy cross-section for the pair production process, provided that the created hard pair moves in the direction of the initial electron three-momentum. The factor 2 takes into account the production of a hard pair moving in the direction of the initial positron beam.

The contribution to the cross-section of the small-angle Bhabha scattering connected with the real soft (with energy lower than $\Delta\varepsilon$) and virtual pair production can be defined [22] by the formula:

$$
\begin{aligned}
\sigma_{\mathrm{soft+virt}} &= \frac{4\alpha^4}{\pi Q_1^2}\int\limits_1^{\rho^2}\frac{\mathrm{d}z}{z^2}\Big\{L^2\Big(\frac{2}{3}\ln\Delta+\frac{1}{2}\Big)+\mathcal{L}\Big(-\frac{17}{6}+\frac{4}{3}\ln^2\Delta \\
&- \frac{20}{9}\ln\Delta-\frac{4}{3}\zeta_2\Big)\Big\}. \qquad (80)
\end{aligned}
$$

Using eqs. (78) and (80) it is easy to verify that the auxiliary parameter Δ is cancelled in the sum $\sigma_{\mathrm{pair}}=\sigma_{\mathrm{hard}}+\sigma_{\mathrm{soft+virt}}$. We can, therefore, write the total contribution σ_{pair} as

$$
\begin{aligned}
&\sigma_{\mathrm{pair}} = \frac{2\alpha^4}{\pi Q_1^2}\int\limits_1^{\rho^2}\frac{\mathrm{d}z}{z^2}\Big\{L^2(1+\frac{4}{3}\ln(1-x_c)-\frac{2}{3}\int\limits_{x_c}^{1}\frac{\mathrm{d}x}{1-x}\bar{\Theta})+\mathcal{L}\Big[-\frac{17}{3} \qquad (81)\\
&-\frac{8}{3}\zeta_2-\frac{40}{9}\ln(1-x_c)+\frac{8}{3}\ln^2(1-x_c)+\int\limits_{x_c}^{1}\frac{\mathrm{d}x}{1-x}\bar{\Theta}\cdot(\frac{20}{9}-\frac{8}{3}\ln(1-x))\Big] \\
&+\int\limits_{x_c}^{1}\mathrm{d}x[L^2(1+\Theta)\bar{R}(x)+\mathcal{L}(\Theta C_1(x)+C_2(x))]\Big\}, \quad \bar{R}(x)=R(x)-\frac{2}{3(1-x)}, \\
&\bar{\Theta}=1-\Theta.
\end{aligned}
$$

The right-hand side of Eq. (81) gives the contribution to the small-angle Bhabha scattering cross-section for pair production. It is finite and can be used for numerical estimations. The leading term can be described by the electron structure function $D_e^{\bar{e}}(x)$ [19].

8 Terms of $\mathcal{O}(\alpha\mathcal{L})^3$

In order to evaluate the leading logarithmic contribution represented by terms of the type $(\alpha\mathcal{L})^3$, we use the iteration up to β^3 of the master equation obtained in Ref. [18]. To simplify the analytical expressions we adopt here a realistic assumption about the smallness of the threshold for the detection of the hard subprocess energy and neglect terms of the order of:

$$x_c^n\,(\frac{\alpha}{\pi}\mathcal{L})^3 \sim 10^{-4}, \qquad n=1,2,3\,. \tag{82}$$

We may, therefore, limit ourselves to consider the emission by the initial electron and positron. Three photons (virtual and real) contribution to Σ have the form:

$$\begin{aligned}\Sigma^{3\gamma} &= \frac{1}{4}\,(\frac{\alpha}{\pi}\mathcal{L})^3 \int\limits_1^{\rho^2}\frac{\mathrm{d}z}{z^2}\int\limits_{x_c}^{1}\mathrm{d}x_1\int\limits_{x_c}^{1}\mathrm{d}x_2\;\Theta(x_1x_2-x_c)\,\Big[\frac{1}{6}\delta(1-x_2)\;P^{(3)}(x_1) \\ &\times\;\;\Theta(x_1^2\rho^2-z)+\frac{1}{2}P^{(2)}(x_1)P(x_2)\Theta_1\Theta_2\Big]\;(1+\mathcal{O}(x_c^3)),\end{aligned} \tag{83}$$

where $P(x)$ and $P^{(2)}(x)$ are given by eqs. (33) and (57) correspondingly:

$$\begin{aligned}\Theta_1\Theta_2 &= \Theta(z-\frac{x_2^2}{x_1^2})\;\Theta(\rho^2\frac{x_2^2}{x_1^2}-z),\\ P^{(3)}(x) &= \delta(1-x)\;\Delta_t+\Theta(1-x-\Delta)\;\Theta_t,\\ \Delta_t &= 48\Big[\frac{1}{2}\zeta_3-\frac{1}{2}\zeta_2\Big(\ln\Delta+\frac{3}{2}\Big)+\frac{1}{6}\Big(\ln\Delta+\frac{3}{2}\Big)^3\Big],\\ \Theta_t &= 48\Big\{\frac{1}{2}\,\frac{1+x^2}{1-x}\Big[\frac{9}{32}-\frac{1}{2}\zeta_2+\frac{3}{4}\ln(1-x)-\frac{3}{8}\ln x+\frac{1}{12}\ln^2(1-x)\\ &+\;\frac{1}{12}\ln^2 x-\frac{1}{2}\ln x\ln(1-x)\Big]+\frac{1}{8}(1+x)\ln x\ln(1-x)-\frac{1}{4}(1-x)\ln(1-x)\\ &+\;\frac{1}{32}(5-3x)\ln x-\frac{1}{16}(1-x)-\frac{1}{32}(1+x)\ln^2 x+\frac{1}{8}(1+x)\mathrm{Li}_2(1-x)\Big\}.\end{aligned} \tag{84}$$

The contribution to Σ of the process of pair production accompanied by photon emission when both, pair and photons, may be real and virtual has the form (with respect to Ref. [16] we include also the non-singlet mechanism of pair production):

$$\begin{aligned}\Sigma^{e^+e^-\gamma} &= \frac{1}{4}(\frac{\alpha}{\pi}\mathcal{L})^3\int\limits_1^{\rho^2}dz\;z^{-2}\int\limits_{x_c}^{1}dx_1\int\limits_{x_c}^{1}dx_2\;\theta(x_1x_2-x_c)\\ &\{\frac{1}{3}[R^P(x_1)-\frac{1}{3}R^s(x_1)]\;\delta(1-x_2)\theta(x_1^2\rho^2-z)+\frac{1}{2}\;P(x_2)R(x_1)\;\theta_1\theta_2\},\end{aligned}$$

where

$$R(x)=R^s(x)+\frac{2}{3}P(x), \qquad R^s(x)=\frac{1-x}{3x}(4+7x+4x^2)+2(1+x)\ln x, \tag{85}$$

$$\begin{aligned}R^P(x) &= R^s(x)(\frac{3}{2}+2\ln(1-x))+(1+x)(-\ln^2 x-4\int\limits_0^{1-x}dy\frac{\ln(1-y)}{y})\\ &+\frac{1}{3}(-9-3x+8x^2)\ln x+\frac{2}{3}(-\frac{3}{x}-8+8x+3x^2).\end{aligned}$$

The total expression for Σ in Eq. (20) is the sum of the contributions in eqs. (21), (32), (56), (60), (66) and (68). The quantity Σ is a function of the parameters x_c, ρ and Q_1^2.

