Massive Two-Dimensional Quantum Chromodynamics

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Abstract

In this work we study the zero-charge sector of massive two-dimensional Quantum Chromodynamics in the decoupled formulation. We find that some general features of the massless theory, concerning the constraints and the right- and left-moving character of the corresponding BRST currents, survive in the massive case. The implications for the integrability properties previously valid in the massless case, and the structure of the Hilbert space are discussed.

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1. Introduction

The recent formulation of two-dimensional Quantum Chromodynamics (QCD_2) with massless fermions in terms of positive and negative level Wess–Zumino–Witten (WZW) fields, ghosts and massive bosonic excitations, has led to interesting insights into the characteristics of the model,[1] such as its integrability,[2] degeneracy of the vacuum,[3] and higher symmetry algebras related to some operators in the theory.[4,5] Although the fields of this equivalent, effective bosonic theory, obtained by making use of the representation of the fermionic determinant in terms of a Wess–Zumino–Witten action[6] seem decoupled at the Lagrangian level, the corresponding sectors are connected by BRST constraints[7,8] operating on the conformally invariant sector of the theory described by a topological WZW-coset model.[9] The solution of the corresponding cohomology problem for $SU(2)^{10}$ was shown[3] to imply a two-fold degeneracy of the ground state. The S-matrix of the massive sector, expected to describe the physical excitations of the theory, factorizes[11] as a consequence of the infinite number of conservation laws, associated with a particular right-moving "current" of the model, implying the conservation of individual momenta in particle scattering amplitudes.

In this paper we extend these investigations to the case where the fermions are massive. In this case we continue to have two BRST nilpotent charges, as in massless QCD_2, but one of them is modified by the presence of the mass term. The corresponding BRST currents are again found to be right- and left-moving. As expected on general grounds[12] there are first-class constraints associated with these currents. They also depend only on the light-cone coordinates $x^- = (x^0 - x^1)$ and $x^+ = (x^0 + x^1)$, respectively, and thus are constraints on the zero-mass sector of the theory. This is consistent with the well-known results for the Abelian case,[13] the massive Schwinger model (MSM), where the longitudinal part of the fermionic current plays the role of these constraints.[14]

In the MSM, the BRST currents involve the right- and left-moving parts of the longitudinal current, and the BRST condition on the physical Hilbert space implies that the positive- and negative-metric interacting bosonic fields can occur in a linear combination corresponding to a zero-mass free field. This is again a reflection of the fact that the BRST currents are either right- or left-moving. In the considerably more complicated non-Abelian case, a similar situation is found to occur; however, the massless particles in the Hilbert space are now given as linear combinations of composites of the massive, group-valued fields in the theory, rendering the methods applied to the MSM obsolete, while the procedure used in the massless case is insufficient to handle the massive modes appearing in the definition of the constraints.

As for the integrability condition obtained in ref. [2] for massless QCD_2, it is found to be spoiled by the mass term. This is consistent with the non-integrability of the MSM, equivalently described by a sine-Gordon theory perturbed by the mass term. We conjecture that in the case of massive fermions the higher conservation laws referred to above are replaced by a constraint on the zero-mass sector.

2. BRST currents and constraint structure

Two-dimensional QCD has several features that are not yet understood. However, when fermions are integrated over in favour of a bosonic field, several features become transparent. In the massless case the procedure has explicitly been carried out in ref. [2]. The main steps are summarized in the Appendix. In the non-local formulation,[2] the partition function of two-dimensional QCD with massless fermions is given by[*]

$$\mathcal{Z} = \int \mathcal{D}g \mathcal{D}\beta \mathcal{D}\Sigma \mathcal{D}C \, e^{iS} \quad , \tag{2.1a}$$

the bosonized action being given in terms of the WZW action, ghosts and a Yang–Mills term,

$$S = \Gamma[g] + \Gamma[\beta] - (c_V + 1)\Gamma[\Sigma] + S_{gh} + S_{YM}[\beta, C] \quad ; \tag{2.1b}$$

the Yang–Mills action $S_{YM}[\beta, C]$, the ghost term and the WZW functional $\Gamma[g]$, are respectively given by

$$S_{YM} = \int d^2x \, tr \left(\frac{1}{2}(\partial_+ C)^2 + \lambda C(\beta^{-1} i \partial_+ \beta) \right) \quad , \tag{2.2a}$$

$$S_{gh} = \int d^2x \, tr \, (c_- i \partial_+ b_- + c_+ i \partial_- b_+) \quad , \tag{2.2b}$$

