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BPS Spectra and Non-Perturbative Gravitational Couplings in $N = 2, 4$ Supersymmetric String Theories

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ABSTRACT

We study the BPS spectrum in $D = 4, N = 4$ heterotic string compactifications, with some emphasis on intermediate $N = 4$ BPS states. These intermediate states, which can become short in $N = 2$ compactifications, are crucial for establishing an $S - T$ exchange symmetry in $N = 2$ compactifications. We discuss the implications of a possible $S - T$ exchange symmetry for the $N = 2$ BPS spectrum. Then we present the exact result for the 1-loop corrections to gravitational couplings in one of the heterotic $N = 2$ models recently discussed by Harvey and Moore. We conjecture this model to have an $S - T$ exchange symmetry. This exchange symmetry can then be used to evaluate non-perturbative corrections to gravitational couplings in some of the non-perturbative regions (chambers) in this particular model and also in other heterotic models.

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1 Introduction

Recently, some major progress has been obtained in the understanding of non-perturbative dynamics in field theories and string theories with extended supersymmetry [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. One important feature of these theories is the existence of BPS states. These BPS states play an important role in understanding duality symmetries and non-perturbative effects in string theory in various dimensions. They are, for instance, essential to the resolution of the conifold singularity in type II string theory [13]. BPS states also play a central role in 1-loop threshold corrections to gauge and gravitational couplings in $N = 2$ heterotic string compactifications, as shown recently in [14].

In the context of $D = 4, N = 4$ compactifications, BPS states also play a crucial role in tests [15] of the conjectured strong/weak coupling $SL(2, \mathbf{Z})_S$ duality [16, 17, 18] in toroidal compactifications of the heterotic string. Moreover, the conjectured string/string/string triality [19] interchanges the BPS spectrum of the heterotic theory with the BPS spectrum of the type II theory. In an $N = 4$ theory, BPS states can either fall into short or into intermediate multiplets. In going from the heterotic to the type IIA side, for example, the four dimensional axion/dilaton field S gets interchanged with the complex Kähler modulus T of the 2-torus on which the type IIA theory has been compactified on. Thus, it is under the exchange of S and T that the BPS spectrum of the heterotic and the type IIA string gets mapped into each other. The BPS mass spectrum of the heterotic(type IIA) string is, however, not symmetric under this exchange of S and T. This is due to the fact that BPS masses in $D = 4, N = 4$ compactifications are given by the maximum of the 2 central charges $|Z_1|^2$ and $|Z_2|^2$ of the $N = 4$ supersymmetry algebra [20].

On the other hand, states, which from the $N = 4$ point of view are intermediate, are actually short from the $N = 2$ point of view. This then leads to the possibility that the BPS spectrum of certain $N = 2$ heterotic compactifications is actually symmetric under the exchange of S and T. If such symmetry exists a lot of information about the BPS spectrum at strong coupling can be obtained, in particular about those BPS states which can become massless at specific points in the moduli space. Assuming that the contributions to the associated gravitational couplings are due to BPS states only (as was shown to be the case at 1-loop for some classes of compactifications in [14]), it follows that these gravitational couplings should also exhibit such an $S \leftrightarrow T$ exchange symmetry. The evaluation of non-perturbative corrections to gravitational couplings is, however, very difficult. The existence of an exchange symmetry $S \leftrightarrow T$ is extremly helpful in that it allows for the evaluation of non-perturbative corrections to gravitational couplings in some of the non-perturbative regions (chambers) in moduli space. This is achieved by taking the known result for the 1-loop correction in some perturbative region (chamber) of moduli space and applying the exchange symmetry to it. Three examples will be discussed in this paper, namely the 2 parameter model $P_{1,1,2,2,6}(12)$ of [21], the 3 parameter model $P_{1,1,2,8,12}(24)$ [7, 12] (for these two models an exchange symmetry $S \leftrightarrow T$ has been observed in [9]) and the $s = 0$ model of [14] (for this example we conjecture that there too is such an exchange symmetry).

The paper is organised as follows. In section 2 we introduce orbits for short and intermediate multiplets in $D = 4, N = 4$ heterotic string compactifications and we show how they get mapped into each other under string/string/string triality. In section 3 we discuss BPS states in the context of $D = 4, N = 2$ heterotic string compactifications and show that states, which from the $N = 4$ point of view are intermediate, actually play an important role in the correct evaluation of non-perturbative effects such as non-perturbative monodromies. We also discuss exchange symmetries of the type $S \leftrightarrow T$ in the 2 and 3 parameter models $P_{1,1,2,2,6}(12)$ and $P_{1,1,2,8,12}(24)$. In section 4 we introduce an $N = 4$ free energy as a sum over $N = 4$ BPS states and suggest that it should be identified with the partition function of topologically twisted $N = 4$ string compactifications. In section 5 we introduce an $N = 2$ free energy as a sum over $N = 2$ BPS states and argue that it should be identified with the heterotic holomorphic gravitational function $\mathcal{F}_{grav}$. We discuss 1-loop corrections to the gravitational coupling and compute them exactly in the $s = 0$ model of [14]. We then argue that this model possesses an $S \leftrightarrow T$ exchange symmetry and use it to compute non-perturbative corrections to the gravitational coupling in some non-perturbative regions of moduli space. We also discuss the 2 parameter model $P_{1,1,2,2,6}(12)$ of [21] and compute the associated holomorphic gravitational coupling in the decompactification limit $T \to \infty$. Finally, appendices A and B contain a more detailed discussion of some of the issues discussed in section 2.

2 The $N = 4$ BPS spectrum

2.1 The truncation of the mass formula

In this section we recall the BPS mass formulae for four-dimensional string theories with $N = 4$ space-time supersymmetry [17, 19]. Specifically, we first consider the heterotic string compactified on a six-dimensional torus. In $N = 4$ supersymmetry, there are in general two central charges Z_1 and Z_2. There exist two kinds of massive BPS multiplets, namely first the short multiplets which saturate two BPS bounds (the associated soliton background solutions preserve 1/2 of the supersymmetries in $N = 4$), i.e.

$$m_S^2 = |Z_1|^2 = |Z_2|^2; \tag{2.1}$$

the short vector multiplets contain maximal spin one. Second there are the intermediate multiplets which saturate only one BPS bound and contain maximal spin 3/2 (the associated solitonic backgrounds preserve only one supersymmetry in $N = 4$), i.e.

$$m_I^2 = \mathrm{Max}(|Z_1|^2, |Z_2|^2). \tag{2.2}$$

The BPS masses are functions of the moduli parameter as well as functions of the dilaton-axion field $S = \frac{4\pi}{g^2} - i\frac{\theta}{2\pi} = e^{-\phi} - ia$. Specifically, the two central charges $Z_{1,2}$ have the following form [22, 19]

$$|Z_{1,2}|^2 = \vec{Q}^2 + \vec{P}^2 \pm 2\sqrt{\vec{Q}^2\vec{P}^2 - (\vec{Q}\cdot\vec{P})^2}, \tag{2.3}$$

where $\vec{Q}$ and $\vec{P}$ are the (6-dimensional) electric and magnetic charge vectors which depend on the moduli and on ϕ, a. One sees that for short vector multiplets, for with $|Z_1| = |Z_2|$, the square root term in (2.3) must be absent, which is satisfied for parallel electric and magnetic charge vectors. In this case the BPS masses agree with the formula of Schwarz and Sen [17].

In a general compactification on a six-dimensional torus T^6 the moduli fields locally parametrize a homogeneous coset space $SO(6,22)/(SO(6)\times SO(22))$. In terms of these moduli fields, the two central charges are then given[2] by [19]

$$|Z_{1,2}|^2 = \frac{1}{16}\left(\gamma^T\mathcal{M}(M+L)\gamma \pm \sqrt{(\gamma^T\epsilon\gamma)_{ab}(\gamma^T\epsilon\gamma)_{cd}(M+L)_{ac}(M+L)_{bd}}\right) \tag{2.4}$$

where $\gamma^T = (\alpha, \beta)$. Let us from now on restrict the discussion by considering only an $SO(2,2)$ subspace which corresponds to two complex moduli fields T and U. This means that we will only consider the moduli degrees of freedom of a two-dimensional two-torus T_2. ($\vec{Q}$ and $\vec{P}$ are now two-dimensional vectors.) Then, converting to a basis where L has diagonal form, $\check{L} = T^{-1}LT, \check{M} = T^{-1}MT, \check{M} + \check{L} = 2\phi\phi^T = \varphi\varphi^\dagger + \bar{\varphi}\varphi^T$, the two central charges can be written as

$$|Z_{1,2}|^2 = \frac{1}{16}\left(\check{\gamma}^T\mathcal{M}(\varphi\varphi^\dagger + \bar{\varphi}\varphi^T)\check{\gamma} \pm 2\sqrt{(\check{\gamma}^T\epsilon\check{\gamma})_{ab}(\check{\gamma}^T\epsilon\check{\gamma})_{cd}\mathcal{R}_{ac}\mathcal{R}_{bd}}\right) \tag{2.5}$$

where $\check{\gamma}^T = (\check{\alpha}, \check{\beta}) = (T^{-1}\alpha, T^{-1}\beta)$ and where $\mathcal{R}_{ac} = \frac{1}{2}(\varphi\varphi^\dagger + \bar{\varphi}\varphi^T)_{ac}$. Using that $(\check{\gamma}^T\epsilon\check{\gamma})_{ab} = \check{\alpha}_a\check{\beta}_b - \check{\alpha}_b\check{\beta}_a$ it follows that

$$|Z_{1,2}|^2 \;=\; \frac{1}{16}\left(\check{\gamma}^T\mathcal{M}(\varphi\varphi^\dagger + \bar{\varphi}\varphi^T)\check{\gamma} \pm 4i\check{\alpha}^T\mathcal{I}\check{\beta}\right)$$

[2]We are using the notation of [18, 19].

$$
\begin{aligned}
&= \frac{1}{4(S+\bar{S})}\Big(\check{\alpha}^T\mathcal{R}\check{\alpha} + S\bar{S}\check{\beta}^T\mathcal{R}\check{\beta} + i(S-\bar{S})\check{\alpha}^T\mathcal{R}\check{\beta} \\
&\pm\ i(S+\bar{S})\check{\alpha}^T\mathcal{I}\check{\beta}\ \Big)
\end{aligned} \tag{2.6}
$$

where $\mathcal{I} = \frac{1}{2}(\varphi\varphi^\dagger - \bar{\varphi}\varphi^T)$. The central charges $|Z_{1,2}|^2$ can finally also be rewritten into

$$
\begin{aligned}
|Z_{1,2}|^2 &= \frac{1}{4(S+\bar{S})}\Big(\check{\alpha}^T\mathcal{R}\check{\alpha} + S\bar{S}\check{\beta}^T\mathcal{R}\check{\beta} \pm i(S-\bar{S})\check{\alpha}^T\mathcal{R}\check{\beta} \pm i(S+\bar{S})\check{\alpha}^T\mathcal{I}\check{\beta}\Big) \\
&= \frac{1}{4(S+\bar{S})(T+\bar{T})(U+\bar{U})}|\mathcal{M}_{1,2}|^2 \\
\mathcal{M}_1 &= \Big(\check{M}_I + iS\check{N}_I\Big)\check{P}^I \\
\mathcal{M}_2 &= \Big(\check{M}_I - i\bar{S}\check{N}_I\Big)\check{P}^I
\end{aligned} \tag{2.7}
$$

where

$$
\begin{aligned}
\check{P}^0 = T + U \quad &, \quad \check{P}^1 = i(1+TU) \\
\check{P}^2 = T - U \quad &, \quad \check{P}^3 = -i(1-TU)
\end{aligned} \tag{2.8}
$$

and where $\check{M} = \check{\alpha}, \check{N} = \check{\beta}$. Here, the $\check{M}_I$ $(I = 0, \ldots, 3)$ are the integer electric charge quantum numbers of the Abelian gauge group $U(1)^4$ and the $\check{N}_I$ are the corresponding integer magnetic quantum numbers.

Note that $|Z_2|^2$ can be obtained from $|Z_1|^2$ by $S \leftrightarrow \bar{S}, \check{N}_I \to -\check{N}_I$. This amounts to complex conjugating $\check{M}_I + iS\check{N}_I$.

Finally, rotating the $\check{P}^I$ into $\hat{P} = (1, -TU, iT, iU)^T$

$$
\check{P} = A\hat{P} \quad , \quad A = i\begin{pmatrix} 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & 0 & 0 \end{pmatrix} \tag{2.9}
$$

gives that

$$
\begin{aligned}
|Z_{1,2}|^2 &= \frac{1}{4(S+\bar{S})(T+\bar{T})(U+\bar{U})}|\mathcal{M}_{1,2}|^2 \\
\mathcal{M}_1 &= \Big(\hat{M}_I + iS\hat{N}_I\Big)\hat{P}^I \\
\mathcal{M}_2 &= \Big(\hat{M}_I - i\bar{S}\hat{N}_I\Big)\hat{P}^I
\end{aligned} \tag{2.10}
$$

where $\hat{M} = A^T\check{M}, \hat{N} = A^T\check{N}$.

Note that

$$\begin{aligned}\Delta Z^2 &= |Z_1|^2 - |Z_2|^2 = 4\sqrt{\vec{Q}^2\vec{P}^2 - (\vec{Q}\cdot\vec{P})^2} = i\frac{(\hat{N}_J\hat{P}^J)(\hat{M}_I\bar{\hat{P}}^I)}{4(T+\bar{T})(U+\bar{U})} - \text{h.c.} \\ &= i\frac{(\hat{N}_0 - \hat{N}_1 TU + i\hat{N}_2 T + i\hat{N}_3 U)(\hat{M}_0 - \hat{M}_1\bar{T}\bar{U} - i\hat{M}_2\bar{T} - i\hat{M}_3\bar{U})}{4(T+\bar{T})(U+\bar{U})} - \text{h.c.}\end{aligned} \tag{2.11}$$

is independent of S and only depends on the moduli T and U.

The BPS mass formula (2.10) is invariant under the perturbative T-duality group $SL(2,\mathbf{Z})_T \times SL(2,\mathbf{Z})_U \times \mathbf{Z}_2^{T\leftrightarrow U}$; for example $SL(2,\mathbf{Z})_T$, $T \to \frac{aT-ib}{icT+d}$, acts on the electric and magnetic charges as

$$\begin{pmatrix}\hat{M}_2\\ \hat{M}_0\end{pmatrix} \to \begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}\hat{M}_2\\ \hat{M}_0\end{pmatrix}, \tag{2.12}$$

where the vectors $\begin{pmatrix}\hat{M}_1\\ \hat{M}_3\end{pmatrix}$, $\begin{pmatrix}\hat{N}_2\\ \hat{N}_0\end{pmatrix}$ and $\begin{pmatrix}\hat{N}_1\\ \hat{N}_3\end{pmatrix}$ transform in the same way. The mirror symmetry $T \leftrightarrow U$ is also perturbative in the heterotic string; it transforms the electric charges $\hat{M}_I$ into electric charges and the magnetic charges $\hat{N}_I$ into magnetic ones:

$$\hat{M}_2 \leftrightarrow \hat{M}_3, \qquad \hat{N}_2 \leftrightarrow \hat{N}_3. \tag{2.13}$$

In addition, the BPS mass formula (2.10) is invariant under the non-perturbative S duality group $SL(2,\mathbf{Z})_S$ which transforms $S \to \frac{aS-ib}{icS+d}$ and mixes the electric and magnetic charges as

$$\begin{pmatrix}\hat{N}_I\\ \hat{M}_I\end{pmatrix} \to \begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}\hat{N}_I\\ \hat{M}_I\end{pmatrix}. \tag{2.14}$$

As discussed in [19] there is furthermore an $S-T-U$ triality symmetry, which is related to the string-string duality symmetries among the heterotic, type IIA and type IIB $N=4$ four-dimensional strings. Specifically, exchanging the S-field with the modulus T amounts to performing the following electric magnetic duality transformation:

$$\hat{M}_2 \leftrightarrow \hat{N}_0, \qquad \hat{M}_1 \leftrightarrow \hat{N}_3. \tag{2.15}$$

This exchange corresponds to the string-string duality transformation between the heterotic string and the type IIA string. (The four-dimensional $N=4$ type IIA string is obtained by compactifying the ten-dimensional IIA string on $K_3 \times T_2$.) In the type IIA string the moduli of T_2 are given by S and U, whereas T corresponds to the string coupling constant. Thus $\hat{M}_1$ and $\hat{M}_2$ are magnetic charges in the type IIA case, whereas $\hat{N}_0$ and $\hat{N}_3$ are electric charges.

