View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by CERN Document Server

CERN-TH/95-294, DFUPG 111-95
UGVA-DPT 1995/11-907

arXiv:hep-th/9511168 v1 23 Nov 1995

Gauge Theories of Josephson Junction Arrays

M. C. Diamantini[1]

Theory Division, CERN, CH-1211 Geneva 23, Switzerland

P.Sodano

I.N.F.N. and Department of Physics, University of Perugia
via A. Pascoli, I-06100 Perugia, Italy

C. A. Trugenberger[2]

Department of Theoretical Physics, University of Geneva
24, quai E. Ansermet, CH-1211 Geneva 4, Switzerland

We show that the zero-temperature physics of planar Josephson junction arrays in the self-dual approximation is governed by an Abelian gauge theory with *periodic* mixed Chern-Simons term describing the charge-vortex coupling. The periodicity requires the existence of (Euclidean) topological excitations which determine the quantum phase structure of the model. The electric-magnetic duality leads to a quantum phase transition between a superconductor and a superinsulator at the self-dual point. We also discuss in this framework the recently proposed quantum Hall phases for charges and vortices in presence of external offset charges and magnetic fluxes: we show how the periodicity of the charge-vortex coupling can lead to transitions to anyon superconductivity phases. We finally generalize our results to three dimensions, where the relevant gauge theory is the so-called BF system with an antisymmetric Kalb-Ramond gauge field.

November 1995

[1] Supported by I.N.F.M., Section of Perugia, Italy

[2] Supported by a Profil 2 fellowship of the Swiss National Science Foundation

1. Introduction

Gauge fields can be used to model the long distance behaviour of several condensed matter systems [1], a connection which has been particularly exploited for planar systems [2] . In a nutshell, the idea is that charge fluctuations around a given ground state are decribed by a conserved current j^μ, which in (2+1) dimensions can be represented in terms of a gauge field B_μ according to $j^\mu \propto \epsilon^{\mu\alpha\nu}\partial_\alpha B_\nu$. For a wide class of systems the effective action governing the dynamics of the charge fluctuations is quadratic in the gauge fields B_μ at long distances [1]. Clearly this effective action is also gauge invariant, reflecting the original gauge invariance of the definition of the current j^μ: one obtains thus an effective gauge theory at long distances (which is not necessarily relativistic). The ground states of a wide class of planar condensed matter systems [3] can thus be classified according to the lowest derivative term appearing in their effective gauge theory at long distances. This way Chern-Simons terms describe incompressible quantum fluids (quantum Hall states) and chiral spin liquids [4] while the Maxwell term describes a (2-dim.) superfluid (superconductor) [1] [2].

In this paper we shall investigate a further connection between Abelian gauge theories and certain condensed matter systems, namely *Josephson junction arrays* [5] . In a recent publication [6] we studied non-perturbative features of the (2+1)-dimensional gauge theory with mixed Chern-Simons term [7]

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{2\pi}A_\mu\epsilon^{\mu\alpha\nu}\partial_\alpha B_\nu - \frac{1}{4g^2}f_{\mu\nu}f^{\mu\nu} \tag{1.1}$$

when the gauge symmetries associated with the two Abelian gauge fields A_μ and B_μ are compact. We also pointed out the relevance of (1.1) to the zero-temperature physics of planar Josephson junction arrays. Here we will derive and study this connection in detail.

After reviewing in section 2 the basic physics of (1.1) and our lattice notation, we shall show in section 3 that the zero-temperature partition function of Josephson junction arrays in the *self-dual approximation* coincides with the Euclidean partition function of the lattice version of (1.1) with *periodic* mixed Chern-Simons coupling. This means that the two gauge fields are compact variables only as far as their coupling is concerned. The periodicity is implemented by two types of topological excitations [8] which constitute electric and magnetic closed loops with short-range interactions. The two energy scales of Josephson junction arrays, the charging energy E_C and the Josephson coupling E_J are directly related to the two massive parameters e^2 and g^2 of (1.1).

In section 4 we investigate the non-perturbative structure of this Chern-Simons lattice gauge model. The phase structure is determined by the 3-dimensional statistical mechanics of the topological excitations and reflects the self-duality of the model. We find three possible phases at zero temperature. For small e/g there is a *superconducting* phase with logarithmic confinement of magnetic fluxes; in this phase the original R_A gauge symmetry of (1.1) is broken down to Z_A so that the full symmetry is given by $Z_A \times R_B$. Correspondingly, one of the two massive excitations of (1.1) becomes massless. The dual phase is realized for large e/g. In this phase we have logarithmic confinement of electric charges and symmetry $R_A \times Z_B$, with a corresponding massless excitation. An infinite energy (voltage) is required to separate the charge dipoles and produce a current through the sample: we call this phase with infinite resistance a *"superinsulator"*. Depending on the details of the lattice, a third phase can open up between the superconductor and the superinsulator. In this phase the topological excitations are irrelevant, the symmetry is $R_A \times R_B$ and both excitations are massive. The amount of energy required to produce a current through the sample is exponentially small. In [6] we called this phase the Chern-Simons phase: in presence of dissipation it would actually correspond to a *"metallic"* phase of the model [9] . The superconductor-insulator quantum phase transition is actually observed experimentally in planar Josephson junction arrays at very low temperatures [5].

Recently it has been suggested that Josephson junction arrays in presence of n_q offset charges and n_ϕ external magnetic fluxes per plaquette might have quantum Hall phases [3] for either charges [10] or fluxes [11] [12] , depending on the ratios n_q/n_ϕ and E_C/E_J. In section 5 we discuss these purely two-dimensional quantum Hall states in the framework of the gauge theory representation. Specifically, we show that they can be described by additional pure Chern-Simons terms for either one of the two gauge fields A_μ or B_μ. In this phases the charges and vortices combine to form an *incompressible quantum fluid* [13] of charge-flux composites with short-range interactions. Localized excitations are charge and flux carrying anyons [14] .

We then investigate how one of the distinctive feature of Josephson junction arrays, namely the periodicity of charge-vortex couplings affects these quantum Hall states. We find that this periodicty can induce two types of phase transitions. The charge-flux fluid corresponding to the charge quantum Hall phase can either expel the flux and form a charge superfluid corresponding to a conventional superconductor or condense into a charge-flux superfluid. Correspondingly, the flux-charge fluid corresponding to the vortex quantum Hall phase can either expel the charge and form a flux superfluid corresponding to a

superinsulator or condense into a flux-charge superfluid. These superfluids of charge-flux composites are (logarithmic) *oblique confinement phases* [15] [16] corresponding to *anyon superconductors* [17] . We thus conclude that Josephson junction arrays might provide the first explicit realization of the anyon superconductivity mechanism.

In section 6 we generalize our results to three dimensions (even if three-dimensional Josephson junction arrays have not yet been fabricated). In this case, one of the two gauge fields becomes an antisymmetric Kalb-Ramond tensor gauge field [18] and the (3+1)-dimensional gauge theory we obtain is the so-called BF-model [19] . This is an Abelian gauge model with a conventional Maxwell gauge field and a Kalb-Ramond gauge field coupled by a topological mass term. In three dimensions the magnetic topological excitations become compact surfaces on the lattice and self-duality is lost. The zero-temperature phase structure is determined by the statistical mechanics of a model of coupled random loops and random surfaces in four Euclidean dimensions: this can also be viewed as the Euclidean partition function for a lattice model of particles interacting with closed Nielsen-Olesen type strings. While the statistical mechanics of random loops is by now well developed [8] [20] there is no corresponding amount of analytical results for random surfaces [21] . Assuming three distinct phases as in (2+1) dimensions, with condensation of electric loops, no condensation of topological excitations and condensation of magnetic surfaces we can identify the first two again with superconducting and metallic phases, respectively. In the phase with condensation of magnetic surfaces the charge dipoles are bound by $1/r$ potentials, which are long-range but not confining. Therefore only a finite amount of energy is required to separate them and the system behaves as an insulator (as opposed to a superinsulator in two dimensions).

2. The lattice Chern-Simons model

Our model (1.1) can be rewritten in terms of the dual field strengths

$$\begin{aligned} F^\mu &\equiv \frac{1}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta} \ , \qquad & F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu \ , \\ f^\mu &\equiv \frac{1}{2}\epsilon^{\mu\alpha\beta}f_{\alpha\beta} \ , \qquad & f_{\mu\nu} &\equiv \partial_\mu B_\nu - \partial_\nu B_\mu \ , \end{aligned} \tag{2.1}$$

as follows *

$$\mathcal{L}_{CS} = -\frac{1}{2e^2}\left(\frac{1}{\eta}F_0F^0 + F_iF^i\right) + \frac{\kappa}{2\pi}A_\mu\epsilon^{\mu\alpha\nu}\partial_\alpha B_\nu - \frac{1}{2g^2}\left(\frac{1}{\eta}f_0f^0 + f_if^i\right) \ . \tag{2.2}$$

* Throughout this paper we use units such that $c = 1$ and $\hbar = 1$.

For later convenience we have introduced a magnetic permeability η, equal for the two gauge fields. The coupling constants e^2 and g^2 have dimension mass, whereas the coefficient κ of the mixed Chern-Simons term is dimensionless. Note that we take B_μ to represent a pseudovector gauge field, so that the mixed Chern-Simons term does not break the discrete symmetries of parity and time reversal.

The action corresponding to (2.2) is separately invariant under the two Abelian gauge transformations

$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu \lambda \ , \\ B_\mu &\to B_\mu + \partial_\mu \omega \ , \end{aligned} \tag{2.3}$$

with gauge groups R_A and R_B, respectively. Moreover, the action is also invariant under the *duality transformation*

$$\begin{aligned} A_\mu &\leftrightarrow B_\mu \ , \\ e &\leftrightarrow g \ , \end{aligned} \tag{2.4}$$

so that the model is *self-dual*.

The Lagrangian (2.2) can be easily diagonalized by the linear transformation

$$\begin{aligned} A_\mu &= \sqrt{\frac{e}{g}}\,(a_\mu + b_\mu) \ , \\ B_\mu &= \sqrt{\frac{g}{e}}\,(a_\mu - b_\mu) \ . \end{aligned} \tag{2.5}$$

In terms of these new variables the model (2.2) describes a free theory,

$$\mathcal{L}_{CS} = -\frac{1}{eg}\left(\frac{1}{\eta}G_0 G^0 + G_i G^i\right) + \frac{\kappa}{2\pi} a_\mu \epsilon^{\mu\alpha\nu}\partial_\alpha a_\nu - \frac{1}{eg}\left(\frac{1}{\eta}g_0 g^0 + g_i g^i\right) - \frac{\kappa}{2\pi} b_\mu \epsilon^{\mu\alpha\nu}\partial_\alpha b_\nu \ , \tag{2.6}$$

where G^μ and g^μ are the dual field strengths for the new gauge fields a_μ and b_μ, respectively. This Lagrangian describes a doublet of excitations with topological mass [22]

$$m = \frac{|\kappa| eg}{2\pi} \ , \tag{2.7}$$

and spectrum

$$E(\mathbf{q}) = \sqrt{m^2 + \frac{1}{\eta}|\mathbf{q}|^2} \ . \tag{2.8}$$

In the following we shall formulate a Euclidean lattice version of the above Chern-Simons model. To this end we introduce a three-dimensional rectangular lattice with lattice spacings l_μ in the three directions. In particular we shall take the lattice spacings

$l_1 = l_2 \equiv l$ and identify l_0 with the spacing in the Euclidean time direction. Lattice sites are denoted by the three-dimensional vector x; the gauge fields $A_\mu(x)$ and $B_\mu(x)$ are associated with the links (x, μ) between the sites x and $x + \hat{\mu}$, where $\hat{\mu}$ denotes a unit vector in direction μ on the lattice.

