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Poisson-Lie T-duality and Loop Groups of Drinfeld Doubles

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Abstract

A duality invariant first order action is constructed on the loop group of a Drinfeld double. It gives at the same time the description of both of the pair of σ-models related by Poisson-Lie T-duality. Remarkably, the action contains a WZW-term on the Drinfeld double not only for conformally invariant σ-models. The resulting actions of the models from the dual pair differ just by a total derivative corresponding to an ambiguity in specifying a two-form whose exterior derivative is the WZW three-form. This total derivative is nothing but the Semenov-Tian-Shansky symplectic form on the Drinfeld double and it gives directly a generating function of the canonical transformation relating the σ-models from the dual pair.

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1 Introduction

T-duality is a discrete symmetry of string theory whose physical interpretation leads to a remarkable equivalence of seemingly very different string backgrounds. Using the σ-model language, a natural questions arise when a σ-model admits a dual one or when two given σ-models form a dual pair. Of course, truly relevant setting for asking these questions is the quantum (conformal) field theory but already at the classical level a nontrivial preselection of the σ-models admitting duals can be made. In fact, a natural criterion of the classical equivalence of σ-models is an existence of a canonical transformation between the phase spaces of the theories preserving the hamiltonians [1, 2, 3]. Having such a pair of the σ-models it is then natural to ask whether it can be promoted to a pair of conformal σ-models.

Much activity was devoted in the past few years to the study of the so-called Abelian T-duality [4]–[10]. In this case, all involved σ-models posses an Abelian isometry of their targets, which plays crucial role in the prescription how to perform duality transformation leading from one model to its dual. In particular, the Abelian isometry makes the task of finding the generating function of the canonical transformation between the models of a dual pair quite easy [1, 2, 3]. Starting from the work [11] (see also [12, 13]), there is a growing interest [14] in a non-Abelian generalization of the Abelian T-duality, which is essentially a duality between σ-models defined on targets being a Lie group and its Lie (co)algebra respectively. The model on the Lie group target does possess the non-Abelian isometry (the right multiplication of the target by the elements of the group) and the duality transformation is realized by the standard procedure of the gauging the isometry and imposing a zero-curvature condition on the gauge field by adding the corresponding Lagrange multipliers. Integrating out the gauge fields the dual model is obtained where the Lie (co)algebra coordinates are just the Lagrange multipliers. This procedure by no means insures an isometry of the dual target, in fact, the dual is generically *not* isometric and it was not known how to perform a 'duality' transformation back to the original model so the very notion of duality was questionable.

Recently, the present authors have shown that the σ-models on the Lie group and its (co)algebra are indeed *dual* to each other from the point of view of the so-called Poisson-Lie T-duality [15]. The crucial ingredience of our approach was the abandoning of the requirement of the isometry of the target space as *the* condition of dualizability. Actually we have shown that the non-Abelian duality exists between σ-models defined on the targets of arbitrary two Lie groups whose Lie algebras satisfy the Lie bialgebra condition [16]. A particular example of such a pair of groups is just any Lie group and its coalgebra (viewed as the commutative additive group) mentioned above. The duality transformation simply exchanges the roles of the two groups and, in general, none of the two dual models possesses an isometry. We have also shown that the Poisson-Lie duality is the canonical transformation[1] thus the classical criterion of the equivalence of the σ-models is indeed fulfilled.

The idea of T-duality without isometry is becoming quite popular right now [3, 20] though concrete examples, except those of the Poisson-Lie T-duality, are rare[2]. An attempt to formulate a mathematical criterion which would select the dualizable σ-models was formulated in [3] where it was also illustrated how the Abelian T-duality saturates the criterion. We would like to stress that the σ-models related by the Poisson-Lie T-duality all fulfil the criterion of [3]. The modular space of such σ-models is in fact huge. With arbitrary manifold which has a the structure of the principal bundle with the fiber being a Drinfeld double we can associate dualizable σ-models. They are parametrized by an infinite space of sections of certain Grassmannian bundle which shares the same base manifold with the principal Drinfeld double bundle [15]. Because this is true for arbitrary Drinfeld double we see, indeed, that the modular space is extremely big. Only its small corner consists of the σ-models admitting the standard non-Abelian duals in the sense of [11, 12, 13]; those corresponds to the Drinfeld doubles in which one of the two algebras forming the Lie bialgebra is commutative.