9 Estimates of neglected terms and numerical results

Let us now estimate the terms that were not taken into account here according to the required accuracy:

a) Weak radiative corrections:

$$\Sigma^{\text{w.r.c.}} \sim \frac{\alpha Q_1^2}{\pi M_z^2} \lesssim 10^{-5} \quad . \tag{86}$$

b) Electromagnetic corrections to weak contributions, including interference terms :

$$\Sigma_W^{\text{h.o.}} \sim \delta_{\text{weak}}|_{\theta=\theta_1} \frac{\alpha}{\pi} \mathcal{L} \lesssim 10^{-4} \quad . \tag{87}$$

Here δ_{weak} is given by Eq. (7).

c) Radiative corrections to the annihilation mechanism, including its interference with the scattering mechanism

$$\Sigma_{\text{st}}^{\text{r.c.}} \sim \theta_1^2 \frac{\alpha}{\pi} \mathcal{L} \lesssim 10^{-4} \quad . \tag{88}$$

Our explicit expressions for Σ^γ, without annihilation terms, coincide numerically with the results obtained at the same order in Ref. [19] by using exact matrix elements.

d) The interference between photon emissions by electron and positron

$$\Sigma_{\text{int}} \sim \theta_1^2 \frac{\alpha}{\pi} \lesssim 10^{-5} \quad . \tag{89}$$

This contribution is connected with terms violating the eikonal form [10] in the expression:

$$A(s,t) = A_0(s,t) e^{i\phi(t)} + O(\frac{\alpha t}{\pi s}) \quad . \tag{90}$$

e) Creation of heavy pairs $(\mu\mu, \tau\tau, \pi\pi, ...)$ is at least one order of magnitude smaller than the corresponding contribution due to the light particle production and is therefore not essential.

f) Higher-order corrections, including soft and collinear multi photon contributions, can be neglected since they only give contributions of the type $(\alpha\,\mathcal{L}/\pi)^n$ for $n \geq 4$.

Let us define Σ_0^0 to be equal to $\Sigma_0|_{\Pi=0}$ (see Eq. (21)), which corresponds to the Born cross-section obtained by switching off the vacuum polarization contribution $\Pi(t)$. For the experimentally observable cross-section we obtain:

$$\sigma = \frac{4\pi\alpha^2}{Q_1^2}\Sigma_0^0\,(1+\delta_0+\delta^\gamma+\delta^{2\gamma}+\delta^{e^+e^-}+\delta^{3\gamma}+\delta^{e^+e^-\gamma}), \tag{91}$$

where

$$\Sigma_0^0 = \Sigma_0|_{\Pi=0} = 1-\rho^{-2}+\Sigma_W+\Sigma_\theta|_{\Pi=0} \tag{92}$$

and

$$\delta_0 = \frac{\Sigma_0-\Sigma_0^0}{\Sigma_0^0}\ ;\ \delta^\gamma = \frac{\Sigma^\gamma}{\Sigma_0^0}\ ;\ \delta^{2\gamma} = \frac{\Sigma^{2\gamma}}{\Sigma_0^0}\ ;\ \cdots \quad . \tag{93}$$

The numerical results are presented below in table 1.

Table 1: The values of δ^i in per cent for $\sqrt{s}$ = 91.161 GeV, Θ_1 = 1.61°, Θ_2 = 2.8°, $\sin^2\Theta_W = 0.2283$, $\Gamma_Z = 2.4857$ GeV.

x_c	δ_0	δ^γ	$\delta^{2\gamma}_{\text{leading}}$	$\delta^{2\gamma}_{\text{non-leading}}$	$\delta^{e^+e^-}$	$\delta^{e^+e^-\gamma}$	$\delta^{3\gamma}$	$\sum\delta^i$
0.1	4.120	-8.918	0.657	0.162	-0.016	0.012	-0.067	-4.050
0.2	4.120	-9.226	0.636	0.156	-0.027	0.009	-0.056	-4.386
0.3	4.120	-9.626	0.615	0.148	-0.033	0.008	-0.051	-4.820
0.4	4.120	-10.147	0.586	0.139	-0.039	0.007	-0.048	-5.382
0.5	4.120	-10.850	0.539	0.129	-0.044	0.006	-0.045	-6.145
0.6	4.120	-11.866	0.437	0.132	-0.049	0.005	-0.041	-7.262
0.7	4.120	-13.770	0.379	0.130	-0.057	0.005	-0.035	-9.228
0.8	4.120	-17.423	0.608	0.089	-0.069	0.004	-0.025	-12.694
0.9	4.120	-25.269	1.952	-0.085	-0.085	0.007	-0.014	-19.374

Each of these contributions to σ has a sign that can change as a result of the interplay between real and virtual corrections. The cross-section corresponding to the Born diagrams for producing a real particle is always positive, whereas the sign of the radiative corrections depends on the order of perturbation theory. For the virtual corrections at odd orders it is negative, and at even orders it is positive. When the aperture of the counters is small the compensation between real and virtual corrections is not complete. In the limiting case of zero aperture only the virtual contributions remain, giving a negative result.

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Appendix A

Infinite momentum frame kinematics

In this appendix we will consider the kinematics we use to obtain the electron–positron and photon distributions. Due to the peculiar range of momenta and angles of the reaction, it is particularly convenient to use the Sudakov parametrization or infinite momentum frame kinematics. For the reaction

$$e^+(p_2) + e^-(p_1) \to e^+(q_2) + e^-(q_1) + \gamma(k) \tag{A.1}$$

let us introduce the Sudakov decomposition:

$$\begin{aligned} q_1 &= \alpha_1 \tilde{p}_2 + \beta_1 \tilde{p}_1 + q_1^\perp, \qquad q_2 = \alpha_2 \tilde{p}_2 + \beta_2 \tilde{p}_1 + q_2^\perp \\ k &= \alpha \tilde{p}_2 + \beta \tilde{p}_1 + k^\perp, \end{aligned} \tag{A.2}$$

where $\tilde{p}_{1,2}$ are almost light-like four-vectors:

$$q_i^\perp p_1 = q_i^\perp p_2 = 0, \qquad q_i^2 = -(\boldsymbol{q}_i^\perp)^2 < 0, \tag{A.3}$$

$$\begin{aligned} \tilde{p}_1 &= p_1 - \frac{m^2}{s} p_2\,, \quad \tilde{p}_2 = p_2 - \frac{m^2}{s} p_2, \\ p_1^2 &= p_2^2 = q_1 = q_2 = m^2, \quad k^2 = 0, \quad \tilde{p}_1^2 = \tilde{p}_2^2 = \frac{m^6}{s^2}\,, \\ s &= 2p_1 p_2 = 2\tilde{p}_1 \tilde{p}_2 = 2\tilde{p}_1 p_2 = 2\tilde{p}_2 p_1 \gg m^2, \end{aligned} \tag{A.4}$$

where $\boldsymbol{q}_i^\perp$ are Euclidean two-dimensional vectors in the centre-of-mass reference frame.

We consider the kinematical configuration when the photon is emitted in the direction close to the initial electron. We have the mass-shell conditions:

$$\begin{aligned} q_1^2 &= s\alpha_1\beta_1 - (\boldsymbol{q}_1^\perp)^2 = m^2, \qquad \alpha_1 = \frac{(\boldsymbol{q}_1^\perp)^2 + m^2}{\beta_1}\,, \\ (q_2')^2 &= s\alpha_2\beta_2 - (\boldsymbol{q}_2^\perp)^2 = m^2, \qquad \beta_2 = \frac{(\boldsymbol{q}_2^\perp)^2 + m^2}{s\alpha_2}\,, \\ k^2 &= s\alpha\beta - (\boldsymbol{k}^\perp)^2 = 0, \qquad s\alpha = \frac{(\boldsymbol{k}^\perp)^2}{\beta}\,, \\ \alpha_2 &= 1, \quad |\beta_2| \sim |\alpha_1| \sim |\alpha| \ll 1, \quad \beta_1 \sim \beta \sim 1. \end{aligned} \tag{A.5}$$

The components along $\tilde{p}_1$ of the jets containing $e^-(q_1)$ and $\gamma(k)$ have a value of $\mathcal{O}(1)$. The phase volume decomposition with $\mathrm{d}^4 q_1 = \frac{s}{2}\,\mathrm{d}\alpha_1 \mathrm{d}\beta_1 \mathrm{d}^2 \boldsymbol{q}_1^\perp$ is:

$$\begin{aligned} \mathrm{d}\phi &= \frac{\mathrm{d}^3\boldsymbol{q}_1 \mathrm{d}^3\boldsymbol{q}_2 \mathrm{d}^3\boldsymbol{k}}{2q_1^0 2q_2^0 2\omega} \delta^{(4)}(p_1 + p_2 - q_1 - q_2 - k) \\ &= \frac{1}{4s\beta\beta_1} \mathrm{d}\beta \mathrm{d}\beta_1 \delta(1 - \beta - \beta_1) \mathrm{d}^2\boldsymbol{k}^\perp \mathrm{d}^2\boldsymbol{q}_1^\perp \mathrm{d}^2\boldsymbol{q}_2^\perp \delta^{(2)}(\boldsymbol{q}_1^\perp + \boldsymbol{q}_2^\perp + \boldsymbol{k}^\perp). \end{aligned} \tag{A.6}$$