The transformation $S \leftrightarrow U$, which corresponds to the string-string duality between the heterotic and type IIB string, is obtained is an analogous way:

$$\hat{M}_1 \leftrightarrow \hat{N}_2, \qquad \hat{M}_3 \leftrightarrow \hat{N}_0. \tag{2.16}$$

In the IIB string the moduli of T_2 correspond to S and T, whereas the string coupling constant is denoted by U. These two transformations $S \leftrightarrow T$ and $S \leftrightarrow U$, are thus of non-perturbative nature since electric charges and magnetic charges are exchanged. However, as we will discuss in the following, the exchange $S \leftrightarrow T$ is not a true symmetry of the heterotic string. The BPS mass spectrum of the the heterotic (IIA, IIB) string is not symmetric under the exchange $S \leftrightarrow T$ ($T \leftrightarrow U$, $S \leftrightarrow U$), since the BPS masses are given by the maximum of $|Z_1|^2$ and $|Z_2|^2$. These operations just exchange the spectrum of the heterotic string with the spectrum of the type IIA, IIB strings.

2.2 The short $N = 4$ BPS multiplets

As already said, the BPS mass formula (2.10) is valid for intermediate as well as for short $N = 4$ supermultiplets. Let us first consider the short $N = 4$ multiplets. Short BPS multiplets are multiplets for which $\Delta Z^2 = |Z_1|^2 - |Z_2|^2 = 0$ at generic points in the moduli space. Namely Z_1 and Z_2 agree in the heterotic case provided that the electric and magnetic charge vectors are parallel:

$$\vec{Q}^{\text{het}} || \vec{P}^{\text{het}}. \tag{2.17}$$

That is, short multiplets are multiplets for which $\hat{M}_I \propto \hat{N}_I$, and the states which satisfy this constraint are characterized by the following condition which we call the S-orbit or also the heterotic orbit (a general discussion about orbits of duality groups can be found in appendix A):

$$s\hat{M}_I = p\hat{N}_I, \qquad s, p \in \mathbf{Z}. \tag{2.18}$$

This condition can be also expressed as

$$\hat{M}_I \hat{N}_J - \hat{M}_J \hat{N}_I = 0. \tag{2.19}$$

Let us plug in the condition (2.18) into the BPS mass formula (2.10). Then the short multiplets have the following holomorphic masses [17, 18]

$$\mathcal{M} = (s + ipS)(m_2 - im_1 U + in_1 T - n_2 UT), \tag{2.20}$$

where we have made the following identification:

$$\begin{aligned}\hat{M}_0 &= sm_2, \quad \hat{M}_1 = sn_2, \quad \hat{M}_2 = sn_1, \quad \hat{M}_3 = -sm_1, \\ \hat{N}_0 &= pm_2, \quad \hat{N}_1 = pn_2, \quad \hat{N}_2 = pn_1, \quad \hat{N}_3 = -pm_1.\end{aligned} \tag{2.21}$$

We see that now the BPS masses factorize into an S-dependent term and into a moduli dependent piece. Thus for the case of short multiplets, (2.20) shows that the quantum numbers m_i and n_i are to be thought of as the momentum and winding numbers associated with the 2-torus parametrised by the T, U-moduli, whereas the quantum numbers s and p denote the electric and magnetic quantum numbers associated with the S-modulus. The short multiplets which fall into the orbit (2.18) clearly contain all elementary, electric heterotic string states with magnetic charge $p = 0$. For the elementary BPS states, the BPS mass is determined by the right-moving T^2 lattice momentum: $M^2 \sim p_R^2$; furthermore the elementary BPS states have to satisfy $N_R + h_R = 1/2$, where N_R is the right-moving oscillator number and is h_R the right-moving internal conformal dimension. The heterotic level matching condition for elementary states reads

$$\frac{1}{2}p_L^2 - \frac{1}{2}p_R^2 = m_1n_1 + m_2n_2 = N_R + h_R - N_L + \frac{1}{2}. \tag{2.22}$$

In the limit $S \to \infty$ an infinite number of elementary string states with $p = 0$, s arbitrary become massless. Similarly for $S \to 0$, an infinite tower of magnetic monoples with $s = 0$, p arbitray become light.

The orbit (2.21) further decomposes into (still reducible) suborbits $m_1n_1+m_2n_2 = a \in \mathbf{Z}$, as follows.

The suborbit (i) $m_1n_1+m_2n_2 = 0$ contains the Kaluza-Klein excitations of the elementary states and the Kaluza-Klein monopoles. However this suborbit does not contain any states which become massless for finite values of T and U.

The second suborbit (ii) $m_1n_1 + m_2n_2 = 1$ contains the elementary states which become massless within the T, U moduli space. Specifically one gets the following critical lines/points (modulo T, U duality transformations) (for a more detailed discussion see [10]):

(1) $T = U$: this is the line of enhanced $SU(2)$ gauge symmetry; the additional massless field carry the following momentum and winding numbers: $m_1 = n_1 = \pm 1$, $m_2 = n_2 = 0$.

(2) $T = U = 1$: here there is an enhanced $SU(2)^2$ gauge symmetry where the four additional vector multiplets carry the charges $m_1 = n_1 = \pm 1$, $m_2 = n_2 = 0$ or $m_1 = n_1 = 0$, $m_2 = n_2 = \pm 1$.

(3) $T = U^{-1} = \rho = e^{i\pi/6}$: this is the point of enhanced $SU(3)$ gauge symmetry with six additional massless vector multiplets of charges $m_1 = n_1 = \pm 1$, $m_2 = n_2 = 0$ or $m_1 = n_1 = m_2 = \pm 1$, $n_2 = 0$ or $m_1 = n_1 = -n_2 = \pm 1$, $m_2 = 0$.

In addition, this suborbit (ii) contains also the socalled H monopoles.

2.3 The intermediate $N = 4$ BPS multiplets

Let us now investigate the structure of the intermediate $N = 4$ BPS multiplets. Intermediate BPS multiplets are multiplets for which $\Delta Z^2 \neq 0$ at generic points in the moduli space. Inspection of (2.11) shows that intermediate multiplets are dyonic and that the vectors $\hat{M}$ and $\hat{N}$ are not proportional to each other. Heterotic intermediate orbits can be characterized as follows

$$\hat{M}_I \hat{N}_J - \hat{M}_J \hat{N}_I \neq 0. \tag{2.23}$$

In analogy to the constraint (2.21) for the short heterotic multiplets let us consider a constraint which leads to a BPS mass formula which factorizes into a T-dependent and into a S, U-dependent term. Specifically this constraint, the T-orbit or type IIA orbit, has the form

$$\begin{aligned} \hat{M}_0 &= sm_2, \quad \hat{M}_1 = -pm_1, \quad \hat{M}_2 = pm_2, \quad \hat{M}_3 = -sm_1, \\ \hat{N}_0 &= sn_1, \quad \hat{N}_1 = pn_2, \quad \hat{N}_2 = pn_1, \quad \hat{N}_3 = sn_2, \end{aligned} \tag{2.24}$$

and the BPS mass formula (2.10) in the heterotic case can be written as

$$\begin{aligned} \mathcal{M}_1 &= (s + ipT)(m_2 - im_1 U + in_1 S - n_2 US), \\ \mathcal{M}_2 &= (s + ipT)(m_2 - im_1 U - in_1 \bar{S} + n_2 U\bar{S}). \end{aligned} \tag{2.25}$$

This formula and the constraint (2.24) are invariant under $SL(2, \mathbf{Z})_S \times SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U$. Clearly, the constraints (2.24) and (2.18) are just related by the $S \leftrightarrow T$ transformation given in eq.(2.15). The states satisfying the constraint (2.24) are short[3] and also intermediate $N = 4$ multiplets in the heterotic string theory. However, using the string-string duality between the heterotic string and the type IIA string, these states are short $N = 4$ multiplets in the dual type IIA theory. This means that the orbit condition (2.24) is satisfied for electric and magnetic charge vectors which are parallel

[3] As shown in appendix A, the short multiplets are precisely those which are simultanously in the T and in the S orbit (and therefore in the STU orbit).

in the type IIA theory: $\vec{Q}^{\rm IIA}||\vec{P}^{\rm IIA} \Leftrightarrow \hat{\mathbf{M}}^{(A)} \wedge \hat{\mathbf{N}}^{(A)} = 0$.[4] Then the transformations $SL(2,\mathbf{Z})_S \times SL(2,\mathbf{Z})_U \times \mathbf{Z}_2^{U\leftrightarrow S}$ are perturbative in the IIA theory, whereas $SL(2,\mathbf{Z})_T$ is of non-perturbative origin. Thus, one can just repeat the analyis of the additional massless states for the type IIA theory. Specifically, in the type IIA theory there is a critical line $S = U$ with two additional massless fields, a critical point $S = U = 1$ with four additional massless points, and a critical point $S = U^{-1} = \rho$ with six additional massless fields. In the case of being electric ($p = 0$) these states lead to a gauge symmetry enhancement in the type IIA theory. The corresponding charges immediately follow from our previous discussion. Note, however, that the additional massless gauge bosons are not elementary in the type II string but of solitonic nature [5].

Switching again back to the heterotic theory, there are no massless intermediate multiplets within this orbit at the line $S = U$ or points $S = U = 1$, $S = U^{-1} = \rho$. The reason is that we have to remind ourselves that the correct BPS masses are given by the maximum of $|Z_1|^2$ and $|Z_2|^2$. To illustrate this, take $S = U$ and consider as an example the state with $p = m_2 = n_2 = 0$, $m_1 = n_1 = 1$ and s arbitrary, i.e. $\hat{M}_0 = \hat{M}_1 = \hat{M}_2 = \hat{N}_1 = \hat{N}_2 = \hat{N}_3 = 0$, $\hat{M}_3 = -s$, $\hat{N}_0 = s$. The BPS mass of this intermediate state is given by $m^2_{\rm BPS} = |Z_2|^2 = \frac{s^2}{4(T+\bar{T})}$. Thus we see that the heterotic BPS mass formula is not symmetric under $S \leftrightarrow T$.

Of course, there exists another constraint, the U or type IIB orbit,

$$\begin{aligned} \hat{M}_0 &= sm_2, \quad \hat{M}_1 = pn_1, \quad \hat{M}_2 = sn_1, \quad \hat{M}_3 = pm_2, \\ \hat{N}_0 &= -sm_1, \quad \hat{N}_1 = pn_2, \quad \hat{N}_2 = sn_2, \quad \hat{N}_3 = -pm_1, \end{aligned} \tag{2.26}$$

for which the corresponding BPS mass formula factorises into

$$\begin{aligned} \mathcal{M}_1 &= (s + ipU)(m_2 - im_1 S + in_1 T - n_2 ST), \\ \mathcal{M}_2 &= (s + ipU)(m_2 + im_1 \bar{S} + in_1 T + n_2 \bar{S} T). \end{aligned} \tag{2.27}$$

The discussion of this case is completely analogous to the previous one; the states which satisfy the constraint (2.26) correspond to the short $N = 4$ BPS multiplets in the dual type IIB theory with $\vec{Q}^{\rm IIB}||\vec{P}^{\rm IIB} \Leftrightarrow \hat{\mathbf{M}}^{(B)} \wedge \hat{\mathbf{N}}^{(B)} = 0$.[5]

As discussed above, the orbits (2.24) and (2.26) do not contain additional massless intermediate states in the heterotic theory. There are, however, further lines in the moduli space at which intermediate multiplets with spin 3/2 components appear to become

[4] See the discussion given in appendix A.

[5] See the discussion in appendix A.

massless, as it was already observed in [22].[6] Additional massless spin 3/2 multiplets are clearly only physically acceptable if they lead to a consistent enhancement of the local $N = 4$ supersymmetry to higher supergravity such as $N = 5, 6, 8$. However, we do not find a non-perturbative enhancement of $N = 4$ supersymmetry at the lines of possible massless intermediate multiplets. Moreover it is absolutely not clear whether these massless spin 3/2 fields really exist as physical soliton solutions. In fact there are some additional good reasons to reject these states from the physical BPS spectrum. First the explicitly known [22] heterotic soliton solutions for massless intermediate states are singular. Second an argument against the existence of massless spin 3/2 multiplets could be the fact that such states do not exist in any fundamental string at weak coupling. Finally, in the next chapter will argue that these kind of massless states also do not appear in $N = 2$ heterotic strings. Nevertheless we think it is useful to further investigate the interesting problem of non-perturbative supersymmetry enhancement in the future. Therefore we list the possible massless spin 3/2 multiplets, i.e. the zeroes of the BPS mass formula, in appendix B.

3 The $N = 2$ BPS Spectrum

3.1 General formulae

Let us now discuss the spectrum of BPS states in four-dimensional strings with $N = 2$ supersymmetry. These masses are dermined by the complex central charge Z of the $N = 2$ supersymmetry algebra: $m^2_{\rm BPS} = |Z|^2$. In $N = 2$ supergravity the states that saturate this BPS bound belong either to short $N = 2$ hyper multiplets or to short $N = 2$ vector multiplets. In general the mass formula as a function of n Abelian massless vector multiplets ϕ^i $(i = 1, \ldots, n_V)$ is given by the following expression [24, 25, 26]

$$m^2_{\rm BPS} = e^K |M_I P^I + iN^I Q_I|^2 = e^K |\mathcal{M}|^2. \tag{3.1}$$

Here K is the Kähler potential, the M_I $(I = 0, \ldots, n_V)$ are the electric quantum numbers of the Abelian $U(1)^{n_V+1}$ gauge group and the N^I are the magnetic quantum numbers. $\Omega = (P^I, iQ_I)^T$ denotes a symplectic section or period vector; the mass formula (3.1) is

[6] A massive intermediate spin 3/2 multiplet saturating one central charge has the following component structure: $(1 \times$ Spin $3/2, 6 \times$ spin $1, 14 \times$ spin $1/2, 14 \times$ spin $0)$, where the components transform as representations of $USp(6)$. A massless spin 3/2 multiplet has the following structure $(1 \times$ spin $3/2, 4 \times$ spin $1, (6+1) \times$ spin $1/2, (4+4) \times$ spin $0)$. Then, if the intermediate multiplet becomes massless at special points in the moduli space, the 'Higgs' effect works such that 1 massive spin 3/2 multiplet splits into a massless spin 3/2 plus 2 massless vector multiplets.

invariant under the following symplectic $Sp(2n_V+2,\mathbf{Z})$ transformations, which act on the period vector Ω as

$$\begin{pmatrix} P^I \\ iQ_I \end{pmatrix} \to \Gamma \begin{pmatrix} P^I \\ iQ_I \end{pmatrix} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} P^I \\ iQ_I \end{pmatrix}, \tag{3.2}$$

where the $(n_V+1)\times(n_V+1)$ sub-matrices U,V,W,Z have to satisfy the symplectic constraints $U^TV-W^TZ=V^TU-Z^TW=1$, $U^TW=W^TU$, $Z^TV=V^TZ$. Thus the target space duality group Γ, perturbatively as well non-perturbatively, is a certain subgroup of $Sp(2n_V+2,\mathbf{Z})$.

The holomorphic section Ω is determined by the vacuum expectation values and couplings of the n_V+1 massless vector multiplets X^I belonging to the Abelian gauge group $U(1)^{n_V+1}$. (The field X^0, which belongs to the graviphoton $U(1)$ gauge group, has no physical scalar degree of freedom; in special coordinates it will simply be set to one: $X^0=1$; then one has $\phi^i=X^i$.) Specifically, in a certain coordinate system [23], one can simply set $P^I=X^I$ and the Q_I can be expressed in terms of the first derivative of an holomorphic prepotential $F(X^I)$ which is an homogeneous function of degree two: $Q_I=F_I=\frac{\partial F(X^I)}{\partial X^I}$. The gauge couplings as well as the Kähler potential can be also expressed in terms of $F(X^I)$; for example the Kähler potential is given by

$$K=-\log(-i\Omega^\dagger \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \Omega) = -\log\left(X^I\bar{F}_I+\bar{X}^IF_I\right) \tag{3.3}$$

which is, like $\mathcal{M}$, again a symplectic invariant.

