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QCD Pressure at Two Loops in the Temporal Gauge

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Abstract

We apply the method of zeta functions, together with the n^*_μ-prescription for the temporal gauge, to evaluate the thermodynamic pressure in QCD at finite temperature T. Working in the imaginary-time formalism and employing a special version of the unified-gauge prescription, we show that the pure-gauge contribution to the pressure at two loops is given by $P_2^{\text{gauge}} = -(g^2/144)N_c N_g T^4$, where N_c and N_g denote the number of colours and gluons, respectively. This result agrees with the value in the Feynman gauge.

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1 Introduction

The temporal gauge is a physical, i.e. ghost-free gauge which belongs to the class of axial-type gauges characterized by the constraint

$$n^\mu A^a_\mu(x) = 0, \quad \mu = 0, 1, 2, 3, \ldots, \tag{1}$$

where $A^a_\mu(x)$ is a massless Yang-Mills field, and $a = 1, 2, \ldots, N^2 - 1$, for SU(N). These axial-type gauges also include the pure axial gauge ($n^2 < 0$), the planar gauge ($n^2 < 0$), and the light-cone gauge ($n^2 = 0$). In the temporal gauge, the fixed vector n_μ is time-like: $n^2 = n_0^2 - \mathbf{n}^2 > 0$. Today, the temporal gauge is as useful as it was 65 years ago in the quantization of the Maxwell-Dirac field by Weyl in 1929 [**?**] and Heisenberg and Pauli in 1930 [**?**]. Only the degree of complexity has increased: today's temporal gauge is applied in such sophisticated areas as the vacuum tunneling by instantons [**?**], Nicolai maps [**?**, **?**], one-loop thermodynamic potentials [**?**], and especially in quark-gluon plasma studies [**?**] at *finite* temperature T. Of course, there are good reasons for this popularity.

Due to the space-time asymmetry at finite temperature in the imaginary-time formalism, the thermodynamic equations are no longer symmetric in 4-space, but are only invariant under spatial rotations. Accordingly, since the temporal gauge-fixing condition $n_0 A^a_0(x) = 0$ leaves the spatial components of $A^a_\mu(x)$ unconstrained, the temporal gauge may indeed be regarded as the "appropriate gauge" for this particular boundary-value problem.

During the last ten years, finite-temperature calculations in the temporal gauge have been performed both in the real-time formalism and in the imaginary-time formalism. Kapusta and Kajantie [**?**], for instance, employed the imaginary-time formalism to evaluate the leading terms in the $1/T$-expansion for the two-point function (static limit). The finite-temperature case was also examined by Landshoff and James [**?**], as well as by Brandt, Frenkel and Taylor [**?**]. Working in the Feynman gauge, Brandt, Frenkel and Taylor utilized the method of ζ-functions to derive the complete $1/T$-expansion for the one-loop Yang-Mills self-energy.

In 1994, the present authors developed a general procedure [**?**] for doing perturbative calculations in the temporal gauge at both zero ($T = 0$) and finite temperature ($T \neq 0$). The procedure hinged on a special version of the n^*_μ-prescription, originally designed for the spurious poles of the temporal-gauge propagator (cf. Eq. (2)). No problems were encountered for zero temperature, but for $T \neq 0$ a different approach was needed. In order to simplify the computation of Matsubara frequency sums, it became necessary to replace the traditional contour method by the method of zeta functions. The latter method is generally known to facilitate the computation of high-temperature expansions to any order in $1/T$.

Application of the ζ-function technique, together with the n^*_μ-prescription for the temporal gauge, enabled us to evaluate the complete $1/T$-expansion for the self-energy $\Pi^{ab}_{00}(k_4 = 0, \mathbf{k})$[?], and to derive the Debye chromo-electric screening length $m_{\rm el}$ in the infrared limit, $m^2_{\rm el} \equiv \Pi^{ab}_{00}(0, \mathbf{k})\big|_{\mathbf{k}^2=m^2_{\rm el}}$[?].

The purpose of the present article is to apply this successful procedure [?] to another important quantity, namely the thermodynamic pressure. Working in the imaginary-time formalism, we shall evaluate the thermodynamic pressure at finite temperature to one and two loops.