$$\Gamma[g] = \frac{1}{8\pi} \int d^2x \, tr \, \partial_\mu g^{-1} \partial^\mu g + \frac{1}{12\pi} \int d^3y \epsilon^{\alpha\beta\gamma} \, tr \, \left(g^{-1} \partial_\alpha g g^{-1} \partial_\beta g g^{-1} \partial_\gamma g \right) \quad , \tag{2.2c}$$

with a similar expression for $\Gamma[\beta]$. The parameter λ is given in terms of the charge e, and the Casimir c_V by $\lambda = \frac{c_V+1}{2\pi}e$, where c_V is normalized according to $f^{abc}f^{dbc} = \frac{1}{2}c_V\delta^{ad}$. The action (2.1) contains the conformally invariant WZW field g corresponding to the bosonized version of the massless free fermionic excitation, Σ describes the negative metric excitations, while the (β, C) system corresponds to the massive sector. As shown in refs. [2, 7, 8], the apparently decoupled g, β, Σ and C sectors are actually connected via BRST constraints.

In the case where the fermions are massive, the functional determinant of the Dirac operator, an essential ingredient for arriving at the bosonized form (2.1a) of the QCD$_2$ partition function, can no longer be computed in closed form, and one must resort to the so-called adiabatic principle of form invariance. Equivalently, one can start with a perturbative expansion in powers of the mass, as given by

$$\sum \frac{1}{n!} M^n \left[\int d^2x \overline{\psi}\psi \right]^n \quad ,$$

use the (massless) bosonization formulae and reexponentiate the result. In this approach, the mass term is given in terms of the bosonic field g of the massless theory by[15,16]

$$S_m = M \int tr \, \overline{\psi}\psi = M\mu \int tr \, (g + g^{-1}) \quad ,$$

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[*] As compared to refs. [1, 2] we drop the tildes in g and Σ, since no confusion arises here. Moreover, C_-, b_{++}, b_{--}, c_+ and c_- of refs. [1, 2] correspond here to C_-, b_+, b_-, c_- and c_+, respectively.

where μ is an arbitrary massive parameter whose value depends on the renormalization prescription for the mass operator.[13]

Defining $m^2 = M\mu$, we reexponentiate the mass term. Going through the changes of variable of ref. [2], one arrives at the following expression in terms of the fields of the non-local formulation:

$$S_m = -m^2 \int \text{tr} \left(g\Sigma^{-1}\beta + \beta^{-1}\Sigma g^{-1} \right) \quad . \tag{2.3}$$

After such a procedure, the effective action of massive QCD$_2$ reads

$$S = \Gamma[g] + \Gamma[\beta] - (c_V + 1)\Gamma[\Sigma] + S_{gh} + S_{YM}[\beta, C] + S_m[g, \beta, \Sigma] \quad , \tag{2.4}$$

and the partition function no longer has a factorized form. Nevertheless, there still exist BRST currents which are either right- or left-moving.

We wish to construct the BRST currents associated with the above action. This action exhibits various symmetries of the BRST type; however, not all of them lead to nilpotent charges. In the case of massless fermions these symmetries were found to be associated with the transformations

a) $g \to gY$, $\Sigma \to \Sigma Y$,
b) $g \to Xg$, $\Sigma \to X\Sigma$,
c) $\Sigma \to X\Sigma$, $\beta \to X\beta$.

In the massive case, transformation b) must be supplemented by $\beta \to X\beta X^{-1}$ in order to leave the mass term invariant.

The respective BRST-type transformations, leaving the action (2.1) invariant are easily found. Corresponding to the right transformation of the g and Σ fields in item a) above, we obtain a transformation similar to the massless case, since such a mapping by itself leaves the mass invariant:

a) $\delta g = \epsilon g c_+$, $\delta\Sigma = \epsilon\Sigma c_+$,

       $\delta C = 0$, $\delta\beta = 0$,

       $\delta c_- = 0$, $\delta c_+ = \dfrac{\epsilon}{2}\{c_+, c_+\}$,

       $\delta b_- = 0$,

$$\delta b_+ = \epsilon \left(\frac{1}{4\pi}g^{-1}i\partial_+ g - \frac{c_V + 1}{4\pi}\Sigma^{-1}i\partial_+\Sigma + \{b_+, c_+\} \right) \quad . \tag{2.5a}$$

Corresponding to the mapping (b), supplemented by the above-mentioned transformation $\beta \to X\beta X^{-1}$ in order to leave the mass term invariant, we obtain

b) $\delta g = \epsilon c_- g$, $\delta\Sigma = \epsilon c_-\Sigma$,

       $\delta C = \epsilon[c_-, C]$, $\delta\beta = \epsilon[c_-, \beta]$,

       $\delta c_- = \dfrac{\epsilon}{2}\{c_-, c_-\}$, $\delta c_+ = 0$,

       $\delta b_+ = 0$,

$$\delta b_- = \epsilon \left(\frac{1}{4\pi}gi\partial_- g^{-1} - \frac{c_V + 1}{4\pi}\Sigma i\partial_-\Sigma^{-1} + \{b_-, c_-\} \right) + \epsilon\mathcal{B} \quad , \tag{2.5b}$$