We see several reasons why to pursue the program of investigating the structure of the Poisson-Lie T-duality further. At the purely classical level, we have a huge laboratory for studying the T-duality without isometry and at the level of quantum field theory (QFT) there arises a possibility of existence of non-trivial discrete symmetry relating a weak coupling regime of one QFT to the strong coupling regime of the dual theory. At the level of statistical physics there is a good chance of obtaining a natural generalization of the

[1] For the standard non-Abelian duality between the $SU(2)$-group and its coalgebra this was shown already in [17] and between the arbitrary simple group and its coalgebra in [18] and partially in [19]. All these cases are special cases of our construction in [15].

[2] Very recently a new approach to T-duality was advanced in [20] where a simple example of T-duality without isometry was constructed.

standard Kramers-Wanier duality [21] and last, but not least, a way to string theory applications is open, since examples of the pairs of conformal σ-models related by the Poisson-Lie T-duality have been already constructed [22].

T-duality can be interpreted as a discrete symmetry of some conformal field theory (CFT) whose σ-model description is ambiguous. We may say, in a sense, that the both of the dual pair of the σ-models are 'present' in the CFT and they show up if we interpret CFT from different (dual) points of view. The picture of CFT in terms of its operator algebra and its Hilbert space we may interpret as the duality invariant description. Eventually, we aim to show that the Poisson-Lie T-duality is a symmetry of some CFT in the same sense. A possible strategy for reaching this goal could consist first in finding a classical analogue of this duality invariant description and then in attempting a path integral or other type of quantization.

In this contribution, we shall show that this duality invariant description is a given in terms of a first order (Hamiltonian) action functional on the phase space which is the loop groop of the Drinfeld double. If we eliminate one 'half' of the phase space variables in terms of the remaining 'half' we obtain the standard second order σ-model action on the target being one of the (isotropic) groups forming the Drinfeld double. If we choose a different description of the phase space and again eliminate one 'half' of the new variables in terms of the other 'half' we obtain the dual σ-model living on the target of the dual group. The change of the old variables to the new ones turns out to be nothing but a canonical transformation. The generating function(al) of the canonical transformation then follows directly from the formalism, because it is given by a total derivative term by which the actions of the mutually dual models differ. This total derivative term comes from the ambiguity in specifying the two-form whose exterior derivative is the WZW three-form and , rather remarkably, it is given by the Semenov-Tian-Shansky symplectic form on the double.

In section 2 we shall describe a detailed form and properties of the first order duality invariant action and show how the dual pair of σ-models can be extracted from it. In section 3 we shall show how the formalism directly gives the generating function(al) of the canonical transformation relating the models.

2 Poisson Lie T-duality

2.1 'Atomic' duality

For the description of the Poisson-Lie duality we need the crucial concept of the Drinfeld double, which is simply a Lie group D such that its Lie algebra $\mathcal{D}$, viewed as the linear space, can be decomposed into a direct sum of vector spaces which are themselves maximally isotropic subalgebras with respect to a non-degenerate invariant bilinear form on $\mathcal{D}$ [16]. An isotropic subspace of $\mathcal{D}$ is such that the value of the invariant form on any two vectors belonging to the subspace vanishes (maximally isotropic means that this subspace cannot be enlarged while preserving its isotropy). Any such decomposition of the double into a pair of maximally isotropic subalgebras $\mathcal{G} + \tilde{\mathcal{G}} = \mathcal{D}$ is usually referred to as the Manin triple. The Lie groups corresponding to the Lie algebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ we denote as G and $\tilde{G}$.

For the sake of clarity, it is convenient first to consider an 'atomic' Poisson-Lie T-duality which means that the targets of the σ-models forming the dual pair are the groups G and $\tilde{G}$ respectively. As we have already mentioned the phase space is the loop group of the Drinfeld double (the elements of the loop group are loops $l(\sigma)$ in the Drinfeld double and the multiplication is simply the point-wise one). The first order Lagrangian is a functional of the fields $l(\tau,\sigma)$ and it is given by

$$\mathcal{L} = \frac{1}{2}\langle \partial_\sigma l\ l^{-1}, \partial_\tau l\ l^{-1}\rangle + \frac{1}{12} d^{-1}\langle dl\ l^{-1}, [dl\ l^{-1}, dl\ l^{-1}]\rangle + \frac{1}{2}\langle \partial_\sigma l\ l^{-1}, A\partial_\sigma l\ l^{-1}\rangle \tag{1}$$