2 Basic Tools

The temporal-gauge propagator for Yang-Mills theory at finite temperature reads as follows [?, ?]:

$$G^{ab}_{\mu\nu}(p) = \frac{-i\delta^{ab}}{(2\pi)^{2\omega}(p^2+i\epsilon)}\Big[g_{\mu\nu} - \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n} + \frac{n^2 p_\mu p_\nu}{(p\cdot n)^2}\Big], \quad n^2 > 0,\ \epsilon > 0, \tag{2}$$

where 2ω denotes the dimensionality of complex space-time; in four dimensions, $g_{\mu\nu} =$ diag(+1, -1, -1, -1), $\mu, \nu = 0, 1, 2, 3$. For $n_\mu = (1, 0, 0, 0)$, the propagator (2) reduces to

$$\begin{aligned} G^{ab}_{ij}(p) &= \frac{-i\delta^{ab}}{(2\pi)^{2\omega}(p^2+i\epsilon)}\Big[-\delta_{ij} + \frac{p_i p_j}{(p_0)^2}\Big], \quad i, j = 1, 2, 3, \\ G^{ab}_{00}(p) &= G^{ab}_{0i}(p) = G^{ab}_{i0}(p) = 0. \end{aligned} \tag{3}$$

The spurious double pole at $p_0 = 0$ in Eq. (3) may be treated by a special version of the unified-gauge prescription for axial-type gauges [?, ?], namely

$$\begin{aligned} \frac{1}{p_0}\Big|^{\rm temp} &= \lim_{\epsilon\to 0}\frac{p_0}{p_0^2+i\epsilon}, \quad \epsilon > 0; \\ \frac{1}{(p_0)^2}\Big|^{\rm temp} &= \lim_{\epsilon\to 0}\Big(\frac{p_0}{p_0^2+i\epsilon}\Big)^2, \quad \epsilon > 0. \end{aligned} \tag{4}$$

Note that prescription (4) is causal and well defined for all $p_0 = 2\pi i n T$, $n = 0, 1, 2, \ldots$, T being the temperature. In particular, no ambiguities arise for the case $n = 0$.

In addition to the formulas (3) and (4), loop calculations at finite temperature also require the replacement

$$\int \frac{d^4p}{(2\pi)^4} \Longrightarrow \frac{i}{\beta} \sum_{n=-\infty}^{n=+\infty} \int \frac{d^3\mathbf{p}}{(2\pi)^3}. \tag{5}$$

Here $\beta \equiv 1/(kT)$, and $p_0 \equiv i\omega_n$, with $\omega_n = 2\pi nT$; it is customary to equate Boltzmann's constant k to unity. Finally, we note that the components of $n_\mu = (n_0, n_1, n_2, n_3)$ have the following structure in the temporal gauge [**?**]:

$$\begin{aligned} n_\mu &= (n_0, \mathbf{n}_\perp, -i|\mathbf{n}_\perp|), \quad n_0 \neq 0, \;\; \mathbf{n}_\perp = (n_1, n_2); \\ n^*_\mu &= (n_0, \mathbf{n}_\perp, +i|\mathbf{n}_\perp|), \end{aligned} \tag{6}$$

so that

$$\lim_{|\mathbf{n}_\perp| \to 0} n_\mu = (1,0,0,0), \quad n_0 \equiv 1. \tag{7}$$

3 The Q.C.D. Pressure to One Loop

Since the pressure P is defined in terms of the partition function Z,

$$P = T\frac{\partial \ln Z}{\partial V}, \quad T \equiv \text{temperature}, \quad V \equiv \text{volume}, \tag{8}$$

we first need to calculate Z for pure gauge theory [**?**]:

$$Z_{\text{gauge}} = \int [dA_\mu] \det(n\cdot\partial) e^S; \tag{9}$$

[ ] denotes a product over all field configurations, with the action S given by

$$\begin{aligned} S &= \int_0^\beta d\tau \int d^3x \Big[-\frac{1}{4} F^{\mu\nu}F_{\mu\nu} - \frac{(n\cdot A)^2}{2\alpha} \Big], \\ &= \frac{1}{2}\int_0^\beta d\tau \int d^3x A^\mu \Big(\partial^2 \delta_{\mu\nu} - \partial_\mu\partial_\nu - \frac{n_\mu n_\nu}{\alpha\beta^2} \Big) A^\nu. \end{aligned} \tag{10}$$

