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**Resistive Wall Impedance  
as Derivative of the Electric Capacitance  
for a Beam Pipe of Arbitrary Cross Section**

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ABSTRACT

We derive a general formula expressing the resistive wall impedance in the ultrarelativistic limit for a beam pipe of arbitrary cross section through the ‘normal derivative’ of its electric capacitance. An application to the case of rectangular cross section yields a closed form expression of the corresponding longitudinal impedance in terms of elliptic integrals.

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# 1 Introduction and summary

The electromagnetic field associated with an ultrarelativistic bunch of charged particles travelling in a perfectly conducting pipe of arbitrary, but constant, cross section can be determined by solving a two-dimensional electrostatic problem. Specifically, the electric field  $\mathbf{E} = -\nabla_{\perp}\varphi$  is derivable from a scalar potential  $\varphi$  satisfying Poisson's equation in the transverse plane, with equipotential boundary conditions at the metallic beam pipe. Indeed, thanks to the translation invariance of the pipe cross section, the electromagnetic field is obtained by a Lorentz transformation of the purely electrostatic field in the rest frame of the bunch. In the extreme relativistic limit, the bunch becomes infinitely long in its rest frame and the electrostatic potential at a given point is only determined by the 'slice' of beam charge in the corresponding transverse plane.

For a pipe of finite resistivity, the electromagnetic field is no longer purely transverse and the Fourier transform of the longitudinal electric field on the beam axis, responsible for the parasitic loss, is associated with the longitudinal resistive wall impedance  $Z_L$ . In the case of a thick pipe with uniform resistivity  $\rho$ , one can treat the effect of resistivity as a small perturbation and assume that the transverse fields can be approximated by those obtained for a perfectly conducting pipe. Each Fourier component of the longitudinal wall current  $I_z$ , equal to the tangential magnetic field  $H_t$  and flowing in the direction opposite to the beam current  $I_o$ , is therefore proportional to  $\nabla_{\perp}\varphi$  at the metallic boundary. On the other hand, using Ohm's law, the longitudinal electric field can be approximately written as  $E_z = Z_w I_z$ , where the wall surface impedance  $Z_w = (1 - i)\rho/\delta$  depends on the skin depth  $\delta$ . The power  $Z_L|I_o|^2$  lost by the beam is equal to the outgoing flux of the complex Poynting vector  $\mathbf{E} \times \mathbf{H}^*$  across the pipe wall and the longitudinal impedance per unit length  $Z_L/L$  can therefore be expressed as a contour integral of  $E_z I_z^* = Z_w |I_z|^2$  over the pipe perimeter [1]

$$\frac{Z_L}{L} = \frac{Z_w}{|I_o|^2} \oint |H_t|^2 d\ell = Z_w \frac{\varepsilon_o^2}{|\lambda_o|^2} \oint |\nabla_{\perp}\varphi|^2 d\ell. \quad (1)$$

Here  $\varepsilon_o$  is the permittivity of free space and  $\lambda_o = I_o/c$  the linear charge density of the beam. For a pencil beam located at  $\mathbf{r}_{\perp} = \mathbf{r}_b$  in the transverse plane, the electrostatic potential  $\varphi$  satisfies the two-dimensional Poisson equation

$$\nabla_{\perp}^2 \varphi = -\frac{\lambda_o}{\varepsilon_o} \delta(\mathbf{r}_{\perp} - \mathbf{r}_b).$$

Therefore the ratio  $\Phi = \varphi/\lambda_o$  is real and depends only on the pipe geometry and on the beam position. A similar expression holds also for the transverse (dipole) impedance  $Z_T$ , provided the electrostatic problem is solved using a dipole source term [1].

In the next section we show that the contour integral of  $(\nabla_{\perp}\Phi)^2$ , required to compute the resistive wall impedance, is proportional to the 'normal derivative' of the electrostatic energy stored in the region between the beam and the surrounding pipe (see Fig. 1). This electrostatic energy can be expressed through the specific capacitance  $\mathcal{C} = C/L$  of the system beam-pipe and, denoting by  $\delta n$  the infinitesimal outward displacement along the normal to the pipe surface, we arrive at the following expression for the longitudinal resistive wall impedance:

$$\frac{Z_L}{L} = Z_w \frac{\delta}{\delta n} \left( \frac{\varepsilon_o}{\mathcal{C}} \right). \quad (2)$$

Note that the specific capacitance  $\mathcal{C}$  has the same dimensions as the permittivity of free space (i.e., farad/metre), so that the ratio  $\epsilon_o/\mathcal{C}$  is dimensionless and its normal derivative times the wall surface impedance yields an impedance per unit length<sup>1</sup>.