Here $\langle .,. \rangle$ denotes the non-degenerate invariant bilinear form on the Lie algebra of the double. In the second term in the r.h.s. we recognize the two-form potential of the WZW three-form on the double and A is a linear (idempotent) map from the Lie algebra $\mathcal{D}$ of the double into itself. It has two eigenvalues $+1$ and -1, the corresponding eigenspaces $\mathcal{R}_+$ and $\mathcal{R}_-$ have the same dimension $dimG$, they are perpendicular to each other in the sence of the invariant form on the double and they are given by the following recipe:

$$\mathcal{R}_+ = Span\{t + R(t,.), t \in \tilde{\mathcal{G}}\}, \quad \mathcal{R}_- = Span\{t - R(.,t), t \in \tilde{\mathcal{G}}\}. \tag{2}$$

Thus the modular space of such actions is described by (non-degenerate) bilinear forms $R(.,.)$ (matrices) on the algebra $\tilde{\mathcal{G}}$ [3]. For a better orientation of an interested reader we stress that the first two terms in (1) give together the standard WZW Lagrangian on the double if we interpret τ and σ as the 'light-cone' variables. These two first terms play the role of the 'polarization' term pdq in the first order variational principle

$$S = \int L = \int pdq - Hdt. \tag{3}$$

The remaining third term of the action (1) plays the role of the Hamiltonian H. The Lagrangian (1) has also a small gauge symmetry $l \to lk(\tau)$ where $k(\tau)$ is arbitrary τ-dependent function on the double. Strictly speaking this means that the phase space is the right coset LD/D of the loop group of D by the right action of D itself. This gives some restrictions on possible zero modes of the strings. We shall comment the role of this fact soon.

Every element of $l \in D$ can be written as

$$l = g\tilde{h}, \quad g \in G, \tilde{h} \in \tilde{G}. \tag{4}$$

By inserting this decompositions in the Lagrangian (1) and using the Polyakov-Wiegmann formula [23], we obtain the following expression

$$L = \langle \Lambda, g^{-1}\partial_\tau g \rangle + Ad_g G(\Lambda, \Lambda)$$

$$+Ad_g G^{-1}(g^{-1}\partial_\sigma g + Ad_g(B+\Pi(g))(\Lambda, .), g^{-1}\partial_\sigma g + Ad_g(B+\Pi(g))(\Lambda, .)). \tag{5}$$

Here $\Lambda = \partial_\sigma \tilde{h}\, \tilde{h}^{-1}$ and we used a compact notation in order not to burden the formula with too many indices: $G(.,.)$ and $B(.,.)$ denote respectively the symmetric and the antisymmetric part of the bilinear form $R(.,.)$ on the Lie algebra $\tilde{\mathcal{G}}$ (see Eq. (2)). $G^{-1}(.,.)$ is, in turn, the inverse bilinear form to $G(.,.)$ and, as such, it is defined on the Lie algebra $\mathcal{G}$. Ad_g means the adjoint action of the group G on the bilinear forms and $\Pi(g)$ is a g-dependent antisymmetric bivector (it acts on the elements from the algebra of $\tilde{\mathcal{G}}$)[4]. $\Pi(g)$ is given by an explicit formula

$$\Pi(g) = b(g)a(g)^{-1}, \tag{6}$$

[3]Since $R(.,.)$ is non-degenerate, there exists the inverse bilinear form defined on the dual $\mathcal{G}$ of $\tilde{\mathcal{G}}$, hence such description of the modular space does not break the duality.

[4]In fact, $\Pi(g)$ is nothing but the famous Poisson bracket on G which makes G the Poisson-Lie group with respect to the dual group $\tilde{G}$.