The conservation law reads (we introduce a new four-momentum q of the exchanged photon):

$$p_1 + q = q_1 + k, \qquad p_2 = q_2 + q. \tag{A.7}$$

The inverse propagators are (here and further we use $\beta_1 = x$):

$$\begin{aligned} &(p_1 - k)^2 - m^2 = \frac{-1}{1-x} d_1\,, \qquad (p_1 + q)^2 - m^2 = \frac{1}{x(1-x)} d, \\ &q^2 = -(\boldsymbol{q}_2^\perp)^2, \quad d = m^2(1-x)^2 + (\boldsymbol{q}_1^\perp + \boldsymbol{q}^\perp x)^2, \quad d_1 = m^2(1-x)^2 + (\boldsymbol{q}_1^\perp + \boldsymbol{q}^\perp)^2. \end{aligned} \tag{A.8}$$

The matrix element reads

$$\begin{aligned} M &= \frac{g^{\mu\nu}}{q^2} \bar{v}(p_2)\gamma_\mu v(q_2)\bar{u}(q_1) O_\nu u(p_1) \\ O^\nu &= \gamma^\nu \frac{\hat{p}_1 - \hat{k} + m}{(p_1 - k)^2 - m^2}\hat{e} + \hat{e}\frac{\hat{p}_1 + \hat{q} + m}{(p_1 + q)^2 - m^2}\gamma^\nu. \end{aligned} \tag{A.9}$$

The following decomposition of the metric tensor $g_{\mu\nu}$ is used:

$$g_{\mu\nu} = g_{\mu\nu}^\perp + \frac{p_1^\mu p_2^\nu + p_1^\nu p_2^\mu}{p_1 p_2} \simeq \frac{2p_1^\mu p_2^\nu}{s}\Big(1 + \mathcal{O}(\frac{\boldsymbol{q}^{\perp 2}}{s})\Big). \tag{A.10}$$

We use also the identity

$$p_2^\nu \bar{u}(q_1) O_\nu u(p_1) \equiv \bar{u}(q_1)\hat{v}_\rho u(p_1) e_\rho(k). \tag{A.11}$$

The generalized vertex v_ρ has the form [12]:

$$v_\rho = s\gamma_\rho x(1-x)\left(\frac{1}{d} - \frac{1}{d_1}\right) - \frac{\gamma_\rho \hat{k}\hat{p}_2}{d}x(1-x) - \frac{\hat{p}_2\hat{k}\gamma_\rho}{d_1}(1-x). \tag{A.12}$$

The evaluation of the spin sum of the squared matrix element gives

$$\sum_{\text{spin}} |\bar{v}(q_2)\hat{p}_1 v(p_2)|^2 = \text{Tr}\ \hat{p}_2\hat{p}_1\hat{p}_2\hat{p}_1 = 2s^2, \tag{A.13}$$

The squared matrix element for the single photon radiation is given by

$$\begin{aligned} R &= -\frac{1}{4s^2}\ \text{Tr}\ (\hat{p}_1 + m)\hat{v}_\mu(\hat{p}_1 + \hat{k} - \hat{q} + m)\hat{v}_\mu \\ &= x[-2xm^2(d - d_1)^2 + (\boldsymbol{q}_2^\perp)^2(1 + x^2)dd_1]\frac{1}{d^2 d_1^2}\,. \end{aligned} \tag{A.14}$$

Finally we obtain that

$$\mathrm{d}\sigma^{e^+e^-\to e^+(e^-\gamma)} = 2\alpha^3 \frac{\mathrm{d}^2\boldsymbol{q}_1^\perp \mathrm{d}\boldsymbol{q}_2^\perp \mathrm{d}x(1-x)}{\pi^2((\boldsymbol{q}_2^\perp)^2)^2(dd_1)^2}[-2xm^2(d-d_1)^2 + (\boldsymbol{q}_2^\perp)^2(1+x^2)dd_1]. \qquad \text{(A.15)}$$

In the same way we may obtain the cross-section for the process of the double bremsstrahlung in the opposite directions:

$$\begin{aligned} \frac{d\sigma^{e^+e^-\to e^+\gamma e^-\gamma}}{\mathrm{d}^2\boldsymbol{q}_1^\perp \mathrm{d}^2\boldsymbol{q}_2^\perp \mathrm{d}x_1 \mathrm{d}x_2} &= \frac{\alpha^4(1+x_1^2)(1+x_2^2)}{\pi^4(1-x_1)(1-x_2)} \int \frac{\mathrm{d}^2\boldsymbol{q}^\perp}{((\boldsymbol{q}^\perp)^2)^2} \qquad \text{(A.16)} \\ &\times \left[\frac{(\boldsymbol{q}^\perp)^2(1-x_1)^2}{d_1 d_2} - \frac{2x_1}{1+x_1^2}\frac{m^2(1-x_1)^2(d_1-d_2)^2}{d_1^2 d_2^2}\right] \\ &\times \left[\frac{(\boldsymbol{q}^\perp)^2(1-x_2)^2}{\tilde{d}_1\tilde{d}_2} - \frac{2x_2}{1+x_2^2}\frac{m^2(1-x_2)^2(\tilde{d}_2-\tilde{d}_1)^2}{\tilde{d}_1^2\tilde{d}_2^2}\right], \end{aligned}$$

where x_1, $\boldsymbol{q}_1^\perp$ and x_2, $\boldsymbol{q}_2^\perp$ are the energy fractions and the components transverse to the beam axis of the scattered electron and positron, respectively; $\boldsymbol{q}^\perp$ is the transverse two-dimensional momentum of the exchanged photon;

$$\begin{aligned} d_1 &= (1-x_1)^2m^2 + (\boldsymbol{q}_1^\perp - \boldsymbol{q}^\perp x_1)^2, \quad d_2 = (1-x_1)^2m^2 + (\boldsymbol{q}_1^\perp - \boldsymbol{q}^\perp)^2, \qquad \text{(A.17)} \\ \tilde{d}_1 &= (1-x_2)^2m^2 + (\boldsymbol{q}_2^\perp + \boldsymbol{q}^\perp x_2)^2, \quad \tilde{d}_2 = (1-x_2)^2m^2 + (\boldsymbol{q}_2^\perp + \boldsymbol{q}^\perp)^2. \end{aligned}$$

Let us now discuss the restrictions on the $\mathrm{d}^2\boldsymbol{q}_1^\perp$, $\mathrm{d}^2\boldsymbol{q}_2^\perp$ integration imposed by experimental conditions of the electron and positron tagging. We consider the emission of a hard photon along the electron direction. We will consider the symmetric case:

$$\begin{aligned} &\theta_1 < \theta_e = \frac{|\boldsymbol{q}_1^\perp|}{x\varepsilon} < \theta_2\,, \qquad \theta_e = \boldsymbol{p}_1\hat{\ }\boldsymbol{q}_1, \qquad \text{(A.18)} \\ &\theta_1 < \theta_{\bar{e}} = \frac{|\boldsymbol{q}_2^\perp|}{\varepsilon} < \theta_2\,, \qquad \theta_{\bar{e}} = \boldsymbol{p}_2\hat{\ }\boldsymbol{q}_2. \end{aligned}$$

Here θ_1 and θ_2 are the minimal and maximal angles of aperture for the counters. It is convenient to introduce dimensionless quantities $\rho = \theta_2/\theta_1$, $z_{1,2} = (\boldsymbol{q}_{1,2})^2/Q_1^2$ $(Q_1 = \varepsilon\theta_1)$.