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where
$$\mathcal{B} = \frac{1}{4\pi}\beta i\partial_-\beta^{-1} + \frac{1}{4\pi\lambda}\partial_+\partial_-C - \lambda[\beta, C\beta^{-1}] + i[C, \partial_+C]. \qquad (2.6)$$

The term $\epsilon\mathcal{B}$ arises from the transformation $\beta \to X\beta X^{-1}$, which is unnecessary in the massless case, and which leads to the transformation law (2.5b) coupling the Yang-Mills sector of the model to the remaining sectors. Finally, corresponding to the third transformation c) we have

c)
$$\delta g = 0 \quad , \quad \delta\Sigma = \epsilon c_-\Sigma \quad ,$$
$$\delta C = 0 \quad , \quad \delta\beta = \epsilon c_-\beta \quad ,$$
$$\delta c_- = \frac{\epsilon}{2}\{c_-, c_-\} \quad , \quad \delta c_+ = 0 \quad ,$$
$$\delta b_+ = 0 \quad ,$$
$$\delta b_- = \epsilon\left(\frac{1}{4\pi}\beta i\partial_-\beta^{-1} - \frac{c_V+1}{4\pi}\Sigma i\partial_-\Sigma^{-1} - \lambda\beta C\beta^{-1} + \{b_-, c_-\}\right) . \quad (2.5c)$$

This symmetry transformation is again analogous to the one found in the massless case.

The equations of motion are obtained from action (2.4) by computing its variation. We obtain

$$\frac{1}{4\pi}\partial_+(g\partial_-g^{-1}) = m^2(g\Sigma^{-1}\beta - \beta^{-1}\Sigma g^{-1}) \quad , \quad (2.7a)$$

$$-\frac{c_V+1}{4\pi}\partial_+(\Sigma\partial_-\Sigma^{-1}) = m^2(\Sigma g^{-1}\beta^{-1} - \beta g\Sigma^{-1}) \quad , \quad (2.7b)$$

$$\frac{1}{4\pi}\partial_+(\beta\partial_-\beta^{-1}) + i\lambda\partial_+(\beta C\beta^{-1}) = m^2(\beta g\Sigma^{-1} - \Sigma g^{-1}\beta^{-1}) \quad , \quad (2.7c)$$

$$-\frac{1}{4\pi}\partial_-(\beta^{-1}\partial_+\beta) + i\lambda[\beta^{-1}\partial_+\beta, C] + i\lambda\partial_+C = m^2(g\Sigma^{-1}\beta - \beta^{-1}\Sigma g^{-1}) \quad , \quad (2.7d)$$

$$\partial_+^2 C = \lambda(\beta^{-1}i\partial_+\beta) \quad , \qquad (2.7e)$$

$$\partial_\pm b_\mp = 0 \quad , \quad \partial_\pm c_\mp = 0 \quad . \qquad (2.7f)$$

Notice the form of the mass term, which can be transformed, from one equation to another, by a suitable conjugation. Making use of eqs. (2.7), the Noether currents are constructed in the standard fashion. The only subtlety in this procedure concerns the WZW term, which only contributes off shell to the variation. The three conserved Noether currents are found to be

$$J_+^{(1)} = \text{tr}\, c_+\left(\frac{1}{4\pi}g^{-1}i\partial_+g - \frac{c_V+1}{4\pi}\Sigma^{-1}i\partial_+\Sigma + \frac{1}{2}\{b_+, c_+\}\right) \quad , \qquad (2.8a)$$

$$J_-^{(2)} = \text{tr}\, c_-\left(\frac{1}{4\pi}gi\partial_-g^{-1} - \frac{c_V+1}{4\pi}\Sigma i\partial_-\Sigma^{-1} + \frac{1}{2}\{b_-, c_-\} + \mathcal{B}\right) \quad , \qquad (2.8b)$$

$$J_-^{(3)} = \text{tr}\, c_-\left(\frac{1}{4\pi}\beta i\partial_-\beta^{-1} - \frac{c_V+1}{4\pi}\Sigma i\partial_-\Sigma^{-1} - \lambda\beta C\beta^{-1} + \frac{1}{2}\{b_-, c_-\}\right) \quad , \qquad (2.8c)$$