The matrices $a(g)$ and $b(g)$ are defined by

$$g^{-1}T^i g \equiv a(g)^i{}_l T^l, \qquad g^{-1}\tilde{T}_j g \equiv b(g)_{jl}T^l + d(g)_j{}^l \tilde{T}_l, \tag{7}$$

in (7) the adjoint action of G is understood to take place in the Drinfeld double algebra $\mathcal{D}$ and the pair of bases T^i and $\tilde{T}_i$ in the algebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ respectively, satisfy the duality condition

$$\langle T^i, \tilde{T}_j \rangle = \delta^i_j. \tag{8}$$

Note that Λ appears quadratically in the Lagrangian (5). One might think that it could be directly integrated from the Lagrangian even in the path integral sense but the story is more complicated , however. The reason is that our strings $l(\tau,\sigma)$ in the double are closed. This fact gives the following constraint of the unit monodromy on Λ:

$$P \exp \int_\gamma \Lambda = \tilde{e}, \tag{9}$$

where P stands for the path-ordered exponential, γ is a closed path around the string worldsheet and $\tilde{e}$ is the unit element of the dual group. Thus, though the path integral with the Lagrangian (5) seems to be Gaussian, the non-local constraint (9) makes it more difficult to compute. In fact, we have not found a way how to compute it yet though we believe it can be eventually managed in future. Here we just remark that we can proceed at the level of equations of motions. Indeed, let us vary Λ while preserving the constraint (9) . We obtain Λ as a function of g. Inserting this function $\Lambda(g)$ back in (5) we get a σ-model Lagrangian on the target G:

$$L = (R + \Pi(g))^{-1}(\partial_+ g g^{-1}, \partial_- g g^{-1}), \tag{10}$$

where

$$R(.,.) = G(.,.) + B(.,.), \qquad \partial_\pm = \partial_\tau \pm \partial_\sigma. \tag{11}$$

We should keep in mind, however, that the solution of the equations of motion coming from the Lagrangian (10) should be subject to the constraint (9) expressed in terms of $\Lambda(g)$. It is precisely this constraint which from the point of view of the σ-model (10) on G entails the factorization LD/D mentioned before. Note that the model (10) possesses an isometry with respect to the right action on the group manifold only in case when the dual group is commutative (the Poisson bracket $\Pi(g)$ then vanishes).

Now we choose the 'dual' parametrization of an element l of the double:

$$l = \tilde{g}h, \quad \tilde{g} \in \tilde{G}, h \in G. \tag{12}$$

Whole procedure of integrating h away can be repeated without any change and we arrive at the obviously equivalent dual σ-model

$$\tilde{L} = (R^{-1} + \tilde{\Pi}(\tilde{g}))^{-1}(\partial_+\tilde{g}\tilde{g}^{-1}, \partial_-\tilde{g}\tilde{g}^{-1}). \tag{13}$$

The pair of the mutually dual σ-models (10) and (13) was constructed in [15] by using a geometric picture of lifting the extremal strings of the model (10) to the double and then projecting them on the extremal configurations of the dual model (13). The derivation presented in the present note clearly shows the common 'roof' (1) of the two models. So we may conclude that the Poisson-Lie T-duality just exchange the two group manifolds and the matrix R by its inverse. In case the both groups dual to each other are the same (Abelian case and also the case of the so-called Borelian Drinfeld doubles) the duality amounts just in exchanging the matrix R by the matrix R^{-1}. We shall give some concrete examples of the dual pairs of model after the discussion of the canonical transformations in the general case.

Note that for every decomposition of the Lie algebra of the double $\mathcal{D} = \mathcal{K} + \tilde{\mathcal{K}}$ into two subalgebras with the property $\langle\mathcal{K}, \mathcal{K}\rangle = \langle\tilde{\mathcal{K}}, \tilde{\mathcal{K}}\rangle = 0$, we again generate a pair of the dual σ-models on the corresponding group targets K and $\tilde{K}$. Thus the action (1) decribes a whole space of the equivalent σ-models; the space is simply given by the set of all such isotropic decompositions of the algebra of the double. In the case of the Abelian duality it is given by the orbit of the group $O(d, d; Z)$. In the next section we shall demonstrate how to use our results to find the generating functions of the canonical transformations among the equivalent models.