The region in the z_1, z_2 plane that gives the largest contribution to Σ is made by two narrow strips along the lines $z_1 = z_2$ and $z_1 = x^2 z_2$. Therefore the leading logarithmic contribution will appear only in the cases where these lines cross the rectangle defined by $x^2 < z_1 < \rho^2 x^2$, $1 < z_2 < \rho^2$. Note that the line $z_1 = x^2 z_2$, which corresponds to the emission of one hard photon along the momentum of the scattered electron, is the diagonal of the rectangle defined above. As for the line $z_1 = z_2$, which corresponds to the emission along the initial electron momentum, it crosses the rectangle only if $x^2\rho^2 > z_2$, $x\rho > 1$. This last condition defines the appearance of leading contributions to Σ^H.

For the contribution from the photon emission by the initial electron we have:

$$
\begin{aligned}
F_1 &= \Theta(1-\rho x)\int\limits_1^{\rho^2}\frac{\mathrm{d}z_2}{z_2^2}\int\limits_{x^2}^{x^2\rho^2}\frac{\mathrm{d}z_1\; z_2(1-x)}{(z_1-xz_2)(z_2-z_1)} \\
&+ \Theta(x\rho-1)\int\limits_{x^2\rho^2}^{\rho^2}\frac{\mathrm{d}z_2}{z_2^2}\int\limits_{x^2}^{x^2\rho^2}\frac{\mathrm{d}z_1\; z_2(1-x)}{(z_1-xz_2)(z_2-z_1)} \\
&+ \Theta(x\rho-1)\int\limits_1^{x^2\rho^2}\frac{\mathrm{d}z_2}{z_2^2}\Biggl\{\int\limits_{x^2}^{z_2-\eta}\frac{\mathrm{d}z_1\; z_2(1-x)}{(z_1-xz_2)(z_2-z_1)}+\int\limits_{z_2+\eta}^{x^2\rho^2}\frac{\mathrm{d}z_1\; z_2(1-x)}{(z_1-xz_2)(z_1-z_2)} \\
&+ \int\limits_{z_2-\eta}^{z_2+\eta}\frac{\mathrm{d}z_1}{\sqrt{R}}-\frac{2x\sigma^2}{1+x^2}\int\limits_{z_2-\eta}^{z_2+\eta}\frac{2\mathrm{d}z_1\; z_2}{\sqrt{R^3}}\Biggr\}, \quad R=(z_2-z_1)^2+4\sigma^2 z_2,
\end{aligned} \tag{A.19}
$$

where we introduced the auxiliary parameter η, $\sigma^2 \ll \eta \ll 1$. Neglecting the terms of order η we obtain:

$$
\begin{aligned}
F_1 &= \int\limits_1^{\rho^2}\frac{\mathrm{d}z}{z^2}\Biggl\{\Theta(\rho x-1)\Theta(x^2\rho^2-z)\Bigl(L-\frac{2x}{1+x^2}\Bigr) \\
&+ \Theta(x^2\rho^2-z)L_2+\Theta(z-x^2\rho^2)L_3\Biggr\},
\end{aligned} \tag{A.20}
$$

where L_i are given in eq. (31) and we used the identity $\Theta(1-\rho x)+\Theta(\rho x-1)\Theta(z-x^2\rho^2)=\Theta(z-x^2\rho^2)$.

In the same way we obtain for the final electron emission:

$$
F_2 = \int\limits_1^{\rho^2}\frac{\mathrm{d}z}{z^2}\Bigl\{L-\frac{2x}{1+x^2}+L_1\Bigr\}. \tag{A.21}
$$

The total contribution due to one hard photon emission in small-angle Bhabha scattering therefore reads:

$$
\Sigma^H = \frac{\alpha}{\pi}\int\limits_{x_c}^{1-\Delta}\mathrm{d}x\;\frac{1+x^2}{1-x}(F_1+F_2). \tag{A.22}
$$

Appendix B

The contribution to Σ from the semi-collinear region of emission of two hard photons in the same direction

An alternative way to use the quasi-real electron approximation is to compute the cross-section directly. To logarithmic accuracy we may restrict ourselves to considering only two regions i) the one with photon with momentum k_1 emitted along the momentum direction of the initial electron inside a narrow cone with opening angle $\theta_0 \ll 1$, and ii) the region with the photon emitted along the scattered electron. Taking into account the identity of photons with the statistical factor $\frac{1}{2!}$ we obtain the cross-section:

$$\begin{aligned} \mathrm{d}\sigma_{SC}^{HH} &= \frac{\alpha^4}{2\pi}\int\frac{\mathrm{d}^2\boldsymbol{q}_2^\perp}{\pi((\boldsymbol{q}_2^\perp)^2)^2}\int\frac{\mathrm{d}^2\boldsymbol{q}_1^\perp}{\pi}\int\limits_{x_c}^{1-2\Delta}\mathrm{d}x \\ &\times \int\limits_{\Delta}^{1-x-\Delta}\frac{\mathrm{d}x_1\mathrm{d}x_2}{x_1x_2x}\delta(1-x_1-x_2-x)\int R\frac{\mathrm{d}^2\boldsymbol{k}_1^\perp}{\pi}\,, \end{aligned} \tag{B.1}$$

where

$$\begin{aligned} \int R\frac{\mathrm{d}^2\boldsymbol{k}_1^\perp}{\pi} &= 2(\boldsymbol{q}_2^\perp)^2Q_1^4\int\frac{\mathrm{d}^2\boldsymbol{k}_1^\perp}{\pi}\left\{\frac{[1+(1-x_1)^2][x^2+(1-x_1)^2]}{x_1(1-x_1)^2(2p_1k_1)(2p_1k_2)(2q_1k_2)}\bigg|_{\boldsymbol{k}_1\|\boldsymbol{p}_1}\right. \\ &+ \left.\frac{x[1+(1-x_2)^2][x^2+(1-x_2)^2]}{x_1(1-x_2)^2(2q_1k_1)(2p_1k_2)(2q_1k_2)}\bigg|_{\boldsymbol{k}_1\|\boldsymbol{q}_1}\right\}. \end{aligned} \tag{B.2}$$

It is convenient to specify the kinematics: in the case of the emission of the collinear photon with momentum $\boldsymbol{k}_1$ parallel to $\boldsymbol{p}_1$ we have

$$\begin{aligned} 2p_1k_1 &= \frac{Q_1^2}{x_1}[(\boldsymbol{k}_1^\perp)^2+\sigma^2x_1^2], \quad 2p_1k_2 = \frac{Q_1^2}{x_2}(\boldsymbol{k}_2^\perp)^2, \\ 2q_1k_2 &= \frac{Q_1^2}{x_2x}[x\boldsymbol{q}_2^\perp-(1-x_1)\boldsymbol{q}_1^\perp]^2, \qquad \boldsymbol{k}_2^\perp = -\boldsymbol{q}_2^\perp-\boldsymbol{q}_1^\perp\,; \end{aligned} \tag{B.3}$$

in the case when the photon is emitted along $\boldsymbol{q}_1$ we have

$$\begin{aligned} 2k_1q_1 = \frac{Q_1^2}{x_1x}[\sigma^2x_1^2+(x\boldsymbol{k}_1^\perp-\boldsymbol{q}_1^\perp)^2], \qquad 2p_1k_2 = \frac{Q_1^2}{x_2}(\boldsymbol{k}_2^\perp)^2, \\ 2q_1k_2 = \frac{Q_1^2}{x_2x}(\boldsymbol{q}_1^\perp-x\boldsymbol{q}_2^\perp)^2, \qquad \boldsymbol{k}_2^\perp = \boldsymbol{k}_2^\perp-\boldsymbol{q}_1^\perp\frac{1-x_2}{x}\,, \end{aligned} \tag{B.4}$$

where $Q_1^2=\epsilon^2\theta_1^2$, $\sigma^2=m^2/Q_1^2$, and we introduced two-dimensional vectors $\boldsymbol{k}_2^\perp$, $\boldsymbol{q}_1^\perp$ and $\boldsymbol{q}_2^\perp$ so that $(\boldsymbol{q}_1^\perp)^2=z_1$, $(\boldsymbol{q}_2^\perp)^2=z_2$ and $\widehat{\boldsymbol{q}_1^\perp\boldsymbol{q}_2^\perp}=\phi$.