To be specific we will now consider an heterotic string which is obtained from six dimensions as a compactification on a two-dimensional torus T_2. The corresponding physical vector fields are defined as $S=i\frac{X^1}{X^0}$, $T=-i\frac{X^2}{X^0}$, $U=-i\frac{X^3}{X^0}$ and the graviphoton corresponds to X^0. Thus there is an Abelian gauge group $U(1)^4$.

3.2 The classical $N=2$ BPS spectrum

Let us start by discussing the form of the classical BPS spectrum. The classical heterotic prepotential is given by [25, 27, 28]

$$F=i\frac{X^1X^2X^3}{X^0}=-STU. \tag{3.4}$$

This classical prepotential is obviously invariant under the full exchange of all vector fields $S\leftrightarrow T\leftrightarrow U$. When considering the classical gauge Lagrangian [23], which follows from this prepotential, one finds a complete 'democracy' among the three fields S, T

and U. Specifically, we will discuss three types of symplectic bases (the discussion about these bases is quite analogous to the discussion about the three S,T,U orbits given in the previous chapter).

First, consider a choice of symplectic basis (we call this the S-basis) in which the S-field plays its conventional role as the loop counting parameter. The weak coupling limit, i.e. the limit when all gauge couplings become simultaneously small, is given by the limit $S \to \infty$. As explained in [25, 27, 28], the period vector (X^I, iF_I) $(F_I = \frac{\partial F}{X^I})$, that follows from the prepotential (3.4), does not lead to classical gauge couplings which all become small in the limit of large S. Specifically, the gauge couplings which involve the $U(1)_S$ gauge group are constant or even grow in the string weak coupling limit $S \to \infty$ like $(S+\bar{S})^{-1}$, whereas the couplings for $U(1)_T \times U(1)_U$ behave in the standard way as being proportional to $S+\bar{S}$. In order to choose a period vector, with all gauge couplings being proportional to $S+\bar{S}$, one has to replace $F^S_{\mu\nu}$ by its dual which is weakly coupled in the large S limit. This is achieved by the following symplectic transformation $(X^I, iF_I) \to (P^I, iQ_I)$ where[7]

$$P^1 = iF_1,\ Q_1 = iX^1,\ \text{and}\ P^i = X^i,\ Q_i = F_i \quad \text{for} \quad i = 0,2,3. \tag{3.5}$$

In the S-basis the classical period vector takes the form

$$\Omega^T = (1, TU, iT, iU, iSTU, iS, -SU, -ST), \tag{3.6}$$

where $X^0 = 1$. One sees that after the transformation (3.5) all electric period fields P^I depend only on T and U, whereas the magnetic period fields Q_I are all proportional to S. In this basis Ω the holomorphic BPS masses (3.1) become[8]

$$\mathcal{M} = M_0 + M_1 TU + iM_2 T + iM_3 U + iS(N_0 TU + N_1 + iN_2 U + iN_3 T) \tag{3.7}$$

Let us compare these $N=2$ BPS masses with the $N=4$ BPS masses discussed in section 2. Specifically, comparing with eq.(2.10) we recognize that the classical $N=2$ mass formula and the $N=4$ mass formula $\mathcal{M}_1$ agree upon the trivial substitution $M_1 = -\hat{M}_1$, $N_0 = -\hat{N}_1$, $N_1 = \hat{N}_0$. (Substituting S by $\bar{S}$ and setting $M_0 = -\hat{M}_0$, $M_1 = \hat{M}_1$, $M_2 = -\hat{M}_2$, $M_3 = -\hat{M}_3$, $N_0 = -\hat{N}_1$, $N_1 = \hat{N}_0$ the $N=2$ BPS masses agree with $\mathcal{M}_2$.) In contrast to $N=4$, eq.(3.7) directly gives the correct BPS masses

[7] Note however that the new coordinates P^I are not independent and hence there is no prepotential $Q(P^I)$ with the property $Q_I = \frac{\partial Q}{\partial P^I}$.

[8] We call this the classical BPS spectrum, since it is computed by using the tree level prepotential. Nevertheless this BPS spectrum contains non-perturbative solitons, and this formula refers to their 'classical', i.e. weak coupling, masses.

without one having to take the maximum of two in general different central charges. The reason for this is the fact that in $N=2$ all BPS states belong to short (vector or hyper) multiplets. In fact, when truncating the $N=4$ heterotic string down to $N=2$, the short as well as the intermediate $N=4$ multiplets become short in the $N=2$ context. This observation potentially leads to new $N=2$ massless BPS multiplets which will be a genuine $N=2$ effect as we discuss in the following.

The classical $U(1)^4$ gauge Lagrangian in the S-basis and the classical $N=2$ BPS mass formula are invariant under the perturbative duality symmetries $SL(2,\mathbf{Z})_T \times SL(2,\mathbf{Z})_U \times \mathbf{Z}_2^{T\leftrightarrow U}$. As discussed above [25, 27, 28], these transformations can be written as specific $Sp(8,\mathbf{Z})$ transformations $\Gamma^{\rm classical}$ with the property that $W^{\rm classical} = Z^{\rm classical} = 0$, $U^{{\rm classical}^T} V^{\rm classical} = 1$. In addition, the field equations in the S-basis and the classical BPS mass formula are also invariant under $SL(2,\mathbf{Z})_S$ and, in contrast to the $N=4$ heterotic case, are also invariant under the transformations $S \leftrightarrow T$ and $S \leftrightarrow U$. Of course, whether the BPS spectrum is really invariant under the symmetries $S \leftrightarrow T$ and $S \leftrightarrow U$ depends on the non-perturbative dynamics and cannot be read off from the BPS mass formula. The point is that, unlike the $N=4$ case, the $N=2$ BPS mass formula in principle allows for an $S \leftrightarrow T \leftrightarrow U$ symmetric spectrum. Indeed there exist some good indications that specific models are $S \leftrightarrow T \leftrightarrow U$ symmetric even after taking into account all non-perturbative corrections. The non-perturbative duality transformations are given by specific $Sp(8,\mathbf{Z})$ transformations with group elements, that have in general non-zero submatrices W and Z. For example, the transformation $S \leftrightarrow T$ corresponds to the following non-perturbative symplectic $Sp(8,\mathbf{Z})$ transformation:

$$P^1 \leftrightarrow -iQ_3, \qquad P^2 \leftrightarrow iQ_1. \tag{3.8}$$

Analogously the transformation $S \leftrightarrow U$ is induced by

$$P^1 \leftrightarrow -iQ_2, \qquad P^3 \leftrightarrow iQ_1. \tag{3.9}$$

The symplectic transformations which correspond to $SL(2,\mathbf{Z})_S$ can be, for example, found in [25, 27].

Let us now define a second symplectic basis, the T-basis, in which the T-field plays the role of the loop counting parameter. In the T-basis all gauge couplings go to zero for large T. As we will see, the T-basis is related to the standard S-basis essentially by a non-perturbative $S \leftrightarrow T$ transformation, i.e. by an exchange of certain electric and magnetic fields [19]. In exact analogy to the S-field dependence of the classical gauge couplings, the prepotential (3.4) leads to gauge couplings of $U(1)_T$ which are constant or grow in the limit $T \to \infty$. In order to obtain a uniform T-dependence of all gauge couplings one has to perform an electric magnetic duality transformation for $U(1)_T$, as follows

$$\tilde{P}^2 = iF_2,\ \tilde{Q}_2 = iX^2,\ \text{and}\ \tilde{P}^i = X^i,\ \tilde{Q}_i = F_i \quad \text{for} \quad i = 0,1,3. \tag{3.10}$$

Then the new classical period vector in the T-basis reads $\tilde{\Omega}^T = (1, iS, -SU, iU, iSTU, TU, -iT, -ST)$. We recognize that all electric periods $\tilde{P}^I$ do not depend on T, whereas the magnetic periods $\tilde{Q}_I$ are propotional to T. Clearly the period vector $\tilde{\Omega}$ is just obtained by an $S \leftrightarrow T$ transformation from the period vector Ω together with some trivial relabeling of electric and magnetic charges. In the T-basis the classical gauge Lagrangian as well as the BPS mass formula are invariant under the transformations $SL(2,\mathbf{Z})_S \times SL(2,\mathbf{Z})_U \times \mathbf{Z}_2^{S\leftrightarrow U}$. These transformations are of perturbative nature in the T-basis and correspond to sympletic matrices $\tilde{\Gamma}$ with $\tilde{W} = \tilde{Z} = 0$, $\tilde{U}^T\tilde{V} = 1$. On the other hand the transformations $SL(2,\mathbf{Z})_T \times \mathbf{Z}_2^{T\leftrightarrow U} \times \mathbf{Z}_2^{S\leftrightarrow T}$ are of non-perturbative nature with in general $\tilde{W}, \tilde{Z} \neq 0$.

It is obvious that one can finally choose another period vector, the U-basis, which leads to classical gauge couplings which have a uniform dependence on U and vanish in the limit of $U \to \infty$. The corresponding formula look analogous to the one just discussed and can be easily written down.

Next let us discuss the form of the classical $N = 2$ BPS spectrum with special focus on the appearance of massless states. Specifically, the singular loci of additional massless states in the classical moduli space fall into three different classes:

(i) First there are the elementary states which become massless at $T = U$, $T = U = 1$ and $T = U = \rho$, for all values of S. At these lines (points) the $U(1)_L^2$ gauge symmetries are classically enhanced to $SU(2)$, $SU(2)^2$ or $SU(3)$ respectively. In the 'standard' S-basis the corresponding BPS states carry only electric charges; however, when seen in the T,U-basis, these states are dyonic.

(ii) Second, suppose that the $S \leftrightarrow T \leftrightarrow U$ symmetry is present in the BPS spectrum. Then there are massless BPS states at the lines (points) $S = T$, $S = T = 1$, $S = T = \rho$ for arbitray U and analogously at $S = U$, $S = U = 1$, $S = U = \rho$ for arbitray T. In case of a perfect dynamical realization of the triality symmetry $S \leftrightarrow T \leftrightarrow U$, the BPS states are $N = 2$ vectormultiplets, and the Abelian gauge symmetries are again enhanced to $SU(2)$, $SU(2)^2$ or $SU(3)$ respectively. In the S-basis these BPS states are non-perturbative dyons, whereas in the T respectively U-basis these states are purely electric. Thus, in the S-basis, $U(1)$ factors, which are magnetic, are enhanced at these special points in the S,T,U moduli space. The possible appearance of these additional massless BPS fields for special values of the S-field is a genuine $N = 2$ effect not being possible in $N=4$.

(iii) Third, there are massless dyons for strong or weak coupling $S=0$ or $S=\infty$ at the lines eqs.(9.6) and (9.7) and, for all S, at $T=U=1$, $T=U=\rho$. These states belong to $N=2$ BPS multiplets, which originate from $N=4$ intermediate multiplets, and are not related to an enhancement of the Abelian gauge symmetries. In case of a $S \leftrightarrow T$, $S \leftrightarrow U$ symmetry there will be also analogous massless BPS states at the transformed lines/points.

3.3 The quantum $N=2$ BPS spectrum

Of course, in general there will be non-perturbative corrections which change the classical BPS spectrum in a crucial way. In the following we will argue that for finite S, after taking into account the non-perturbative corrections, the classical singular lines in (i) split into lines of massless monopoles and dyons a la Seiberg and Witten. With respect to the massless states in (ii), we will conjecture that in models, which are completely $S \leftrightarrow T \leftrightarrow U$ symmetric, there is a non-perturbative gauge symmetry enhancement for large T or large U. Moreover we conjecture that for finite T, U respectively, these lines of massless gauge bosons are again split into lines of massless monopoles and dyons. However we will find no sign of massless states of type (iii) in the non-perturbative spectrum.

In order to consider the form of the BPS spectrum after perturbative as well of non-perturbative corrections, we make the following ansatz for the prepotential in the S-basis

$$F = i\frac{X^1X^2X^3}{X^0} + (X^0)^2\left(f^1(T,U) + f^{\rm NP}(e^{-2\pi S},T,U)\right) \tag{3.11}$$

Here $f^1(T,U)$ denotes the one-loop prepotential [27, 28] which cannot, by simple power counting arguments, depend on S. Clearly, for large S one gets back the tree level prepotential. From the prepotential (3.11) we obtain the following non-perturbative period vector $\Omega^T=(P,iQ)$

$$\begin{aligned}\Omega^T &= (1, TU - f_S^{\rm NP}, iT, iU, iSTU + 2i(f^1+f^{\rm NP}) - iT(f_T^1+f_T^{\rm NP}) - iU(f_U^1+f_U^{\rm NP})\\ &- iSf_S^{\rm NP}, iS, -SU + f_T^1 + f_T^{\rm NP}, -ST + f_U^1 + f_U^{\rm NP})\end{aligned} \tag{3.12}$$

This leads to the following non-perturbative mass formula for the BPS states

$$\begin{aligned}\mathcal{M} &= M_IP^I + iN^IQ_I = M_0 + M_1(TU - f_S^{\rm NP}) + iM_2T + iM_3U + iN^0(STU\\ &+ 2(f^1+f^{\rm NP}) - T(f_T^1+f_T^{\rm NP}) - U(f_U^1+f_U^{\rm NP}) - Sf_S^{\rm NP}) + iN^1S\\ &+ iN^2(iSU - if_T^1 - if_T^{\rm NP}) + iN^3(iST - if_U^1 - if_U^{\rm NP})\end{aligned} \tag{3.13}$$

We see that all states with $M_1 \neq 0$ or $N^I \neq 0$ undergo a non-perturbative mass shift. We also recognize that electric states with $N^I = 0$ do not get a mass shift at the perturbative 1- loop level. However the masses of states with magnetic charges $N^I \neq 0$ are already shifted at the 1-loop level.

The 1-loop prepotential f^1 exhibits logarithmic singularities exactly at the lines (points) of the classically enhanced gauge symmetries and is therefore not a single valued function when transporting the moduli fields around the singular lines (see [27, 28, 10] for all the details). For example around the singular $SU(2)$ line $T = U \neq 1, \rho$ the function f^1 must have the following form [27, 28, 10]

$$f^1(T,U) = \frac{1}{\pi}(T-U)^2 \log(T-U) + \Delta(T,U), \tag{3.14}$$

where $\Delta(T,U)$ is finite and single valued at $T = U \neq 1, \rho$. Around the point $(T,U) = (1,1)$ the prepotential takes the form [27, 28, 10]

$$f^1(T,U=1) = \frac{1}{\pi}(T-1)\log(T-1)^2 + \Delta'(T) \tag{3.15}$$

and around $(T,U) = (\rho,\bar{\rho})$ [27, 28, 10]

$$f^1(T,U=\bar{\rho}) = \frac{1}{\pi}(T-\rho)\log(T-\rho)^3 + \Delta''(T), \tag{3.16}$$

where $\Delta'(T)$, $\Delta''(T)$ are finite at $T = 1$, $T = \rho$ respectively. It follows that, when moving around these critical lines via duality transformations, one has non-trivial monodromy properties. Hence at one-loop, the perturbative duality transformations $SL(2,\mathbf{Z})_T \times SL(2,\mathbf{Z})_U \times \mathbf{Z}_2^{T\leftrightarrow U}$, called Γ^∞, are given in terms of $Sp(8,\mathbf{Z})$ matrices with $U^\infty = U^{\text{classical}}$, $W^\infty \neq 0$, but still $Z^\infty = 0$. This results in non-trivial shifts of the θ-angles at 1-loop. In contrast to $\Gamma^{\text{classical}}$, the 1-loop duality matrices [28] do not preserve the short orbit condition eq.(2.18). This means that, from the $N = 4$ point of view, the $N = 2$ 1-loop monodromies in general mix $N = 2$ BPS states which originate from $N = 4$ vectormultiplets with hypermultiplets which are truncated $N = 4$ intermediate multiplets.