We shall keep n_μ general for now (cf. Eqs. (6)), taking the limit $|\mathbf{n}_\perp| \to 0$ at a later time. Notice that periodicity of the boundary condition causes the boundary term in Eq. (10) to disappear. Introducing the Grassmann variables C and $\bar{C}$ ($\bar{C}$ is the hermitian conjugate of C), we may write the determinant $\det(n\cdot\partial)$ as follows:

$$\det(n\cdot\partial) = \int [d\bar{C}][dC] \exp\Big(\int_0^\beta d\tau \int d^3x \bar{C}\partial\cdot nC \Big). \tag{11}$$

It is convenient now to apply a discrete Fourier transformation in Euclidean space (the continuous limit will be taken later in the calculation):

$$C(\mathbf{x},\tau)=\frac{1}{V^{\frac{1}{2}}}\sum_n\sum_{\mathbf{p}}e^{i(\mathbf{p}\cdot\mathbf{x}+\omega_n\tau)}\tilde{C}_n(\mathbf{p}),\quad \omega_n=(2n+1)\pi T, \tag{12}$$

where $\tilde{C}_n(\mathbf{p})$ is the Fourier transform of $C(\mathbf{x},\tau)$, $\mathbf{p}$ and n label the momentum and energy quantum numbers, respectively, and where the volume factor V is necessary for proper normalization of the transformation. The frequency ω_n is odd here, because C is a fermion field. For the gauge field $A^\mu(\mathbf{x},\tau)$, we have:

$$A^\mu(\mathbf{x},\tau)=\sqrt{\frac{\beta}{V}}\sum_n\sum_{\mathbf{p}}e^{i(\mathbf{p}\cdot\mathbf{x}+\omega_n\tau)}A^\mu_n(\mathbf{p}),\quad \omega_n=2n\pi T. \tag{13}$$

The extra factor of $\sqrt{\beta}$ is needed to keep the action dimensionless. Thus

$$\begin{aligned}
Z_{\text{gauge}} &= \int[dA^\mu][d\bar{C}][dC]\exp\Big\{\int_0^\beta d\tau\int d^3x\bar{C}n\cdot\partial C\Big\} \\
&\quad \exp\Big\{\int_0^\beta d\tau\int d^3x\frac{\beta}{2V}\sum_{n,\mathbf{p}}\sum_{n',\mathbf{p}'}e^{i(\omega_n+\omega_{n'})\tau}e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}}A^\mu_n(\mathbf{p})A^\nu_{n'}(\mathbf{p}') \\
&\quad \Big(-p^2\delta_{\mu\nu}+p_\mu p_\nu-\frac{n_\mu n_\nu}{\alpha\beta^2}\Big)\Big\},
\end{aligned} \tag{14}$$

$$\begin{aligned}
&= \int[dA^\mu][d\bar{C}][dC]\exp\Bigg\{\int_0^\beta d\tau\int d^3x\frac{1}{V}\sum_{n,\mathbf{p}}\sum_{n',\mathbf{p}'}e^{-i(\mathbf{p}\cdot\mathbf{x}+\omega_n\tau)} \\
&\quad \bar{\tilde{C}}_n(\mathbf{p})(i\mathbf{p}')e^{i(\mathbf{p}'\cdot\mathbf{x}+\omega'_n\tau)}\tilde{C}_{n'}(\mathbf{p}')\Bigg\}\exp\Bigg\{\frac{\beta}{2V}(\beta V)\sum_{n,\mathbf{p}}A^\mu_n(\mathbf{p})A^\nu_{-n}(-\mathbf{p}) \\
&\quad \Big(-p^2\delta_{\mu\nu}+p_\mu p_\nu-\frac{n_\mu n_\nu}{\alpha\beta^2}\Big)\Bigg\},
\end{aligned} \tag{15}$$