The integration over $\mathrm{d}^2\boldsymbol{k}_1^\perp$ can be done with single logarithmic accuracy:

$$Q_1^2 \int \frac{\mathrm{d}^2\boldsymbol{k}_1^\perp}{\pi(2p_1k_1)}\bigg|_{\boldsymbol{k}_1\|\boldsymbol{p}_1} = x_1 L, \quad Q_1^2 \int \frac{\mathrm{d}^2\boldsymbol{k}_1^\perp}{\pi(2q_1k_1)}\bigg|_{\boldsymbol{k}_1\|\boldsymbol{q}_1} = \frac{x_1}{x} L. \tag{B.5}$$

It is also necessary, here, to consider the kinematical restrictions on the integration variables ϕ and z_1. When the photon is emitted within an angle θ_0 along the direction of the momentum of the initial electron, θ_0 represents the angular range to be filled by collinear kinematics events. We assign to the semi-collinear kinematics the events characterized by

$$i)\ \left|\frac{\boldsymbol{k}_2^\perp}{x_2}\right| > \theta_0\,, \qquad ii)\ \left|\frac{\boldsymbol{q}_1^\perp}{x} - \frac{\boldsymbol{k}_2^\perp}{x_2}\right| > \theta_0, \tag{B.6}$$

where the region $i)$ the photon with four-momentum k_2 escapes the narrow cone with opening angle θ_0 along the momentum direction of the initial electron. In the region $ii)$ the same happens for the final electron.

We can rewrite the conditions above in terms of the variables z_1 and ϕ as follows:

$$\begin{aligned}
i)&\quad 1 > \cos\phi > -1 + \frac{\lambda^2 - (\sqrt{z_1} - \sqrt{z_2})^2}{2\sqrt{z_1 z_2}}, \qquad |\sqrt{z_1} - \sqrt{z_2}| < \lambda, \\
ii)&\quad 1 > \cos\phi > -1, \qquad |z_1 - z_2| > 2\sqrt{z_2}\lambda, \\
iii)&\quad 1 > \cos\phi > -1 + \frac{\frac{x^2}{(1-x_1)^2}\lambda^2 - (\sqrt{z_1} - \frac{x}{1-x_1}\sqrt{z_2})^2}{2\sqrt{z_1 z_2}\frac{x}{1-x_1}}, \\
&\qquad |\sqrt{z_1} - \frac{x\sqrt{z_2}}{1-x_1}| < \lambda\frac{x}{1-x_1}, \\
iv)&\quad 1 > \cos\phi > -1, \qquad |z_1 - \frac{x^2}{(1-x_1)^2} z_2| > 2\lambda\sqrt{z_2}\frac{x^2}{(1-x_1)^2},
\end{aligned}$$

where $\lambda = x_2\theta_0/\theta_1$. In our calculation we take the parameter $\lambda \ll 1$. Indeed, the resctrictions on θ_0 for collinear kinematics calculations are $\varepsilon\theta_0 \gg m$ or $\theta_0 \gg 10^{-5}$ at LEP energies. On the other hand the experimental conditions on θ_1 are $\theta_1 > 10^{-2}$. Therefore we can take $\lambda \ll 1$ within our accuracy.

Analogous considerations can be made for the case when a photon with momentum k_1 is emitted along the direction of the final electron. In regions $ii)$ and $iv)$ we may do the integration over the azimuthal angle:

$$\int\limits_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi(2p_1k_2)(2q_1k_2)}\bigg|_{\boldsymbol{k}_1\|\boldsymbol{q}_1} = \frac{x_2 x Q_1^{-4}}{(1-x_1)z_1 - x z_2}\left[\frac{1}{|z_2 - z_1|} - \frac{x(1-x_1)}{|x^2 z_2 - (1-x_1)^2 z_1|}\right], \tag{B.7}$$

$$\int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi(2p_1k_2)(2q_1k_2)}\bigg|_{\boldsymbol{k}_1\|\boldsymbol{p}_1} = \frac{x_2 x^3(1-x_2)^{-2} Q_1^{-4}}{z_1 - z_2 x^2/(1-x_2)}\left[\frac{1}{|z_1 - z_2\frac{x^2}{(1-x_2)^2}|} - \frac{1-x_2}{|z_1 - x^2 z_2|}\right]. \tag{B.8}$$

The integration of regions $i)$, $iii)$ has the form

$$\begin{aligned}\mathcal{I} &= \int \mathrm{d}z_1 \int \frac{\mathrm{d}\phi}{2\pi(z_1+z_2+2\sqrt{z_1z_2}\,\cos\phi)}\Big|_{|\sqrt{z_1}-\sqrt{z_2}|<\lambda} \\ &= \frac{2}{\pi}\int \frac{dz}{|z_1-z_2|}\mathrm{arctg}\Big\{\frac{(\sqrt{z_1}-\sqrt{z_2})^2}{|z_1-z_2|}\mathrm{tg}\frac{\phi_0}{2}\Big\},\end{aligned} \tag{B.9}$$

where

$$\phi_0 = \arccos\Big(-1+\frac{\lambda^2-(\sqrt{z_1}-\sqrt{z_2})^2}{2\sqrt{z_1z_2}}\Big). \tag{B.10}$$

The result reads

$$\mathcal{I} = 2\,\ln 2. \tag{B.11}$$

We give here the complete contribution of the semi-collinear region:

$$\begin{aligned}\mathrm{d}\sigma^{HH}_{\text{s-coll}} &= \frac{\alpha^2\mathcal{L}}{4\pi^2}\int\limits_{x_c}^{1-2\Delta}\mathrm{d}x\int\limits_{\Delta}^{1-x-\Delta}\frac{\mathrm{d}x_1\mathrm{d}x_2\delta(1-x-x_1-x_2)}{x_1x_2(1-x_1)^2}[1+(1-x_1)^2][x^2+(1-x_1)^2] \\ &\times \int\limits_1^{\rho^2}\frac{\mathrm{d}z}{z^2}\Big\{\ln\frac{z\theta_1^2}{\theta_0^2}[1+\Theta(\rho^2x^2-z)+2\Theta(\rho^2(1-x_1)^2-z)] \\ &+ \Theta(\rho^2x^2-z)\ln\frac{(z-x^2)(\rho^2x^2-z)}{x^2(z-x(1-x_1))(\rho^2x(1-x_1)-z)} \\ &+ \Theta(z-\rho^2(1-x_1)^2)\Big[\ln\frac{(z-\rho^2(1-x_1)x)(z-(1-x_1)^2)}{(\rho^2(1-x_1)^2-z)(z-x(1-x_1))} \\ &+ \ln\frac{(\rho^2(1-x_1)-z)(z-(1-x_1)^2)}{(\rho^2(1-x_1)^2-z)(z-(1-x_1))}\Big] + \Theta(z-\rho^2x^2)\ln\frac{z-\rho^2x(1-x_1)(z-x^2)}{(\rho^2x^2-z)(z-x(1-x_1))} \\ &+ \Theta(\rho^2(1-x_1)^2-z)\Big[\ln\frac{(z-(1-x_1)^2)(\rho^2(1-x_1)^2-z)}{(\rho^2x(1-x_1)-z)(z-x(1-x_1))(1-x_1)^2} \\ &+ \ln\frac{(z-(1-x_1)^2)(\rho^2(1-x_1)^2-z)}{(\rho^2(1-x_1)-z)(z-(1-x_1))(1-x_1)^2}\Big] + \ln\frac{(z-1)(\rho^2-z)}{(z-(1-x_1))(\rho^2(1-x_1)-z)}\Big\}.\end{aligned}$$

To see the cancellation of the auxiliary parameter θ_0/θ_1 we give here the relevant part of the contribution for the collinear region :

$$\begin{aligned}\Sigma^{HH}_{\text{coll}} &= \frac{\alpha^2}{4\pi^2}\int\limits_{x_c}^{1-2\Delta}\mathrm{d}x\int\limits_{\Delta}^{1-x-\Delta}\frac{\mathrm{d}x_1\,\mathrm{d}x_2\delta(1-x-x_1-x_2)}{x_1x_2(1-x_1)^2}[1+(1-x_1)^2][x^2+(1-x_1)^2] \\ &\times \int\limits_1^{\rho^2}\frac{\mathrm{d}z}{z^2}\Big(L+2L\ln\frac{\theta_0^2}{z\theta_1^2}\Big)\Big[\frac{1}{2}+\frac{1}{2}\Theta(\rho^2x^2-z)+\Theta(\rho^2(1-x_1)^2-z)\Big]+\dots\,.\end{aligned}$$

We see from the above expression that the dependence on θ_0/θ_1 disappears in the sum of the contributions for the collinear and semi-collinear regions. The total sum is given by Eq. (54).