Now, taking into account the non-perturbative effects with $e^{-2\pi S} \neq 0$, the non-Abelian gauge symmetries are never restored, and each perturbative critical line splits into two lines of massless monopoles and dyons respectively [1, 10, 11]. It follows that each semiclassical, i.e. 1-loop, monodromy around the lines of enhanced gauge symmetries are given by the product of two monodromies around the singular monopole and dyon lines, i.e. $\Gamma^\infty = \Gamma^{\text{monopole}} \times \Gamma^{\text{dyon}}$ with $\Gamma^{\text{monopole}}, \Gamma^{\text{dyon}} \in Sp(8)$ and $W^{\text{monopole}}, Z^{\text{monopole}}, W^{\text{dyon}}, Z^{\text{dyon}} \neq 0$. Thus, only in the limit $S \to \infty$ is the theory still symmetric under the perturbative duality group $SL(2,\mathbf{Z})_T \times SL(2,\mathbf{Z})_U \times \mathbf{Z}_2^{T\leftrightarrow U}$. Making an reasonable ansatz for $\Gamma^{\rm monopole}$ and $\Gamma^{\rm dyon}$, one can show [10, 11] that this splitting can be performed in such a way that in the rigid limit one precisely recovers the results of Seiberg and Witten. In addition the correct rigid limit was confirmed [12, 11] by directly computing the non-perturbative monodromies $\Gamma^{\rm monopole}$ and $\Gamma^{\rm dyon}$ in type II Calabi-Yau compactifications with $h_{11}=2$, i.e. for models with two vector fields S and T.

Consider for example the splitting of the critical line $T=U$ with classically enhanced gauge group $SU(2)$; the associated magnetic monopole has non vanishing magnetic quantum numbers $N^3=-N^2$. Like the massless gauge bosons before, this magnetic monopole corresponds to a short $N=4$ vector multiplet, i.e. it belongs to the first orbit (2.21). Using (3.13) its mass vanishes for $Q_2=Q_3$, which leads to following singular monopole locus

$$iS(T-U)-i(f_T^1-f_U^1)-i(f_T^{\rm NP}-f_U^{\rm NP})=0 \tag{3.17}$$

Similarly, the locus of massless dyons with charges $M_2=-M_3=N^3$, $N^2=-N^3$ has the form $T-U=Q_2-Q_3$. Like Γ^∞, $\Gamma^{\rm monopole}$ and $\Gamma^{\rm dyon}$ do not preserve the heterotic short orbit condition eq.(2.18).

Let us now suppose that the full non-perturbative theory is symmetric under the exchange symmetry $S\leftrightarrow T$. In fact the existence of this type of quantum symmetry was already observed in models with only two fields S and T [9, 12, 11]. If this symmetry is exact we expect that in the 'weak coupling limit' $T\to\infty$ one finds an enhancement of the Abelian gauge group at special points in the S,U moduli space. Specifically, at $S=U$ the enhanced gauge group should be $SU(2)$, at $S=U=1$ one has $SU(2)^2$ and at $S=U^{-1}=\rho$ one should find $SU(3)$. In the limit $T\to\infty$ the non-perturbative prepotential, written in the symplectic T-basis, then takes the form

$$f(S,U)=\frac{1}{\pi}(S-U)^2\log(S-U)+\ldots; \tag{3.18}$$

at the point $(S,U)=(1,1)$ the prepotential takes the form

$$f(S,U=1)=\frac{1}{\pi}(S-1)\log(S-1)^2+\ldots \tag{3.19}$$

and around $(S,U)=(\rho,\bar\rho)$

$$f(S,U=\bar\rho)=\frac{1}{\pi}(S-\rho)\log(S-\rho)^3+\ldots. \tag{3.20}$$

It follows that, when moving around these critical lines via duality transformations, one has non-trivial monodromy properties just like at one loop for large S. At large T the theory is symmetric under the duality transformations $SL(2,\mathbf{Z})_S \times SL(2,\mathbf{Z})_U \times \mathbf{Z}_2^{S\leftrightarrow U}$, called $\tilde{\Gamma}^\infty$, which are then given in terms of $Sp(8,\mathbf{Z})$ matrices with $\tilde{U}^\infty \tilde{V}^T = 1$, $\tilde{W}^\infty \neq 0$, $\tilde{Z}^\infty = 0$.

What will happen if we turn on the coupling $e^{-2\pi T}$ in a $S \leftrightarrow T$ symmetric theory? In the spirit of Seiberg and Witten we expect that the lines of enhanced gauge symmetries at $T = \infty$ again split into two lines of massless monopoles and dyons for finite $e^{-2\pi T}$. The corresponding monopole and dyon monodromies $\tilde{\Gamma}^{\rm monopole}$ and $\tilde{\Gamma}^{\rm dyon}$ are just given by conjugating $\Gamma^{\rm monopole}$ and $\Gamma^{\rm dyon}$ by the generator of the $S \leftrightarrow T$ exchange symmetry. An analogous discussion of course applies for the non-perturbative symmetry $S \leftrightarrow U$.

In order to make the existence of the $S \leftrightarrow T$, $S \leftrightarrow U$ symmetries in certain type of models more plausibel it is very useful to utilize the (conjectured, however already quite well established) duality [7, 9, 11, 12, 8, 34, 35, 36] between heterotic $N = 2$ strings and type II $N = 2$ strings on a suitably choosen Calabi-Yau backgrounds. Specifically consider a Calabi-Yau background characterized by the two Hodge numbers h_{11} and h_{21}. In the type IIA models, h_{11} must agree with the number of massless vector multiplets n_V in the heterotic model. The number of hypermultiplets n_H is given by $h_{21}+1$ where the extra 1 accounts for the type II dilaton. Since the type II dilaton sits in an $N = 2$ hypermultiplet and does not couple to the vector multiplets, the classical type II prepotential is exact. It follows that BPS spectrum of the form (3.1) is exact in the type II case as well. For the type IIA models, the Calabi-Yau world-sheet instanton effects then correspond to the target space instanton effects on the heterotic side [8].

After performing the mirror map from IIA to IIB, the number of massless fields in the IIB Calabi-Yau compactification is determined as $n_H = h_{11} + 1$, $n_V = h_{21}$. In the type IIB case the holomorphic prepotential receives no world sheet instanton corrections; it becomes singular at the socalled conifold points in the Calabi-Yau moduli space and at some other isolated points. Then, within the string-string duality picture the type IIB singular locus just corresponds to the locus of massless magnetic monopoles or dyons on the heterotic side (see the discussion below).

Let us first consider as the most simple example the case with only two vector fields S and T. Specifically we consider the Calabi-Yau space, constructed as a hypersurface of degree 12 in $WP_{1,1,2,2,6}(12)$, with $h_{11} = 2$ and $h_{21} = 128$ [21]. It was observed in [9, 12] that this model indeed possesses an exchange symmetry $S \leftrightarrow T$ at the non-perturbative level. This symmetry can be recognized by looking at the instanton expansions, as done in [9]. Specifically, the transformation $q_1 \to q_1 q_2, q_2 \to 1/q_2$ ($q_1 = e^{i2\pi t_1} = e^{-2\pi T}$, $q_2 = e^{i2\pi t_2} = e^{-2\pi(S-T)}$) can be traced back to the monodromy considerations of [21].

Under this transformation $n_{j,k}q_1^j q_2^k \to n_{j,k}q_1^j q_2^{j-k}$, so that the non-perturbative symmetry should come from $n_{j,k} = n_{j,j-k}$ where the $n_{j,k}$ are world-sheet instanton numbers of genus zero. Indeed, it was shown in [21] (there in the model $P_{1,1,2,2,2}(8)$, but this makes no difference here) that the homology type of the holomorphic image of the worldsheet Σ changes as $\Sigma_{j,k} = jh + kl \to j(h+l) + k(-l) = j'h + k'l$ with $j' = j$, $k' = j - k$ under the monodromy $T_\infty : (t_1, t_2) \to (t_1 + t_2, -t_2 + 1)$ on the periods (t_1, t_2).

The singular discriminant locus of this Calabi-Yau, on which certain BPS states become massless, is given by the following equation [7]

$$\Delta = (1-y)((1-x)^2 - x^2 y). \tag{3.21}$$

The conventional weak coupling limit is given by $y = e^{-2\pi S_{\rm inv}} = 0$. In this limit one recovers the perturbative duality symmetry $SL(2,\mathbf{Z})_T$ ($S_{\rm inv}$ is the 1-loop redefined S-field, invariant under the perturbative duality group), and the parameter x can be expressed [9, 12] in terms of modular functions as $x = 1728/j(T)$. In the limit $y = 0$, Δ degenerates into the quadratic factor $(1-x)^2$, and at $x = 1$, i.e. $T = 1$, one finds the classical $SU(2)$ gauge symmetry enhancement. For $y \neq 0$ this line splits into the two lines of massless monopoles and dyons [9, 12].

In addition, the discriminant locus has a second 'weak coupling' limit, where Δ quadratically degenerates: $x = 1/2$. We suggest to identify this limit with $T \to \infty$; in this limit $SL(2,\mathbf{Z})_S$ should be a symmetry of the theory. Then Δ takes the form $\Delta \sim (1-y)^2$ which signals a gauge symmetry enhancement at $y = 1$. Thus we conjecture to make the following identification for $x = 1/2$: $y = 1728/j(S)$. Observe that for large S, this y is exponential in S: $y \to e^{-2\pi S}$. Turning on the coupling $\delta x = x - 1/2$, the quadratic degeneracy is again lifted, and we expect that the large T gauge group enhancement is replaced by the existence of a massless monopole, dyon pair. Recall that in the weak coupling limit $y = 0$ the appearance of the modular function $j(T)$ originates from the fact that the underlying Calabi-Yau space can be constructed as a K_3-fibration [9], where S plays the role of the size of the base space [36]. In analogy, the $S \leftrightarrow T$ symmetric picture could then mean that there exist a dual 'quantum K_3 fibration' with T being the modulus of the base space, implying the appearance of the modular function $j(S)$ in the limit $T \to \infty$.

Now let us investigate the case of three moduli S, T, U. The singular loci of massless BPS states for this type of $N = 2$ string models was recently derived [12] from a type IIB compactification on a Calabi-Yau space $WP_{1,1,2,8,12}(24)$ with $h_{21} = 3$ and $h_{11} = 243$ leading to 244 hypermultiplets (including the type II dilaton multiplet). In ref.[9] some arguments were given supporting the conjecture that this model is symmetric under the exchange $S \leftrightarrow T$, $S \leftrightarrow U$. The discriminant locus of the $P_{1,1,2,8,12}(24)$ Calabi-Yau is [7, 9]

$$\Delta = (y-1) \times \frac{(1-z)^2 - yz^2}{z^2} \times \frac{((1-x)^2 - z)^2 - yz^2}{z^2} = \Delta_y \times \Delta_z \times \Delta_x. \tag{3.22}$$

x, y, z are functions of the three vector fields S, T, U.

Let us now consider three differents limits where Δ degenerates into quadratic factors that signal an enhancement of the Abelian gauge symmetry at special (boundary) points in the moduli space.

(i) First consider the conventional classical limit $y = e^{-2\pi S_{\rm inv}} = 0$. In this limit one recovers the perturbative duality symmetry $SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U \times \mathbf{Z}_2^{T \leftrightarrow U}$ (again, $S_{\rm inv}$ is the 1-loop redefined S-field, invariant under the perturbative duality group), and the parameters x, z can be expressed in terms of the fields T, U as follows [9, 29, 12]:

$$\begin{aligned} x &= \frac{1}{864} \frac{j(T)j(U) + \sqrt{j(T)j(U)(j(T)-1728)(j(U)-1728)}}{j(T)+j(U)-1728}, \\ z &= 864^2 \frac{x^2}{j(T)j(U)}. \end{aligned} \tag{3.23}$$

In this limit the two equations $\Delta_x = 0$ and $\Delta_z = 0$ are completely equivalent. They both correspond to the classical enhancement of one Abelian $U(1)$ gauge group to $SU(2)$. Both equations are solved only by the relation $j(T) = j(U)$, the line of enhanced $SU(2)$ gauge symmetry. More exactly, Δ_x and Δ_z are double valued functions in terms of $j(T), j(U)$. For $j(T) = j(U)$ the branch points are at $j(T) = 0$ and $j(T) = 1728$, i.e. at $T = \rho$, $T = 1$ respectively and at all the points obtained by duality transformations of these two points. With $j(T) = j(U)$ one obtains in the first branch that $z = 1$, $x = \frac{1}{864} j(T)$, $\sqrt{\Delta_x} = \frac{4j(T)(j(T)-1728)}{1728^2}$, $\sqrt{\Delta_z} = \frac{(j(T)-j(U))^2}{4j(T)(j(T)-1728)} = 0$. The points $x = 0$, $x = 2$, where Δ further degenerates, correspond to the points of enhanced gauge symmetries $SU(3)$ or $SU(2)^2$ respectively. In the second branch $j(T) = j(U)$ belongs to $z = \frac{1728^2}{(2j(T)-1728)^2}$, $x = 1 \pm \sqrt{z}$, $\sqrt{\Delta_x} = \frac{(j(T)-j(U))^2}{4j(T)(j(T)-1728)} = 0$, $\sqrt{\Delta_z} = \frac{4j(T)(j(T)-1728)}{1728^2}$. However the product $\sqrt{\Delta_x \Delta_z}$ is single valued, and one obtains as an identity in the limit $y = 0$ [38]: $\sqrt{\Delta_x}\sqrt{\Delta_z} = \frac{(j(T)-j(U))^2}{1728^2}$. In summary, in the classical limit $y = 0$ one precisely finds the lines (points) of enhanced gauge symmetries, namely first $SU(2)$ with $j(T) = j(U)$ corresponding to $T = U$ (plus all dual equivalent lines), second $SU(2)^2$ with $j(T) = j(U) = 1728$ corresponding to $T = U = 1$ and third $SU(3)$ with $j(T) = j(U) = 0$ corresponding to $T = U = \rho$.

(ii) There exists a second limit where Δ degenerates into quadratic factors: $x = 0$. We conjecture that this limit corresponds to $T \to \infty$ and make in this limit the identification $x = e^{-2\pi T_{\rm inv}}$. In this limit the theory is invariant under $SL(2, \mathbf{Z})_S \times SL(2, \mathbf{Z})_U \times \mathbf{Z}_2^{S \leftrightarrow U}$ and $T_{\rm inv}$ is a redefined modulus, invariant under this group. The discriminant locus then becomes

$$\Delta = (y-1)\Big[\frac{(1-z)^2}{z^2} - y\Big]^2 \tag{3.24}$$

Thus $\Delta = 0$ at the line $y = \frac{(1-z)^2}{z^2}$ which should be the line of enhanced $SU(2)$ gauge symmetry for large T. Analogous to the previous case, $y = 0$, one gets a further degeneration at the two points $z = \frac{1}{2}$, $z = \infty$ where $\Delta \sim (y-1)^3$. It is tempting to conjecture that these two points correspond to the further enhancement of the gauge group to $SU(2)^2$, $SU(3)$ respectively. Thus if everything concerning the conjectured $S \leftrightarrow T$ symmetry in these type of models goes through, the discriminant locus should take the following form in the limit $x = 0$: $\sqrt{\Delta} \sim (j(S) - j(U))^2$. If $x \neq 0$ the quadratic degeneracy is lifted, and we expect that the solutions of $\Delta = 0$ correspond to lines of massless monoples and dyons. Unfortunately we are at the moment not ready to prove all these conjectures. It would require a complete reorganisation of the instanton sums in the type II mirror map.

Finally there exists also a third quadratic degeneration of Δ, namely in the limit $z = 1/2$. We believe that this limit corresponds to $U \to \infty$ with $SL(2,\mathbf{Z})_S \times SL(2,\mathbf{Z})_T \times \mathbf{Z}_2^{S\leftrightarrow T}$ duality symmetry. The discriminant locus takes the form

$$\Delta \sim (y-1)^2[((1-x)^2 - \frac{1}{2})^2 - \frac{y}{4}]. \tag{3.25}$$

Thus for $y = 1$ there should be the gauge symmetry enhancement to $SU(2)$. Like before we find two additional special points, $x = 0$ and $x = 2$, where further degeneration and further gauge symmetry enhancement to $SU(2)^2$ or $SU(3)$ takes place. In the limit $z = 1/2$, $\sqrt{\Delta}$ should be writable as $(j(S) - j(T))^2$.