On the lattice we introduce the following forward and backward derivatives and shift operators:

$$\begin{aligned} d_\mu f(x) &\equiv \frac{f(x + l_\mu \hat{\mu}) - f(x)}{l_\mu} \ , & S_\mu f(x) &\equiv f(x + l_\mu \hat{\mu}) \ , \\ \hat{d}_\mu f(x) &\equiv \frac{f(x) - f(x - l_\mu \hat{\mu})}{l_\mu} \ , & \hat{S}_\mu f(x) &\equiv f(x - l_\mu \hat{\mu}) \ . \end{aligned} \tag{2.9}$$

Summation by parts on the lattice interchanges both the two derivatives (with a minus sign) and the two shift operators; gauge transformations are defined using the forward lattice derivative. Corresponding to the two derivatives in (2.9), we can define also two lattice analogues of the Chern-Simons operators $\epsilon_{\mu\alpha\nu}\partial_\alpha$ [23] [6]:

$$k_{\mu\nu} \equiv S_\mu \epsilon_{\mu\alpha\nu} d_\alpha \ , \qquad \hat{k}_{\mu\nu} \equiv \epsilon_{\mu\alpha\nu} \hat{d}_\alpha \hat{S}_\nu \ , \tag{2.10}$$

where no summation is implied over equal indices μ and ν. Summation by parts on the lattice interchanges also these two operators (without an extra minus sign). The operators (2.10) are both local and gauge invariant, in the sense that

$$k_{\mu\nu} d_\nu = \hat{d}_\mu k_{\mu\nu} = 0 \ , \qquad \hat{k}_{\mu\nu} d_\nu = \hat{d}_\mu \hat{k}_{\mu\nu} = 0 \ , \tag{2.11}$$

and their product reproduces the relativistic, Euclidean lattice Maxwell operator:

$$k_{\mu\alpha} \hat{k}_{\alpha\nu} = \hat{k}_{\mu\alpha} k_{\alpha\nu} = -\delta_{\mu\nu} \nabla^2 + d_\mu \hat{d}_\nu \ , \tag{2.12}$$

where $\nabla^2 \equiv \hat{d}_\mu d_\mu$ is the three-dimensional Laplace operator. Using $k_{\mu\nu}$ we can also define the lattice dual field strengths as

$$\begin{aligned} F_\mu &\equiv \hat{k}_{\mu\nu} A_\nu \ , \\ f_\mu &\equiv k_{\mu\nu} B_\nu \ . \end{aligned} \tag{2.13}$$

The identity (2.12) then tells us that we can simply write the relativistic, Euclidean lattice Maxwell terms as $\sum_x F_\mu F_\mu$ and $\sum_x f_\mu f_\mu$.

Using all these definitions we can now write the Euclidean lattice partition function of our model (2.2) as follows:

$$\begin{aligned} Z_{CS} &= \int \mathcal{D}A_\mu \int \mathcal{D}B_\mu \, \exp(-S_{CS}) \ , \\ S_{CS} &= \sum_x \frac{l_0 l^2}{2e^2}\left(\frac{1}{\eta}F_0F_0 + F_iF_i\right) - i\frac{l_0 l^2 \kappa}{2\pi}A_\mu k_{\mu\nu}B_\nu + \frac{l_0 l^2}{2g^2}\left(\frac{1}{\eta}f_0f_0 + f_if_i\right) \ , \end{aligned} \tag{2.14}$$

where we have introduced the notation $\mathcal{D}A_\mu \equiv \prod_{(x,\mu)} dA_\mu(x)$ and gauge fixing is understood.

For later convenience we introduce also the finite difference operators

$$\Delta_\mu \equiv l_\mu d_\mu \ , \qquad\qquad \hat{\Delta}_\mu \equiv l_\mu \hat{d}_\mu \ , \tag{2.15}$$

where no summation over equal indices is implied. Correspondingly, we introduce also the finite difference analogue of the operators $k_{\mu\nu}$ and $\hat{k}_{\mu\nu}$:

$$K_{\mu\nu} \equiv S_\mu \epsilon_{\mu\alpha\nu}\Delta_\alpha \ , \qquad\qquad \hat{K}_{\mu\nu} \equiv \epsilon_{\mu\alpha\nu}\hat{\Delta}_\alpha \hat{S}_\nu \ . \tag{2.16}$$

These satisfy equations analogous to (2.11) and (2.12) with all derivatives substituted by finite differences.

3. Josephson junction arrays

Josephson junction arrays [5] are quadratic, planar arrays of spacing l of superconducting islands with nearest neighbours Josephson couplings of strength E_J. Each island has a capacitance C_0 to the ground; moreover there are also nearest neighbours capacitances C. The Hamiltonian characterizing such systems is thus given by

$$H = \sum_{\mathbf{x}} \frac{C_0}{2} V_{\mathbf{x}} + \sum_{<\mathbf{xy}>} \left(\frac{C}{2}\left(V_{\mathbf{y}} - V_{\mathbf{x}}\right)^2 + E_J\left(1 - \cos\, N\left(\Phi_{\mathbf{y}} - \Phi_{\mathbf{x}}\right)\right)\right) \ , \tag{3.1}$$

where boldface characters denote the sites of the two-dimensional array, $<\mathbf{xy}>$ indicates nearest neighbours, $V_{\mathbf{x}}$ is the electric potential of the island at $\mathbf{x}$ and $\Phi_{\mathbf{x}}$ the phase of its order parameter. For generality we allow for any integer N in the Josephson coupling, so that the phase has periodicity $2\pi/N$: obviously $N = 2$ for the real systems.

With the notation introduced in the previous section the Hamiltonian (3.1) can be rewritten as

$$H = \sum_{\mathbf{x}} \frac{1}{2} V \left(C_0 - C\Delta \right) V + \sum_{\mathbf{x},i} E_J \left(1 - \cos N \left(\Delta_i \Phi \right) \right) \ , \tag{3.2}$$

where $\Delta \equiv \hat{\Delta}_i \Delta_i$ is the two-dimensional finite difference Laplacian and we have omitted the explicit location indeces on the variables V and Φ.

The phases $\Phi_{\mathbf{x}}$ are quantum-mechanically conjugated to the charges $Q_{\mathbf{x}}$ on the islands: these are quantized in integer multiples of N (Cooper pairs for $N = 2$):

$$Q = q_e N p_0 \ , \qquad\qquad p_0 \in Z \ , \tag{3.3}$$

where q_e is the electron charge. The Hamiltonian (3.2) can be expressed in terms of charges and phases by noting that the electric potentials $V_{\mathbf{x}}$ are determined by the charges $Q_{\mathbf{x}}$ via a discrete version of Poisson's equation:

$$\left(C_0 - C\Delta \right) V_{\mathbf{x}} = Q_{\mathbf{x}} \ . \tag{3.4}$$

Using this in (3.2) we get

$$H = \sum_{\mathbf{x}} N^2 E_C \ p_0 \frac{1}{\frac{C_0}{C} - \Delta} p_0 + \sum_{\mathbf{x},i} E_J \left(1 - \cos N \left(\Delta_i \Phi \right) \right) \ , \tag{3.5}$$

where $E_C \equiv q_e^2 / 2C$. The integer charges p_0 interact via a two-dimensional Yukawa potential of mass $\sqrt{C_0/C}/l$. In the nearest-neighbours capacitance limit $C \gg C_0$, which is accessible experimentally, this becomes essentially a two-dimensional Coulomb law. From now on we shall consider the limiting case $C_0 = 0$. In this case the *charging energy* E_C and the *Josephson coupling* E_J are the two relevant energy scales in the problem. These two massive parameters can also be traded for one massive parameter $\sqrt{2N^2 E_C E_J}$, which represents the *Josephson plasma frequency* and one massless parameter E_J / E_C.

The zero-temperature partition function of the Josephson junction array admits a (phase-space) path-integral representation [24] . Since the variables p_0 are integers, the imaginary-time integration has to be performed stepwise; we introduce therefore a lattice spacing l_0 also in the imaginary-time direction. This has to be just smaller of the typical

time scale on which the integers p_0 vary, in the present case the inverse of the Josephson plasma frequency: $l_0 \leq O\left(1/\sqrt{2N^2 E_C E_J}\right)$. We thus get the following partition function:

$$\begin{aligned} Z &= \sum_{\{p_0\}} \int_{-\pi/N}^{+\pi/N} \mathcal{D}\Phi \, \exp(-S) \; , \\ S &= \sum_x -iN \; p_0 \Delta_0 \Phi + N^2 E_C l_0 \; p_0 \frac{1}{-\Delta} p_0 + \sum_{x,i} \; l_0 E_J \left(1 - \cos \; N \left(\Delta_i \Phi\right)\right) \; , \end{aligned} \tag{3.6}$$

where now the sum in the action S extends over the three-dimensional lattice with spacing l_0 in the imaginary time direction and l in the spatial directions.

In the next step we introduce vortex degrees of freedom by replacing the Josephson term by its Villain form [25] :

$$\begin{aligned} Z &= \sum_{\substack{\{p_0\} \\ \{v_i\}}} \int_{-\pi/N}^{+\pi/N} \mathcal{D}\Phi \, \exp(-S) \; , \\ S &= \sum_x -iN \; p_0 \Delta_0 \Phi + N^2 E_C l_0 \; p_0 \frac{1}{-\Delta} p_0 + N^2 l_0 \frac{E_J}{2} \left(\Delta_i \Phi + \frac{2\pi}{N} v_i\right)^2 \; . \end{aligned} \tag{3.7}$$

Strictly speaking, this substitution is valid only for $l_0 E_J \gg 1$; however the Villain approximation retains all most relevant features of the Josephson coupling for the whole range of values of the coupling E_J [25] and therefore we shall henceforth adopt it.

We now represent the Villain term as a Gaussian integral over real variables p_i and we transform also p_0 to a real variable by introducing new integers v_0 via the Poisson summation formula

$$\sum_{k=-\infty}^{k=+\infty} \exp(i2\pi k z) = \sum_{n=-\infty}^{n=+\infty} \delta(z - n) \; . \tag{3.8}$$

By grouping together the real and integer p and v variables into three-vectors p_μ and v_μ, $\mu = 0, 1, 2$ we can write the partition function as

$$\begin{aligned} Z &= \sum_{\{v_\mu\}} \int \mathcal{D}p_\mu \int_{-\pi/N}^{+\pi/N} \mathcal{D}\Phi \, \exp(-S) \; , \\ S &= \sum_x -iN p_\mu \left(\Delta_\mu \Phi + \frac{2\pi}{N} v_\mu\right) + N^2 E_C l_0 \; p_0 \frac{1}{-\Delta} p_0 + \frac{p_i^2}{2 l_0 E_J} \; . \end{aligned} \tag{3.9}$$

Following [8] we use the longitudinal part of the integer vector field v_μ to shift the integration domain of Φ. To this end we decompose v_μ as follows:

$$v_\mu = \Delta_\mu m + \Delta_\mu \alpha + K_{\mu\nu} \psi_\nu \; , \tag{3.10}$$

where $m \in Z$, $|\alpha| < 1$ and $K_{\mu\nu}$ defined in (2.16). Here the vectors ψ_μ are not integer, but they are nonetheless restricted by the fact that the combinations $q_\mu \equiv \hat{K}_{\mu\nu} v_\nu = \hat{K}_{\mu\alpha} K_{\alpha\nu} \psi_\nu$ must be integers. The original sum over the three independent integers $\{v_\mu\}$ can thus be traded for a sum over the four integers $\{m, q_\mu\}$ subject to the constraint $\hat{\Delta}_\mu q_\mu = 0$. The sum over the integers $\{m\}$ can then be used to shift the Φ integration domain from $[-\pi/N, +\pi/N)$ to $(-\infty, +\infty)$. The integration over Φ is now trivial and enforces the constraint $\hat{\Delta}_\mu p_\mu = 0$:

$$\begin{aligned} Z &= \sum_{\{q_\mu\}} \delta_{\hat{\Delta}_\mu q_\mu, 0} \int \mathcal{D}p_\mu \; \delta\left(\hat{\Delta}_\mu p_\mu\right) \; \exp(-S) \; , \\ S &= \sum_x -i2\pi \; p_\mu K_{\mu\nu} \psi_\nu + N^2 l_0 E_C \; p_0 \frac{1}{-\Delta} p_0 + \frac{p_i^2}{2 l_0 E_J} \; . \end{aligned} \tag{3.11}$$