2.2 Buscher's duality

In this paragraph we give a very brief description of Buscher's duality with the purpose just to illustrate to an interested reader that the Poisson-Lie duality is not just duality between two group targets but it applies in much more general setting. The corresponding formalism is more cumbersome than in the case of the atomic duality and we plan to give its most detailed account in a separate publication. Here we sketch just the results. Consider some manifold M and some coordinates x^μ on it. Consider then its cotangent bundle T^*M and the coordinates x^μ, p^μ which are the Darboux coordinates in which the canonical symplectic form ω on T^*M looks like $\omega = dp_\mu \wedge dx^\mu$. We may construct the following first order Lagrangian on a phase space formed by closed loops in the manifold $T^*M \times D$:

$$\mathcal{L} = \frac{1}{2}\langle \partial_\sigma l\ l^{-1}, \partial_\tau l\ l^{-1}\rangle + \frac{1}{2}d^{-1}\langle dl\ l^{-1}[dl\ l^{-1}, dl\ l^{-1}]\rangle + p_\mu \partial_\tau x^\mu + \frac{1}{2}\langle J, A(x)J\rangle. \tag{14}$$

Here the bracket in the last term in the r.h.s. $\langle .,. \rangle$ denotes a non-degenerate bilinear form on the tangent spaces to $T^*M \times D$ at points living on the submanifold $M \times D$ [5]. $A(x)$ is a linear (idempotent) map from the space $T_x^*M + T_xM + \mathcal{D}$ into itself. It has two eigenvalues $+1$ and -1, the corresponding eigenspaces $\mathcal{R}_+(x)$ and $\mathcal{R}_-(x)$ have the same dimension $dimG + dimM$, they are perpendicular to each other in the sence of the just described bilinear form on $T_x^*M + T_xM + \mathcal{D}$ and they are given by the following recipe:

$$\begin{aligned} \mathcal{R}_+(x) &= Span\{t + R(x)(t,.), t \in T_x^*M + \tilde{\mathcal{G}}\}, \\ \mathcal{R}_-(x) &= Span\{t - R(x)(.,t), t \in T_x^*M + \tilde{\mathcal{G}}\}. \end{aligned} \tag{15}$$

Thus the modular space of such actions is described by (non-degenerate) bilinear forms $R(x)(.,.)$ (matrices) on the space $T_x^*M + \tilde{\mathcal{G}}$ [6] or, in other words, by sections of a Grassmannian bundle over the base manifold M. Finally, it remains to explain what is J in (14).Of course, it is an element of $T_{x(\sigma)}M + T_{x(\sigma)}^*M + \mathcal{D}$ given by

$$J = (\partial_\sigma x^\mu(\sigma), p_\mu(\sigma), \partial_\sigma l l^{-1}), \tag{16}$$

[5] If S, S' are some two elements from such a tangent space at some point x of the submanifold $M \times D$; for instance $S = (t, \beta, \rho); t \in T_xM, \beta \in T_x^*M, \rho \in \mathcal{D}$ and $S' = (t', \beta', \rho'); t' \in T_xM, \beta' \in T_x^*M, \rho' \in \mathcal{D}$ then

$$\langle S, S'\rangle = \langle \beta, t'\rangle + \langle \beta', t\rangle + \langle \rho, \rho'\rangle.$$

Here the first two brackets on the r.h.s. mean the standard pairing between T_x^*M and T_xM and the third bracket is the standard bilinear form (8) on the double.

[6] Since $R(x)(.,.)$ is non-degenerate, there exists the inverse bilinear form defined on the dual space $T_xM + \mathcal{G}$ of $T_x^*M + \tilde{\mathcal{G}}$, hence such description of the modular space does not break the duality.

where $x^\mu(\sigma), p_\mu(\sigma), l(\sigma)$ is an element of the phase space of the string.

As in the case of the atomic duality, we first write $l = g\tilde{h}$ and solve away from the action the fields p_μ and $\tilde{h}$. We obtain the following σ-model Lagrangian

$$L = (R(x) + \Pi(g))^{-1}(j_+, j_-), \tag{17}$$

where (with a slight abuse of the notation) $\Pi(g)$ now denotes a bilinear form on the space $T^*_x M + \tilde{\mathcal{G}}$ which on $\tilde{\mathcal{G}}$ acts as $\Pi(g)$ in (6) before and on $T^*_x M$ it vanishes; $j_\pm$ are elements of $T_x M + \mathcal{G}$ given by

$$j_\pm = (\partial_\pm x^\mu, \partial_\pm g g^{-1}). \tag{18}$$

We may also write $l = \tilde{g}h$ and solve away the fields p_μ and h. The result is the Lagrangian of the dual σ-model