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and are either "right" or "left" moving, that is, the equations of motion read

$$\partial_- J_+^{(1)} = 0 \quad , \quad \partial_+ J_-^{(2)} = 0 \quad , \quad \partial_+ J_-^{(3)} = 0 \quad . \tag{2.9}$$

It is convenient to write these currents in the form

$$J_+^{(1)} = \mathrm{tr}\left(c_+ \Omega^{(1)} - \frac{1}{2}b_+\{c_+, c_+\}\right) \quad , \tag{2.10a}$$

$$J_-^{(r)} = \mathrm{tr}\left(c_- \Omega^{(r)} - \frac{1}{2}b_-\{c_-, c_-\}\right) \quad , \quad r = 2, 3 \quad , \tag{2.10b}$$

where $\Omega^{(r)}$ are seen to be given by

$$\Omega^{(1)} = \frac{1}{4\pi}g^{-1}i\partial_+ g - \frac{c_V + 1}{4\pi}\Sigma^{-1}i\partial_+\Sigma + \{b_+, c_+\} \quad , \tag{2.11a}$$

$$\Omega^{(2)} = \frac{1}{4\pi}gi\partial_- g^{-1} - \frac{c_V + 1}{4\pi}\Sigma i\partial_-\Sigma^{-1} + \{b_-, c_-\} + \mathcal{B} \quad , \tag{2.11b}$$

$$\Omega^{(3)} = \frac{1}{4\pi}\beta i\partial_-\beta^{-1} - \frac{c_V + 1}{4\pi}\Sigma i\partial_-\Sigma^{-1} - \lambda\beta C\beta^{-1} + \{b_-, c_-\} \quad . \tag{2.11c}$$

These operators obey simple equations as a consequence of the current conservation equations, namely $\Omega^{(1)}$ is right-moving while $\Omega^{(2,3)}$ are left-moving. Indeed, making use of the equation of motion (2.7) one readily checks that the operators $\Omega^{(1)}$, $\Omega^{(2)}$, and $\Omega^{(3)}$ satisfy

$$\partial_- \Omega^{(1)} = 0 \quad , \quad \partial_+ \Omega^{(2)} = 0 \quad , \quad \partial_+ \Omega^{(3)} = 0 \quad , \tag{2.12}$$

consistent with the conservation laws (2.9).

In order that the corresponding charges $Q^{(r)}$ be nilpotent, the operators $\Omega^{(r)}$ should be first class. We examine this question in the following section.

3. Anomalous constraints, physical subspace

To establish the first- and second-class character of the operators $\Omega^{(r)}$, $r = 1, 2, 3$, we rewrite the operators (2.11) in terms of canonical phase-space variables. Following the canonical quantization procedure of ref. [17], we have

$$\Omega^{(1)} = -i\hat{\Pi}^g g + \frac{i}{4\pi}g^{-1}g' - i\hat{\Pi}^\Sigma \Sigma - i\frac{c_V + 1}{4\pi}\Sigma^{-1}\Sigma' + j_+^{gh} \quad , \tag{3.1a}$$

$$\Omega^{(2)} = ig\hat{\Pi}^g + \frac{i}{4\pi}g'g^{-1} + i\Sigma\hat{\Pi}^\Sigma - i\frac{c_V + 1}{4\pi}\Sigma'\Sigma^{-1} + j_-^{gh}$$

$$+ i\beta\hat{\Pi}^\beta + \frac{i}{4\pi}\beta'\beta^{-1} - i\hat{\Pi}^\beta\beta + \frac{i}{4\pi}\beta^{-1}\beta' - \frac{1}{2\pi\lambda}\Pi'_C + i[C, \Pi_C] \quad , \tag{3.1b}$$

$$\Omega^{(3)} = i\beta\hat{\Pi}^\beta + \frac{i}{4\pi}\beta'\beta^{-1} + i\Sigma\hat{\Pi}^\Sigma - i\frac{c_V + 1}{4\pi}\Sigma'\Sigma^{-1} + j_-^{gh} \quad . \tag{3.1c}$$

where

$$j_\pm^{gh} = \{b_\pm, c_\pm\}. \tag{3.1d}$$

Although in the bosonized formulation quantum anomalies arising from one-loop fermion graphs are already incorporated on the semi-classical level, the commutators of the operators $\Omega^{(r)}$ may still be non-canonical[18] due to the presence of other types of anomalies.[*] For the operators $\Omega^{(1)}$ and $\Omega^{(3)}$ this situation does not occur. Therefore we are able in this case to compute their Poisson brackets using the canonical formalism. Concerning $\Omega^{(1)}$, it is straightforward to verify that it obeys a Kac–Moody algebra with vanishing central term, being therefore first class with respect to itself. Moreover, since the Poisson brackets respect chirality, it also has vanishing Poisson brackets with the other operators, $\Omega^{(2)}$ and $\Omega^{(3)}$. Since the computation does not involve any ambiguity, $\Omega^{(3)}$ weakly commutes with itself; the operators are again the usual Kac–Moody currents. Hence $\Omega^{(3)}$ is also first class with respect to itself, and moreover it commutes with $\Omega^{(1)}$. Summarizing, we have,