We now solve the two constraints by introducing a real gauge field b_μ and an integer gauge field a_μ:

$$\begin{aligned} p_\mu &\equiv K_{\mu\nu} b_\nu \; , \qquad & b_\mu &\in R \; , \\ q_\mu &\equiv \hat{K}_{\mu\nu} a_\nu \; , \qquad & a_\mu &\in Z \; . \end{aligned} \tag{3.12}$$

By inserting the first of these two equations and by summing by parts, the first term in the action (3.11) reduces to $\sum_x -i2\pi b_\mu q_\mu$. By inserting the second of the above equations and by summing by parts again, this term of the action finally reduces to the mixed Chern-Simons coupling $\sum_x -i2\pi \; a_\mu K_{\mu\nu} b_\nu$. Using the Poisson formula (3.8) we can finally make a_μ also real at the expense of introducing a set of integer link variables $\{Q_\mu\}$ satisfying the constraint $\hat{\Delta}_\mu Q_\mu$, which guarantees gauge invariance:

$$\begin{aligned} Z &= \sum_{\{Q_\mu\}} \int \mathcal{D}a_\mu \int \mathcal{D}b_\mu \; \exp(-S) \; , \\ S &= \sum_x -i2\pi \; a_\mu K_{\mu\nu} b_\nu + N^2 l_0 E_C \; p_0 \frac{1}{-\Delta} p_0 + \frac{p_i^2}{2 l_0 E_J} + i2\pi a_\mu Q_\mu \; . \end{aligned} \tag{3.13}$$

In this representation $K_{\mu\nu} b_\nu$ represents the conserved three-current of charges, while $\hat{K}_{\mu\nu} a_\nu$ represents the conserved three-current of vortices. Note that, actually, both these conserved currents are integers (the factors of N are explicit): indeed, the summation over $\{Q_\mu\}$ makes a_μ (and therefore also $\hat{K}_{\mu\nu} a_\nu$) an integer, and then the summation over $\{a_\mu\}$ makes $K_{\mu\nu} b_\nu$ an integer. The third term in the action (3.13) contains two parts: the longitudinal part $\left(p_i^L\right)^2$ describes the Josephson currents and represents a kinetic term for

the charges; the transverse part $\left(p_i^T\right)^2$ can be rewritten as a Coulomb interaction term for the vortex density q_0 by solving the Gauss law enforced by the Lagrange multiplier b_0.

The partition function (3.13) displays a high degree of symmetry between the charge and the vortex degrees of freedom. The only term which breaks this symmetry (apart from the integers Q_μ) is encoded in the kinetic term for the charges (Josephson currents). This near-duality between charges and vortices has already been often invoked in the literature [5] to explain the experimental quantum phase diagram at very low temperatures. Here we introduce what we call the *self-dual approximation* of Josephson junction arrays. This consists in adding to the action in (3.13) a bare kinetic term for the vortices * and combining this with the Coulomb term for the charges into $\sum_x \frac{\pi^2}{N^2 l_0 E_C} q_i^2$. The coefficient is chosen so that the transverse part of this term reproduces exactly the Coulomb term for the charges upon solving the Gauss law enforced by the Lagrange multiplier a_0. The longitudinal part, instead, represents the additional bare kinetic term for the vortices. Given that now the gauge field a_μ has acquired a kinetic term, we are also forced to introduce new integers M_μ via the Poisson formula to guarantee that the charge current $K_{\mu\nu} b_\nu$ remains an integer:

$$\begin{aligned} Z_{SD} &= \sum_{\substack{\{Q_\mu\} \\ \{M_\mu\}}} \int \mathcal{D}a_\mu \int \mathcal{D}b_\mu \, \exp(-S_{SD}) \ , \\ S_{SD} &= \sum_x -i2\pi \ a_\mu K_{\mu\nu} b_\nu + \frac{p_i^2}{2 l_0 E_J} + \frac{\pi^2 q_i^2}{N^2 l_0 E_C} + i2\pi a_\mu Q_\mu + i2\pi b_\mu M_\mu \ , \end{aligned} \tag{3.14}$$

where the new integers satisfy the constraint $\hat{\Delta}_\mu M_\mu = 0$ to guarantee gauge invariance. After a rescaling

$$\begin{aligned} A_0 &\equiv \frac{2\pi}{\sqrt{N} l_0} a_0 \ , & A_i &\equiv \frac{2\pi}{\sqrt{N} l} a_i \ , \\ B_0 &\equiv \frac{2\pi}{\sqrt{N} l_0} b_0 \ , & B_i &\equiv \frac{2\pi}{\sqrt{N} l} b_i \ , \end{aligned} \tag{3.15}$$

we obtain finally

$$\begin{aligned} Z_{SD} &= \sum_{\substack{\{Q_\mu\} \\ \{M_\mu\}}} \int \mathcal{D}A_\mu \int \mathcal{D}B_\mu \, \exp(-S_{SD}) \ , \\ S_{SD} &= \sum_x \frac{l_0 l^2}{2e^2} \ F_i F_i - i \frac{l_0 l^2 \kappa}{2\pi} A_\mu k_{\mu\nu} B_\nu + \frac{l_0 l^2}{2g^2} \ f_i f_i \\ &\qquad + i\sqrt{\kappa} \left(l_0 Q_0 A_0 + l Q_i A_i \right) + i\sqrt{\kappa} \left(l_0 M_0 B_0 + l M_i B_i \right) \ , \end{aligned} \tag{3.16}$$

* Note that such a kinetic term is anyhow induced by integrating out the charge degrees of freedom.

where F_i and f_i are defined in (2.13) and

$$e^2 = 2NE_C \ , \qquad \kappa = N \ , \qquad g^2 = \frac{4\pi^2}{N} E_J \ . \tag{3.17}$$

This is exactly the partition function of our lattice Chern-Simons model (2.14) in the limit of infinite magnetic permeability $\eta = \infty$ and with additional, integer-valued link variables Q_μ and M_μ coupled to the two gauge fields. Note that, with the above identifications, the topological Chern-Simons mass (2.7) coincides with the Josephson plasma frequency:

$$m = \sqrt{2N^2 E_C E_J} \ . \tag{3.18}$$

In the physical case $N = 2$ this reduces to $m = \sqrt{8E_C E_J}$. From the kinetic terms in (3.16) we can also read off the charge and vortex masses:

$$\begin{aligned} m_q &= \frac{1}{l^2 g^2} = \frac{N}{4\pi^2 l^2 E_J} \ , \\ m_\phi &= \frac{1}{l^2 e^2} = \frac{1}{2N l^2 E_C} \ . \end{aligned} \tag{3.19}$$

In the regime $ml \leq O(1)$, which is typically experimentally relevant, we can choose $l_0 = l$: in this case the infinite magnetic permeability constitutes the only non-relativistic effect in the physics of Josephson junction arrays in the self-dual approximation. However, we expect this non-relativistic effect to be irrelevant as far as the phase structure and the charge-vorticity assignements are concerned. Therefore, for simplicity, we shall henceforth consider the relativistic model, by setting $l_0 = l$ and $\eta = 1$, although it is not hard to incorporate a generic value of η into our subsequent formalism:

$$\begin{aligned} Z_{SD} &= \sum_{\substack{\{Q_\mu\} \\ \{M_\mu\}}} \int \mathcal{D}A_\mu \int \mathcal{D}B_\mu \, \exp(-S_{SD}) \ , \\ S_{SD} &= \sum_x \frac{l^3}{2e^2} F_\mu F_\mu - i\frac{l^3\kappa}{2\pi} A_\mu k_{\mu\nu} B_\nu + \frac{l^3}{2g^2} f_\mu f_\mu + il\sqrt{\kappa} A_\mu Q_\mu + il\sqrt{\kappa} B_\mu M_\mu \ . \end{aligned} \tag{3.20}$$

Josephson junction arrays in the self-dual approximation constitute thus a further, experimentally accessible example of the ideas presented in [1] and [2]. The action in (3.20) provides in fact a pure gauge theory representation of a model of interacting charges and vortices, represented by the conserved currents

$$\begin{aligned} q_\mu^{\text{charge}} &\equiv \frac{\kappa^{\frac{3}{2}}}{2\pi} \ k_{\mu\nu} B_\nu \ , \\ \phi_\mu^{\text{vortex}} &\equiv \frac{1}{2\pi\kappa^{\frac{1}{2}}} \ \hat{k}_{\mu\nu} A_\nu \ , \end{aligned} \tag{3.21}$$

where the prefactors are chosen so that the quantum of charge is given by κ, while the quantum of vorticity is given by $1/\kappa$ (factors of q_e and 2π are absorbed in the definitions of the gauge fields and the coupling constants).

In this framework, the mixed Chern-Simons term represents both the Lorentz force caused by vortices on charges (coupling of $q_\mu^{\rm charge}$ to the "electric" gauge field A_μ) and, by a summation by parts, the Magnus force [26] caused by charges on vortices (coupling of $\phi_\mu^{\rm vortex}$ to the "magnetic" gauge field B_μ). The integer-valued link variables Q_μ and M_μ represent the (Euclidean) *topological excitations* [8] in the model. They satisfy the constraints

$$\begin{aligned} \hat{d}_\mu Q_\mu &= 0 \ , \\ \hat{d}_\mu M_\mu &= 0 \ . \end{aligned} \tag{3.22}$$

In a dilute phase they constitute closed electric (Q_μ) and magnetic (M_μ) loops on the lattice; in a dense phase there is the additional possibility of infinitely long strings. Due to the constraints (3.22) we can choose to represent these topological excitations as

$$\begin{aligned} Q_\mu &\equiv l k_{\mu\nu} Y_\nu \ , \qquad & Y_\nu &\in Z \ , \\ M_\mu &\equiv l \hat{k}_{\mu\nu} X_\nu \ , \qquad & X_\mu &\in Z \ , \end{aligned} \tag{3.23}$$

and reabsorb them in the mixed Chern-Simons term as follows:

$$S_{SD} = \sum_x \ldots - i \frac{l^3 \kappa}{2\pi} \left(A_\mu - \frac{2\pi}{l\sqrt{\kappa}} X_\mu \right) k_{\mu\nu} \left(B_\nu - \frac{2\pi}{l\sqrt{\kappa}} Y_\mu \right) + \ldots \ . \tag{3.24}$$

In this representation it is clear that the topological excitations render the charge-vortex coupling *periodic* under the shifts

$$\begin{aligned} A_\mu &\to A_\mu + \frac{2\pi}{l\sqrt{\kappa}} \ a_\mu \ , \qquad & a_\mu &\in Z \ , \\ B_\mu &\to B_\mu + \frac{2\pi}{l\sqrt{\kappa}} \ b_\mu \ , \qquad & b_\mu &\in Z \ . \end{aligned} \tag{3.25}$$

In physical terms, the topological excitations implement the well-known [3] periodicity of the charge dynamics under the addition of an integer multiple of the flux quantum $1/\kappa$ per plaquette and the (less-known) periodicity of the vortex dynamics under the addition of an integer multiple of the charge quantum κ per site.