$$\tilde{L} = (\tilde{R}(x) + \tilde{\Pi}(\tilde{g}))^{-1}(\tilde{j}_+, \tilde{j}_-), \tag{19}$$

where

$$\tilde{j}_\pm = (\partial_\pm x^\mu, \partial_\pm \tilde{g}\tilde{g}^{-1}) \tag{20}$$

and $\tilde{R}(x)$ is given in terms of $R(x)$ as

$$\tilde{R}(x) = (A + R(x)B)^{-1}(C + R(x)D). \tag{21}$$

Here (Id means the identity matrix)

$$A = D = \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = C = \begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix} \tag{22}$$

and the block matrices are understood in terms of the decomposition $T^*_x M + \tilde{\mathcal{G}}$. Needless to say, in the standard Abelian case ($D = U(1) \times U(1)$) the formula (21) gives just the Buscher transformations [4].

3 Canonical transformations

As we have already mentioned we can interpret the duality invariant first order action as the sum of the WZW action (which corresponds to the polarization form pdq on the phase space of the model) and the expression quadratic in the Drinfeld double currents (which plays the role of the Hamiltonian). We can parametrize the first order action in terms of the fields $g, \tilde{h}$

and $\tilde{g}, h$ respectively such that $g\tilde{h} = \tilde{g}h = l$ and we obtain, after eliminating $\tilde{h}$ or h, the dual pair of the σ-models (10) and (13). The transformation between those two parametrizations is obviously canonical transformation because it respects the form $pdq - Hdt$ of the variational principle, moreover, it preserves the Hamiltonian itself. Hence, by definition, the total derivative of the generating function is given by the difference $PdQ - pdq$ where the capital characters denote the new coordinates $(\tilde{g}(\sigma), h(\sigma))$ and the small characters the old coordinates $(g(\sigma), \tilde{h}(\sigma))$. In our case we have used the Polyakov-Wiegmann formula[7]

$$I(\tilde{g}h) = \int \langle \partial_\sigma h h^{-1}, \tilde{g}^{-1}\partial_\tau \tilde{g}\rangle d\tau d\sigma; \tag{23}$$

$$I(g\tilde{h}) = \int \langle \partial_\sigma \tilde{h}\tilde{h}^{-1}, g^{-1}\partial_\tau g\rangle d\tau d\sigma \tag{24}$$

to obtain the explicit form of the terms in the actions corresponding to polarization forms PdQ and pdq. We know already by construction that the two expressions in the r.h.s. of (23) and (24) at most differ by a total derivative (they may not be identical because of the ambiguity in defining d^{-1} of the WZW three-form; this ambiguity is a total derivative). In principle, it is now easy task to evalute the generating function(al) $F(g, \tilde{g})$ of the canonical transformation between $\tilde{g}, h$ and $g, \tilde{h}$. It is given by

$$\int dF = \int \langle \partial_\sigma h h^{-1}, \tilde{g}^{-1}\partial_\tau \tilde{g}\rangle - \int \langle \partial_\sigma \tilde{h}\tilde{h}^{-1}, g^{-1}\partial_\tau g\rangle, \quad h = h(g,\tilde{g}), \tilde{h} = \tilde{h}(g,\tilde{g}). \tag{25}$$

The dependence of $h, \tilde{h}$ on $g, \tilde{g}$ is, of course, given by the requirement encountered before: $g\tilde{h} = \tilde{g}h$. Thus we have obtained an implicit expression for the generating function(al) F; to make it explicit for a concrete Drinfeld double we only have to express g and $\tilde{g}$ in terms of h and $\tilde{h}$. Remarkably, however, it is possible to write a more explicit formula for F still considering a generic Drinfeld double. It was derived already in [15] (that time we did not know yet the duality invariant action (1) on the double). The result is

$$F(g, \tilde{g}) = \int_{D(l)} \Omega. \tag{26}$$

Here $g(\sigma), \tilde{g}(\sigma)$ parametrize the elements $l(\sigma)$ of the phase space LD of the model in such a way that $h(g, \tilde{g})$ and $\tilde{h}(g, \tilde{g})$ interpolate between g and $\tilde{g}$, i.e.

[7] Note that the WZW action I of a *single* argument $g, \tilde{g}, h$ and $\tilde{h}$ vanishes.