At the end of this section let us also mention that in the quantum case we did not find any trace of those massless states which we discussed under point (iii) at the classical level. If they would exist they should have shown up for large S in Δ, since they were classically present for any S and hence in particular for weak coupling. However the Calabi-Yau discriminant locus has no more zeroes at large S in addition to the ones we just discussed. This observation may be one more argument against the existence of the corresponding massless intermediate states in the $N = 4$ heterotic string.

4 $N = 4$ BPS sums

4.1 The $N = 4$ free energy

In the next sections we will discuss the topological string partition function as a sum over BPS states. A similar type of partition function was introduced in [24]. More recently the sum over BPS states was also discussed by Vafa in [32]. Concretely, let us define the following partition function Z[9]

$$\log Z = \sum_{\text{BPS states}} \log m_{\text{BPS}}^2 \tag{4.1}$$

In the following, we will discuss the non-perturbative partition function obtained by summing over the heterotic $N = 4$ BPS spectrum. Specifically, we will consider the following holomorphic free energy

$$\mathcal{F} = \sum_{\hat{M}_I, \hat{N}_I} \log \mathcal{M}_{1,2}, \tag{4.2}$$

where the holomorphic BPS masses are given in (2.10). The holomorphic free energy and the non-holomorphic partition function are related as

$$Z = e^{\mathcal{F}+\bar{\mathcal{F}}} e^{K}, \tag{4.3}$$

where K is the Kähler potential of the moduli fields S, T, U (we are restricting the discussion to an $SO(2,2)$-coset subspace of the toroidal moduli space). K is given by

$$K = -\log[(S+\bar{S})(T+\bar{T})(U+\bar{U})], \tag{4.4}$$

which transforms under $SL(2,\mathbf{Z})_S$, $S \to \frac{aS-ib}{icS+d}$, as $K \to K + \log(icS+d) + \log(-ic\bar{S}+d)$ and likewise for $SL(2,\mathbf{Z})_T$ and $SL(2,\mathbf{Z})_U$. It follows that $e^{\mathcal{F}}$ must be a modular function of modular weight -1 under $SL(2,\mathbf{Z})_S$, $SL(2,\mathbf{Z})_T$ and $SL(2,\mathbf{Z})_U$ in order for Z being completely duality invariant. $\mathcal{F}$ and Z are clearly non-perturbative expressions since they involve the summation over elementary string states as well as over soliton states like magnetic monopoles etc. Thus $Z, \mathcal{F}$ will exhibit the non-perturbative dilaton dependence of the string partition function. Note that by demanding Z to be completely duality invariant we are requiring the absence of non-perturbative duality anomalies, in particular the absence of S-duality anomalies.

The sum eq.(4.2) can be more conveniently computed by selecting some specific summation orbits. One criterion of selecting the relevant summation orbits is that at least all singularities of the free energy have to be contained in the correct way; in other words, this means that the sum has to contain all possible states which can become massless at certain points in the moduli space. In addition, the duality invariance of the free energy must not be destroyed by summing over specific orbits. Let us start by first summing over the three orbits (2.21), (2.24) and (2.26), which are related by the triality exchange

[9] Here, Z is not to be confused with the central charge.

$S \leftrightarrow T \leftrightarrow U$. (Each of these summation orbits will contain further suborbits.) $\mathcal{F}_{T\leftrightarrow U}$ sums over the BPS states in the first orbit (2.21) and is invariant under $T \leftrightarrow U$. Therefore $\mathcal{F}_{T\leftrightarrow U}$ is summing over the short heterotic $N=4$ vector multiplets; using eq.(2.20) $\mathcal{F}_{T\leftrightarrow U}$ becomes

$$\mathcal{F}_{T\leftrightarrow U} = \sum_{\{\hat{M}_I,\hat{N}_I|C_{KL}=0\}} \log(\hat{M}_0 - \hat{M}_1 TU + i\hat{M}_2 T + i\hat{M}_3 U + i\hat{N}_0 S - i\hat{N}_1 STU - \hat{N}_2 ST - \hat{N}_3 SU) \tag{4.5}$$

In order to perform the sum one must solve the six equations $C_{KL} = 0$ in terms of unconstrained summation variables. This can be done generalizing a method described in [39]. Consider the three equations $C_{0i} = 0$ first. Setting $\hat{N}_1 = \hat{N}_2 = \hat{N}_3 = 0$ they are fulfilled if either (i) $\hat{N}_0 = 0$ or (ii) $\hat{M}_1 = \hat{M}_2 = \hat{M}_3 = 0$. In case (ii) the other three equations are already solved. It only remains to sum over two unconstrained variables $s := \hat{M}_0$ and $p := \hat{N}_0$. Thus the first contribution to the sum is $\sum_{(s,p)\neq(0,0)} \log(s + ipS)$. In case (i) we are left with four unconstrained variables $m_2 := \hat{M}_0$, $n_2 := \hat{M}_1$, $n_1 := \hat{M}_2$ and $m_1 := -\hat{M}_3$ resulting in a second contribution $\sum_{(m_1,m_2,n_1,n_2)\neq(0,0,0,0)} \log(m_2 - im_1 U + in_1 T - n_2 TU)$. Summarizing we have succeded in writing $\mathcal{F}_{T\leftrightarrow U}$ as an unconstrained sum

$$\mathcal{F}_{T\leftrightarrow U} = \sum_{(s,p)\neq(0,0)} \log(s+ipS) + \sum_{(m_1,m_2,n_1,n_2)\neq(0,0,0,0)} \log(m_2 - im_1 U + in_1 T - n_2 TU) \tag{4.6}$$

which splits into a non–perturbative part, which only involves S, and into a perturbative part only depending on the moduli T and U.

Consider the first term in (4.6). The regularized sum [39, 24] over the electric and magnetic charges s, p leads to the following contribution: $\sum_{s,p} \log(s+ipS) = \log \eta(S)^{-2}$, where η is the Dedekind function. This term describes the non-perturbative S dependence of $\mathcal{F}_{T\leftrightarrow U}$. $e^{\mathcal{F}}_{T\leftrightarrow U}$ transforms as a modular function of modular weight -1 under $SL(2,\mathbf{Z})_S$. $\mathcal{F}_{T\leftrightarrow U}$ diverges linearly for large S as well as for small S. These divergences reflect the appearance of infinitely many massless electric or magnetic states for $S \to \infty, 0$ respectively. Now, in order to evaluate the second term in expression (4.6) we split the sum into the two further suborbits, namely (i): $m_1 n_1 + m_2 n_2 = 0$, (ii): $m_1 n_1 + m_2 n_2 = 1$. This choice is dictated by the appearance of the massless fields (see [33] for a detailled discussion). The suborbit (i) contains no states which become massless for finite T and U but infinitely many states (Kaluza-Klein and winding modes) which become massless in the degeration limits $T, U \to 0, \infty$. Summing over the suborbit (i) leads to a term $\log \eta(T)^{-2} \eta(U)^{-2}$ with linear divergences for $T \to \infty, 0$ due to the massless Kaluza Klein states or winding modes in this limit. The second suborbit (ii) contains the finite number of states which are massless at the critical points in the moduli space $T = U$, $T = U = 1$

and $T = U = \rho$. Then the suborbit (ii) leads to [33] $\log(j(T) - j(U)$,[10] where j is the absolute modular invariant function. This expression is logarithmically divergent at the critical lines (points) where the residues of the poles correctly agree with the number of massless fields at the symmetry enhancement points. Thus collecting the different terms (the higher orbits $m_1 n_1 + n_2 m_2 > 1$ do not give new terms) we obtain the following holomorphic free energy

$$\mathcal{F}_{T\leftrightarrow U} = \log(\eta(S)^{-2}\eta(T)^{-2}\eta(U)^{-2}(j(T) - j(U))^r). \tag{4.7}$$

The coefficient r is undetermined at this stage and corresponds to the overall number of states becoming massless at the specific lines (points). Clearly $e^{\mathcal{F}_{T\leftrightarrow U}}$ transform as a modular function of modular weight -1 under $SL(2,\mathbf{Z})_S \times SL(2,\mathbf{Z})_T \times SL(2,\mathbf{Z})_U$, and it is invariant under $T \leftrightarrow U$ (up to a possible extra overall $\pm$ sign).

Next let us discuss the sum $\mathcal{F}_{S\leftrightarrow U}$ over the orbit eq.(2.24). At the first glimpse one could believe that this sum is just obtained by performing $S \leftrightarrow T$ exchange in $\mathcal{F}_{T\leftrightarrow U}$. This conclusion would be true, if there were intermediate massless states in the second orbit for $S = U$, $S = U = 1$, $S = U = \rho$ in the heterotic string theory. They however do not exist. Thus we conclude that the suborbit (ii) does not lead to singularities for finite S, T, U. The fact that $\mathcal{F}_{T\leftrightarrow U}$ and $\mathcal{F}_{S\leftrightarrow U}$ do not agree reflects the non-invariance of the heterotic BPS spectrum under the exchange $S \leftrightarrow T$.

Finally, the discussion about the sum over the orbit eq.(2.26) is completely analogous to the previous case.

In case that there exist massless intermediate states at specific points/lines in the moduli space, these states would also contribute to the free energy. However, as we have discussed in section 2.3 there are many good reasons to discard these massless spin 3/2 BPS soliton states. Thus we take eq.(4.7) as the complete result for the $N = 4$ heterotic free energy. The associated non-perturbative partition function is invariant under $SL(2,\mathbf{Z})_S \times SL(2,\mathbf{Z})_T \times SL(2,\mathbf{Z})_U \times \mathbf{Z}_2^{T\leftrightarrow U}$. It is very similar to the ordinary bosonic string partition function. The type IIA (IIB) partition function is finally obtained by the exchange $S \leftrightarrow T$ ($S \leftrightarrow U$) in eq.(4.7).

[10] Here we have assumed that the regularization procedure is modular invariant. This assumption however may not hold, and the regularization procedure may possess a kind of modular anomaly which destroys the duality covariance of the sum; thus non-modular invariant, but completely finite terms may be added to the regularized sum. These finite terms can be absorbed by a redefinition of the dilaton field [27].

4.2 Absence of $N = 4$ thresholds and the role of the $N = 4$ free energy

In the $N = 4$ case the free energy does not correspond to threshold corrections in the low energy effective action, since loop corrections are absent in $N = 4$, even at the non-perturbative level. In the following we will, for example, first recall the absence of 1-loop gravitational threshold corrections in $N = 4$ heterotic strings.

In $N = 4$ compactifications of the heterotic string the dilaton $S = \frac{1}{g^2} - i\frac{\theta}{8\pi^2}$ parametrises a Kählerian $SU(1, 1)$-coset, whereas the non-Kählerian $SO(6, 22)$-coset is parametrised by moduli Φ^I.

In a string calculation, 1-loop corrections to gravitational couplings in $N = 4$ heterotic compactifications should, if present, be of the form

$$\frac{1}{g_{grav}^2} = 12(S + \bar{S}) + \frac{b_{grav}}{16\pi^2} \log \frac{M_{string}^2}{p^2} + \Delta(\Phi^I) \tag{4.8}$$

$\Delta(\Phi)$ denotes the moduli dependent 1-loop corrections due to both massless and massive modes in the theory. b_{grav}, on the other hand, denotes the gravitational beta function coefficient computed from the massless fields. Note that the scale appearing in the logarithm in (4.8) is the string scale, as it should for a string calculation.

In a field theory calculation, on the other hand, it is the Planck scale which should appear in a 1-loop calculation. Thus, consider rewriting (4.8) as

$$\frac{1}{g_1^2} = 12(S + \bar{S}) + \frac{b_{grav}}{16\pi^2}(\log \frac{M_{Planck}^2}{p^2} + K) + \Delta(\Phi^I) \tag{4.9}$$

where $K = -\log(S+\bar{S})$. Here K denotes the Kähler potential for the Kählerian $SU(1, 1)$-coset parametrised by the dilaton field S. We have used that $M_{Planck}^2 \propto (S + \bar{S})M_{string}^2$. Actually, a field theory calculation would a priori give that

$$\frac{1}{g_1^2} = 12(S + \bar{S}) + \frac{b_{grav}}{16\pi^2} \log \frac{M_{Planck}^2}{p^2} + \frac{c_{grav}}{16\pi^2} K + \Delta(\Phi^I) \tag{4.10}$$

with some coefficient c_{grav}. The term proportional to $\frac{b_{grav}}{16\pi^2} \log \frac{M_{Planck}^2}{p^2}$ arises from a 1-loop graph with 2 external gravitational legs sticking out and massless fields running in the loop. The term proportional to $c_{grav}K$ arises from a triangle graph with 2 gravitational legs and one a_m leg sticking out and with massless fields running in the loop. Indeed, as shown in [41], every fermion in the $N = 4$ theory couples to the "Kähler connection" $a_m \propto \partial_S K\ \partial_m S - c.c$ associated to the $SU(1, 1)$-coset (note again that the $SO(6, 22)$-coset is not Kählerian and hence there is no "Kähler connection" associated to it). If the field theory calculation is to match the string calculation (4.8), then one has to find that $b_{grav} = c_{grav}$ in the field theory calculation.

Consider now calculating b_{grav} and c_{grav} in field theory. b_{grav} is nothing but the sum over the trace anomalies of the massless multiplets in the theory. At generic points in the $SO(6,22)$-moduli space, the massless multiplets around are the $N = 4$ supergravity multiplet and 22 abelian $N = 4$ vector multiplets. The trace anomaly for an $N = 4$ vector multiplet is zero, as it is wellknown. What about the trace anomaly of the $N = 4$ supergravity multiplet? For an $N = 4$ compactification of the heterotic string, the axion is not really a scalar degree of freedom but rather an antisymmetric tensor degree of freedom.[11] Taking into account the following trace anomaly contributions (in units where a real scalar degree of freedom contributes an amount of 1) [42], namely 1 from a real scalar field, $\frac{7}{4}$ from a Weyl fermion, -13 from a vector field, 212 from a graviton, $-\frac{233}{4}$ from a gravitino and 91 from an antisymmetric tensor, then gives that $b_{grav} = 0$ for an $N = 4$ heterotic compactification.

Since it must be that $b_{grav} = c_{grav}$, it follows that one should for consistency also find that $c_{grav} = 0$ in a field theory calculation. c_{grav} is nothing but the Kähler anomaly coefficent. Using the $N = 1$ assignments for the Kähler charges one has that the fermions in the $N = 4$ gravitational multiplet carry charges $+1$, whereas the gauginos in the $N = 4$ vector multiplets carry charges -1. Then it follows that indeed $c_{grav} = 4(21 + 1 - 22) = 0$.

The fact that $b_{grav} = c_{grav} = 0$ indicates that there are no 1-loop corrections to g^2_{grav} in $N = 4$ heterotic compactifications at all, as indeed shown by string scattering amplitude calculations in the context of orbifold compactifications [43].

Thus, the $N = 4$ holomorphic free energy cannot correspond to threshold corrections in the low energy effective action. What role then does the $N = 4$ non-holomorphic free energy discussed in the previous section play in the context of $N = 4$ heterotic strings? We conjecture that it is the S-duality invariant partition function of topologically twisted $N = 4$ heterotic string compactifications. A priori one might expect the partition function of the topologically twisted theory to be holomorphic in the moduli fields. However, it was pointed out in [44] that, at least in the context of topologically twisted $N = 4$ super Yang-Mills theory on four-manifolds, there are examples where this is not the case due to the appearence of holomorphic anomalies. Hence, it is possible that the non-holomorphicity of the $N = 4$ free energy is again a manifestation of the appearance of holomorphic anomalies in the twisted version of $N = 4$ string compactifications.