Appendix C

Virtual corrections to single photon emission cross-section

The cross-section for single hard photon bremsstrahlung containing virtual and real soft photon corrections may be written as follows:

$$\mathrm{d}\sigma^{H(S+V)} = \frac{\alpha^3 \mathrm{d}x \mathrm{d}^2\boldsymbol{q}_2^\perp \mathrm{d}^2\boldsymbol{q}_1^\perp}{2\pi^2 x(1-x)(\boldsymbol{q}_2^\perp)^4} R, \qquad R = \lim_{(2p_1p_2)\to\infty} \frac{4p_{2\rho}p_{2\sigma}k_{\rho\sigma}}{(2p_1p_2)^2}\,. \tag{C.1}$$

The tensor $k_{\rho\sigma}$ entering in R is connected with a matrix element $\epsilon_\rho M_\rho$ of the Compton scattering process (see the definition of invariants in Section 5). Taking ϵ_ρ as the polarization of the heavy photon, this reads

$$\begin{aligned} k_{\rho\sigma} &= \sum_{\text{spin}} M_\rho M_\sigma^* = \tilde{g}_{\rho\sigma}k_g + \tilde{p}_{1\rho}\tilde{p}_{1\sigma}k_{11} + \tilde{q}_{1\rho}\tilde{q}_{1\sigma}k_{22} + \tilde{q}_{1\rho}\tilde{p}_{1\sigma}k_{21} + \tilde{p}_{1\rho}\tilde{q}_{1\sigma}k_{12} \\ &\equiv B_{\rho\sigma} + \frac{\alpha}{\pi}T_{\rho\sigma}\,, \end{aligned} \tag{C.2}$$

where

$$\tilde{g}_{\rho\sigma} = g_{\rho\sigma} - \frac{q_\rho q_\sigma}{q^2}\,, \quad \tilde{p}_{1\rho} = p_{1\rho} - \frac{p_1q}{q^2}q_\rho\,, \quad \tilde{q}_{1\rho} = q_{1\rho} - \frac{q_1q}{q^2}q_\rho \tag{C.3}$$

are explicitly gauge-invariant combinations of momenta $k_{\rho\sigma}q_\rho = k_{\rho\sigma}q_\sigma = 0$.

In the case under consideration, R has the form:

$$\begin{aligned} R &= (1+\frac{\alpha}{2\pi}\rho)(B_{11}(s_1,t_1) + x^2B_{11}(t_1,s_1)) + \frac{\alpha}{2\pi}T, \\ T &= T_{11} + x^2T_{22} + x(T_{12}+T_{21}). \end{aligned} \tag{C.4}$$

The exact expressions for T_{ik} are given in [14]. We need only limited values of T_{ik} in the cases of $s_1 \ll |t_1|$ and $|t_1| \ll s_1$ at fixed q^2 and $u_1 = -2p_1q_1$.

In the case of small s_1 we have $s_1 \equiv s = [m^2(1-x)^2 + (\boldsymbol{q}_2^\perp x + \boldsymbol{q}_1^\perp)^2]/[x(1-x)]$ (we omit in the remaining part of this Appendix the subscript 1 in the notation of invariants of the Compton subprocess).

Taking into account that, at small s, $q^2 = -(\boldsymbol{q}_2^\perp)^2$, $t = -(1-x)(\boldsymbol{q}_2^\perp)^2$ and $u = -(\boldsymbol{q}_2^\perp)^2 x$, we derive the following expressions for ρ and T in this limit:

$$\begin{aligned} \rho_{s\ll|t|} &= 2(L-1+\ln x)(2\ln\Delta - \ln x) + 3L - \ln^2 x - \frac{9}{2}\,, \\ T_{s\ll|t|} &= \frac{2}{s(1-x)}\Big\{4(1+x^2)\Big[\ln x\ \ln\frac{(\boldsymbol{q}_2^\perp)^2}{s} - \mathrm{Li}_2(1-x)\Big] \\ &- 1 + 2x + x^2\Big\} - \frac{16m^2}{s^2}\ln xL. \end{aligned} \tag{C.5}$$

In the case of small $|t|$ we have:

$$\begin{aligned}
\rho_{|t|\ll s} &= 2(L-1-\ln x)(2\ln\Delta-\ln x)+3L-\ln^2 x-\frac{9}{2}\,, \\
T_{|t|\ll s} &= \frac{2x}{t(1-x)}\Big\{4(1+x^2)\Big[\ln x\;\ln\frac{(\boldsymbol{q}_2^\perp)^2}{-t}-\frac{1}{2}\ln^2 x-\mathrm{Li}_2(1-x)\Big] \\
&- 1-2x+x^2\Big\}+\frac{16m^2x^2}{t^2}\ln x\,L.
\end{aligned} \tag{C.6}$$

The further integration is straightforward. We show here the most important moments. The contribution of the ρ containing terms gives (in close analogy with the Born contribution):

$$\begin{aligned}
\Sigma_\rho^{H(S+V)} &= \frac{1}{2}\Big(\frac{\alpha}{\pi}\Big)^2\int\limits_1^{\rho^2}\frac{\mathrm{d}z}{z^2}L\int\limits_{x_c}^{1-\Delta}\mathrm{d}x\frac{1+x^2}{1-x}\Big\{(1+\Theta(\rho^2x^2-z)) \\
&\times \Big[L(2\ln\Delta-\ln x+\frac{3}{2})+(2\ln\Delta-\ln x)(\ln x-2)-\frac{1}{2}\ln^2 x-\frac{15}{4}\Big] \\
&+ (2\ln\Delta-\ln x+\frac{3}{2})\,k(x,z)-2\ln x(2\ln\Delta-\ln x)\Theta(\rho^2x^2-z)\Big\}.
\end{aligned} \tag{C.7}$$

To obtain the contributions from T we consider at first new types of integrals:

$$\begin{aligned}
I_{s\{t\}} &= Q_1^2\int\frac{\mathrm{d}^2\boldsymbol{q}_2^\perp}{\pi(\boldsymbol{q}_2^\perp)^4}\int\frac{\mathrm{d}^2\boldsymbol{q}_1^\perp}{\pi s\{t\}}\ln\frac{(\boldsymbol{q}_2^\perp)^2}{s\{-t\}}\,, \\
i_{s\{t\}} &= Q_1^2\int\frac{\mathrm{d}^2\boldsymbol{q}_2^\perp}{\pi(\boldsymbol{q}_2^\perp)^4}\int\frac{\mathrm{d}^2\boldsymbol{q}_1^\perp}{\pi s\{t\}}\,,\quad m_{s\{t\}}=Q_1^2\int\frac{\mathrm{d}^2\boldsymbol{q}_2^\perp}{\pi(\boldsymbol{q}_2^\perp)^4}\int\frac{\mathrm{d}^2\boldsymbol{q}_1^\perp m^2}{\pi s^2\{t^2\}}\,.
\end{aligned} \tag{C.8}$$