If we would require that the full action (including charge-charge and vortex-vortex interactions) (3.20) be periodic under the shifts (3.25), then we would obtain the compact Chern-Simons model studied in [6]. In this case the relevant topological excitations would be essentially iX_μ and iY_μ: since these can also describe finite open strings, there is the additional possibility of electric and magnetic monopoles [8]. As we showed in [6], these monopoles play a crucial role in the regime $ml \ll 1$.

4. Phase structure analysis

In this section we investigate symmetry aspects and non-perturbative features of the model (3.20) due to the periodicity of the charge-vortex interactions encoded in the mixed Chern-Simons term. As expected, these depend entirely on the topological excitations which enforce the periodicity.

Upon a Gaussian integration the partition function (3.20) factorizes readily as

$$Z_{SD} = Z_{CS} \cdot Z_{\rm Top} \ , \tag{4.1}$$

where Z_{CS} is the pure gauge part defined in (2.14) and

$$\begin{aligned} Z_{\rm Top} &= \sum_{\substack{\{Q_\mu\} \\ \{M_\mu\}}} \exp\left(-S_{\rm Top}\right) \ , \\ S_{\rm Top} &= \sum_x \frac{e^2\kappa}{2l}\ Q_\mu \frac{\delta_{\mu\nu}}{m^2-\nabla^2} Q_\nu + \frac{g^2\kappa}{2l}\ M_\mu \frac{\delta_{\mu\nu}}{m^2-\nabla^2} M_\nu \\ &\qquad + i\frac{2\pi m^2}{l}\ Q_\mu \frac{k_{\mu\nu}}{\nabla^2\left(m^2-\nabla^2\right)} M_\nu \ , \end{aligned} \tag{4.2}$$

with m defined in (2.7), describes the contribution due to the topological excitations. The phase structure of our model is thus determined by the statistical mechanics of a coupled gas of closed or infinitely long electric and magnetic strings with short-range Yukawa interactions. The scale $(1/m)$ represents the width of these strings. In our case it is of the order of the lattice spacing l. The third term in the action (4.2), describing the topological Aharonov-Bohm interaction of electric and magnetic strings, vanishes for strings separated by distances much bigger than $(1/m)$: in this case the denominator reduces to $m^2\nabla^2$ and, by using either one of the two equations in (3.23) and the constraints (3.22) one recognizes immediately that the whole term in the action reduces to ($i2\pi$integer), which is equivalent to 0 *.

4.1. Free energy arguments

In order to establish the phase diagram of our model we use the free energy arguments for strings introduced in [27] and extensively used for the analysis of four-dimensional self-dual models [16].

* This reflects the fact that the original charges and vortices satisfy the Dirac quantization condition.

The usual argument for strings with Coulomb interactions [27] is that interactions between strings are unimportant for the phase structure because small strings interact via short-range dipole interactions, while large strings have most of their multipole moments canceled by fluctuations. This argument is even stronger in our case, where the interaction is anyway short-range. Therefore one retains only the self-energy of strings, which is proportional to their length, and phase transitions from dilute to dense phases appear when the entropy of large strings, also proportional to their length, overwhelms the self-energy. We shall also neglect the interaction term between electric and magnetic strings (imaginary term in the action (4.2)). This is clearly a good approximation if both types of topological excitations are dilute.

Thus, one assigns a free energy

$$F = \left(\frac{le^2\kappa}{2} G(ml)\ Q^2 + \frac{lg^2\kappa}{2} G(ml) M^2 - \mu \right)\ N \tag{4.3}$$

to a string of length $L = lN$ carrying electric and magnetic quantum numbers Q and M, respectively. Here $G(ml)$ is the diagonal element of the lattice kernel $G(x-y)$ representing the inverse of the operator $l^2\left(m^2 - \nabla^2\right)$. Clearly $G(ml)$ is a function of the dimensionless parameter ml. The last term in (4.3) represents the entropy of the string: the parameter μ is given roughly by $\mu = \ln 5$, since at each step the string can choose between 5 different directions. In (4.3) we have neglected all subdominant functions of N, like a $\ln N$ correction to the entropy.

The condition for condensation of topological excitations is obtained by minimizing the free energy (4.3) as a function of N. If the coefficient of N in (4.3) is positive, the minimum is obtained for $N = 0$ and topological excitations are suppressed. If, instead, the same coefficient is negative, the minimum is obtained for $N = \infty$ and the system will favour the formation of large closed loops and infinitely long strings. Topological excitations with quantum numbers Q and M condense therefore if

$$\frac{le^2\kappa G(ml)}{2\mu}\ Q^2 + \frac{lg^2\kappa G(ml)}{2\mu}\ M^2 < 1\ . \tag{4.4}$$

If two or more condensations are allowed by this condition one has to choose the one with the lowest free energy.

The condition (4.4) describes the interior of an ellipse with semi-axes $2\mu/(le^2\kappa G(ml))$ and $2\mu/(lg^2\kappa G(ml))$ on a square lattice of integer electric and magnetic charges. The phase diagram is obtained by investigating which points of the integer lattice lie inside the

ellipse as its semi-axes are varied. We find it convenient to present the results in terms of the dimensionless parameters lm and e/g:

$$\begin{aligned}
&\frac{mlG(ml)\pi}{\mu} < 1 \rightarrow \begin{cases} \frac{e}{g} < 1 \ , & \text{electric condensation} \ , \\ \frac{e}{g} > 1 \ , & \text{magnetic condensation} \ , \end{cases} \\
&\frac{mlG(ml)\pi}{\mu} > 1 \rightarrow \begin{cases} \frac{e}{g} < \frac{\mu}{mlG(ml)\pi} \ , & \text{electric condensation} \ , \\ \frac{\mu}{mlG(ml)\pi} < \frac{e}{g} < \frac{mlG(ml)\pi}{\mu} \ , & \text{no condensation} \ , \\ \frac{e}{g} > \frac{mlG(ml)\pi}{\mu} \ , & \text{magnetic condensation} \ . \end{cases}
\end{aligned} \tag{4.5}$$

As expected, these condensation patterns are symmetric around the the point $e/g = 1$, reflecting the self-duality of the model. In first approximation the electric (magnetic) condensation phase is characterized by the fact that $\{Q_\mu\}$ ($\{M_\mu\}$) fluctuate freely, while all $M_\mu = 0$ ($Q_\mu = 0$). Within this approximation it is clearly consistent to neglect altogether the interaction term between electric and magnetic strings in (4.3). Taking into account small loop corrections [25] in the various phases can lead to a renormalization of coupling constants and masses and, correspondingly, to a shift of the critical couplings $(ml)_{\rm crit}$ and $(e/g)_{\rm crit}$ for the phase transitions. A notable exception is the case in which there is only one phase transition: in this case the critical coupling is $(e/g)_{\rm crit} = 1$ due to self-duality.

4.2. Wilson and 't Hooft loops

In order to distinguish the various phases we introduce the typical order parameters of lattice gauge theories [8][28], namely the *Wilson loop* for an electric charge q and the *'t Hooft loop* for a vortex ϕ:

$$\begin{aligned}
L_W &\equiv \exp\left(i \frac{q}{\kappa^{\frac{1}{2}}} \sum_x l q_\mu A_\mu \right) \ , \\
L_H &\equiv \exp\left(i \phi \kappa^{\frac{3}{2}} \sum_x l \phi_\mu B_\mu \right) \ ,
\end{aligned} \tag{4.6}$$

where q_μ and ϕ_μ vanish everywhere but on the links of the closed loops, where they take the value 1. Since the loops are closed they satisfy

$$\hat{d}_\mu q_\mu = \hat{d}_\mu \phi_\mu = 0 \ . \tag{4.7}$$

The expectation values $\langle L_W \rangle$ and $\langle L_H \rangle$ can be used to characterize the various phases. First of all they measure the interaction potential between static, external test charges q

and $-q$ and vortices ϕ and $-\phi$, respectively [8]. Secondly, by representing the closed loops q_μ and ϕ_μ as

$$\begin{aligned} q_\mu &\equiv l k_{\mu\nu} A^q_\nu \ , \\ \phi_\mu &\equiv l \hat{k}_{\mu\nu} A^\phi_\nu \ , \end{aligned} \tag{4.8}$$

we can rewrite the Wilson and 't Hooft loops as

$$\begin{aligned} L_W &= \exp\left(i \frac{q}{\kappa^{\frac{1}{2}}} \sum_x l^2 A^q_\mu F_\mu \right) \ , \\ L_H &= \exp\left(i \kappa^{\frac{3}{2}} \phi \sum_x l^2 A^\phi_\mu f_\mu \right) \ , \end{aligned} \tag{4.9}$$

which is a lattice version of Stoke's theorem, the integers A^q_μ and A^ϕ_μ $(= 0, \pm 1)$ representing the area elements of the surfaces spanned by the closed loops. The second terms of the expansions of $\langle L_W \rangle$ and $\langle L_H \rangle$ in powers of q and ϕ measure therefore the gauge invariant correlation functions $\langle F_\mu(x) F_\nu(y) \rangle$ and $\langle f_\mu(x) f_\nu(y) \rangle$. Third, if we represent ϕ_μ as

$$\phi\phi_\mu \equiv \frac{l^2}{2\pi} \ \hat{k}_{\mu\nu} A^{\text{e.m.}}_\nu \ , \tag{4.10}$$

we can also rewrite the 't Hooft loop as

$$L_H = \exp\left(i \sum_x l^3 A^{\text{e.m.}}_\mu q^{\text{charge}}_\mu \right) \ . \tag{4.11}$$

With the interpretation of $A^{\text{e.m.}}_\mu$ as an external electromagnetic gauge potential the expectation value of the 't Hooft loop measures the *electromagnetic response* of the system in the various phases. An analogous relation clearly holds for the Wilson loop.

The expectation values of the Wilson and 't Hooft loops are easily obtained by combining the definitions (4.6) with (3.20):

$$\begin{aligned} \langle L_W \rangle &= \frac{Z_{\text{Top}}\left(Q_\mu + \frac{q}{\kappa} q_\mu, M_\mu\right)}{Z_{\text{Top}}\left(Q_\mu, M_\mu\right)} \ , \\ \langle L_H \rangle &= \frac{Z_{\text{Top}}\left(Q_\mu, M_\mu + \phi\kappa\phi_\mu\right)}{Z_{\text{Top}}\left(Q_\mu, M_\mu\right)} \ , \end{aligned} \tag{4.12}$$

where the notation is self-explanatory. In the following we shall analyze these expressions in the various phases obtained in (4.5). We shall mostly only indicate the form of small loop corrections: a full renormalization group analysis is beyond the scope of the present paper and we won't be able to predict the orders of the phase transitions.

Let us begin with the *electric condensation phase*. In this phase the ground state contains many infinitely long electric strings Q_μ. These have a crucial effect on the gauge symmetry associated with the gauge field A_μ. To see this let us consider a gauge transformation $A_\mu \to A_\mu + d_\mu \Lambda$, where, for simplicity, we take Λ as a function of the component x^1 only. If we choose the usual boundary conditions $F_\mu = f_\mu = 0$ at infinity, the change of the action (3.20) under the above gauge transformation is given by

$$\Delta S_{SD} = \sum_{x^0,x^2} i\sqrt{\kappa}\left(\Lambda(x^1=+\infty)Q_1(x^1=+\infty) - \Lambda(x^1=-\infty)Q_1(x^1=-\infty)\right) \ . \quad (4.13)$$

In a dilute phase, with only small closed loops, $Q_1(x^1=+\infty) = Q_1(x^1=-\infty) = 0$ and the action is automatically gauge invariant. In a dense phase, with many infinitely long strings, $Q_1(x^1=+\infty)$ and $Q_1(x^1=-\infty)$ are generically different from zero. Gauge invariance requires that ΔS_{SD} vanishes modulo $i2\pi$. In the dense phase this is realized only if Λ takes the values

$$\Lambda = \frac{2\pi}{\sqrt{\kappa}} n \ , \qquad\qquad n \in Z \ , \quad (4.14)$$

at infinity. This means that, in the electric condensation phase, the *global* gauge symmetry is spontaneously broken down to the discrete gauge group Z, so that the total (global) symmetry of this phase is $Z_A \times R_B$.