$l = g\tilde{h} = \tilde{g}h$; $D(l)$ is an arbitrary two-dimensional surface embedded in the double whose boundary is just the loop $l(\sigma)$ and Ω is the symplectic (hence closed) two-form on the double constructed by Semenov-Tian-Shansky[8][24]. It is easy to describe the form Ω in terms of the Poisson bracket on the double to which it gives rise. Define $(\nabla_L f)_a, (\nabla_L f)^a, (\nabla_R f)_a$ and $(\nabla_R f)^a$ as follows

$$df = (\nabla_L f)_a (dll^{-1})^a + (\nabla_L f)^a (dll^{-1})_a = (\nabla_R f)_a (l^{-1}dl)^a + (\nabla_R f)^a (l^{-1}dl)_a, \tag{27}$$

where f is some function on the double. Clearly, the upper and lower indices for the forms dll^{-1} (or $l^{-1}dl$) mean:

$$dll^{-1} = (dll^{-1})_a T^a + (dll^{-1})^a \tilde{T}_a. \tag{28}$$

Then the Semenov-Tian-Shansky Poisson bracket is given by

$$\{f, f'\} = \Omega^{-1}(df, df') = (\nabla_L f)_a (\nabla_L f')^a - (\nabla_R f)^a (\nabla_R f')_a \tag{29}$$

for arbitrary functions f, f' on the double.

The formula (26) gives the maximally explicit description of the generating function of the canonical transformation which can be obtained for a *generic* double. It uses only the canonical structure on a Drinfeld double such as the Semenov-Tian-Shansky form certainly is. In concrete examples of the Drinfeld doubles with concrete parametrizations of their group manifolds the formula (26) for the generating function look in general very cumbersome. We feel, however, that it is its rather positive than negative feature of the formula (26). Indeed, we have some doubts that it would be easy to find the formula for the generating function without having the geometric understanding ot T-duality which has led to the conceptually simple formula (26). It is also worth mentioning that the dual σ-models (in the sense of the Buscher duality) (17) and (19) are also connected by a canonical transformation whith the same generating function $F(g, \tilde{g})$ given by (26). In other words: the structure of the 'attached' manifold M with the coordinates x^μ is highly irrelevant for the duality transformation.

The duality invariant first order action (1) enables us to find the formula for the generating function of the canonical transformation between arbitrary

[8]The form Ω has the important property that the action of the double with the standard *Poisson* structure on the double with the Semenov-Tian-Shansky *symplectic* structure is the Poisson action.

pair of mutually equivalent σ-models coming from the action (1) for all possible choices $(\mathcal{K},\tilde{\mathcal{K}})$ of the isotropic decompositions of the Lie algebra of the double (see the discussion in the previous section). Consider e.g. some two decompositions: $\mathcal{D}=\mathcal{G}+\tilde{\mathcal{G}}$ and $\mathcal{D}=\mathcal{K}+\tilde{\mathcal{K}}$ and find the generating function of the canonical transformation between the σ-models, say, on target G and target K. In order to do that we parametrize $l\in D$ as $l=g\tilde{h}=k\tilde{m}$ where $g\in G$,$\tilde{h}\in\tilde{G}$, $k\in K$ and $\tilde{m}\in\tilde{K}$. The generating function $F(g,k)$ is then given by

$$\int dF=\int\langle\partial_\sigma\tilde{m}\tilde{m}^{-1},k^{-1}\partial_\tau k\rangle-\int\langle\partial_\sigma\tilde{h}\tilde{h}^{-1},g^{-1}\partial_\tau g\rangle,\quad \tilde{m}=\tilde{m}(g,k),\tilde{h}=\tilde{h}(g,k). \tag{30}$$

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Note added: In the few hours' gap between completing the present paper and submitting it to hep-th we received a preprint [25] where an interesting and differently looking form of a Poisson-Lie duality invariant action was also presented. We hope to elucidate the relation between those two actions in near future. We believe, however, that another main result of [25] which should be the path integral derivation of the Poisson-Lie T-duality requires more rigour. The reason is that the authors of [25] did not comment that the fields over which they integrate are also constrained by the highly non-linear and non-local unit monodromy constraint (9) and therefore the process of continual integration over such fields should become more non-trivial. This is also the reason why we hesitate to claim that we already have a path integral derivation of the Poisson-Lie T-duality.

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