In practice, to find the resistive wall impedance one has first to solve the two-dimensional electrostatic problem for a uniform beam with unit linear charge density and to compute the corresponding electric potential difference  $\Delta\Phi = 1/\mathcal{C}$  from the beam (assumed of infinitesimal, but constant radius) to the pipe (assumed equipotential). The calculation is then repeated for a pipe of slightly larger dimensions, each surface element being displaced by a constant amount  $\delta n$  along the normal to the surface: this yields the ‘normal derivative’ of the potential difference, proportional to that of the pipe electric capacitance. With the help of this prescription, one avoids the contour integral of  $(\nabla_{\perp}\Phi)^2$  and performs instead a simple derivative. Moreover, the numerical estimate of the capacitance (and of its normal derivative) for complicated beam pipe cross sections can be improved using variational techniques.

In section 3, we first apply our result to the known case of a beam in a circular pipe and then to the more complicated case of a centred beam in a rectangular pipe, for which only a series expansion of the impedance is available [1]. Using Eq. (2), we arrive at a closed form expression of the longitudinal impedance in terms of elliptic integrals.

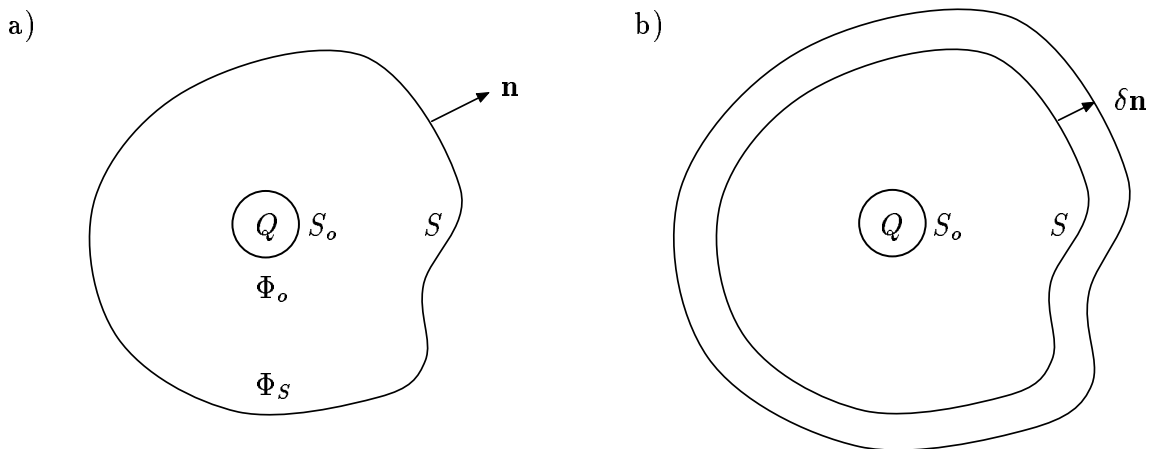


Figure 1: (a) Two closed equipotential curves  $S_o$  and  $S$ , the former enclosing the beam charge  $Q$  and the latter representing the pipe cross section with unit normal vector  $\mathbf{n}$ . (b) Normal variation of the beam pipe geometry: the vector  $\delta\mathbf{n} = \mathbf{n} \delta n$  has constant norm  $\delta n$ .