$$[\Omega^{(1)a}(x), \Omega^{(1)b}(y)] = if^{abc}\Omega^{(1)c}(x)\delta(x-y) \quad , \tag{3.2a}$$

$$[\Omega^{(3)a}(x), \Omega^{(3)b}(y)] = if^{abc}\Omega^{(3)c}(x)\delta(x-y) \quad , \tag{3.2b}$$

$$[\Omega^{(1)a}(x), \Omega^{(3)b}(y)] = 0 \quad . \tag{3.2c}$$

As for $\Omega^{(2)}$ the situation is more delicate. It is convenient to write $\Omega^{(2)}$ in the form

$$\Omega^{(2)} = \Omega^{(3)} + \widetilde{\Omega}^{(2)} \quad , \tag{3.3a}$$

where now

$$\widetilde{\Omega}^{(2)} = \frac{1}{4\pi\lambda}\partial_+\partial_- C + \lambda C + i[C, \partial_+ C] + \frac{i}{4\pi}g\partial_- g^{-1} \quad . \tag{3.3b}$$

Our discussion can be restricted to $\widetilde{\Omega}^{(2)}$, which is also right-moving. Unlike the previous case, the computation of the Poisson bracket of this operator with itself involves quantum corrections, arising from the presence of the algebraic commutator $[C, \Pi_C]$, which contribute to the central charge. These quantum corrections are obtained via the short-distance expansion

$$[C(x), \Pi_C(x)]_{ij}[C(y), \Pi_C(y)]_{kl} = \frac{2N}{(x-y)^2}\left(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}\right) \quad , \tag{3.4}$$

valid for a symmetry group $G = SU(N)$. Thus we arrive at the result

$$[\widetilde{\Omega}^{(2)a}(x), \widetilde{\Omega}^{(2)b}(y)] = f^{abc}\widetilde{\Omega}^{(2)c}(x)\delta(x-y) + \left[\frac{N}{\pi}\right]\delta'(x-y)\delta^{ab} \quad . \tag{3.5}$$

Similarly

$$[\Omega^{(2)a}(x), \Omega^{(2)b}(y)] = f^{abc}\Omega^{(2)c}(x)\delta(x-y) + \left[\frac{2N+1}{2\pi}\right]\delta'(x-y)\delta^{ab} \quad . \tag{3.6}$$

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[*] In the absence of further anomalies these commutators are identical with i times their respective Poisson brackets.

Therefore these operators are second class. This is analogous to the case of QCD_2 with massless quarks. Thus only the charges associated with $J^{(1)}$ and $J^{(3)}$ are nilpotent and lead to bona fide BRST charges.

Using the Karabali–Schnitzer (KS) method we find that $\Omega^{(1)}$ and $\Omega^{(3)}$ are constrained to vanish. This is also consistent with the BRST conditions, which require that the physical states be annihilated by the BRST charges:

$$Q^{(1)}|\text{Phys}\rangle = 0 , \quad \text{and} \quad Q^{(3)}|\text{Phys}\rangle = 0 \quad . \tag{3.7}$$

In the massless case the solution of the corresponding cohomology problem only involves the WZW fields g, Σ, and then ghosts, being equivalent to a G/G topological field theory, for the vacuum sector. This problem has been explicitly solved for the gauge group $SU(2)$ using the Wakimoto representation for the currents,[19] and the representation theory of affine algebras.[20] In the massive case the problem is more involved, since the partition function no longer factorizes.

4. Abelian case

It is curious that there exists no KS gauging[7] of the action $(2.1b)$, which would establish $\widetilde{\Omega}^{(2)} \approx 0$ as one further constraint. Of course, since $\widetilde{\Omega}^{(2)}$ is second class with respect to itself, the associated charge $\widetilde{Q}^{(2)}$ is not nilpotent, and there is no compelling reason for $\widetilde{\Omega}^{(2)}$ to be constrained to vanish. Nevertheless, in the massless case there does exist one further constraint. In order to gain some insight into this curious fact, it is instructive to specialize the above results to the Abelian case (massive Schwinger model (MSM)), where a plethora of results are available in the literature.[13]

In order to allow for a simple comparison with the standard results on the massive Schwinger model, we parametrize the non-Abelian fields as follows

$$g = e^{i2\sqrt{\pi}\varphi} , \quad \tilde{\Sigma} = e^{-i2\sqrt{\pi}\eta} , \quad \beta = e^{-i2\sqrt{\pi}E} \quad . \tag{4.1}$$