Denoting $\sigma^2(1-x)^2+(\boldsymbol{q}_2^\perp x-\boldsymbol{q}_1^\perp)^2/Q_1^2$ as $a+b\cos\phi$ and using the expressions

$$\begin{aligned}
\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{\mathrm{d}\phi}{a+b\cos\phi} &= \frac{1}{\sqrt{a^2-b^2}}\,, \\
\frac{1}{2\pi}\int\limits_0^{2\pi}\mathrm{d}\phi\frac{\ln(a+b\cos\phi)}{a+b\cos\phi} &= \frac{1}{\sqrt{a^2-b^2}}\ln\frac{2(a^2-b^2)}{a+\sqrt{a^2-b^2}}
\end{aligned} \tag{C.9}$$

with $a^2-b^2=(z_1-x^2z_2)^2+4\sigma^2(1-x)^2x^2z_2\,,\;\;\sigma^2=m^2/Q_1^2$ and $z_{1,2}=(\boldsymbol{q}_{1,2}^\perp)^2/Q_1^2$ we derive that

$$\begin{aligned}
I_s &= (1-x)x\int\limits_1^{\rho^2}\frac{\mathrm{d}z_2}{z_2^2}\int\limits_{x^2}^{x^2\rho^2}\mathrm{d}z_1 \\
&\times \frac{\ln(z_2^2(1-x)x^3)-\ln[(z_1-z_2x^2)^2+4\sigma^2x^2(1-x)^2z_2]}{\sqrt{(z_1-x^2z_2)^2+4\sigma^2x^2(1-x)^2z_2}}\,.
\end{aligned} \tag{C.10}$$

Since we are evaluating with logarithmic accuracy, we may consider the contribution of the region $|z_1 - x^2 z_2| < \eta$, $\sigma^2 \ll \eta \ll 1$ when integrating over z_1. The result reads

$$I_s = x(1-x)\int\limits_1^{\rho^2} \frac{\mathrm{d}z_2}{z_2^2} L\Big[\frac{1}{2}L + \ln\frac{x}{1-x}\Big]. \tag{C.11}$$

The rest of the integrals can be calculated in the same way, and we have:

$$\begin{aligned} I_t &= -(1-x)\int\limits_1^{\rho^2 x^2} \frac{\mathrm{d}z_2}{z_2^2} L\Big[\frac{1}{2}L + \ln\frac{1}{1-x}\Big], \quad i_t = -(1-x)\int\limits_1^{\rho^2 x^2} \frac{\mathrm{d}z_2}{z_2^2} L, \\ i_s &= x(1-x)\int\limits_1^{\rho^2} \frac{\mathrm{d}z_2}{z_2^2} L, \quad m_t = \int\limits_1^{\rho^2 x^2} \frac{\mathrm{d}z_2}{z_2^2}, \quad m_s = x^2 \int\limits_1^{\rho^2 x^2} \frac{\mathrm{d}z_2}{z_2^2}. \end{aligned} \tag{C.12}$$

Using (C.11) and (C.12) we may represent the final result for the contribution to $\Sigma^{H(S+V)}$ due to the T term as

$$\begin{aligned} \Sigma_T^{H(S+V)} &= \frac{1}{2}\Big(\frac{\alpha}{\pi}\Big)^2 \int\limits_1^{\rho^2} \frac{\mathrm{d}z}{z^2}\, L \int\limits_{x_c}^{1-\Delta} \mathrm{d}x \frac{1+x^2}{1-x}\Big\{\Big(\frac{1}{2}L + \ln\frac{x}{1-x}\Big)\ln x \\ &+ \zeta_2 - \mathrm{Li}_2(x) + \frac{x^2+2x-1}{4(1+x^2)} - \frac{2x\ln x}{1+x^2} - \Theta(x^2\rho^2 - z)\Big[\Big(\frac{1}{2}L + \ln\frac{1}{1-x}\Big)\ln x \\ &- \frac{1}{2}\ln^2 x - \mathrm{Li}_2(1-x) - \frac{1+2x-x^2}{4(1+x^2)} - \frac{2x\ln x}{1+x^2}\Big]\Big\}. \end{aligned} \tag{C.13}$$

The total contribution to $\Sigma^{H(S+V)}$ (one-side hard photon emission with virtual and soft photon corrections) is the sum of (C.7) and (C.13):

$$\Sigma^{H(S+V)} = \Sigma_\rho^{H(S+V)} + \Sigma_T^{H(S+V)}. \tag{C.14}$$

This quantity is given in Eq. (51).

Appendix D

Leading logarithmic contribution to $\Sigma^{\gamma\gamma}$

Here we show that the main logarithmic terms can be summed according to the renormalization group. The sum, as given in the text, may be written as:

$$
\begin{aligned}
S &= 2(\frac{\alpha}{\pi})^2 \int\limits_1^{\rho^2} \frac{\mathrm{d}z}{z^2} L^2 \Big\{ \int\limits_{x_c}^{1} \mathrm{d}x\, \delta(1-x) \Big(\ln^2\Delta + \frac{3}{2}\ln\Delta + \frac{9}{16} \Big) \qquad \text{(D.1)} \\
&\times (1+\Theta(x^2\rho^2 - z)) + \frac{1}{2}\int_{x_c}^{1-\Delta} \frac{1+x^2}{1-x}(2\ln\Delta - \ln x + \frac{3}{2})(1+\Theta(x^2\rho^2-z)) \\
&+ \frac{1}{4}\int\limits_{x_c}^{1-2\Delta} \mathrm{d}x \Big[2\frac{1+x^2}{1-x}\ln\frac{1-x-\Delta}{\Delta} + \frac{1}{2}(1+x)\ln x - 1 + x \Big][1+3\Theta(x^2\rho^2-z)] \\
&+ \frac{1}{4}\int\limits_{x_c}^{1} \mathrm{d}x \Big[2\frac{1+x^2}{1-x}\ln\Big(\frac{(1-x-\Delta)(\rho-\sqrt{z})}{\Delta(\sqrt{z}-\rho x)}\sqrt{x} \Big) + x - 1 \\
&- \frac{1}{2}(1+x)\ln\frac{\rho^2}{z} + \frac{\sqrt{z}}{\rho} - \frac{x\rho}{\sqrt{z}} \Big] \Theta(z - x^2\rho^2) \Big\}.
\end{aligned}
$$

One can see that the dependence on the parameter Δ disappears in the expression above.

We will now show that eq. (D.1) is equivalent to eq. (57). Let us transform eq. (57) using the substitution

$$
\Theta(t^2\rho^2 - z) = \frac{1}{2}(1+\Theta(x^2\rho^2-z)) + \frac{1}{2}\Theta(z - x^2\rho^2) - \Theta(z - t^2\rho^2), \qquad \text{(D.2)}
$$

and changing the order of integration in the last term:

$$
\int\limits_x^1 \mathrm{d}t \int\limits_{\rho^2 t^2}^{\rho^2} \mathrm{d}z = \int\limits_{\rho^2 x^2}^{\rho^2} \mathrm{d}z \int\limits_x^{\sqrt{z}/\rho} \mathrm{d}t = \int\limits_1^{\rho^2} \mathrm{d}z\Theta(z-\rho^2x^2) \int\limits_x^{\sqrt{z}/\rho} \mathrm{d}t. \qquad \text{(D.3)}
$$

By evaluating the integral over t, and using the explicit expressions for the splitting functions one can verify the coincidence of eqs. (D.1) and (57). In an analogous way one can prove the validity of the representation (59) for Σ^{γ}_{γ}.

Using the representation in Eq. (58) for the function $P^{(2)}$ one can see that the above expression is equivalent to Eq. (D.1). In an analogous way one can prove the validity of representation in Eq. (59) for Σ^{γ}_{γ}.

Appendix E

Cancellation of the Δ dependence in the non-leading contributions to $\Sigma^{2\gamma}$

Let us consider the singular non-leading terms in $\Sigma^{2\gamma}$ in the limiting case $\Delta \to 0$. Dropping the common factor $(\alpha/\pi)^2\mathcal{L}\int \mathrm{d}z/z^2$, we give below the various contributions separately.

Let us consider first $\Sigma^{\gamma\gamma}$. The contributions from the soft photon radiation and virtual corrections are:

$$(\Sigma^{VV+VS+SS})_\Delta = \ln\Delta(-7-4\ln\Delta)(1,\rho^2), \tag{E.1}$$

where we denote by (a,b) the limits of the integration over z: $(a,b)=\Theta(z-a)\Theta(b-z)$.