## 2 Normal variation of the electrostatic energy

We consider the electrostatic potential  $\Phi$  corresponding to a uniform beam with unit linear charge density travelling in a perfectly conducting pipe of arbitrary cross section. The charge

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<sup>1</sup>The dimensionless ratio  $\epsilon_o/\mathcal{C}$  is also equal to the ratio  $\mathcal{L}/\mu_o$  between the specific inductance  $\mathcal{L}$  (of the transmission line consisting of the beam and of its surrounding pipe) and the permeability  $\mu_o$  of free space [2]. However, it is more natural to express the electrostatic energy in terms of capacitance.

density induced on the inner surface of the beam pipe is

$$\frac{1}{L} \frac{dQ}{d\ell} = -\varepsilon_o E$$

and the electrostatic force on a surface element of length  $L$  and tangential extent  $d\ell$  is

$$d\mathbf{F} = \frac{1}{2} \mathbf{E} dQ,$$

where the factor  $1/2$  is due to the fact that the electric field vanishes inside the conductor. Since  $\mathbf{E} = -\nabla_{\perp} \Phi = -\mathbf{n} \nabla_{\perp} \Phi$ , where  $\mathbf{n}$  denotes the unit vector normal to the surface and oriented in the outward direction (see Fig. 1), we obtain

$$\frac{d\mathbf{F}}{d\ell} = -\frac{1}{2} \nabla_{\perp} \Phi \frac{dQ}{d\ell} = -\frac{\varepsilon_o L}{2} (\nabla_{\perp} \Phi)^2 \mathbf{n}.$$

The work required to modify the cross section of the beam pipe by an infinitesimal amount  $\delta\mathbf{n}$  in the outward direction is<sup>2</sup>

$$\delta U = - \oint d\ell \frac{d\mathbf{F}}{d\ell} \cdot \delta\mathbf{n} = \frac{\varepsilon_o L}{2} \oint d\ell (\nabla_{\perp} \Phi)^2 \delta n.$$

From the principle of energy conservation, the corresponding variation of the electrostatic energy  $U$  for a *uniform* normal variation  $\delta n$  is thus proportional to the contour integral of the square of the electric field over the pipe perimeter

$$\frac{\delta U}{\delta n} = \frac{\varepsilon_o L}{2} \oint d\ell (\nabla_{\perp} \Phi)^2.$$

For a beam of unit linear charge density  $Q/L = 1$ , the specific electrostatic energy  $\mathcal{U} = U/L$  stored in the region between the beam and the metallic pipe can be expressed in terms of the electric capacitance per unit length  $\mathcal{C} = C/L$  as follows:

$$U = \mathcal{U}L = \frac{1}{2} \frac{Q^2}{C} \quad \rightarrow \quad \mathcal{U} = \frac{1}{2\mathcal{C}}.$$

Therefore

$$\oint d\ell (\nabla_{\perp} \Phi)^2 = \frac{1}{\varepsilon_o} \frac{\delta}{\delta n} \left( \frac{1}{\mathcal{C}} \right) \quad (3)$$

and, recalling that  $\Phi = \varphi/\lambda_o$  is real, from Eq. (1) we obtain expression (2) for the resistive wall impedance. Since the electrostatic energy is also given by  $U = Q\Delta\Phi/2$ , the inverse specific capacitance  $1/\mathcal{C}$  equals the electric potential difference  $\Delta\Phi$  between the beam and the surrounding pipe.

In this context, the effect of resistivity can be interpreted as a longitudinal *friction force*, proportional to the normal electrostatic force  $\mathbf{F}$  on the pipe surface: the friction coefficient depends on the frequency  $\omega$  through the skin depth  $\delta = \sqrt{2\rho/(\omega\mu_o)}$ , appearing in the wall surface impedance  $Z_w$ .

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<sup>2</sup>This work does not include the contribution of the force on the opposite charge induced on the outer surface of the beam pipe: such contribution vanishes for an infinitely thick or for a grounded pipe.

### 3 Examples

As a simple application of our result Eq. (2), we first compute the longitudinal impedance of a circular pipe with a centred beam and then consider a transverse beam offset. Finally we discuss the more complicated case of a centred beam in a rectangular pipe.