If indeed the $N = 4$ free energy is to be identified with the partition function of topologically twisted $N = 4$ string compactifications, then this implies that, whereas the

[11] The associated $N = 4$ supergravity multiplet will thus contain 1 graviton, 4 gravitini, 6 graviphotons, 4 Weyl fermions, 1 antisymmetric tensor and one real scalar.

holomorphic gravitational coupling $\mathcal{F}_{grav} = 24S$ of the untwisted model doesn't receive perturbative or non-perturbative corrections, the holomorphic coupling $\mathcal{F}_1$ of the twisted model is more complicated and given by (4.7). Something similar happens in the case of twisted $N = 4$ super Yang-Mills theory on four-manifolds. There, it was found [44] that S-duality invariance of the twisted partition function only holds provided that there are certain non-minimal couplings in the Lagrangian of the form $\log \eta(S)\chi$ that involve the background gravitational field, where χ denotes the Euler characteristic of the four-manifold ($\chi \propto \int GB$, where GB denotes the Gauss-Bonnet combination). Namely, the partition function $Z[S]$ for the topologically twisted $N = 4$ super Yang-Mills theory transforms like a modular form with modular weight w under $S \to \frac{1}{S}$

$$Z[S] \to S^{w\chi} Z[S] \tag{4.11}$$

(ignoring the issue of holomorphic anomalies). The following modified partition function

$$\hat{Z}[S] = e^{-\mathcal{F}_1 \chi} Z[S] \tag{4.12}$$

however, is invariant under $S \to \frac{1}{S}$ provided that $\mathcal{F}_1 \propto \log \eta(S)$.

5 $N = 2$ BPS sums

5.1 The $N = 2$ free energy

Let us again define the $N = 2$ holomorphic free energy $\mathcal{F}$ as the sum over the $N = 2$ BPS states (3.1), that is

$$\mathcal{F} = \sum_{M_I, N^I} \log(M_I P^I + i N^I Q_I). \tag{5.1}$$

This formula was introduced in [24] in the context of string compactifications on Calabi-Yau spaces. Like in the previous $N = 4$ case it is useful to split this sum into sums over the different orbits of the relevant duality group Γ. Since the $N = 2$ Kähler potential changes under duality transformations $\Gamma = \begin{pmatrix} U & Z \\ W & V \end{pmatrix}$ as

$$K \to K + \log |U_I^0 P^I / P^0|^2 \tag{5.2}$$

the holomorphic $N = 2$ free has to transform as

$$\mathcal{F} \to \mathcal{F} - \log U_I^0 P_I / P^0. \tag{5.3}$$

The non-perturbative heterotic $N = 2$ free energy based on the non-perturbative BPS mass formula (3.13) is in general very difficult to compute. It is clear that $\mathcal{F}$ will diverge at those loci in the non-perturbative moduli space where BPS states become massless. These are the loci of massless magnetic monopoles and massless dyons plus other singular lines at strong coupling. Using the string-string duality between the $N = 2$ heterotic and type IIA/B strings, the non-perturbative heterotic free energy is identical to the classical free energy of the type II strings, where one sums over the classical BPS spectrum. Thus $\mathcal{F}$ is singular precisely on the discriminant locus Δ of the (mirror) Calabi-Yau which, for the particular IIB model with $h_{21} = 3$ and $h_{11} = 243$ for example, is given in (3.22). In the next chapter we will identify the $N = 2$ BPS sum $\mathcal{F}$ with the gravitational threshold function on the heterotic side; on the type II side this is given by the known topological function F_1^{II} [30].

5.2 Perturbative and non-perturbative $N = 2$ gravitational threshold corrections

In $N = 2$ supergravity a particular combination of higher derivative curvature terms (namely of C^2 and $\mathcal{R}\tilde{\mathcal{R}}$) resides in the square of the chiral Weyl superfield. Its coupling to the abelian vector multiplets is governed by a holomorphic function $\mathcal{F}_{grav}$. Below, $\mathcal{F}_{grav}$ will be identified with the $N = 2$ holomorphic free energy $\mathcal{F}$. We will, in the following, focus on the dependence of $\mathcal{F}_{grav}$ on S, T and U. The discussion given below can, in principle, also be extended to the dependence of $\mathcal{F}_{grav}$ on additional Wilson line moduli.

In $N = 2$ heterotic string compactifications one has at tree-level that $\mathcal{F}_{grav} = 24S$, where $S = \frac{1}{g^2} - i\frac{\theta}{8\pi^2}$. The gravitational coupling g_{grav}^{-2} is then given by $g_{grav}^{-2} = \Re\mathcal{F}_{grav} = 24\Re S$. At the 1-loop level, on the other hand, $\mathcal{F}_{grav}$ reads [33, 34, 37, 38]

$$\mathcal{F}_{grav} = 24S_{inv} + \frac{b_{grav}}{8\pi^2}\log\eta^{-2}(T)\eta^{-2}(U) + \frac{2}{4\pi^2}\log(j(T) - j(U)) \tag{5.4}$$

where $b_{grav} = 46+2(n_H-n_V) = 48-\chi, \chi = 2(n_V-(n_H-1))$.[12] n_V denotes the number of massless vector multiplets (not including the graviphoton) and n_H the number of massless hyper multiplets in the $N = 2$ heterotic string compactification. Here, $S_{inv} = S+\sigma(T,U)$ denotes the invariant dilaton field [27]. It was shown in [27] that $\sigma = -\frac{1}{2}\partial_T\partial_U h^{(1)} - \frac{1}{8\pi^2}\log(j(T)-j(U))$. The term proportional to $\log(j(T)-j(U))$ in (5.4) reflects the fact that there are points of symmetry enhancement in the classical (T,U)-moduli space at which additional BPS states become massless [33]. $\mathcal{F}_{grav}$ has the correct modular weight to render the perturbative gravitational coupling g_{grav}^2 invariant under the perturbative

[12] χ is the Euler number of the associated CY manifold in the dual Type IIA formulation of the theory (assuming that there exists such a dual formulation).

duality group $SL(2,\mathbf{Z})_T \times SL(2,\mathbf{Z})_U \times \mathbf{Z}_2^{T\leftrightarrow U}$

$$\frac{1}{g^2_{grav}} = \Re\mathcal{F}_{grav} + \frac{b_{grav}}{16\pi^2}(\log\frac{M^2_{Planck}}{p^2} + K) + \frac{12(3-n_V)}{16\pi^2}\log(S+\bar{S}) \tag{5.5}$$

where K denotes the tree-level Kähler potential $K = -\log(S+\bar{S})(T+\bar{T})(U+\bar{U})$. Note that there is an additional dependence on $\log(S+\bar{S})$ in (5.5).[13] The 1-loop corrected gravitational coupling (5.5) can also be written as follows

$$\begin{aligned}\frac{1}{g^2_{grav}} &= 12\left(S+\bar{S}+V_{GS}\right) + \frac{b_{grav}}{16\pi^2}\log\frac{M^2_{string}}{p^2} + \frac{12(3-n_V)}{16\pi^2}\log(S+\bar{S}) \\ &+ \Delta_{grav}\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}\Delta_{grav} &= 12(-V_{GS}+\sigma+\bar{\sigma}) + \frac{b_{grav}}{16\pi^2}\hat{K} \\ &+ \Re\left(\frac{b_{grav}}{8\pi^2}\log\eta^{-2}(T)\eta^{-2}(U) + \frac{2}{4\pi^2}\log(j(T)-j(U))\right)\end{aligned} \tag{5.7}$$

Here, $M^2_{Planck} \propto (S+\bar{S})M^2_{string}$ and $\hat{K} = -\log(T+\bar{T})(U+\bar{U})$. V_{GS} denotes the Green-Schwarz term and $S+\bar{S}+V_{GS}$ denotes the true loop counting parameter of the heterotic string. Finally, note that (5.6) can also be written as

$$\frac{1}{g^2_{grav}} = \frac{b_{grav}}{16\pi^2}\log\frac{M^2_{Planck}}{p^2} + \frac{1}{16\pi^2}F_1 \tag{5.8}$$

where

$$F_1 = \log\{\exp[(\frac{17}{3}+\frac{5}{3}n_V+\frac{1}{3}n_H)K]\det K^{-2}_{i\bar{j}}e^{8\pi^2(\mathcal{F}_{grav}+\bar{\mathcal{F}}_{grav})}\} \tag{5.9}$$

Here, $K_{i\bar{j}}$ denotes the tree-level Kähler metric of the massless vector multiplets.

As explained in section (4.1), the term proportional to $\log\eta^{-2}(T)\eta^{-2}(U)$ arises from BPS states laying on the orbit $m_1n_1+m_2n_2=0$, whereas the term proportional to $\log(j(T)-j(U))$ arises from BPS states laying on the orbit $m_1n_1+m_2n_2=1$. Thus, it is natural to conjecture [24, 32, 14] that $\mathcal{F}_{grav}$ is obtained by summing over suitable orbits of BPS states, that is

$$\mathcal{F}_{grav} \propto \mathcal{F} = (\sum_{M_I,N^I}^{\text{vector}} - \sum_{M_I,N^I}^{\text{hyper}})\log(M_IP^I + iN^IQ_I) \tag{5.10}$$

Here, the period vector (P^I, iQ_I) entering in (5.10) is given by the classical period vector (3.6). Comparing (5.10) with (5.4) shows that the tree-level piece $\mathcal{F}_{grav} = 24S$ should be

[13] We thank Jan Louis for pointing this out to us.

due to BPS states as well, that is it should arise from (5.10) when taking $S \to \infty$. For instance, it could arise from a term in $\mathcal{F}_{grav}$ of the type[14] $\log \eta^{-2}(\check{S}) = \sum_{(s,p)\neq(0,0)} \log(s + ip\check{S})$ in the limit $\check{S} \to \infty$. Inspection of the mass formula (2.20) shows that such a term could indeed arise.

$\mathcal{F}_{grav}$ will, in general, receive non-perturbative corrections. It is natural to conjecture that the non-perturbatively corrected $\mathcal{F}_{grav}$ will be given as in (5.10), where this time the period vector (P^I, iQ_I) is the non-perturbative period vector (3.12). Finally note that, whereas on the heterotic side F_1 describes the gravitational threshold function, it is the known topological function F_1^{II} on the type II side [30].

Consider the 1-loop corrected gravitational coupling (5.6). In general, it is difficult to compute Δ_{grav} exactly at the 1-loop level. For the s=0 model (which has a gauge group $G = E_8 \times E_7 \times U(1)^4$ at generic points in the moduli space) discussed recently in [14], however, this can be done using the technology introduced there, as follows.

It was shown in [14] that the Green-Schwarz term V_{GS} is given by $V_{GS} = \frac{2}{16\pi^2}\Delta_{univ}$ with Δ_{univ} given in equation (4.4) of [14], that is[15]

$$\begin{aligned} V_{GS} &= \frac{2}{-(\Re y)^2}\Re\left(h^{(1)} - y_1^a \partial_{y^a} h^{(1)}\right) \\ &= \frac{2(h^{(1)} + \bar{h}^{(1)}) - (y^a + \bar{y}^a)(\partial_{y^a} h^{(1)} + \partial_{\bar{y}^a}\bar{h}^{(1)})}{(T+\bar{T})(U+\bar{U})} \end{aligned} \tag{5.11}$$

where $y = (y_+, y_-) = (T, U), y_1 = \Re y$. It is convenient to introduce a coupling $\tilde{S} = S - \frac{1}{2}\partial_T\partial_U h^{(1)}$. Then, it was shown in [14] that V_{GS} and $\tilde{S}$ satisfy the following differential equation

$$\begin{aligned} \frac{1}{2}\left(-V_{GS} + \tilde{S} - S + \bar{\tilde{S}} - \bar{S}\right) &= -\frac{1}{12}\frac{1}{16\pi^2}\left(\tilde{I}_{2,2} - I_{2,2}\right) + \frac{1}{8\pi^2}(\log\Psi + \log\bar{\Psi}) \\ &+ \frac{b(E_8)}{16\pi^2}\log(-y_1^2) \end{aligned} \tag{5.12}$$

Inserting (5.12) into Δ_{grav} in (5.7) gives that

$$\begin{aligned} \Delta_{grav} &= 24\left(-\frac{1}{12}\frac{1}{16\pi^2}\left(\tilde{I}_{2,2} - I_{2,2}\right) + \frac{1}{8\pi^2}(\log\Psi + \log\bar{\Psi}) + \frac{b(E_8)}{16\pi^2}\log(-y_1^2)\right) \\ &- 12\left(\frac{1}{8\pi^2}\log(j(T) - j(U)) + \frac{1}{8\pi^2}\log(j(\bar{T}) - j(\bar{U}))\right) \\ &+ \frac{b_{grav}}{16\pi^2}\hat{K} + \Re\left(\frac{b_{grav}}{8\pi^2}\log\eta^{-2}(T)\eta^{-2}(U) + \frac{2}{4\pi^2}\log(j(T) - j(U))\right) \end{aligned} \tag{5.13}$$

[14]Here, $\check{S} = 4\pi S = \frac{4\pi}{g^2} - i\frac{\theta}{2\pi}$. Then, under the axionic shift $\theta \to \theta + 2\pi$, $\check{S} \to \check{S} - i$.

[15]Here, $h^{(1)}$ denotes the 1-loop correction to the prepotential F given in (3.11), that is $h^{(1)} = f^1$.

Using that [14]

$$
\begin{aligned}
\log\Psi &= \frac{1}{2}\log(j(T)-j(U)) + \frac{1}{2}b(E_8)\log\eta^2(T)\eta^2(U)\\
I_{2,2} &= -c_3(0)\log(T+\bar{T})(U+\bar{U}) - 2\log|j(T)-j(U)|^2\\
&- c_3(0)\log|\eta^2(T)\eta^2(U)|^2 + \text{constant}
\end{aligned}
\tag{5.14}
$$

and inserting (5.14) into (5.13) gives that

$$
\Delta_{grav} = -\frac{2}{16\pi^2}\tilde{I}_{2,2} \tag{5.15}
$$

where we have used that $b_{grav} = -2\tilde{c}_1(0) = 528, c_3(0) = -984, b(E_8) = -60$. $\tilde{I}_{2,2}$ is given by [14]

$$
\tilde{I}_{2,2} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}[Z_{2,2}\frac{E_4E_6}{\eta^{24}}(E_2 - \frac{3}{\pi\tau_2}) - (-2\tilde{c}_1(0))] \tag{5.16}
$$

Using the results of [14], it is straightforward to show that $\tilde{I}_{2,2}$ is related to the "new supersymmetric index" of [40] as follows

$$
\tilde{I}_{2,2} = \frac{i}{2}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}[\frac{1}{\eta^2}Tr_R J_0(-1)^{J_0} q^{L_0-22/24}\bar{q}^{\bar{L}_0-9/24}(E_2 - \frac{3}{\pi\tau_2}) - b_{grav}] \tag{5.17}
$$

Expression (5.17), on the other hand, was shown in [14] to be due to BPS states only. That is, Δ_{grav} is indeed due to BPS states, only. The integral $\tilde{I}_{2,2}$ was explicitly evaluated in [14] and is given by

$$
\begin{aligned}
\tilde{I}_{2,2} &= 4\Re\left(\sum_{r>0}[\tilde{c}_1(-\frac{r^2}{2})Li_1(e^{-2\pi r\cdot y}) + \frac{6}{\pi y_1^2}c_1(-\frac{r^2}{2})\mathcal{P}(r\cdot y)]\right)\\
&+ \tilde{c}_1(0)\left(-\log[-y_1^2] - \kappa\right) + \frac{1}{y_1^2}[\tilde{d}^{2,2}_{abc}y_1^a y_1^b y_1^c + \delta]
\end{aligned}
\tag{5.18}
$$

Note that $\tilde{I}_{2,2}$ possesses a $T-U$ chamber dependence, i.e. $\tilde{I}_{2,2}(\Re T > \Re U) \neq \tilde{I}_{2,2}(\Re U > \Re T)$. The exact expression for the 1-loop corrected gravitational coupling (5.6) then follows from (5.18) and from the explicit evaluation of V_{GS} given in [14] [16]

$$
V_{GS} = \frac{1}{y_1^2}\frac{2}{(2\pi)^3}\Re\left(\sum_{r>0}c_1(-\frac{r^2}{2})\mathcal{P}(r\cdot y)\right) + \frac{1}{96\pi^2}\frac{1}{y_1^2}\left(\tilde{d}^{2,2}_{abc}y_1^a y_1^b y_1^c + \delta\right) \tag{5.19}
$$

Similarly, the exact expression for the 1-loop corrected holomorphic coupling $\mathcal{F}_{grav}$ (5.4) is given by

$$
\mathcal{F}_{grav} = 24\left(S - \frac{1}{768\pi^2}\partial_T\partial_U\left(\tilde{d}^{2,2}_{abc}y^a y^b y^c\right) - \frac{1}{8\pi^2}\log(j(T)-j(U))\right.
$$

[16]Note that the polynomial term is missing in equation (4.28) of [14].

$$+ \quad \frac{1}{8\pi^2}\sum_{r>0} c_1(kl)klLi_1(e^{-2\pi r\cdot y})\Big) + \frac{b_{grav}}{8\pi^2}\log\eta^{-2}(T)\eta^{-2}(U)$$

$$+ \quad \frac{2}{4\pi^2}\log(j(T)-j(U)) \tag{5.20}$$

Now, consider taking the limit $T \to \infty$ of (5.20) (keeping U finite). Then, using that $\log\eta^{-2}(T) \to \frac{\pi}{6}T, \log j(T) \to 2\pi T$, it is straightforward to show that

$$\mathcal{F}_{grav} \to 24S \tag{5.21}$$

Note that a possible linear T-dependence drops out in this limit! (5.21) is nothing but the holomorphic gravitational coupling of an $N=4$ heterotic compactification. Thus, it is suggestive to interprete the limit $\Re S > \Re T \to \infty$ in this model as a limit in which one obtains an $N=4$ like situation.