$$\begin{aligned}
&= \int[d\bar{C}][dC]\exp\Bigg\{\frac{\beta V}{V}\sum_{n,\mathbf{p}}\bar{\tilde{C}}_n(\mathbf{p})(i\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}+\omega_n\tau)}\tilde{C}_n(\mathbf{p})\Bigg\} \\
&\quad \int[dA^\mu]\exp\Bigg\{\frac{1}{2}\beta^2\sum_{n,\mathbf{p}}A^\mu_n(\mathbf{p})A^{\nu*}_n(\mathbf{p}) \\
&\quad \Big(-p^2\delta_{\mu\nu}+p_\mu p_\nu-\frac{n_\mu n_\nu}{\alpha\beta^2}\Big)\Bigg\},
\end{aligned} \tag{16}$$

where we have used $A^\mu_{-n}(-\mathbf{p})=A^{\mu*}_n(\mathbf{p})$, since $A^\mu(\mathbf{x},\tau)$ is real. After further simplification, we get:

$$\begin{aligned}
Z &= \int[d\bar{C}][dC]\exp\{(\bar{C},FC)\}\int[dA^\mu]\exp\{-\frac{1}{2}(A^\mu D_{\mu\nu}A^\nu)\}, \\
&= \det(F)(\det(D_{\mu\nu}))^{-\frac{1}{2}};
\end{aligned} \tag{17}$$

here

$$F = i\beta p\cdot n,$$
$$D_{\mu\nu} = (p^2\delta_{\mu\nu} - p_\mu p_\nu + \frac{n_\mu n_\nu}{\alpha\beta^2})\beta^2. \tag{18}$$

The determinants are calculated by summing over space-time indices as well as field indices. Further manipulation leads to

$$\begin{aligned}\ln Z &= \ln\det(\beta p\cdot n) - \frac{1}{2}\ln\det(D_{\mu\nu}),\\ &= \ln\det(\beta\omega_n) - \frac{1}{2}\ln\det(D_{\mu\nu}),\end{aligned} \tag{19}$$

with

$$D_{\mu\nu} = \begin{bmatrix} p^2 - p_0^2 + \frac{1}{\alpha\beta^2} & -p_0p_1 & -p_0p_2 & -p_0p_3 \\ -p_0p_1 & p^2 - p_1^2 & -p_1p_2 & -p_1p_3 \\ -p_0p_2 & -p_1p_2 & p^2 - p_2^2 & -p_2p_3 \\ -p_0p_3 & -p_1p_3 & -p_2p_3 & p^2 - p_3^2 \end{bmatrix}\beta^2. \tag{20}$$

In the temporal gauge,

$$\det(D_{\mu\nu}) = \frac{1}{\alpha}\beta^6 p^4 \omega_n^2. \tag{21}$$

Hence

$$\begin{aligned}\ln Z &= \ln\det(\beta\omega_n) - \frac{1}{2}\ln\det(\frac{1}{\alpha}\beta^6 p^4\omega_n^2),\\ &= Tr\ln(\beta^2\omega_n^2) - Tr\ln(\frac{1}{\alpha}\beta^6 p^4\omega_n^2),\\ &= \Big(\prod_n\prod_{\mathbf{p}}[\beta^2(\omega_n^2 + \mathbf{p}^2)]^{-1}\Big) + \sum_{n,\mathbf{p}}\ln\sqrt{\alpha}.\end{aligned} \tag{22}$$

Absorbing the second term in Eq. (22) into the overall normalization of Z, we are left with the expression

$$\ln Z = -\sum_n\sum_{\mathbf{p}}\ln[\beta^2(\omega_n^2 + \mathbf{p}^2)], \tag{23}$$

which may be further reduced by applying the formulas [**?**]:

$$\begin{aligned}\ln[(2\pi n)^2 + \beta^2\omega^2] &= \int_1^{\beta^2\omega^2}\frac{d\theta^2}{\theta^2 + (2\pi n)^2} + \ln[1 + (2\pi n)^2],\\ \sum_{n=-\infty}^{+\infty}\frac{1}{n^2 + (\frac{\theta}{2\pi})^2} &= \frac{2\pi^2}{\theta}\Big(1 + \frac{2}{e^\theta - 1}\Big).\end{aligned} \tag{24}$$