#### 3.1 Circular pipe

The electrostatic potential  $\Phi$  of a uniform pencil beam with unit linear charge density, travelling at the centre of a circular pipe of radius  $b$ , is

$$\Phi(r) = -\frac{1}{2\pi\epsilon_o} \ln(r) \quad \text{for } \epsilon \leq r \leq b,$$

where  $r$  is the radial distance from the pipe axis and  $\epsilon$  denotes the (infinitesimal) beam radius. The potential difference  $\Delta\Phi$  from the beam to the pipe is therefore

$$\Delta\Phi = \frac{1}{C} = \Phi(\epsilon) - \Phi(b) = \frac{1}{2\pi\epsilon_o} \ln\left(\frac{b}{\epsilon}\right)$$

and, since the normal derivative for a circular pipe corresponds to an ordinary derivative with respect to the radius  $b$ , from Eq. (2) we immediately obtain the well known result

$$\frac{Z_L}{L} = Z_w \frac{\partial}{\partial b} \left( \frac{\epsilon_o}{C} \right) = \frac{Z_w}{2\pi b}. \quad (4)$$

We now consider the case of a beam offset  $a < b$  in the horizontal direction  $x$  and write the potential  $\Phi(x, y)$  in rectangular coordinates, using an opposite image charge at  $x = b^2/a$  to satisfy the equipotential boundary condition at  $x^2 + y^2 = b^2$ :

$$\Phi(x, y) = -\frac{1}{4\pi\epsilon_o} \ln \left[ \frac{(x-a)^2 + y^2}{(x-b^2/a)^2 + y^2} \right].$$

The potential difference  $\Delta\Phi$  from the beam, of infinitesimal radius  $\epsilon$ , to the pipe is

$$\Delta\Phi = \frac{1}{C} = \Phi(a + \epsilon, 0) - \Phi(b, 0) = \frac{1}{4\pi\epsilon_o} \left\{ \ln \left[ \left( \frac{a}{b} \right)^2 \right] - \ln \left[ \left( \frac{\epsilon}{a + \epsilon - b^2/a} \right)^2 \right] \right\}$$

and we obtain

$$\frac{\partial}{\partial b} \left( \frac{\epsilon_o}{C} \right) = -\frac{1}{2\pi} \left\{ \frac{1}{b} + \frac{2b/a}{a + \epsilon - b^2/a} \right\}.$$

In the limit  $\epsilon \rightarrow 0$ , Eq. (2) yields

$$\frac{Z_L}{L} = \frac{Z_w}{2\pi b} \frac{b^2 + a^2}{b^2 - a^2}, \quad (5)$$

in agreement with the known result (see Ref. [3], Exercise 2.31 on p. 118). Taking the limit  $a, b \rightarrow \infty$ , with constant distance  $b - a = d$  from the beam to the pipe, the longitudinal impedance per unit length becomes  $Z_L/L = Z_w/(2\pi d)$ : therefore the parasitic loss is the same for a beam travelling in the centre of a circular pipe of radius  $b$  or parallel to an infinite metallic plane of equal resistivity at a distance  $d = b$ .

### 3.2 Rectangular pipe

We now consider the case of a rectangular pipe with sides  $a$  and  $b$  and write the electrostatic potential  $\Phi(z)$ , using the complex notation  $z = x + iy$ , for a uniform pencil beam of unit charge density, travelling at the centre  $z_o = (a + ib)/2$  of the rectangle, as [4]

$$\Phi(z) = \frac{1}{2\pi\epsilon_o} \operatorname{Re} \left\{ \ln \left[ \frac{\operatorname{sn}^2(Kz/a, k) - \operatorname{sn}^2(Kz_o/a, k)}{\operatorname{sn}^2(Kz/a, k) - \operatorname{sn}^2(Kz_o/a, k)} \right] \right\}. \quad (6)$$

Here  $K = K(k)$  is the complete elliptic integral of the first kind with modulus  $k$ , while  $\operatorname{sn}(u, k)$  denotes the Jacobian elliptic sine-amplitude, again with the same modulus  $k$ : the latter depends on the ratio  $a/b$  between the sides of the rectangle and is implicitly defined by

$$K/K' = a/b, \quad (7)$$

where  $K' = K(\sqrt{1 - k^2})$ . The potential (6) is obtained by the conformal transformation  $w = \operatorname{sn}^2(Kz/a, k)$ , which maps the inside of the rectangle in the  $z$ -plane into the upper half of the  $w$ -plane. The electrostatic problem is then solved by adding an opposite image charge in the lower  $w$ -plane, such that the real  $w$ -axis be at zero potential. Therefore  $\Phi(z)$  vanishes on the pipe boundary and the potential difference  $\Delta\Phi$  between the beam, of infinitesimal radius  $\epsilon$ , and the rectangular pipe is given by the limit of  $\Phi(z_o + \epsilon)$  for  $\epsilon \rightarrow 0$ . As shown in the Appendix, neglecting the divergent self-potential proportional to  $\ln(\epsilon)$ , this limit is

$$\Delta\Phi = \frac{1}{\mathcal{C}} = \frac{1}{4\pi\epsilon_o} \ln \left( \frac{ab}{KK'} \right). \quad (8)$$