The equations of motion (2.7) then reduce to ($\lambda = e/2\pi$)

$$\Box\varphi = -\Box\eta = 4\sqrt{\pi}m^2 \sin 2\sqrt{\pi}(E - \varphi - \eta) \quad , \tag{4.2a, b}$$

$$\Box E + \frac{e}{\sqrt{\pi}}\partial_+ C = -4\sqrt{\pi}m^2 \sin 2\sqrt{\pi}(E - \varphi - \eta) \quad , \tag{4.2c, d}$$

$$\partial_+ \left(\partial_+ C - \frac{e}{\sqrt{\pi}}E\right) = 0 \quad . \tag{4.2e}$$

The constraints $\Omega^{(1)} \approx 0$ and $\Omega^{(3)} \approx 0$ read in this case,

$$2\sqrt{\pi}\Omega^{(1)} = -\partial_+(\varphi + \eta) = -[(\Pi_\varphi + \partial_1\varphi) + (-\Pi_\eta + \partial_1\eta)] \approx 0 \quad , \tag{4.3a}$$

$$2\sqrt{\pi}\Omega^{(3)} = -\partial_-(E - \eta) - \frac{e}{\sqrt{\pi}}C = -[(\Pi_E - \partial_1 E) + (\Pi_\eta + \partial_1\eta)] \approx 0 \quad , \tag{4.3c}$$

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where the canonical momenta are given by the expressions

$$\Pi_\varphi = \partial_0\varphi \quad , \quad \Pi_\eta = -\partial_0\eta \quad , \quad \Pi_C = \partial_+C \quad , \quad \Pi_E = \partial_0 E + \frac{e}{2\sqrt{\pi}}C \quad . \tag{4.4}$$

The constraint $\Omega^{(1)} \approx 0$ defines the physical Hilbert space in the MSM corresponding to positive chirality.[3] For $\widetilde{\Omega}^{(2)}$ one obtains

$$2\sqrt{\pi}\widetilde{\Omega}^{(2)} = \partial_-\varphi + \frac{\sqrt{\pi}}{e}\partial_+\partial_- C + \frac{e}{\sqrt{\pi}}C$$

$$= (\Pi_\varphi - \partial_1\varphi) + (\Pi_E - \partial_1 E) + 2\partial_1\left(E - \frac{\sqrt{\pi}}{e}\Pi_C\right) \quad . \tag{4.5}$$

Making use of (4.3c), and supposing that the equation of motion (4.2e) has only the trivial solution $\Pi_C = \frac{e}{\sqrt{\pi}}E$ (i.e. assuming the operator ∂_+ to be "invertible"), $\widetilde{\Omega}^{(2)}$ reduces to $\frac{1}{2\sqrt{\pi}}\partial_-(\varphi + \eta)$, which, following the method of Karabali–Schnitzer,[7] is easily shown to be constrained to vanish. Thus $\partial_-\widetilde{\Omega}^{(2)} = 0$ is replaced by the constraint $\widetilde{\Omega}^{(2)} = 0$. As is well known,[3,13] the constraint $\partial_-(\varphi + \eta) = 0$ defines the Hilbert space of the MSM corresponding to negative chirality. The constraints $\partial_\pm(\varphi + \eta) \approx 0$ are indeed (first-class) constraints of the MSM.

The "invertibility requirement" referred to above can be appreciated by formally rewriting the constraint (4.3c) in configuration space as:

$$-2\sqrt{\pi}\Omega^{(3)} = \partial_+^{-1}\left\{\left(\Box E + \frac{e}{\sqrt{\pi}}\right) - \Box\eta\right\} \approx 0 \quad . \tag{4.6}$$

Using the equation of motion (4.2) we see that the constraint is guaranteed by the equations of motion, provided the operator ∂_+ is invertible. This corresponds to the requirement that massless states in the combination (4.6) be absent.