Finally, incorporating the β-independent term into the normalization of Z, and replacing $\sum_{\mathbf{p}}$ by

$$\sum_{\mathbf{p}} \longrightarrow \frac{V}{(2\pi)^3}\int d^3\mathbf{p}, \tag{25}$$

we find that

$$\ln Z = 2V \int \frac{d^3\mathbf{p}}{(2\pi)^3}\Big[-\frac{1}{2}\beta\omega - \ln(1-e^{-\beta\omega})\Big], \quad \omega = |\mathbf{p}|. \tag{26}$$

Eq. (26), which is just the usual black-body radiation formula for the two physical gluon polarizations, agrees with a similar calculation in the Feynman gauge [**?**]. We now proceed with the more challenging task of computing the thermodynamic pressure at two loops.

4 The Q.C.D. Pressure at Two Loops

Higher-order terms in the partition function may be evaluated by forming bubble diagrams and applying the usual Feynman rules. The pure gauge contribution to the pressure at two loops is contained in the diagrams of Figure 1. The first of these is called the oyster diagram, the second the bowtie diagram. The thermodynamic pressure was the first gauge-invariant, physical quantity to be calculated in finite-temperature Q.C.D. at the two loop level. The computation was carried out by Kapusta [**?**] in the Feynman gauge who obtained the following result:

$$P_2^{\text{glue}} = -\frac{g^2}{144} N_c N_g T^4, \tag{27}$$

where N_c is the number of colours, and $N_g = {N_c}^2 - 1$ the number of gluons. N_c and N_g arise from counting the number of particles participating in the interaction. As stated in the Introduction, our aim is to calculate the 2-loop pressure in the noncovariant temporal gauge, by using the method of ζ-functions, along with a special version of the unified-gauge prescription.

Applying the Feynman rules of Section 2 to the oyster diagram in Figure 1, we obtain:

$$\begin{aligned} P_2^{\text{oyster}} \;=\; & g^2 N_c N_g T \sum_{n=-\infty}^{+\infty} \int \frac{d^3\mathbf{p}}{(2\pi)^3} T \sum_{l=-\infty}^{+\infty} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{(p+k)^2}\frac{1}{p^2}\frac{1}{k^2} \\ & \cdot\Big[-\delta_{\mu\sigma}(p+2k)_\tau + \delta_{\sigma\tau}(2p+k)_\mu - \delta_{\tau\mu}(p-k)_\sigma\Big] \\ & \cdot\Big[-\delta_{\sigma'\nu}(p+2k)_{\tau'} - \delta_{\nu\tau'}(p-k)_{\sigma'} + \delta_{\tau'\sigma'}(2p+k)_\nu\Big] \\ & \cdot\left[\delta^{\sigma\sigma'} + \frac{(p+k)^\sigma (p+k)^{\sigma'}}{(p_4+k_4)^2}\right] \\ & \cdot\left[\delta^{\tau\tau'} + \frac{p^\tau p^{\tau'}}{{p_4}^2}\right] \\ & \cdot\left[\delta^{\mu\nu} + \frac{k^\mu k^\nu}{{k_4}^2}\right], \end{aligned} \tag{28}$$

where $p_4 = 2\pi nT$, $k_4 = 2\pi lT$, and $\mu, \nu, \tau, \tau', \sigma, \sigma'$ are only summed over 1,2,3, while $n = 1, 2, \ldots$ and $l = 1, 2, \ldots$. Multiplication gives rise to a total of 720 terms. Contracting indices we find that the resulting expression reduces to 63 tadpole-like integrals, many of which are equal to zero. The contribution from the oyster diagram is, therefore, given by

$$P_2^{\text{oyster}} = g^2 N_c N_g T \sum_{n=-\infty}^{+\infty} T \sum_{l=-\infty}^{+\infty} \sum_{i=1}^{63} I_i. \tag{29}$$

The integrals $\{I_i\}$ are summarized in the Appendix. They are related to the familiar tadpole integrals which arise in massless theories such as Q.C.D. Let us briefly discuss this important class of integrals.