The Wilson loop expectation value takes a particularly simple form if the external test charges are multiples of the charge quantum: $q = n\kappa$, $n \in Z$. In fact, since we sum over $\{Q_\mu\}$, the integer loop variables nq_μ can be absorbed by a redefinition of the appropriate Q_μ's, with the result

$$\langle L_W(q = n\kappa)\rangle = 1 \ . \quad (4.15)$$

This indicates that, in this phase, external test charges $q = n\kappa$ are perfectly *screened* by the topological excitations and behave thus freely. In order to compute the Wilson loop expectation value for generic q we have to perform explicitly the sum over $\{Q_\mu\}$. To this end we have to remember the constraint $\hat{d}_\mu Q_\mu = 0$. We solve this constraint by representing $Q_\mu = l k_{\mu\nu} n_\nu$ and summing over $\{n_\mu\}$, with the appropriate gauge fixing. We then use Poisson's formula (3.8) to turn this sum into an integral, by introducing a new set of integer link variables $\{k_\mu\}$ satisfying $\hat{d}_\mu k_\mu = 0$ in order to guarantee the gauge invariance under $n_\mu \to n_\mu + l d_\mu i$. At this point we can perform explicitly the Gaussian integration over $\{n_\mu\}$. In the approximation of neglecting terms proportional to ∇^2/m^2

(keeping such terms would not alter substantially the result) the new integers $\{k_\mu\}$ can be absorbed by a redefinition of the magnetic topological excitations $\{M_\mu\}$, giving the result:

$$\begin{aligned} \langle L_W \rangle &= \frac{Z_{\text{corr.}}\left(q_\mu\right)}{Z_{\text{corr.}}\left(q_\mu = 0\right)} \ , \\ Z_{\text{corr.}}\left(q_\mu\right) &= \sum_{\{M_\mu\}\text{loops}} \exp \sum_x \left(-\frac{g^2\kappa}{2l} \ M_\mu \frac{\delta_{\mu\nu}}{-\nabla^2} M_\nu + i2\pi \frac{q}{\kappa} A^q_\mu M_\mu \right) \ . \end{aligned} \tag{4.16}$$

Since the magnetic topological excitations are in a dilute phase we have to sum only over small closed loops: in this phase the dominant part of $\ln\langle L_W \rangle$ vanishes for generic q and the whole result is given by small loop corrections. These are identical in form to the small loop corrections for the correlation functions in the low-temperature phase of the three-dimensional XY model [25]; correspondingly the Wilson loop expectation value can be computed by exactly the same low-temperature expansion used for the XY model [25]. The first-order term in this expansion is obtained by considering only the smallest possible lattice loops and gives the result

$$\langle L_W \rangle = \exp \left(2\mathrm{e}^{-\frac{g^2\kappa l}{6}} \sum_{x,\mu} \left[\cos\left(2\pi \frac{q}{\kappa} q_\mu \right) - 1 \right] \right) \ . \tag{4.17}$$

The periodicity of this result is a direct consequence of the spontaneous symmetry breaking $R_A \to Z_A$. This implies also that the gauge invariant correlation function reduces to

$$\langle F_\mu(x) F_\nu(y) \rangle \propto \left(\delta_{\mu\nu}\nabla^2 - d_\mu \hat{d}_\nu \right) \ \frac{\delta_{x,y}}{l^3} \ , \tag{4.18}$$

which is essentially a contact term on the scale of the lattice spacing.

The computation of the 't Hooft loop expectation value follows exactly the same lines as the above computation of the Wilson loop. The results is

$$\begin{aligned} \langle L_H \rangle &= \exp \left(-\frac{g^2\kappa^3\phi^2}{2l} \sum_x \phi_\mu \frac{\delta_{\mu\nu}}{-\nabla^2} \phi_\nu \right) \ \frac{Z_{\text{corr}}\left(\phi_\mu\right)}{Z_{\text{corr}}\left(\phi_\mu = 0\right)} \ , \\ Z_{\text{corr}}\left(\phi_\mu\right) &= \sum_{\{M_\mu\}\text{loops}} \exp \left(-\frac{g^2\kappa}{2l} \sum_x M_\mu \frac{\delta_{\mu\nu}}{-\nabla^2} M_\nu + 2\kappa\phi \ M_\mu \frac{\delta_{\mu\nu}}{-\nabla^2} \phi_\nu \right) \ . \end{aligned} \tag{4.19}$$

The first few terms in the expansion of the small loop corrections can again be computed with the same techniques as in the low-temperature phase of the XY model [25]. One finds that their contribution amounts to perturbative corrections of the Coulomb coupling constant $g^2\kappa^3\phi^2/2l$ of the dominant term in (4.19).

From (4.19) we can extract the nature of the electric condensation phase. First of all, by considering, as usual, a rectangular loop of length T in the imaginary time direction and of length R in one of the spatial directions and computing the dominant large-T behaviour of $\ln\langle L_H\rangle$ we find that the interaction potential between external test vortices of strength ϕ and $-\phi$ is proportional to $\ln R$. Vortices are thus logarithmically *confined*, which amounts to the *Meissner effect*. Secondly, by using the representations (4.8) and (4.9), we find the correlation function

$$\langle f_\mu(x) f_\nu(y)\rangle \propto \frac{\delta_{\mu\nu}\nabla^2 - d_\mu \hat{d}_\nu}{\nabla^2}\ \frac{\delta_{x,y}}{l^3}\ , \tag{4.20}$$

which is long-range, indicating that the "B_μ-photon" is *massless*. This is the massless excitation associated with the spontaneous symmetry breaking of the global gauge symmetry $R_A \to Z_A$. Third, by using the representations (4.10) and (4.11), we find that the induced electromagnetic current is given by

$$J_\mu^{\text{e.m.}} \propto \left(\delta_{\mu\nu} - \frac{d_\mu \hat{d}_\nu}{\nabla^2}\right)\ A_\nu^{\text{e.m.}}\ , \tag{4.21}$$

which is the standard *London form*. We thus conclude that the electric condensation phase is actually a *superconducting phase*.

No further computation is needed to extract the nature of the *magnetic condensation phase*: this is the exact dual of the electric condensation phase just described. Specifically, the global gauge symmetry associated with B_μ is spontaneoulsy broken down to Z_B, so that the total symmetry of this phase is $R_A \times Z_B$. Correspondingly, the "A_μ-photon" is *massless* and the $\langle F_\mu(x) F_\nu(y)\rangle$ correlation function is long-range. Electric charges are logarithmically *confined*, which means that an infinite energy (voltage) is required to separate a neutral pair of charges. We call this phase with infinite resistance a *superinsulator*. In real Josephson junction arrays we expect however the conduction gap to be large but finite due to the small ground capacity C_0, resulting in a normal insulator.

If $mlG(ml)\pi/\mu > 1$ a third phase can open up between the superconducting and superinsulating phases. In this third phase both the electric and the magnetic topological excitations are dilute. Far away from the phase transitions and to first approximation we can neglect them altogether. This gives the result

$$\begin{aligned}
\langle L_W\rangle &= \exp\left(-\frac{e^2 q^2}{2l\kappa}\ q_\mu \frac{\delta_{\mu\nu}}{m^2-\nabla^2} q_\nu\right)\ , \\
\langle L_H\rangle &= \exp\left(-\frac{g^2\phi^2\kappa^3}{2l}\ \phi_\mu \frac{\delta_{\mu\nu}}{m^2-\nabla^2}\phi_\nu\right)\ .
\end{aligned} \tag{4.22}$$

Small loop corrections to these results can be obtained by restricting the $\{Q_\mu\}$ and $\{M_\mu\}$ sums in (4.12) to small closed loops and using again the same techniques as in the low-temperature expansion of the XY model [25]. These will lead to perturbative corrections of the coupling constants and masses in (4.22); however the first-order result (4.22) is enough to establish the nature of this phase. The global symmetry characterizing this phase is $R_A \times R_B$ and, corrrespondingly, both "photons" are massive, resulting in short-range correlation functions $\langle F_\mu(x) F_\nu(y)\rangle$ and $\langle f_\mu(x) f_\nu(y)\rangle$. Both charges and vortices interact via short-range Yukawa potentials and behave thus freely when separated by distances larger then the scale $(1/m)$. In presence of any dissipation mechanism (which would not alter the other two phases) this third phase corresponds thus to a *metallic* phase of the Josephson junction array [9].

In conclusion we can represent the phase diagram of our model as follows:

$$\begin{aligned}
&\frac{mlG(ml)\pi}{\mu} < 1 \rightarrow \begin{cases} \frac{e}{g} < 1 \ , & \text{superconductor } (Z_A \times R_B) \ , \\ \frac{e}{g} > 1 \ , & \text{superinsulator } (R_A \times Z_B) \ , \end{cases} \\
&\frac{mlG(ml)\pi}{\mu} > 1 \rightarrow \begin{cases} \frac{e}{g} < \frac{\mu}{mlG(ml)\pi} \ , & \text{superconductor } (Z_A \times R_B) \ , \\ \frac{\mu}{mlG(ml)\pi} < \frac{e}{g} < \frac{mlG(ml)\pi}{\mu} \ , & \text{metal } (R_A \times R_B) \ , \\ \frac{e}{g} > \frac{mlG(ml)\pi}{\mu} \ , & \text{superinsulator } (R_A \times Z_B) \ , \end{cases}
\end{aligned} \tag{4.23}$$

where we have indicated in parenthesis the global symmetries of the various phases. In fig. 1 we plot the (numerically computed) function $mlG(ml)\pi/\mu$ for the value $\mu = \ln 5$. This gives an indication that a window for the metallic phase is open for ml just larger than 1, while in the regime $ml \leq O(1)$, relevant for Josephson junction arrays, a single phase transition from a superconductor to a superinsulator at $(e/g) = 1$ is favoured.

The experimental results for Josephson junction arrays are plotted in fig. 2 . These are essentially resistance measurements as a function of temperature in arrays with $\mathrm{O}(10^4)$ cells. The zero-temperature extrapolation of these results indicates a quantum phase transition between an insulator and a superconductor in the vicinity of the self-dual point $E_J/E_C = 2/\pi^2 \simeq 0.2$.

5. Quantum Hall phases and anyon superconductivity

Recently it has been suggested that, in presence of n_q offset charge quanta per site and n_ϕ external magnetic flux quanta per plaquette in specific ratios, Josephson junction arrays might have incompressible quantum fluid [13] phases corresponding to purely two-dimensional *quantum Hall phases* for either charges [10] or vortices [11] [12].

In analogy with the conventional quantum Hall setting [3] one expects the charge and vortex transport properties to depend on the *filling fractions* (n_q/n_ϕ) and (n_ϕ/n_q), respectively. Due to the *periodicity* of the charge-vortex coupling, however, n_ϕ (n_q) is defined only modulo an integer as far as charge (vortex) transport properties are concerned. Using this freedom one can thus define *effective filling fractions* (we shall assume $n_q \geq 0$, $n_\phi \geq 0$ for simplicity):

$$\begin{aligned} \nu_q &\equiv \frac{n_q}{n_\phi - [n_\phi]^- + [n_q]^+} \ , \qquad & 0 \leq \nu_q \leq 1 \ , \\ \nu_\phi &\equiv \frac{n_\phi}{n_q - [n_q]^- + [n_\phi]^+} \ , \qquad & 0 \leq \nu_\phi \leq 1 \ , \end{aligned} \tag{5.1}$$

where $[n_q]^\pm$ indicate the smallest (greatest) integer greater (smaller) than n_q. These effective filling fractions are always smaller than 1.