$N=4$ string/string/string triality, on the other hand, says that $N=4$ compactifications of the heterotic string and of the type II string are related through exchange symmetries $S \leftrightarrow T$ or $S \leftrightarrow U$ [19]. Thus $N=4$ string/string/string triality together with our discussion in section (3.3) then suggests that there might be an $S \leftrightarrow T, U$ exchange symmetry at the non-perturbative level in the $s=0$ model of [14]! As discussed in section (3.3), such an exchange symmetry is made possible due to those short $N=2$ BPS multiplets which from an $N=4$ point of view are intermediate BPS states. The direct evalution of non-perturbative corrections to $\mathcal{F}_{grav}$ is very hard, because in order to evaluate $\mathcal{F}_{grav} \propto (\sum^{\text{vector}}_{M_I,N^I} - \sum^{\text{hyper}}_{M_I,N^I})\log(M_I P^I + iN^I Q_I)$ knowledge of the non-perturbative period vector Ω is needed. The existence of an exchange symmetry $S \leftrightarrow T$, on the other hand, would allow one to produce quantitative statements about $\mathcal{F}_{grav}$ in a certain strong coupling regime.

How would such an exchange symmetry act on $\mathcal{F}_{grav}$? Consider first the 2 parameter model $P_{1,1,2,2,6}(12)$ of [21]. It was observed in [9] that this model indeed possesses an exchange symmetry $S \leftrightarrow T$ at the non-perturbative level. In this model, the holomorphic gravitational coupling $\mathcal{F}_1^{top} = \frac{2\pi^2}{3}\mathcal{F}_{grav}$ enjoys an instanton expansion of the following type

$$\mathcal{F}_1^{top} = -\frac{2\pi i}{12}c_2\cdot(B+iJ) - \sum_{j,k\geq 0}\left(2d_{jk}\log\eta(q_1^j q_2^k) + \frac{1}{6}n_{jk}\log(1-q_1^j q_2^k)\right) \tag{5.22}$$

where [21, 9] $q_i = e^{2\pi i t_i}$, $t_1 = -iT$, $t_2 = -i(\check{S}-T)$, $q_1^j q_2^k = e^{-2\pi T(j-k)}e^{-2\pi k\check{S}}$ and where $c_2\cdot(B+iJ) = 24t_2 + 52t_1 = -i(24\check{S}+28T)$. Here, $\check{S} = 4\pi S$ denotes the redefined dilaton which enjoys modular properties (see footnote 14). In the weak coupling limit $\Re T < \Re S \to \infty$, $q_1^j q_2^k \to 0$, and so

$$\mathcal{F}_1^{top} \to -\frac{2\pi i}{12}c_2\cdot(B+iJ) = \frac{2\pi}{12}\left(24\check{S}+28T\right) \tag{5.23}$$

This expression agrees with the large T limit of the known 1-loop expression [34]

$$\mathcal{F}_{grav} = 24S_{\rm inv} + \frac{1}{4\pi^2}\log(j(T)-j(1)) - \frac{300}{4\pi^2}\log\eta^2(T) \tag{5.24}$$

provided that $S_{\rm inv} = S + \sigma(T)$ goes in the limit $T \to \infty$ as $S_{\rm inv} \to S - \frac{1}{4\pi}T$.

In the strong coupling limit $\Re T > \Re S \to \infty$, on the other hand, one has that $q_1^j q_2^k$ is only non-vanishing for $k > j$. For $k > j$ one has that $d_{jk} = 0$, $n_{jk} = 2\delta_{j0}\delta_{k1}$. It follows that

$$\sum_{j,k\geq 0} n_{jk}\log(1-q_1^j q_2^k) = 2\log(1-q_2) = 2\log q_2 = 4i\pi t_2 = -4\pi(\check{S}-T) \tag{5.25}$$

Then

$$\mathcal{F}_1^{top} \to -\frac{2\pi i}{12}c_2\cdot(B+iJ) + \frac{4\pi}{6}(S-T) = \frac{2\pi}{12}\left(28\check{S}+24T\right) \tag{5.26}$$

Combining (5.23) and (5.26) gives that as $S\to\infty, T\to\infty$

$$\mathcal{F}_1^{top} = \frac{2\pi}{12}\left((24\check{S}+28T)\theta(\check{S}-T) + (28\check{S}+24T)\theta(T-\check{S})\right) \tag{5.27}$$

which exhibits the exchange symmetry $\check{S} \leftrightarrow T$. [17] Note that (5.27) exhibits an $\check{S}-T$ chamber dependence. Applying the $\check{S}\leftrightarrow T$ exchange symmetry on the 1-loop expression eq.(5.24) we obtain for large T but arbitrary S the non-perturbative gravitational coupling as

$$\mathcal{F}_{grav} = 24T_{\rm inv} + \frac{1}{4\pi^2}\log(j(\check{S})-j(1)) - \frac{300}{4\pi^2}\log\eta^2(\check{S}) \tag{5.28}$$

Let us then assume that the $s=0$ model of [14] also possesses an exchange symmetry $\check{S}\leftrightarrow T$ (and similarly for $\check{S}\leftrightarrow U$). Then, in view of (5.20), the non-perturbative $\mathcal{F}_{grav}$ should in the limit $T\to\infty$ be given to all orders in $\check{S}$ and U by

$$\begin{aligned}
\mathcal{F}_{grav} &= 24\left(\frac{T}{4\pi} - \frac{1}{768\pi^2}\partial_{\check{S}}\partial_U\left(\tilde{d}^{2,2}_{abc}\tilde{y}^a\tilde{y}^b\tilde{y}^c\right) - \frac{1}{8\pi^2}\log(j(\check{S})-j(U))\right.\\
&+ \left.\frac{1}{8\pi^2}\sum_{r>0}c_1(kl)klL_{i_1}(e^{-2\pi r\cdot\tilde{y}})\right) + \frac{b_{grav}}{8\pi^2}\log\eta^{-2}(\check{S})\eta^{-2}(U)\\
&+ \frac{2}{4\pi^2}\log(j(\check{S})-j(U))
\end{aligned} \tag{5.29}$$

where $\tilde{y} = (\check{S}, U)$. The logarithmic singularity at $\check{S}=U$ corresponds to the $SU(2)$ gauge symmetry enhancement along this line, which is further enhanced to $SU(2)^2$, $SU(3)$ at

[17] Taking $T\to\infty$ corresponds to decompactification [31] to 5 dimensions. In 5 dimensions there is a discontinuity at $t=1$ (where t is the 5D modulus) corresponding to the non-perturbative singularity at $\check{S}=T$ in 4 dimensions [31].

$\check{S} = U = 1$ and $\check{S} = U^{-1} = \rho$ respectively. It follows that $g^2_{\rm grav}$ is invariant under $SL(2,\mathbf{Z})_{\check{S}} \times SL(2,\mathbf{Z})_U \times \mathbf{Z}_2^{\check{S}\leftrightarrow U}$ for large T. Taking the limit $\check{S} \to \infty$ of (5.29) yields that

$$\mathcal{F}_{grav} \to \frac{6}{\pi} T \tag{5.30}$$

(5.30) gives the holomorphic gravitational coupling for an $N = 4$ compactification of the type IIA string. Combining both (5.21) and (5.30) yields that

$$\mathcal{F}_{grav} = \frac{6}{\pi} \left(T\theta(T - \check{S}) + \check{S}\theta(\check{S} - T) \right) \tag{5.31}$$

in analogy to (5.27).

Our results contain, as a subcase, the five dimensional results of [31] which discussed the behavior of the model in the two limits $\Re S > \Re T \to \infty$ and $\Re T > \Re S \to \infty$. In five dimensions the θ-function discontinuities in eq.(5.31) are again due to non-perturbative states becoming massless at $\check{S} = T$ and $\check{S} = U$ [31]. However the $\check{S} \leftrightarrow T$ exchange symmetry provides also information in the entire strong coupling region $T \to \infty$ for arbitrary $\check{S}$ (and U.) In particular this symmetry predicts the further gauge symmetry enhancement at the strong coupling points $\check{S} = U = 1$ and $\check{S} = U^{-1} = \rho$.

6 Conclusions

In this paper, we studied the BPS spectrum in $D = 4, N = 4$ heterotic string compactifications. These BPS states can either fall into short or into intermediate multiplets. As pointed out in [19], the string/string/string triality conjecture between $N = 4$ compactifications of the heterotic, the type IIA and the type IIB string implies, for instance, that the BPS spectrum of the heterotic and of the type IIA string are mapped into each other under the exchange $S \leftrightarrow T$. The BPS mass spectrum of the heterotic (type IIA) string is, however, not symmetric under this exchange of S and T. This is due to the fact that BPS masses in $D = 4, N = 4$ compactifications are given by the maximum of the 2 central charges $|Z_1|^2$ and $|Z_2|^2$. On the other hand, states, which from the $N = 4$ point of view are intermediate, are actually short from the $N = 2$ point of view. This then leads to the possibility that the BPS spectrum of certain $N = 2$ heterotic compactifications is actually symmetric under the exchange of S and T. Since contributions to the holomorphic gravitational coupling $\mathcal{F}_{grav}$ arise from BPS states only (as shown in [14] for 1-loop contributions), it follows that $\mathcal{F}_{grav}$ should exhibit a symmetry under the exchange of S and T. As an example of an $N = 2$ compactification we took the $s = 0$ model of [14] and computed the exact 1-loop contribution to the holomorphic gravitational coupling $\mathcal{F}_{grav}$ using the technology introduced in [14]. We then showed that in the decompactification limit $T \to \infty$ at weak coupling one recovers the tree level holomorphic gravitational coupling. This $N=4$ like situation then suggests that the $N=4$ triality exchange symmetries are actually realised as exchange symmetries $S \leftrightarrow T$ and $S \leftrightarrow U$ in the $s=0$ $N=2$ heterotic model. Assuming that there are indeed such exchange symmetries in the $s=0$ model allows one to evaluate non-perturbative corrections to the gravitational couplings in some of the non-perturbative regions (chambers) in this particular heterotic model.

7 Acknowledgement

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8 Appendix A

We will, in this appendix, discuss orbits and invariants of duality groups. For many purposes, like the computation of the BPS sums in the previous sections, it is very useful to consider subsets of BPS states which fall into socalled orbits of the duality group. An orbit of a group on a set is a subset that is invariant under the group action. To divide a set into orbits one must therefore take group invariant constraints. We are interested in finding orbits of the group

$$SL(2,\mathbf{Z})_S \otimes SL(2,\mathbf{Z})_T \otimes SL(2,\mathbf{Z})_U \otimes \mathbf{Z}_2^{(\mathrm{Mirror})} \tag{8.1}$$

and some of its subgroups on the set of (short or intermediate) BPS states. Here $\mathbf{Z}_2^{(\mathrm{Mirror})}$ denotes the perturbative $\mathbf{Z}_2$ group which permutes the two moduli of the theory under consideration, i.e. $T \leftrightarrow U$ for the heterotic theory. In the following we will for definiteness always deal with the $N=4$ heterotic string, if not specified otherwise.

As pointed out in eq.(2.12) and below, the quantities

$$\begin{pmatrix} \hat{M}_2 \\ \hat{M}_0 \end{pmatrix}, \quad \begin{pmatrix} \hat{M}_1 \\ \hat{M}_3 \end{pmatrix}, \quad \begin{pmatrix} \hat{N}_2 \\ \hat{N}_0 \end{pmatrix}, \quad \begin{pmatrix} \hat{N}_1 \\ \hat{N}_3 \end{pmatrix} \tag{8.2}$$

transform as $SL(2,\mathbf{Z})_T$ vectors, i.e. by multiplication with $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Analogously, the quantities

$$\begin{pmatrix} \hat{M}_3 \\ \hat{M}_0 \end{pmatrix}, \quad \begin{pmatrix} \hat{M}_1 \\ \hat{M}_2 \end{pmatrix}, \quad \begin{pmatrix} \hat{N}_3 \\ \hat{N}_0 \end{pmatrix}, \quad \begin{pmatrix} \hat{N}_1 \\ \hat{N}_2 \end{pmatrix} \tag{8.3}$$

transform as $SL(2,\mathbf{Z})_U$ vectors. The non–perturbative $SL(2,\mathbf{Z})_S$ acts by

$$M_S = \begin{pmatrix} d \cdot \mathbf{1}_4 & b \cdot \mathbf{1}_4 \\ c \cdot \mathbf{1}_4 & a \cdot \mathbf{1}_4 \end{pmatrix}, \tag{8.4}$$

where the representation space is now spanned by the vectors $\mathbf{V} = (\hat{M}_0, \ldots, \hat{N}_3)$ consisting of all electric and magnetic quantum numbers. In other words $(\hat{M}_I, \hat{N}_I)$ transforms as a $SL(2,\mathbf{Z})_S$ vector for fixed I.

Let us first discuss orbits and invariants of a single $SL(2,\mathbf{Z})$, which for definiteness we take to be $SL(2,\mathbf{Z})_S$. The eight–dimensional representation on the quantum numbers is of course reducible and decomposes into the four irreducible two–dimensional representations specified above. We begin by looking at orbits and invariants associated to an irreducible two–dimensional representation. In order to characterize orbits, we would like to construct invariants out of vectors $\mathbf{v}$, which could then lable the orbits. As is well known the only invariant tensor of the corresponding continuous group $SL(2,\mathbf{R})$ is the ϵ tensor and the related invariant is nothing but the antisymmetric scalar product

$$(\mathbf{v}, \mathbf{w}) = \epsilon^{ij} v_i w_j \tag{8.5}$$

Due to antisymmetry we cannot construct a non–trivial invariant out of a single vector, since $(\mathbf{v}, \mathbf{v}) = 0$. How then characterize orbits? First note that the vector $(1,0)^T$ can be mapped to any other vector $\mathbf{v} \neq 0$ by an $SL(2,\mathbf{R})$ transformation. Therefore the continuous group has precisely two orbits, namely the zero vector $\{\mathbf{v} = 0\}$ and the punctured plane $\{\mathbf{v} \neq 0\}$. The non–existence of a non–trivial invariant associated to a single vector reflects the fact that all vectors $\mathbf{v} \neq 0$ are related by group transformation. Conversely groups like $SO(2)$ where one has such invariants (the length) have orbits that are labled by the invariant (circles of a given radius).

Clearly the orbit $\{\mathbf{v} \neq 0\}$ becomes highly reducible, when switching to the discrete group $SL(2,\mathbf{Z})$. To see this just note that $(p,0)^T$ and $(q,0)^T$, $p,q \in \mathbf{Z}$ cannot be related by a $SL(2,\mathbf{Z})$ for coprime p,q. However the discrete version of $\mathbf{v} \neq 0$, namely $\{(p,q) \neq$

$(0,0)|p,q \in \mathbf{Z}\}$ is precisely the kind of orbit that one needs, since various modular forms including all Eisenstein series and (using ζ regularization) the Dedeking η function can be expressed as sums over a two dimensional lattice with the origin excluded.