A typical example, at zero temperature, is the tadpole integral I,

$$I = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2}. \tag{30}$$

In the context of dimensional regularization, such integrals may be formally set to zero [**?**]. The <u>finite-temperature</u> version of (30) reads

$$I = T \sum_{n=-\infty}^{+\infty} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{p^2}, \tag{31}$$

with $p_4 = 2\pi nT$. It is instructive to solve Eq. (31) by using both the contour method and the zeta-function method. The contour method gives [**?**, **?**]:

$$I = T \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2\pi i} \oint_c dp_0 \frac{1}{{p_0}^2 - \mathbf{p}^2} \frac{1}{2} \beta \coth\left(\frac{\beta p_0}{2}\right), \quad \beta = 1/kT, k = 1, \tag{32}$$

or

$$\begin{aligned} I &= T\beta \int \frac{d^3\mathbf{p}}{(2\pi)^3} \Bigg\{ \frac{1}{2\pi i} \int_{i\infty-\epsilon}^{-i\infty-\epsilon} dp_0 \frac{1}{{p_0}^2 - \mathbf{p}^2} \Big(-\frac{1}{2} - \frac{1}{e^{-\beta p_0} - 1} \Big) \\ &\quad + \frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} dp_0 \frac{1}{{p_0}^2 - \mathbf{p}^2} \Big(\frac{1}{2} + \frac{1}{e^{\beta p_0} - 1} \Big) \Bigg\}, \\ &= I^{\text{vac}} + I^{\text{matt}}, \end{aligned} \tag{33}$$

where

$$I^{\text{vac}} = \int \frac{d^4p}{(2\pi)^4} \Big(\frac{1}{{p_4}^2 + \mathbf{p}^2} \Big) = 0, \quad p_0 = ip_4, \tag{34}$$

and

$$I^{\text{matt}} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{|\mathbf{p}|} \frac{1}{e^{\beta|\mathbf{p}|} - 1},$$

$$
\begin{aligned}
&= \frac{4\pi}{(2\pi)^3}\int_0^\infty dp\frac{p}{e^{\beta p}-1}, \qquad \text{(integrating over the angles)}\\
&= \frac{4\pi}{(2\pi)^3}\Gamma(2)\zeta(2)\frac{1}{\beta^2},\\
&= \frac{T^2}{12}. \qquad (35)
\end{aligned}
$$

Thus $I = T^2/12$.

Next we compute the tadpole in Eq. (31) by employing the method of zeta-functions [**?**]. Making a Wick rotation to Euclidean space and using the Schwinger representation for the denominators, we get

$$
\begin{aligned}
I &= T\sum_{n=-\infty}^{+\infty}\int_0^\infty d\alpha e^{-\alpha {p_4}^2}\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{-\alpha\mathbf{p}^2},\\
&= T\sum_{n=-\infty}^{+\infty}\int_0^\infty d\alpha e^{-\alpha {p_4}^2}\frac{\pi^{\omega-1/2}}{(2\pi)^{2\omega-1}}\alpha^{1/2-\omega},\\
&= \frac{1}{8\pi^{3/2}}\Gamma(-\frac{1}{2})T\sum_{n=-\infty}^{+\infty}|p_4|, \qquad \omega=2. \qquad (36)
\end{aligned}
$$

The sum over $|p_4|$ looks divergent, but is actually finite in the spirit of analytic continuation. The relevant analytic continuations of the gamma and zeta functions are [**?**]:

$$
\zeta(1-\alpha) = \pi^{-\alpha}2^{1-\alpha}\Gamma(\alpha)\zeta(\alpha)\cos(\frac{\pi\alpha}{2}), \qquad (37)
$$

$$
\Gamma(-m+\frac{1}{2}) = \frac{(-1)^m 2^m\sqrt{\pi}}{1\cdot 3\cdot 5\cdots(2m-1)}, \qquad m=1,2,\ldots. \qquad (38)
$$