The masses of charges and vortices are given in (3.19). For given values of n_q and n_ϕ which admit an incompressible quantum fluid ground state one expects charges to bind vortices [3] and form a charge quantum Hall phase in the regime where charges are heavier than vortices, i.e. $e/g > 1$. For given values of n_q and n_ϕ we thus expect a charge quantum Hall phase at filling ν_q for $e/g > 1$ and a vortex quantum Hall phase at filling ν_ϕ for $e/g < 1$. Correspondingly, these two regimes were analyzed in [10] and [11], respectively. In [12], however, it was pointed out that e/g cannot be too small for the vortex quantum Hall phase, since for $e/g \ll 1$ the effective vortex band mass due to the periodic array becomes exponentially large and vortices loose their mobility.

In the following we shall assume the existence of these quantum Hall phases and discuss them in the framework of the gauge theory representation of Josephson junction arrays in the self-dual approximation. The idea is as follows. For $n_q = n_\phi = 0$ we have derived that the gauge theory describing Josephson junction arrays (in the self-dual approximation) is given by (1.1) with periodic charge-vortex (mixed Chern-Simons) coupling and the identifications (3.21). This gauge theory describes the dynamics of charge and vortex fluctuations of the array in absence of external offset charges and fluxes. Drawing on previous experience [2] [4] with the quantum Hall effect we shall modify this gauge theory in order to describe charge and vortex fluctuations about a *homogeneous ground state* with n_q charges and n_ϕ vortices per plaquette, describing a quantum Hall fluid for either charges or vortices. We shall then analyze how the periodicity of the new Chern-Simons charge-vortex couplings affects this picture. To this end we shall consider the Euclidean

partition function of the new gauge theories, enforcing the periodicity by appropriate topological excitations and we shall study the resulting zero-temperature phase diagram. Given the expected jump in the relevant effective filling fraction at $e/g = 1$, we shall consider two different gauge theories in the regimes $e/g > 1$ and $e/g < 1$.

5.1. Gauge theories for the quantum Hall phases

Let us begin with the charge quantum Hall phase for $e/g > 1$. To this end we consider the (Minkowski space-time) gauge theory with Lagrangian

$$\mathcal{L}_q = -\frac{1}{2e^2} F_\mu F^\mu + \frac{\kappa}{\pi} A_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu - \frac{1}{2g^2} f_\mu f^\mu - \frac{\nu_q}{g^2} F_\mu f^\mu + \frac{\kappa\nu_q}{\pi} A_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha A_\nu \ . \qquad (5.2)$$

The main differences with respect to (1.1) are the addition of a *pure Chern-Simons term* for the A_μ gauge field and a new coupling term proportional to $F_\mu f^\mu$. The Gauss law constraint associated with the A_μ gauge field now assigns a vorticity

$$\phi = -\frac{1}{2\nu_q \kappa^2} q \qquad (5.3)$$

to a charge $q = \int d^2\mathbf{x}\, q_0^{\text{charge}}$ (since all the gauge fields are massive there are no corrections to this equation from boundary terms). The $F_\mu f^\mu$ coupling then associates a corresponding magnetic moment $\mu \propto \nu_q/\kappa g^2$ to these composites. We have also rescaled the coefficient of the mixed Chern-Simons coupling by a factor of 2 (compare with (1.1)) while maintaining the definitions (3.21). This factor of 2 is a well-known aspect of Chern-Simons gauge theories [29] . Indeed, the vorticity (5.3) has a back-reaction on the charges since it also couples to A_μ via the pure Chern-Simons term. With our rescaling, the total current coupling to A_μ is given by $\left(2q_\mu^{\text{charge}} + 2\kappa^2\nu_q \phi_\mu^{\text{vortex}}\right)$ and using (5.3) we see that the total "dressed" charge of the charge-vortex composite is indeed q. The rescaling ensures thus that dressed charges maintain their nominal value.

The effective Lagrangian for the charge degrees of freedom, obtained by integrating out A_μ is given by

$$\mathcal{L}^B_{\text{eff.}} = -\frac{\kappa}{4\pi\nu_q}\ B_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu + \dots \ , \qquad (5.4)$$

where the ellipse stands for higher-derivative terms which are suppressed at long distances by inverse powers of a mass. Following [4] we introduce as external probes a conserved vortex current ϕ_μ and the electromagnetic gauge field $A_\mu^{\text{e.m.}}$:

$$\mathcal{L}^B_{\text{eff.}} + A_\mu^{\text{e.m.}} q^\mu_{\text{charge}} + \kappa^{\frac{3}{2}} B_\mu \phi^\mu = \mathcal{L}^B_{\text{eff.}} + \frac{\kappa^{\frac{3}{2}}}{2\pi}\ A_\mu^{\text{e.m.}} \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu + \kappa^{\frac{3}{2}} B_\mu \phi^\mu \ . \qquad (5.5)$$

Integrating also over the charge gauge field B_μ we find the effective Lagrangian

$$\mathcal{L}_{\text{eff.}}\left(A_\mu^{\text{e.m.}},\phi_\mu\right)=\frac{\kappa^2\nu_q}{4\pi}\ A_\mu^{\text{e.m.}}\epsilon^{\mu\alpha\nu}\partial_\alpha A_\nu^{\text{e.m.}}+\kappa^2\nu_q\ A_\mu^{\text{e.m.}}\phi^\mu+\pi\kappa^2\nu_q\ \phi_\mu\epsilon^{\mu\alpha\nu}\frac{\partial_\alpha}{\partial^2}\phi_\nu\ . \quad (5.6)$$

From this effective Lagrangian we learn two things. First of all, the electromagnetic response of the system is encoded in the induced current

$$\begin{aligned} J^\mu_{\text{ind.}} &\equiv \frac{\delta}{\delta A_\mu^{\text{e.m.}}}S_{\text{eff.}}\left(A_\mu^{\text{e.m.}},\phi_\mu=0\right)=\frac{\kappa^2}{2\pi}\nu_q\ \epsilon^{\mu\alpha\nu}\partial_\alpha A_\nu^{\text{e.m.}}\ , \\ J^i_{\text{ind.}} &= -\frac{\kappa^2}{2\pi}\nu_q\ \epsilon^{ij}E^j\ , \end{aligned} \quad (5.7)$$

where $\mathbf{E}$ is the applied electric field. This represents a *Hall current* with Hall conductivity given by

$$\sigma_H=\frac{\kappa^2}{2\pi}\nu_q\ . \quad (5.8)$$

Secondly, the last two terms in (5.6) tell us that ϕ_μ represent charge and flux carrying anyons [14] with charge-flux relation and fractional statistics given by

$$\begin{aligned} q &= \nu_q\kappa^2\phi\ , \\ \theta &= \nu_q\kappa^2\phi^2\ . \end{aligned} \quad (5.9)$$

An excitation carrying no effective vorticity can be obtained by combining a charge q with a vortex $\phi=+q/2\nu_q\kappa^2$, so that the bare and induced vorticities cancel. This excitation has the standard electromagnetic coupling $A_\mu^{\text{e.m.}}q^\mu$, with q^μ representing its conserved current. The Gauss law following from (5.6) then assigns to this excitation also a magnetic flux $q/\nu_q\kappa^2$. All excitations in the model are therefore anyons satisfying (5.9). Note that the magnetic moment can be written as $\mu\propto 2S/\kappa g^2$, where $S=\nu_q/2$ is the fractional spin associated with the fractional statistics (5.9).

For $\nu_q=p/n$, with p and n coprime, n flux quanta $1/\kappa$ have p units κ of charge. Since in Josephson junction arrays the charge degrees of freedom are bosons (Cooper pairs), this excitation must also have bosonic statistics, i.e. $\theta=$ even integer. This requires that pn must be an even integer. The allowed filling fractions are thus

$$q=\frac{p}{n}\ , \qquad\qquad pn=\text{even integer}\ , \quad (5.10)$$

in accordance with [30] .

Note that there is no Chern-Simons term in the effective action for the vortices, obtained by integrating out B_μ. Indeed, the bare and induced Chern-Simons terms for A_μ cancel exactly. We thus conclude that (5.2) is indeed the appropriate gauge theory to describe the charge quantum Hall phase of Josephson junction arrays.

The gauge theory (5.2) has a *hidden duality*, which can be made manifest by rewriting the Lagrangian as

$$\begin{aligned} \mathcal{L}_q &= -\frac{1}{2e'^2} F_\mu F^\mu + \frac{\kappa}{\pi} A_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha \left(B_\nu + \nu_q A_\nu \right) - \frac{1}{2g^2} \left(f_\mu + \nu_q F_\mu \right) \left(f^\mu + \nu_q F^\mu \right) \ , \\ e' &\equiv \frac{e}{\sqrt{1 - \frac{e^2}{g^2} \nu_q^2}} \ , \end{aligned} \tag{5.11}$$

and defining a new gauge field $B^q_\mu \equiv B_\mu + \nu_q A_\mu$. Indeed, in terms of A_μ and B^q_μ, (5.11) coincides with (1.1) upon substituting $e \to e'$ (and $\kappa \to 2\kappa$). In the sector in which $f_\mu + \nu_q F_\mu = 0$ the only kinetic term of (5.11) is contained in $-(1/2e'^2) F_\mu F^\mu$. Therefore $m_{q\phi} \equiv 1/l^2 e'^2$ is the mass of the anyonic charge-flux composites. The gap for collective oscillations is given by the modified topological Chern-Simons mass

$$M_q \equiv m(e', g, 2\kappa) = \frac{e' g \kappa}{\pi} = \frac{e g \kappa}{\pi \sqrt{1 - \frac{e^2}{g^2} \nu_q^2}} \ . \tag{5.12}$$

In the representation (5.11) it is also manifest that our modified gauge theory can be defined only in the range

$$1 < \frac{e}{g} < \frac{1}{\nu_q} \ . \tag{5.13}$$

For $e/g \to 1/\nu_q$ the anyon mass $m_{q\phi}$ vanishes, while the topological mass M_q diverges.

The gauge theory describing the vortex quantum Hall phase is the dual of (5.2),

$$\mathcal{L}_\phi = -\frac{1}{2e^2} F_\mu F^\mu + \frac{\kappa}{\pi} A_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu - \frac{1}{2g^2} f_\mu f^\mu - \frac{\nu_\phi}{e^2} F_\mu f^\mu + \frac{\kappa \nu_\phi}{\pi} B_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu \ , \tag{5.14}$$

and contains a pure Chern-Simons term for the gauge field B_μ and the corresponding magnetic moment interaction. Again, the rescaling of the mixed Chern-Simons coupling by a factor of 2 ensures that dressed vorticity maintains its nominal value.