The infinitesimal variation of  $\epsilon_o/\mathcal{C}$  can then be written

$$\delta \left( \frac{\epsilon_o}{\mathcal{C}} \right) = \frac{1}{4\pi} \left( \frac{\delta a}{a} + \frac{\delta b}{b} - \frac{\delta(KK')}{KK'} \right)$$

and for a normal variation  $\delta n$  of the rectangular pipe cross section we must require

$$\delta a = \delta b = 2\delta n.$$

Therefore

$$\frac{\delta}{\delta n} \left( \frac{\epsilon_o}{\mathcal{C}} \right) = \frac{1}{4\pi} \left( \frac{2}{a} + \frac{2}{b} - \frac{1}{KK'} \frac{\delta(KK')}{\delta n} \right). \quad (9)$$

In the following, we need the following identities [5]:

$$EK' + E'K - KK' = \frac{\pi}{2}, \quad \text{Legendre identity} \quad (10)$$

$$\frac{dK}{dk} = \frac{1}{k} \left( \frac{E}{1 - k^2} - K \right) \quad \rightarrow \quad \frac{dK'}{dk} = -\frac{k}{1 - k^2} \left( \frac{E'}{k^2} - K' \right), \quad (11)$$

where  $E = E(k)$  is the complete elliptic integral of the second kind and  $E' = E(\sqrt{1 - k^2})$ .

Using Eq. (7), we can now establish a relation between  $\delta n$  and the variation  $\delta k$  of the elliptic modulus. Indeed

$$\delta \left( \frac{a}{b} \right) = \frac{b\delta a - a\delta b}{b^2} = \frac{b-a}{b^2} 2\delta n = \delta \left( \frac{K}{K'} \right).$$

Since, from Eqs. (10) and (11),

$$\delta \left( \frac{K}{K'} \right) = \frac{\pi}{2K'^2} \frac{\delta k}{k(1-k^2)},$$

it follows that

$$\delta n = \frac{\pi}{4K'^2} \frac{b^2}{b-a} \frac{\delta k}{k(1-k^2)}$$

and, using Eq. (7), Eq. (9) can be written

$$\frac{\delta}{\delta n} \left( \frac{\varepsilon_o}{C} \right) = \frac{1}{2\pi} \left[ \frac{1}{a} + \frac{1}{b} - \frac{2}{\pi} \left( \frac{1}{a} - \frac{1}{b} \right) k(1-k^2) \frac{\delta(KK')}{\delta k} \right].$$

In the case of a square pipe of side  $2d$ , i.e., for  $a = b = 2d$ , the term proportional to  $1/a - 1/b$  vanishes and substituting this formula into Eq. (2) yields  $Z_L/L = Z_w/(2\pi d)$ . We thus find that, for a centred beam and for a given wall resistivity, a square pipe has the same longitudinal impedance of the inscribed circular pipe. In the general case, using again Eqs. (10) and (11), the derivative of  $KK'$  with respect to the modulus  $k$  can be expressed in closed form as follows

$$\frac{\delta(KK')}{\delta k} = \frac{EK' - E'K + (2k^2 - 1)KK'}{k(1-k^2)}$$

and the longitudinal impedance per unit length becomes

$$\frac{Z_L}{L} = \frac{Z_w}{2\pi} \left[ \frac{1}{a} + \frac{1}{b} - \frac{2}{\pi} \left( \frac{1}{a} - \frac{1}{b} \right) (EK' - E'K + (2k^2 - 1)KK') \right]. \quad (12)$$