5. The fate of the integrability condition

The operator (3.3b) may be written in the form

$$\widetilde{\Omega}^{(2)} = I_- + \frac{1}{4\pi}gi\partial_- g^{-1} \quad , \tag{5.1}$$

where I_- is the operator

$$I_- = \frac{1}{4\pi\lambda}\partial_+\partial_- C + i[C, \partial_+ C] + \lambda C \quad , \tag{5.2}$$

introduced in ref. [2]. Making use of the equation of motion (2.7a), we have

$$\partial_+ I_- = -m^2(g\Sigma^{-1}\beta - \beta^{-1}\Sigma g^{-1}) \quad . \tag{5.3}$$

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In the massless case, $\partial_+ I_- = 0$, so that I_- is right-moving. In refs. [2,11] this was shown to imply the existence of an infinite number of charges $Q^{(n)}$, acting on asymptotic states $|\vec{p}\rangle$ in the massive β-sector like generators Q_R^a of $SU(N)_R$, with multiplication by the nth power of the momenta p_-:

$$Q^{(n)}|\vec{p}\rangle = p_-^n Q_R^a|\vec{p}\rangle \quad . \tag{5.4}$$

It has further been shown in [11] that the conservation laws implied by (5.4) determine the S-matrix in the β sector, up to bound-states poles. Further results implying integrability of the model were also found by other authors.[21]

For $m \neq 0$ it is evident from (5.3) that the conservation law $\partial_+ I_- = 0$ of the massless theory is being replaced by $\partial_+ \widetilde{\Omega}^{(2)} = 0$. This property is evidently guaranteed by the equation of motion (2.7c). We could require $\widetilde{\Omega}^{(2)}$ itself to vanish. As we illustrated in the preceding section for the case of the MSM, this requirement is easily seen from (2.7c) to reflect the absence of zero modes of the operator ∂_+, and is thus a condition on the zero-mass sector of the theory, rather than an integrability condition. Of course we could require such a constraint also in the massless case. However, in that case we have in each sector two conservation laws, for I_- and $gi\partial_- g^{-1}$ separately!

In principle, the operator $\widetilde{\Omega}^{(2)}$ is not constrained to vanish in the general massive non-Abelian case. However, it is clear that it is a massless field. Even more, it contains only right-moving excitations. We can speculate whether such excitations are remnants of colour states. In that case, it would be rather desirable to require such an operator to vanish, in which case the fields constrained in this way would be free from such zero-mass colour excitations, as a consequence of this requirement. On the other hand, the operator I_-, which was connected, in the massless case, to an infinite number of conservation laws, is now no longer conserved on its own; instead, its conservation is spoiled by the mass term, and the previous conclusions for the integrability of the β sector can no longer be drawn. Instead, a relation to the g sector is established.

6. Conclusions

In this paper we have extended the analysis of massless QCD_2 from refs. [2, 8] to the case where the fermions are massive. In particular we have obtained the BRST currents in the non-local formulation of ref. [2], thereby extending the BRST analysis of ref. [8] to the massive case. It is interesting that these currents again turned out to be either right- or left-moving. As a result, the BRST condition implied restrictions on the massless sector of the physical Hilbert space in particular the existence of two first-class constraints depending only on one of the light-cone coordinates $x^\pm$. These first-class constraints were also obtained by appropriately gauging the partition function, following the method of Karabali and Schnitzer.[7]

In the massless case the solution of the associated cohomology problem for $SU(2)$ revealed a two-fold degeneracy of the right- and left-handed vacuum sector. In that case the physical states could be characterized in each sector by the eigenvalues of the third

component of the isospin of two conformal WZW currents and the ghost current. In the massive case, the situation remains the same for the ghost sector, but we are left with only one combination of these currents for each matter sector, serving to label the states. As a result we expect the ground state to no longer be degenerate. This would be in accordance with the well-known results on the massive Schwinger model. Work in this direction is in progress.

Finally, problems related to the description of physical properties of the model concerning chiral symmetry breaking[22] as well as the issue of screening versus confinement,[23] can be analysed.

Appendix

The implementation of the bosonization techniques of a non-Abelian symmetry is well known. In two dimensions we can locally write the gauge field in terms of two matrix-valued fields U and V as

$$A_+ = \frac{i}{e} U^{-1} \partial_+ U \quad , \quad A_- = \frac{i}{e} V \partial_- V^{-1} \quad . \tag{A1}$$

The effective action $W[A]$ is obtained by integrating the functional differential equations associated with the conservation of the vector current, and the anomaly in the axial vector current. One finds $W[A] = -\Gamma[UV]$, where $\Gamma[UV]$ is the Wess–Zumino–Witten (WZW) functional, obeying the Polyakov–Wiegmann identity

$$\Gamma[UV] = \Gamma[U] + \Gamma[V] + \frac{1}{4\pi} \text{tr} \int \mathrm{d}^2 x \, U^{-1} \partial_+ U V \partial_- V^{-1} \quad . \tag{A2}$$

In order to implement the change of variables $(A1)$, in the quantum theory, we still have to compute its Jacobian, that is

$$J = \det \frac{\delta A_+}{\delta U} \frac{\delta A_-}{\delta V} = \det \nabla = \mathrm{e}^{-ic_V \Gamma[UV]} \quad . \tag{A3}$$