In this case there is no pure Chern-Simons term in the effective action for the charges, while the vortex effective Lagrangian is given by

$$\mathcal{L}^A_{\text{eff.}} = -\frac{\kappa}{4\pi\nu_\phi} A_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha A_\nu + \dots \ , \tag{5.15}$$

at long distances. We probe the vortex response by coupling (5.15) to a gauge field $G_\mu^{\text{ext.}}$ such that $\epsilon^{\mu\alpha\nu}\partial_\alpha G_\nu^{\text{ext.}} = q_{\text{ext.}}^\mu$ describes an external current distribution. We also introduce an additional conserved charge current q_μ coupling to A_μ in order to probe the quantum numbers of excitations:

$$\mathcal{L}_{\text{eff.}}^A + \frac{1}{\kappa^{\frac{1}{2}}} G_\mu^{\text{ext.}} \epsilon^{\mu\alpha\nu}\partial_\alpha A_\nu + \frac{1}{\kappa^{\frac{1}{2}}} A_\mu q^\mu \ . \tag{5.16}$$

Integrating over the vortex gauge field A_μ we find the effective Lagrangian

$$\mathcal{L}_{\text{eff.}}\left(G_\mu^{\text{ext.}}, q_\mu\right) = \frac{\pi\nu_\phi}{\kappa^2}\ G_\mu^{\text{ext.}}\epsilon^{\mu\alpha\nu}\partial_\alpha G_\nu^{\text{ext.}} + \frac{2\pi\nu_\phi}{\kappa^2}\ G_\mu^{\text{ext.}} q^\mu + \frac{\pi\nu_\phi}{\kappa^2}\ q_\mu\epsilon^{\mu\alpha\nu}\frac{\partial_\alpha}{\partial^2}q_\nu \ . \tag{5.17}$$

This leads to the following induced vortex current:

$$\Phi_{\text{ind.}}^\mu \equiv \frac{1}{2\pi}\frac{\delta}{\delta G_\mu^{\text{ext.}}} S_{\text{eff.}}\left(G_\mu^{\text{ext.}}, q_\mu = 0\right) = \frac{\nu_\phi}{\kappa^2}\ q_{\text{ext.}}^\mu \ . \tag{5.18}$$

This equation embodies the quantum Hall effect for vortices. While charges react to external electromagnetic fields, vortices react to external electric currents: in the normal case the induced vortex current is perpendicular to the applied electric current, in the quantum Hall phase it is parallel, with coefficient proportional to ν_ϕ. From the last two terms in (5.17) we read off the flux-charge relation and the fractional statistics of the anyon excitations represented by q_μ:

$$\begin{aligned} \phi &= \frac{\nu_\phi q}{\kappa^2} \ , \\ \theta &= \frac{\nu_\phi q^2}{\kappa^2} \ . \end{aligned} \tag{5.19}$$

For $\nu_\phi = p/n$, n charge quanta κ carry p quanta $1/\kappa$ of flux. Since vortices are also bosons, we find the same allowed values of ν_ϕ as for the charge quantum Hall phase:

$$\nu_\phi = \frac{p}{n} \ , \qquad\qquad pn = \text{even integer} \ . \tag{5.20}$$

The self-dual representation analogous to (5.11) is given by

$$\begin{aligned} \mathcal{L}_\phi &= -\frac{1}{2e^2}\left(F_\mu + \nu_\phi f_\mu\right)\left(F^\mu + \nu_\phi f^\mu\right) + \frac{\kappa}{\pi}\left(A_\mu + \nu_\phi B_\mu\right)\epsilon^{\mu\alpha\nu}\partial_\alpha B_\nu - \frac{1}{2g'^2} f_\mu f^\mu \ , \\ g' &\equiv \frac{g}{\sqrt{1 - \frac{g^2}{e^2}\nu_\phi^2}} \ , \end{aligned} \tag{5.21}$$

and coincides with (1.1) upon introducing a new gauge field $A_\mu^\phi \equiv A_\mu + \nu_\phi B_\mu$ and substituting $g \to g'$ (and $\kappa \to 2\kappa$). In this case $m_{\phi q} \equiv 1/l^2 g'^2$ is the mass of anyonic

flux-charge composites, while the gap for collective oscillations is given by the topological Chern-Simons mass

$$M_\phi \equiv m(e, g', 2\kappa) = \frac{eg'\kappa}{\pi} = \frac{eg\kappa}{\pi\sqrt{1 - \frac{g^2}{e^2}\nu_\phi^2}} \ . \tag{5.22}$$

Clearly, (5.21) is defined only in the range

$$\nu_\phi < \frac{e}{g} < 1 \ . \tag{5.23}$$

Again, for $e/g \to \nu_\phi$ the anyon mass $m_{\phi q}$ vanishes, while the topological mass M_ϕ diverges.

Combining (5.13) and (5.23) we find the overall condition

$$\nu_\phi < e/g < 1/\nu_q \ , \tag{5.24}$$

which we interpret as the regime in which a homogeneous ground state with n_q charges and n_ϕ vortices per plaquette can exist. Presumably, for $e/g < \nu_\phi$ and $e/g > 1/\nu_q$ the ground state consists of an Abrikosov-type cristal for charge-flux composites. In particular, (5.24) tells us that in Josephson junction arrays E_C/E_J cannot be either too large or too small for the existence of quantum Hall phases. Although its origin is different, this condition agrees with the result of [12] (at least for the vortex quantum Hall phase).

5.2. Periodic Chern-Simons terms and phase structure analysis

In the following we shall analyze how the above picture is modified when we impose the distinctive feature of Josephson junction arrays, namely the periodicity of charge-vortex couplings, encoded in our formalism in the Chern-Simons terms. This is achieved by introducing appropriate topological excitations in the Euclidean lattice partition functions of (5.11)and (5.21).

Let us begin with the gauge theory (5.11) for the charge quantum Hall phase. Its Euclidean lattice partition function coincides with (3.20) upon substituting $e \to e'$, $B_\mu \to B^q_\mu = B_\mu + \nu_q A_\mu$ and rescaling the mixed Chern-Simons term by a factor of 2. Therefore we present here only the coupling of the topological excitations enforcing the periodicity of the mixed Chern-Simons term $A_\mu k_{\mu\nu} B^q_\nu$ and the Wilson and 't Hooft loops (4.6):

$$S_q = \sum_x \ldots + ilp\sqrt{\kappa} A_\mu \left(Q_\mu + M_\mu\right) + iln\sqrt{\kappa} B_\mu M_\mu + il\frac{q}{\kappa^{\frac{1}{2}}} A_\mu q_\mu + il\phi\kappa^{\frac{3}{2}} B_\mu \phi_\mu \ , \tag{5.25}$$

where we have used the representation $\nu_q = p/n$. Due to the change $B_\mu \to B^q_\mu$ the periodicities of the two original gauge fields are changed from (3.25) to

$$\begin{aligned} A_\mu &\to A_\mu + \frac{\pi n}{l\sqrt{\kappa}}\, a_\mu \ , \qquad\qquad a_\mu \in Z \ , \\ B_\mu &\to B_\mu + \frac{\pi p}{l\sqrt{\kappa}}\, b_\mu \ , \qquad\qquad b_\mu \in Z \ . \end{aligned} \tag{5.26}$$

The displayed terms in (5.25) can be rearranged as follows:

$$S_q = \sum_x \ldots + ilp\sqrt{\kappa} A_\mu \left(Q_\mu + \frac{q}{\kappa p} q_\mu - \frac{\kappa\phi}{n}\phi_\mu \right) + iln\sqrt{\kappa} B^q_\mu \left(M_\mu + \frac{\kappa\phi}{n}\phi_\mu \right) \ , \tag{5.27}$$

so that the whole model is reformulated in terms of the gauge fields A_μ and B^q_μ and we can use thus the results of the previous section. In particular we obtain

$$\langle L_W L_H \rangle = \frac{Z^q_{\rm Top}\left(Q_\mu + \frac{q}{\kappa p} q_\mu - \frac{\kappa\phi}{n}\phi_\mu, M_\mu + \frac{\kappa\phi}{n}\phi_\mu \right)}{Z^q_{\rm Top}\left(Q_\mu, M_\mu\right)} \ , \tag{5.28}$$

with

$$\begin{aligned} Z^q_{\rm Top} &= \sum_{\substack{\{Q_\mu\} \\ \{M_\mu\}}} \exp\left(-S^q_{\rm Top} \right) \ , \\ S^q_{\rm Top} &= \sum_x \frac{e'^2 p^2 \kappa}{2l}\ Q_\mu \frac{\delta_{\mu\nu}}{M_q^2 - \nabla^2} Q_\nu + \frac{g^2 n^2 \kappa}{2l}\ M_\mu \frac{\delta_{\mu\nu}}{M_q^2 - \nabla^2} M_\nu \\ &\qquad + i\frac{\pi p n M_q^2}{l}\ Q_\mu \frac{k_{\mu\nu}}{\nabla^2\left(M_q^2 - \nabla^2\right)} M_\nu \ , \end{aligned} \tag{5.29}$$

and M_q defined in (5.12).

At this point we can repeat verbatim the analysis of section 4. The phase structure of (5.11) with periodic Chern-Simons term is governed by the topological excitations Q_μ and M_μ. The phase in which both these topological excitations are dilute corresponds to the charge quantum Hall phase discussed above, which constitutes an *incompressible fluid* of charge-flux composites with short-range interactions. The stability of this phase depends entirely on the condensation conditions for the two types of topological excitations. If Q_μ condenses we obtain a phase in which charges $q = \kappa p$ (and multiples thereof) are completely screened, while fluxes ϕ are logarithmically confined: this is a conventional *superconducting phase* with a charge κp condensate. Using the trick (4.10) we can identify the Coulomb law for fluxes with the London form of the electromagnetic response. If M_μ condenses we obtain, instead, a phase in which excitations with quantum numbers $(q/(p\kappa) - \phi\kappa/n)$

interact logarithmically. This means that the only non-confined excitations in the model must carry both charge and flux in the combination

$$\frac{q}{\kappa p} - \frac{\phi\kappa}{n} = 0 \;\Rightarrow \frac{q}{\phi} = \kappa^2 \nu_q \; . \tag{5.30}$$

These excitations are completely screened, while all other combinations of quantum numbers are logarithmically confined. This (logarithmic) *oblique confinement* [15] [16] phase describes thus a *charge-flux superfluid* phase. Since the condensed composites carry charge, this is actually an *anyon superconductivity phase* [17] with a charge $p\kappa$ and vorticity n/κ condensate. Indeed, the electromagnetic response, obtained again by the trick (4.10) has still the London form.

We now have to study the range of parameters in which these three phases are realized. The analysis goes exactly as in section 4, giving the result

$$\begin{aligned}
X_q < 1 \rightarrow & \begin{cases} \frac{e'}{g} < \frac{1}{\nu_q} \; , & \text{conventional superconductor} \; , \\ \frac{e'}{g} > \frac{1}{\nu_q} \; , & \text{anyon superconductor} \quad , \end{cases} \\
X_q > 1 \rightarrow & \begin{cases} \frac{e'}{g} < \frac{1}{\nu_q X_q} \; , & \text{conventional superconductor} \; , \\ \frac{1}{\nu_q X_q} < \frac{e'}{g} < \frac{X_q}{\nu_q} \; , & \text{charge quantum Hall phase} \; , \\ \frac{e'}{g} > \frac{X_q}{\nu_q} \; , & \text{anyon superconductor} \quad , \end{cases} \\
& X_q \equiv \frac{M_q l G\left(M_q l\right) \pi}{\mu} \; \frac{pn}{2} \; .
\end{aligned} \tag{5.31}$$

It is now harder to disentangle the phase diagram in terms of the original (Josephson junction) parameters e and g since e', and consequently M_q depend themselves on the ratio e/g.

As was pointed out in [12], the periodicity of charge-vortex couplings is the distinctive feature of Josephson junction arrays which allows effective filling fractions of order $O(1)$ and which is therefore expected to favour the formation of charge and vortex quantum Hall phases. From the above result, however, it is clear that the same mechanism can also destabilize these quantum Hall phases as follows. The condition for a charge quantum Hall phase is given by $X_q > 1$. The filling fraction parameters p and n enter this condition in two ways. First there is the explicit dependence of X_q on the product pn; secondly there is the dependence of the gap M_q on the ratio $\nu_q = p/n$. The former favours filling fractions with a large product pn; too large numerators and denominators are however presumably suppressed by the same mechanism as in the conventional quantum Hall effect. The latter

has the following effect. Decreasing both ν_q and e/g makes both the gap M_q and the ratio e'/g smaller; the parameter X_q approaches a fixed value, while $e'\nu_q/g$ can decrease indefinitely till it becomes favourable for the system to expel the magnetic flux and form a conventional superconductor. Increasing both ν_q and e/g, instead, makes both the gap and the ratio e'/g larger; for large values of $M_q l$ the parameter X_q tends to zero (see fig. 1), while $e'\nu_q/g$ can grow indefinitely and it quickly becomes favourable for the charge-flux fluid to condense into a superfluid, so that the system becomes an anyon superconductor. For values $e\nu_q/g > 1$ we would expect a charge-flux cristal.