This formula is symmetric in  $a$  and  $b$ , since for an exchange of the sides of the rectangle  $k^2$  is transformed into  $1-k^2$ ,  $K$  into  $K'$  and  $E$  into  $E'$ . The corresponding resistive loss, normalized to that of the inscribed circular pipe, is plotted in Fig. 2: the loss is the same when the sides of the rectangle are equal (square pipe) or when one of them becomes infinite (two parallel plates). Indeed, for  $a \rightarrow \infty$ ,  $k = 1$  and  $K$  is divergent, while  $K'$ ,  $E$  and  $E'$  stay finite [5]; therefore, the Legendre identity Eq. (10) implies that  $E'K = KK'$  and  $EK' = \pi/2$ , so that the square bracket equals  $2/b$ , i.e., the inverse of the radius of the inscribed pipe. When the ratio between the sides of the rectangle is around 1.33, the resistive loss reaches a minimum value about 6% lower than that corresponding to a square pipe.

## 4 Discussion

The result presented in this paper can be considered as a special example of the variational formulation discussed in Ref. [6], where the tune shift due to a gradient perturbation in a

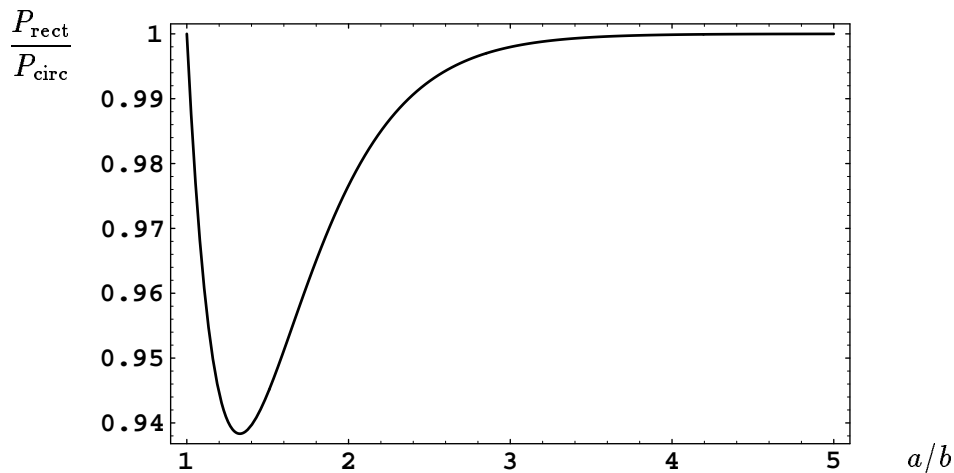


Figure 2: Parasitic loss for a centred beam in a rectangular pipe, normalized to that for the inscribed circular pipe, as a function of the ratio  $a/b$  between the sides of the rectangle.

circular accelerator was obtained by a first order variation of a suitable action integral with respect to the gradient perturbation. The variation with respect to the betatron function vanishes by virtue of the corresponding Euler equation and the stationary value of the action integral coincides with the tune of the accelerator. In the present paper, we consider the volume integral of the square of the electrostatic field over the charge-free region between the beam and its surrounding pipe. The stationary value of this action integral, for an equipotential boundary, corresponds to the stored electrostatic energy. For a normal variation of the boundary, the corresponding variation of the electrostatic energy is proportional to the resistive wall impedance, while the variation due to the change of the electric potential vanishes by virtue of the Laplace equation. It should be noted, however, that in general the variation of the potential is not zero on the original boundary (as one usually assumes when deriving the corresponding Euler equation). Nevertheless, the potential perturbation is harmonic and preserves the flux of the electric field across the boundary; this is enough to prove that the associated first order variation of the action integral vanishes. It is remarkable that such complicated derivation is not necessary, if one makes use of the principle of energy conservation.

We would like to stress that our starting equation (1) is only valid for relatively high frequencies, typically above a few MHz, corresponding to skin depths much smaller than both the pipe thickness and its local radius of curvature. On the other hand, the perturbative treatment of the wall resistivity<sup>3</sup> requires that the frequency be not too high: for example  $\omega/2\pi \ll 10^{12}$  Hz for a cylindrical aluminum pipe, with 5 cm radius, at room temperature (see

<sup>3</sup>The elegant derivation of Eq. (1) presented in Ref. [1] and based on the Lorentz reciprocity theorem may give the impression of an exact result, with the only approximation introduced by the so-called Leóntovich boundary condition, relating the longitudinal electric field to the tangential magnetic field via the wall surface impedance  $Z_w$ . However, the gradients appearing in the Poynting theorem and required to convert the longitudinal impedance into a surface integral can be considered as transverse gradients only for a perfectly conducting pipe, while this is only approximately true for a pipe with wall losses. Such additional approximation is implicitly used in Ref. [1].