It is well known that the invariances of the fermionic part of the QCD Lagrangian under local $SU(N)$, as well as $SU(N)_L \times SU(N)_R$ transformations, $U \rightarrow U w^{-1}$ and $V \rightarrow w^{-1} V$, are not symmetries of the effective action $W[A]$ due to the axial anomaly. As a consequence we find

$$\det i \, \slashed{D} \equiv \mathrm{e}^{iW[A]} = \int \mathcal{D}g \, \mathrm{e}^{iS_F[A,g]} \quad , \tag{A4}$$

where $S_F(A, g)$ plays the role of an equivalent bosonic action

$$S_F[A,g] = \Gamma[g] + \frac{1}{4\pi} \int \mathrm{d}^2 x \, \text{tr} \left[e^2 A^\mu A_\mu - e^2 A_+ g A_- g^{-1} - ei A_+ g \partial_- g^{-1} - ei A_- g^{-1} \partial_+ g \right]$$

$$= \Gamma[UgV^{-1}] - \Gamma[UV] \quad . \tag{A5}$$

Using $(A5)$, we have for the partition function

$$\mathcal{Z} = \int \mathcal{D}g\mathcal{D}A_\mu \, e^{i\int d^2z\left\{S_F[A,g]-\frac{1}{4}\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}\right\}} \quad . \tag{A6}$$

Using further the identity

$$e^{-\frac{i}{4}\int d^2z\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}} = \int \mathcal{D}E \, e^{-i\int d^2z\left[\frac{1}{2}\mathrm{tr}\,E^2+\frac{1}{2}\mathrm{tr}\,EF_{+-}\right]} \quad , \tag{A7}$$

where E is a matrix-valued field, we may rewrite $(A6)$ as

$$\mathcal{Z} = \int \mathcal{D}E\mathcal{D}U\mathcal{D}V\mathcal{D}g e^{i\Gamma[UgV]-i(c_V+1)\Gamma[UV]-i\int d^2z\,\mathrm{tr}\left[\frac{1}{2}E^2+\frac{1}{2}EF_{+-}\right]} \quad . \tag{A8}$$

Defining a new gauge-invariant field $\widetilde{g} = UgV$, and using the invariance of the Haar measure, $\mathcal{D}g = \mathcal{D}\widetilde{g}$, we see that the field $\widetilde{g}$ decouples in the partition function:

$$\mathcal{Z} = \int \mathcal{D}\widetilde{g} \, e^{i\Gamma[\widetilde{g}]} \int \mathcal{D}E\mathcal{D}U\mathcal{D}V\mathcal{D}(\text{ghosts}) e^{-i(c_V+1)\Gamma[UV]-i\int d^2z\,\mathrm{tr}\left[\frac{1}{2}E^2+\frac{1}{2}EF_{+-}\right]+iS_{\text{ghosts}}} . \tag{A9}$$

Introducing the new variable $\Sigma = UV$, we have the identity

$$\mathrm{tr}\,EF_{+-} = \frac{i}{e}\mathrm{tr}\,UEU^{-1}\partial_+(\Sigma\partial_-\Sigma^{-1}) \quad . \tag{A10}$$

It is natural to redefine variables as $\widetilde{E} \approx UEU^{-1}, \mathcal{D}E = \mathcal{D}\widetilde{E}$, where we have used again the invariance of the Haar measure. The partition function $(A9)$ then reduces to the form (we choose the gauge $U = 1$)

$$\mathcal{Z} = \int \mathcal{D}\widetilde{g} \, e^{i\Gamma[\widetilde{g}]}\mathcal{D}(\text{gh}) \, e^{iS_{\text{gh}}}\mathcal{D}\Sigma\mathcal{D}\widetilde{E}e^{-i(c_V+1)\Gamma[\Sigma]-(c_V+1)\mathrm{tr}\int d^2z\partial_+\widetilde{E}\Sigma\partial_-\Sigma^{-1}-2ie^2(c_V+1)^2\int d^2z\,\mathrm{tr}\,\widetilde{E}^2} . \tag{A11}$$

In order to arrive at the partition function (2.1) we perform a further change of variables:

$$\partial_+\widetilde{E} = \frac{i}{4\pi}\beta^{-1}\partial_+\beta \quad , \qquad \mathcal{D}\widetilde{E} = e^{-ic_V\Gamma[\beta]}\mathcal{D}\beta \quad . \tag{A12}$$

The partition function (2.1) is now obtained upon using the Polyakov–Wiegmann identity $(A2)$. When the partition function is written in terms of Σ, eq. $(A11)$, we talk about the local formulation. In the form (2.1) we say that it is the non-local formulation, since formal integration over the auxiliary field C leads to a non-local term in the β-fields.

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