Applying formulas (37) and (38) to Eq. (36), we see that

$$
\begin{aligned}
I &= \frac{1}{8\pi^{3/2}}4T^2\pi\Gamma(-\frac{1}{2})\zeta(-1),\\
&= \frac{T^2}{12}, \qquad (39)
\end{aligned}
$$

since $\Gamma(-1/2) = -2\sqrt{\pi}$ and $\zeta(-1) = -1/12$. Note that Eq. (39) agrees with Eq. (35), and that both results vanish in the limit $T \to 0$. This answer is consistent with the conclusions in refs. [**?**].

There are two other integrals which are similar to tadpoles and arise in the calculation of the thermodynamic pressure. They are included here for completeness. The first of these integrals is calculated as follows:

$$
T\sum_{n=-\infty}^{+\infty}\int\frac{d^3\mathbf{p}}{(2\pi)^3} = T\sum_{n=-\infty}^{+\infty}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{{p_4}^2+\mathbf{p}^2}{{p_4}^2+\mathbf{p}^2},
$$

$$
\begin{aligned}
&= T \sum_{n=-\infty}^{+\infty} \int_0^\infty d\gamma e^{-\gamma {p_4}^2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} ({p_4}^2 + \mathbf{p}^2) e^{-\gamma \mathbf{p}^2}, \\
&= 2T \frac{\pi^{3/2}}{(2\pi)^3} (- 2\sqrt{\pi} + 2\sqrt{\pi})(2\pi T)^3 \zeta(-3), \\
&= 0.
\end{aligned} \tag{40}
$$

The second integral may be decomposed as follows:

$$
T \sum_{n=-\infty}^{+\infty} \int \frac{d^3\mathbf{p}}{(2\pi)^3} p^2 = I_1 + I_2, \tag{41}
$$

where

$$
\begin{aligned}
I_1 &\equiv T \sum_{n=-\infty}^{+\infty} \int \frac{d^3\mathbf{p}}{(2\pi)^3} {p_4}^2, \\
&= T(2\pi T)^2 2 \sum_{n=1}^{\infty} n^2 \int \frac{d^3\mathbf{p}}{(2\pi)^3}, \\
&= T(2\pi T)^2 2\zeta(-2) \int \frac{d^3\mathbf{p}}{(2\pi)^3}, \\
&= 0, \quad \zeta(-2) = 0,
\end{aligned} \tag{42}
$$

while

$$
\begin{aligned}
I_2 &\equiv T \sum_{n=-\infty}^{+\infty} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathbf{p}^2, \\
&= T \sum_{n=-\infty}^{+\infty} \frac{1}{(2\pi)^3} \int_0^\infty d\gamma e^{-\gamma {p_4}^2} \Big(\frac{-\partial}{\partial \gamma}\Big) \int d^3\mathbf{p} ({p_4}^2 + \mathbf{p}^2) e^{-\gamma \mathbf{p}^2}, \\
&= T \sum_{n=-\infty}^{+\infty} \frac{\pi^{3/2}}{(2\pi)^3} \int_0^\infty d\gamma e^{-\gamma {p_4}^2} \Bigg({p_4}^2 (- 3/2) \gamma^{-5/2} + \frac{3}{2} (- 5/2) \gamma^{-7/2} \Bigg), \\
&= T \sum_{n=-\infty}^{+\infty} \frac{1}{8\pi^{3/2}} \frac{\Gamma(-3/2)}{({p_4}^2)^{-5/2}} \Big[\frac{1}{2} - 2 + \frac{3}{2} \Big], \\
&= 0.
\end{aligned} \tag{43}
$$

Thus

$$
T \sum_{n=-\infty}^{+\infty} \int \frac{d^3\mathbf{p}}{(2\pi)^3} p^2 = 0. \tag{44}
$$

We now have enough machinery to compute all 63 integrals in Eq. (29). Their values are listed in the Appendix. Substituting these integrals into Eq. (29), we only need to complete the indicated summations in order to derive P_2^{oyster}.