The contribution due to the virtual corrections to the single hard photon emission gives

$$\begin{aligned}(\Sigma^{H(S+V)})_\Delta &= \frac{1}{2}\ln\Delta\Big\{(1,\rho^2)(16\ln\Delta+14)+4\int\limits_{\tilde{x}_c}^{1}\mathrm{d}x\,\frac{1+x^2}{1-x}[(1,\rho^2)-(1,\rho^2x^2)] \\ &+ 4[-2\ln(1-x_c)-2\ln(1-\tilde{x}_c)+\int\limits_{x_c}^{1}\mathrm{d}x\,(1+x)+\int\limits_{\tilde{x}_c}^{1}\mathrm{d}x\,(1+x)] \\ &+ 2\int\limits_{x_c}^{1}\mathrm{d}x\,\frac{1+x^2}{1-x}[(1,\rho^2)-(1,\rho^2x^2)]\ln x+2\int\limits_{x_c}^{1}\mathrm{d}x\,\frac{1+x^2}{1-x}\,k(x,z)\Big\},\end{aligned} \tag{E.2}$$

where $\tilde{x}_c=\max(x_c,1/\rho)$ and the quantity $k(x,z)$ is defined in Eq. (30). The singular part in the contribution in Eq. (54) due to double hard photon bremsstrahlung reads:

$$\begin{aligned}(\Sigma^{HH})_\Delta &= \ln\Delta\Big\{\int\limits_{x_c}^{1}\mathrm{d}x\,\frac{1+x^2}{1-x}[-(1,\rho^2)L_1-(1,\rho^2x^2)L_2-((1,\rho^2)-(1,\rho^2x^2))L_3] \\ &- \int\limits_{x_c}^{1}\mathrm{d}x\,(3+x)-\int\limits_{\tilde{x}_c}^{1}\mathrm{d}x\,(3+x)-4\ln\Delta+4\ln(1-x_c)+4\ln(1-\tilde{x}_c) \\ &- \int\limits_{\tilde{x}_c}^{1}\mathrm{d}x\,\frac{1+x^2}{1-x}[(1,\rho^2)-(1,\rho^2x^2)]\Big\}.\end{aligned} \tag{E.3}$$

It is possible to verify the cancellation:

$$(\Sigma^{VV+VS+SS})_\Delta+(\Sigma^{H(S+V)})_\Delta+(\Sigma^{HH})_\Delta=0. \tag{E.4}$$

The corresponding contributions to Σ^{γ}_{γ} are:

$$
\begin{aligned}
(\Sigma^{S+V}_{S+V})_\Delta &= \ln\Delta(-14-8\ln\Delta)(1,\rho^2), \qquad\qquad (E.5)\\
(\Sigma^{H}_{S+V}+\Sigma^{S+V}_{H})_\Delta &= \ln\Delta\Big\{2\int_{x_c}^{1}\mathrm{d}x\;\frac{1+x^2}{1-x}\;k(x,z)+(1,\rho^2)[16\ln\Delta+14-8\ln(1-x_c)\\
&- 8\ln(1-\tilde{x}_c)+4\int_{x_c}^{1}\mathrm{d}x\;(1+x)+4\int_{\tilde{x}_c}^{1}\mathrm{d}x\;(1+x)]\\
&+ 4\int_{\tilde{x}_c}^{1}\mathrm{d}x\;\frac{1+x^2}{1-x}[(1,\rho^2)-(1,\rho^2x^2)]\Big\},\\
(\Sigma^{H}_{H})_\Delta &= \ln\Delta\Big\{-8(1,\rho^2)[\ln\Delta-\ln(1-x_c)-\ln(1-\tilde{x}_c)]\\
&- 8\int_{\tilde{x}_c}^{1}\mathrm{d}x\;\frac{(1,\rho^2)-(1,\rho^2x^2)}{1-x}-2\int_{x_c}^{1}\mathrm{d}x\;k(x,z)\frac{1+x^2}{1-x}\\
&- 4\int_{x_c}^{1}\mathrm{d}x\;(1+x)[(1,\rho^2)+(1,\rho^2x^2)]\Big\}.
\end{aligned}
$$

Rearranging the last term in $(\Sigma^{H}_{H})_\Delta$ as

$$
\begin{aligned}
-4\int_{x_c}^{1}\mathrm{d}x\;(1+x)[(1,\rho^2)+(1,\rho^2x^2)] &= -4\int_{x_c}^{1}\mathrm{d}x\;(1+x)(1,\rho^2) \qquad\qquad (E.6)\\
&-4\int_{\tilde{x}_c}^{1}\mathrm{d}x\;(1+x)(1,\rho^2)+4\int_{\tilde{x}_c}^{1}\mathrm{d}x\;(1+x)[(1,\rho^2)-(1,\rho^2x^2)],
\end{aligned}
$$

we can see again the cancellation of the Δ-dependence in the sum:

$$
(\Sigma^{S+V}_{S+V})_\Delta+(\Sigma^{H}_{S+V}+\Sigma^{S+V}_{H})_\Delta+(\Sigma^{H}_{H})_\Delta=0. \qquad (E.7)
$$

Appendix F

Relevant integrals for collinear pair production

We give here a list of the relevant integrals, calculated within the logarithmic accuracy, for the collinear kinematical region of hard pair production.

We use the definitions in Eq. (65) and we imply, in the left-hand side of the expressions below, the general operation:

$$\langle(\ldots)\rangle \equiv \int\limits_0^{z_0} \mathrm{d}z_1 \int\limits_0^{z_0} \mathrm{d}z_2 \int\limits_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi}(\ldots)\,, \tag{F.1}$$

with the conditions $z_0 = (\varepsilon\theta_0/m)^2 \gg 1$, $L_0 = \ln z_0 \gg 1$. The details of the calculations can be found in the Appendix of Ref. [16]. The results are:

$$\begin{aligned}
&\langle\left(\frac{x_2D + (1-x_2)A}{DC}\right)^2\rangle = \frac{L_0}{(1-x_2)^2}\Big\{L_0 + 2\ln\frac{x_1x_2}{x} - 8 \\
&\qquad + \frac{(1-x)^2(1-x_2)^2}{xx_1x_2}\Big\}, \qquad \langle\frac{1}{DC}\rangle = \frac{L_0}{x_1x_2(1-x_2)}[\frac{1}{2}L_0 + \ln\frac{x_1x_2}{x}]\,, \\
&\langle\left(\frac{x_2A_1 - x_1A_2}{AD}\right)^2\rangle = \frac{L_0}{(1-x)^2}\Big\{L_0 + 2\ln\frac{x_1x_2}{x} - 8 + \frac{(1-x)^2}{xx_1x_2} - \frac{4(1-x)}{x}\Big\}\,, \\
&\langle\frac{x_1A_2 - x_2A_1}{AD^2}\rangle = \frac{(x_1-x_2)L_0}{xx_1x_2(1-x)}\,, \qquad \langle\frac{1}{A^2D}\rangle = \frac{-L_0}{(1-x)^3}, \\
&\langle\frac{1}{AD}\rangle = \frac{-L_0}{x_1x_2(1-x)}[\frac{1}{2}L_0 + \ln\frac{x_1x_2}{x}], \qquad \langle\frac{1}{C^2D}\rangle = \frac{-L_0}{x_1(1-x_2)^3}\,, \\
&\langle\frac{1}{AC}\rangle = \frac{-L_0}{x_1x_2^2}[L_0 + 2\ln\frac{x_1x_2}{x} + 2\ln\frac{xx_2}{(1-x)(1-x_2)}], \qquad \langle\frac{1}{D^2}\rangle = \frac{L_0}{xx_1x_2}\,, \\
&\langle\frac{A}{C^2D^2}\rangle = \frac{x_2L_0}{x_1(1-x_2)^4}, \qquad \langle\frac{C}{A^2D^2}\rangle = \frac{-x_2L_0}{(1-x)^4}\,, \\
&\langle\frac{A}{CD^2}\rangle = \frac{-L_0}{x_1(1-x_2)^2}[\frac{1}{2}L_0 + \ln\frac{x_1x_2}{x}] + L_0\frac{x_2x - x_1}{xx_1x_2(1-x_2)^2}\,, \\
&\langle\frac{C}{AD^2}\rangle = \frac{-L_0}{x_1(1-x)^2}[\frac{1}{2}L_0 + \ln\frac{x_1x_2}{x}] - L_0\left(\frac{x_1-x_2}{x_1x_2(1-x)^2} + \frac{1}{xx_2(1-x)}\right)\,.
\end{aligned} \tag{F.2}$$