Let us next discuss the reducible representation of $SL(2,\mathbf{Z})_S$ on the eight electric and magnetic quantum numbers $\hat{M}_I, \hat{N}_I$. Since this decomposes into four irreducible representations we can now construct six non–trivial (this means generically non–vanishing) invariants by taking mutual scalar products between the various irreducible parts. These invariants $\hat{M}_I\hat{N}_J - \hat{N}_I\hat{M}_J$, $I<J$ can be arranged into an antisymmetric invariant matrix:

$$\hat{M}_I\hat{N}_J - \hat{M}_J\hat{N}_I =: C_{IJ} \tag{8.6}$$

Note that this matrix is in fact the exterior product of the electric and magnetic part of the vector of quantum numbers:

$$\hat{\mathbf{M}} \wedge \hat{\mathbf{N}} = \mathbf{C} \tag{8.7}$$

Further note that this product vanishes if and only if the electric and magnetic part are parallel:

$$\hat{\mathbf{M}} \wedge \hat{\mathbf{N}} = 0 \Longleftrightarrow \vec{P}^{\mathrm{het}}||\vec{Q}^{\mathrm{het}} \tag{8.8}$$

The groups $SL(2,\mathbf{Z})_T$ and $SL(2,\mathbf{Z})_U$ can be treated in a similar way. The simplest way to obtain the corresponding invariants is to apply the duality transformations $S \leftrightarrow T$ and $S \leftrightarrow U$, respectively. Obviously the six non–trivial invariants of $SL(2,\mathbf{Z})_T$ ($SL(2,\mathbf{Z})_U$) vanish simultanously if and only if electric and magnetic vector of the IIA (IIB) theory are parallel.

Let us now consider orbits and invariants for products of two $SL(2,\mathbf{Z})$ groups. For defineteness we will take the T duality group $SL(2,\mathbf{Z})_T \otimes SL(2,\mathbf{Z})_U$ of the heterotic string. The eight–dimensional representation space splits under this group into two irreducible four–dimensional representations spanned by the electric and magnetic parts. The invariant tensor $\epsilon \otimes \epsilon$ is represented by a symmetric matrix on each irreducible part which is easily found to be conjugated to the standard $SO(2,2)$ invariant metric, as expected from the local isomorphism $SL(2,\mathbf{R}) \otimes SL(2,\mathbf{R}) \simeq SO(2,2)$. The corresponding invariant is the $SO(2,2)$ scalarproduct $\langle , \rangle$, which reads in our parametrization:

$$\langle \mathbf{V}, \mathbf{W} \rangle = V_0 W_1 + V_1 W_0 - V_3 W_2 - V_2 W_3 \tag{8.9}$$

Therefore one can construct three non–trivial invariants out of the quantum numbers $(\hat{\mathbf{M}}, \hat{\mathbf{N}})^T$ namely the scalar products $\langle \hat{\mathbf{M}}, \hat{\mathbf{M}} \rangle$, $\langle \hat{\mathbf{N}}, \hat{\mathbf{N}} \rangle$ and $\langle \hat{\mathbf{M}}, \hat{\mathbf{N}} \rangle$. $SO(2,2)$ orbits of the form $\langle \hat{\mathbf{M}}, \hat{\mathbf{M}} \rangle = \mathit{const}$ play an important role in perturbative threshold corrections and they are indeed related to $SO(2,2,\mathbf{Z})$ modular forms. Orbits of the form $\langle \hat{\mathbf{M}}, \hat{\mathbf{N}} \rangle = \mathit{const}$

also play some role because they appear as suborbits of $SL(2,\mathbf{Z})_S \otimes SL(2,\mathbf{Z})_T \otimes SL(2,\mathbf{Z})_U$ orbits. Finally note that the $SO(2,2)$ scalar product is also manifestly invariant under the perturbative heterotic mirror symmetry $\mathbf{Z}_2^{(T\leftrightarrow U)}$ and therefore it is invariant under the full perturbative heterotic duality group $SL(2,\mathbf{Z})_T \otimes SL(2,\mathbf{Z})_U \otimes \mathbf{Z}_2^{(T\leftrightarrow U)}$.

Let us now discuss orbits and invariants of the full duality group $SL(2,\mathbf{Z})_S \otimes SL(2,\mathbf{Z})_T \otimes SL(2,\mathbf{Z})_U \times \mathbf{Z}_2^{(T\leftrightarrow U)}$. Under this group our eight–dimensional representation is irreducible and since the invariant tensor $\epsilon \otimes \epsilon \otimes \epsilon$ is antisymmetric we cannot construct an invariant out of the vector $(\hat{\mathbf{M}}, \hat{\mathbf{N}})^T$. This situation is similar to that of the irreducible two–dimensional representation of a single $SL(2,\mathbf{Z})$. However the non–existence of an invariant **number** that one can assign to an orbit does not mean that there are no invariant **equations** that characterize orbits. A closer inspection shows that the six $SL(2,\mathbf{Z})_S$ invariants C_{IJ} are not invariant under $SL(2,\mathbf{Z})_T \otimes SL(2,\mathbf{Z})_U \otimes \mathbf{Z}_2^{(T\leftrightarrow U)}$ for generic values, but that they are invariant if and only if they vanish. Thus either $C_{IJ} = 0$ or $C_{IJ} \neq 0$ are invariant equations which decompose the representation space into disjoint orbits. These conditions are the analogue of $\mathbf{v} = 0$, $\mathbf{v} \neq 0$ in the case of a single $SL(2,\mathbf{Z})$. In geometrical terms one can say that although the 'angle' between the electric and magnetic part is not preserved, parallelity is respected.

Let us therefore summarize that the condition

$$\hat{\mathbf{M}} \wedge \hat{\mathbf{N}} = 0 \tag{8.10}$$

defines an orbit of the full group $SL(2,\mathbf{Z})_S \otimes SL(2,\mathbf{Z})_T \otimes SL(2,\mathbf{Z})_U \otimes \mathbf{Z}_2^{(T\leftrightarrow U)}$, which is singled out by (i) the simultanous vanishing of all $SL(2,\mathbf{Z})_S$ invariants and (ii) by the parallel alignement of the electric and magnetic quantum numbers. This orbit, which we call the S orbit, is clearly a special, non–generic subset of vectors. Note that it contains the short $N = 4$ BPS multiplets of the heterotic theory. Note also that $\hat{\mathbf{M}}$ and $\hat{\mathbf{N}}$ being parallel implies that the quantum numbers are pairwise proportional, that is $s\hat{M}_I = p\hat{N}_I$, $(\exists p, s \in \mathbf{Z})$.

Obviously we can construct two further distinguished orbits of $SL(2,\mathbf{Z})_S \otimes SL(2,\mathbf{Z})_T \otimes SL(2,\mathbf{Z})_U$[18] by applying the transformations $S \leftrightarrow T$ and $S \leftrightarrow U$ to the S orbit. The resulting orbits will be called the T and the U orbit. They are singled out by the simultanous vanishing of the six $SL(2,\mathbf{Z})_T$ $(SL(2,\mathbf{Z})_U)$ invariants and by parallel alignement of the electric and magnetic quantum numbers of the IIA (IIB) theory. Denoting these electric and magnetic quantum numbers by

$$\hat{\mathbf{M}}^{(A)} = (\hat{M}_0, \hat{N}_3, \hat{N}_0, \hat{M}_3)^T,\ \hat{\mathbf{N}}^{(A)} = (\hat{M}_2, \hat{N}_1, \hat{N}_2, \hat{M}_1)^T, \tag{8.11}$$

[18]The $\mathbf{Z}_2^{(\mathrm{Mirror})}$ symmetries will be discussed below.

$$\hat{\mathbf{M}}^{(B)} = (\hat{M}_0, \hat{N}_2, \hat{M}_2, \hat{N}_0)^T, \ \hat{\mathbf{N}}^{(B)} = (\hat{M}_3, \hat{N}_1, \hat{M}_1, \hat{N}_3)^T, \tag{8.12}$$

the T and the U orbit are characterized by

$$\hat{\mathbf{M}}^{(A)} \wedge \hat{\mathbf{N}}^{(A)} = 0, \quad \hat{\mathbf{M}}^{(B)} \wedge \hat{\mathbf{N}}^{(B)} = 0, \tag{8.13}$$

respectively. The perturbative mirror symmetry $\mathbf{Z}_2^{(T \leftrightarrow U)}$ of the heterotic string is mapped to the corresponding perturbative mirror symmetries $\mathbf{Z}_2^{(S \leftrightarrow U)}$ ($\mathbf{Z}_2^{(S \leftrightarrow T)}$) of the IIA (IIB) theory by the non–perturbative duality transformations $S \leftrightarrow T$ and $S \leftrightarrow U$. Thus the T orbit (U orbit) is invariant under the full duality group $SL(2,\mathbf{Z})_S \otimes SL(2,\mathbf{Z})_T \otimes SL(2,\mathbf{Z})_U \otimes \mathbf{Z}_2^{(S \leftrightarrow U)}$ ($SL(2,\mathbf{Z})_S \otimes SL(2,\mathbf{Z})_T \otimes SL(2,\mathbf{Z})_U \otimes \mathbf{Z}_2^{(S \leftrightarrow T)}$) of the IIA (IIB) string.

Another even more special orbit is given by the constraint that the non–trivial invariants of all three $SL(2,\mathbf{Z})$ groups vanish simultanously. This gives the intersection of the S, T, U orbits and will therefore be called the STU orbit. The states in it fulfill

$$\hat{M}_0 \hat{M}_1 = \hat{M}_2 \hat{M}_3 \tag{8.14}$$

on top of $\hat{M}_I$, $\hat{N}_I$ being proportional.

One could also try to define orbits by setting the invariants of only two $SL(2,\mathbf{Z})$ subgroups simultanously to zero. But it turns out that then the invariants of the third $SL(2,\mathbf{Z})$ are automatically also zero and we are back at the STU orbit.

Finally note that the 0 vector is trivially an invariant suborbit of the STU orbit, and that the STU orbit is itself an invariant suborbit of the S,T and U orbit. Therefore disjoint invariant orbits are given bei 0, STU - 0, S - STU, T - STU, U - STU.

9 Appendix B

In this appendix we investigate at which points in the $N = 4$ heterotic moduli space one can obtain, at least in principle, massless intermediate spin 3/2 BPS states. Clearly, at the special points of massless intermediate states one has that $|Z_1|^2 = 0$ and $\Delta Z^2 = 0$. In case that $S + \bar{S} \neq 0$ this further implies that

$$\begin{aligned} \hat{M}_0 - \hat{M}_1 TU + i\hat{M}_2 T + i\hat{M}_3 U &= 0, \\ \hat{N}_0 - \hat{N}_1 TU + i\hat{N}_2 T + i\hat{N}_3 U &= 0. \end{aligned} \tag{9.1}$$

Thus intermediate multiplets may become massless at special lines/points in the T, U moduli space for generic values of S. First consider the line $T = U$. It follows from (9.1) that the only states becoming massless at this line are the states having $\hat{M}_2 = -\hat{M}_3$, $\hat{N}_2 = -\hat{N}_3 = 0$, $\hat{M}_0 = \hat{M}_1 = \hat{N}_0 = \hat{N}_1 = 0$. However, for these states $\hat{M} \propto \hat{N}$, so that these states are actually short, and not intermediate.

Next consider intermediate states becoming massless at the point $T = U = 1$. These are the states for which $\mathcal{M}_{1,2} = 0$ at $T = U = 1$:

$$
\begin{aligned}
\hat{M}_0 - \hat{M}_1 + i(\hat{M}_2 + \hat{M}_3) &= 0 \\
\hat{N}_0 - \hat{N}_1 + i(\hat{N}_2 + \hat{N}_3) &= 0
\end{aligned} \qquad (9.2)
$$

Then, the only intermediate states $\mathbf{V} = (\hat{M}_0, \ldots, \hat{N}_3)$ satisfying (9.2) are as follows (we rescrict the non-vanishing charges to be ± 1):

$$
\begin{aligned}
a)\ \hat{M}_2 &= -\hat{M}_3 = \pm 1, \hat{N}_0 = \hat{N}_1 = \pm 1 \\
b)\ \hat{M}_2 &= -\hat{M}_3 = \pm 1, \hat{N}_0 = \hat{N}_1 = \pm 1, \hat{N}_2 = -\hat{N}_3 = \pm 1 \\
c)\ \hat{M}_0 &= \hat{M}_1 = \pm 1, \hat{N}_2 = -\hat{N}_3 = \pm 1 \\
d)\ \hat{M}_0 &= \hat{M}_1 = \pm 1, \hat{M}_2 = -\hat{M}_3 = \pm 1, \hat{N}_2 = -\hat{N}_3 = \pm 1 \\
e)\ \hat{M}_0 &= \hat{M}_1 = \pm 1, \hat{M}_2 = -\hat{M}_3 = \pm 1, \hat{N}_0 = \hat{N}_1 = \pm 1 \\
f)\ \hat{M}_0 &= \hat{M}_1 = \pm 1, \hat{N}_0 = \hat{N}_1 = \pm 1, \hat{N}_2 = -\hat{N}_3 = \pm 1 \\
g)\ \hat{M}_0 &= \hat{M}_1 = \pm 1, \hat{M}_2 = -\hat{M}_3 = \pm 1, \hat{N}_0 = \hat{N}_1 = \mp 1, \hat{N}_2 = -\hat{N}_3 = \pm 1
\end{aligned} \qquad (9.3)
$$

Next consider the point $T = \bar{U} = \rho$. Here dyons become massless with the following electric magnetic charge vectors

$$
\begin{aligned}
\pm(1,1,0,0;1,0,1,-1) \quad &, \quad \pm(1,1,0,0;-1,0,-1,1), \quad \pm(1,1,0,0;0,-1,1,-1), \\
\pm(1,1,0,0;0,1,-1,1) \quad &, \quad \pm(1,0,1,-1;1,1,0,0), \quad \pm(1,0,1,-1;-1,-1,0,0), \\
\pm(1,0,1,-1;0,-1,1,-1) \quad &, \quad \pm(1,0,1,-1;0,1,-1,1), \quad \pm(0,1,-1,1;1,1,0,0), \\
\pm(0,1,-1,1;-1,-1,0,0) \quad &, \quad \pm(0,1,-1,1;-1,0,-1,1), \\
\pm(0,1,-1,1;1,0,1,-1) \quad &.
\end{aligned} \qquad (9.4)
$$

Next, let us discuss the possible appearance of massless intermediate multiplets for the case of strong couplings, i.e. $S_1 = \mathrm{Re}S = 0$. (Of course, via S-duality one could equivalently consider weak coupling.) Then one gets massless intermediate states if the following condition is satisfied ($S_2 = \mathrm{Im}S$)

$$
\hat{M}_0 - \hat{M}_1 TU + i\hat{M}_2 T + i\hat{M}_3 U = S_2(\hat{N}_0 - \hat{N}_1 TU + i\hat{N}_2 T + i\hat{N}_3 U). \qquad (9.5)
$$

Now consider one of the intermediate states given in (9.3), namely the state $(1,1,0,0;0,0,-1,1)$. In this strong coupling limit this state is not only massless at $T=U=1$ but also on the following critical line

$$U=-i\frac{1+iS_2T}{S_2-iT}. \tag{9.6}$$

This line contains the point $(T,U)=(1,1)$ for all possible values of S_2. For $S_2=0$ this line becomes $T=1/U$. For $S_2=1$ one also obtains a special line in the T,U moduli space: consider, for example, $T_2=\mathrm{Im}T=0$. Then U lies on the unit circle, $\mathrm{Re}U^2+\mathrm{Im}U^2=1$, which is the boundary of the U moduli space. A similar discussion holds for all the other states in eq.(9.3). The associated critical line is obtained from (9.6) by a corresponding T/U-duality transformation.

Next consider a state listed in (9.4), for example $(1,1,0,0;1,0,1,-1)$. In the strong coupling limit it is massless at the line

$$U=\frac{1-S_2-iS_2T}{T-iS_2}, \tag{9.7}$$

which contains the point $T=\bar{U}=\rho$. For $S_2=0$ it becomes $U=1/T$, and for $S_2=1$ this relation is again satisfied for U lying on the unit circle if T lies on the boundary of the moduli space, i.e. $T_2=1/2$.

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