Ref. [3], p. 73). The range of validity of these approximations and thus of our result Eq. (2) is usually wide enough to yield accurate estimates of the parasitic loss in particle accelerators. These considerations can be extended to the case of anomalous skin effect (when the surface impedance  $Z_w$  has a different dependence on frequency [7]) and, to some extent, also to the case of non-uniform resistivity along the pipe perimeter. In the latter case, however, even for an infinitely thick pipe the perturbative treatment of the wall resistivity breaks down at low frequency, when the induced currents tend to redistribute themselves among regions with different resistivity following the path of least dissipation.

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## Appendix

In this appendix we shall prove that, neglecting a divergent term proportional to  $\ln(\epsilon)$ , the limit for  $\epsilon \rightarrow 0$  of the potential  $\Phi(z_o + \epsilon)$ , defined by Eq. (6), is given by Eq. (8). Indeed, replacing  $z = z_o = (a + ib)/2$  in the numerator of expression (6) and expanding the denominator to first order in  $\epsilon$ , we can write

$$\Phi(z_o + \epsilon) = \frac{1}{2\pi\epsilon_o} \operatorname{Re} \left\{ \ln \left[ \frac{\operatorname{sn}^2(u, k) - \operatorname{sn}^2(u^*, k)}{\frac{\partial \operatorname{sn}^2(u, k)}{\partial u} \frac{K}{a} \epsilon} \right] \right\},$$

where

$$u = \frac{K}{a} z_o = \frac{K + iK'}{2}$$

as a consequence of Eq. (7). Therefore

$$\Phi(z_o + \epsilon) = \frac{1}{2\pi\epsilon_o} \left[ \ln \left( \frac{a}{K} \right) - \ln(\epsilon) + \ln \left( \left| \frac{\operatorname{sn}^2(u, k) - \operatorname{sn}^2(u^*, k)}{\frac{\partial \operatorname{sn}^2(u, k)}{\partial u}} \right| \right) \right]$$

and, since the last term in square brackets can be shown to be zero, neglecting the term proportional to  $\ln(\epsilon)$  (which does not contribute to the normal derivative of  $\Delta\Phi$ ) yields

$$\Delta\Phi = \frac{1}{2\pi\epsilon_o} \ln\left(\frac{a}{K}\right) = \frac{1}{4\pi\epsilon_o} \ln\left(\frac{ab}{KK'}\right),$$

where we have used again Eq. (7) to obtain an expression symmetric in  $a$  and  $b$ . This result coincides with Eq. (8).

To prove that the last term in square brackets is zero, we start from the identities [5]

$$\frac{\partial \text{sn}(u, k)}{\partial u} = \text{cn}(u, k) \text{dn}(u, k) = \sqrt{1 - \text{sn}^2(u, k)} \sqrt{1 - k^2 \text{sn}^2(u, k)},$$

$$\text{sn}(u, k) = \frac{\sqrt{1+k} + i\sqrt{1-k}}{\sqrt{2k}}.$$

Here  $\text{cn}(u, k)$  and  $\text{dn}(u, k)$  denote the Jacobian elliptic cosine-amplitude and delta amplitude, respectively, and the first of these identities is generally true, while the second holds as a consequence of our choice of a centred beam, corresponding to the special value of  $u = (K + iK')/2$ . Then

$$\text{sn}^2(u, k) - \text{sn}^2(u^*, k) = \frac{2i}{k} \sqrt{1-k^2}, \quad \frac{\partial \text{sn}^2(u, k)}{\partial u} = -\frac{2i}{k} \sqrt{1-k^2} (k + i\sqrt{1-k^2})$$

and, since the elliptic modulus  $k$  ranges from 0 and 1, the ratio of these two complex quantities has a norm equal to unity: therefore its logarithm is zero.