The analysis of the gauge theory (5.21) for the vortex quantum Hall phase follows exactly the same steps with all "electric quantities" and "magnetic quantities" interchanged. Therefore we present here only the final result:

$$
\begin{aligned}
X_\phi < 1 \rightarrow & \begin{cases} \frac{e}{g'} < \nu_\phi \ , & \text{anyon superconductor} \ , \\ \frac{e}{g'} > \nu_\phi \ , & \text{superinsulator} \ , \end{cases} \\
X_\phi > 1 \rightarrow & \begin{cases} \frac{e}{g'} < \frac{\nu_\phi}{X_\phi} \ , & \text{anyon superconductor} \ , \\ \frac{\nu_\phi}{X_\phi} < \frac{e}{g'} < \nu_\phi X_\phi \ , & \text{vortex quantum Hall phase} \ , \\ \frac{e}{g'} > \nu_\phi X_\phi \ , & \text{superinsulator} \ , \end{cases} \\
& X_\phi \equiv \frac{M_\phi l G\left(M_\phi l\right) \pi}{\mu} \ \frac{pn}{2} \ ,
\end{aligned}
\tag{5.32}
$$

where M_ϕ is defined in (5.22) and we have used the representation $\nu_\phi = p/n$. Starting from the vortex quantum Hall phase and decreasing both g/e and ν_ϕ makes both the gap M_ϕ and the ratio g'/e smaller: it becomes eventually favourable for the system to expel the offset charges and become a superinsulator. In real Josephson junction arrays this phase would presumably be an insulating Abrikosov-type cristal of charges due to the small but finite ground capacitance C_0. Increasing both g/e and ν_ϕ makes both the gap M_ϕ and the ratio g'/e larger; the quantity $1/X_\phi$ tends to infinity (see fig. 1) while e/g' decreases and the flux-charge fluid of the vortex quantum Hall phase condenses again into a flux-charge superfluid, becoming thus an anyon superconductor. For even smaller values of $e/g < \nu_\phi$ we would expect again a flux-charge cristal.

6. Three dimensions

In this section we generalize our results (for zero offset charges and external magnetic fluxes) to three-dimensional Josephson junction arrays. While these are not (yet)

experimentally accessible, we find it nonetheless interesting to construct their gauge theory representation and to study the differences with the two-dimensional case. Clearly, self-duality is lost in three dimensions since the fluctuating degrees of freedom are charges and closed vortex loops. However, it is still possible to introduce the approximation which allows a coupled gauge theory representation, as we now show.

Up to eq. (3.10) the analysis parallels exactly the two-dimensional case. The decomposition analogous to (3.10), however, requires the introduction of the three-dimensional generalizations of the lattice operators $K_{\mu\nu}$ and $\hat{K}_{\mu\nu}$. These are given by the three-index lattice operators

$$\begin{aligned} K_{\mu\nu\rho} &\equiv S_\mu \epsilon_{\mu\alpha\nu\rho} \Delta_\alpha \ , \\ \hat{K}_{\mu\nu\rho} &\equiv \epsilon_{\mu\nu\alpha\rho} \hat{\Delta}_\alpha \hat{S}_\rho \ , \end{aligned} \tag{6.1}$$

where S_μ and $\hat{S}_\mu$ are the shift operators (2.9) and Δ_μ and $\hat{\Delta}_\mu$ are the finite difference operators (2.15). As in the two-dimensional case, these two operators are interchanged (no minus sign) upon summation by parts on the lattice. Moreover they are gauge invariant, in the sense that they obey the following equations:

$$\begin{aligned} K_{\mu\nu\rho}\Delta_\nu = K_{\mu\nu\rho}\Delta_\rho = \hat{\Delta}_\mu K_{\mu\nu\rho} = 0 \ , \\ \hat{K}_{\mu\nu\rho}\Delta_\rho = \hat{\Delta}_\mu \hat{K}_{\mu\nu\rho} = \hat{\Delta}_\nu \hat{K}_{\mu\nu\rho} = 0 \ . \end{aligned} \tag{6.2}$$

Finally they satisfy also the equations

$$\begin{aligned} \hat{K}_{\mu\nu\rho} K_{\rho\lambda\omega} &= -\left(\delta_{\mu\lambda}\delta_{\nu\omega} - \delta_{\mu\omega}\delta_{\nu\lambda}\right)\Delta + \left(\delta_{\mu\lambda}\Delta_\nu\hat{\Delta}_\omega - \delta_{\nu\lambda}\Delta_\mu\hat{\Delta}_\omega\right) \\ &\quad + \left(\delta_{\nu\omega}\Delta_\mu\hat{\Delta}_\lambda - \delta_{\mu\omega}\Delta_\nu\hat{\Delta}_\lambda\right) \ , \\ \hat{K}_{\mu\nu\rho} K_{\rho\nu\omega} &= K_{\mu\nu\rho}\hat{K}_{\rho\nu\omega} = 2\left(\delta_{\mu\omega}\Delta - \Delta_\mu\hat{\Delta}_\omega\right) \ . \end{aligned} \tag{6.3}$$

Using these operators we can decompose v_μ as

$$v_\mu = \Delta_\mu m + \Delta_\mu \alpha + K_{\mu\alpha\beta}\psi_{\alpha\beta} \ , \tag{6.4}$$

with $m \in Z$ and $|\alpha| < 1$. The $\psi_{\alpha\beta}$'s are restricted by the fact that the antisymmetric combinations $q_{\mu\nu} \equiv \hat{K}_{\mu\nu\alpha}v_\alpha = \hat{K}_{\mu\nu\alpha}K_{\alpha\lambda\rho}\psi_{\lambda\rho}$ must be integers. We can thus trade the original sum over the four independent integers $\{v_\mu\}$ for a sum over the seven integers $\{m, q_{\mu\nu}\}$ subject to the constraint $\hat{\Delta}_\mu q_{\mu\nu} = \hat{\Delta}_\nu q_{\mu\nu} = 0$. This constraint eliminates the three longitudinal degrees of freedom of $q_{\mu\nu}$, so that $\{m, q_{\mu\nu}\}$ with the above constraint

describes only four independent integers. After shifting the Φ integration domain using the sum over $\{m\}$ and performing the resulting trivial Φ integration we are left with

$$\begin{aligned} Z &= \sum_{\{q_{\mu\nu}\}} \delta_{\hat{\Delta}_\mu q_{\mu\nu},0} \int \mathcal{D}p_\mu \ \delta\left(\hat{\Delta}_\mu p_\mu\right) \ \exp(-S) \ , \\ S &= \sum_x -i2\pi \ p_\mu K_{\mu\alpha\beta}\psi_{\alpha\beta} + N^2 l_0 E_C \ p_0 \frac{1}{-\Delta} p_0 + \frac{p_i^2}{2l_0 E_J} \ . \end{aligned} \tag{6.5}$$

The constraints are solved by introducing a real *antisymmetric gauge field* $b_{\mu\nu}$ and an integer gauge field a_μ:

$$\begin{aligned} p_\mu &\equiv K_{\mu\alpha\beta} b_{\alpha\beta} \ , \qquad\qquad b_{\alpha\beta} \in R \ , \\ q_{\mu\nu} &\equiv \hat{K}_{\mu\nu\alpha} a_\alpha \ , \qquad\qquad a_\alpha \in Z \ . \end{aligned} \tag{6.6}$$

Repeating the same steps as in the two-dimensional case we find

$$\begin{aligned} Z &= \sum_{\{Q_\mu\}} \int \mathcal{D}a_\mu \int \mathcal{D}b_\mu \ \exp(-S) \ , \\ S &= \sum_x -i2\pi \ a_\mu K_{\mu\alpha\beta} b_{\alpha\beta} + N^2 l_0 E_C \ p_0 \frac{1}{-\Delta} p_0 + \frac{p_i^2}{2l_0 E_J} + i2\pi a_\mu Q_\mu \ , \end{aligned} \tag{6.7}$$

which is the three-dimensional analogue of (3.13). Here, $K_{\mu\alpha\beta}b_{\alpha\beta}$ maintains its interpretation of the conserved four-current of charge fluctuations, while $\hat{K}_{\mu\nu\alpha}a_\alpha$ represents the fluctuations of *closed vortex loops*.

Since the magnetic fluctuations are represented by closed vortex loops we cannot render the partition function self-dual, as in the two-dimensional case. However, it is still possible to introduce a bare kinetic term for the vortex loops with a coefficient tuned so that it can be combined with the Coulomb term for the charges into $\sum_x \frac{\pi^2}{2N^2 l_0 E_C} q_{ij} q_{ij}$. As in two dimensions we have to introduce new integers via the Poisson summation formula to guarantee that the charge current $K_{\mu\alpha\beta}b_{\alpha\beta}$ remains an integer. In three dimensions these are two-index antisymmetric integers $M_{\mu\nu}$ satisfying the constraint $\hat{\Delta}_\mu M_{\mu\nu} = \hat{\Delta}_\nu M_{\mu\nu} = 0$:

$$\begin{aligned} Z &= \sum_{\substack{\{Q_\mu\} \\ \{M_{\mu\nu}\}}} \int \mathcal{D}a_\mu \int \mathcal{D}b_\mu \ \exp(-S) \ , \\ S &= \sum_x -i2\pi \ a_\mu K_{\mu\alpha\beta} b_{\alpha\beta} + \frac{p_i^2}{2l_0 E_J} + \frac{\pi^2 q_{ij}^2}{2N^2 l_0 E_C} + i2\pi a_\mu Q_\mu + i2\pi b_\mu M_{\mu\nu} \ . \end{aligned} \tag{6.8}$$

After a rescaling

$$\begin{aligned} A_0 &\equiv \frac{2\pi}{\sqrt{N}l_0}a_0 \ , \qquad & A_i &\equiv \frac{2\pi}{\sqrt{N}l}a_i \ , \\ B_{0i} &\equiv \frac{2\pi}{\sqrt{N}l_0 l}b_{0i} \ , \qquad & B_{ij} &\equiv \frac{2\pi}{\sqrt{N}l^2}b_{ij} \ , \end{aligned} \tag{6.9}$$

we obtain finally

$$\begin{aligned} Z &= \sum_{\substack{\{Q_\mu\} \\ \{M_{\mu\nu}\}}} \int \mathcal{D}A_\mu \int \mathcal{D}B_\mu \, \exp(-S) \ , \\ S &= \sum_x \frac{l_0 l^3}{4e^2} \ \tilde{F}_{ij}\tilde{F}_{ij} - i\frac{l_0 l^3 \kappa}{2\pi} A_\mu k_{\mu\alpha\beta} B_{\alpha\beta} + \frac{l_0 l^3}{2g^2} \ f_i f_i \\ &\qquad + i\sqrt{\kappa}\left(l_0 Q_0 A_0 + l Q_i A_i\right) + i\sqrt{\kappa}\left(l_0 l M_{0i} B_{0i} + l_0 l M_{i0} B_{i0} + l^2 M_{ij} B_{ij}\right) \ , \end{aligned} \tag{6.10}$$

where $k_{\mu\nu\rho}$ and $\hat{k}_{\mu\nu\rho}$ are the analogues of (6.1) defined in terms of derivative operators rather than finite difference operators (and satisfying the correspondingly modified eq. (6.3)) , $\tilde{F}_{ij}$ and f_i are the spatial components of

$$\begin{aligned} \tilde{F}_{\mu\nu} &\equiv \hat{k}_{\mu\nu\alpha} A_\alpha \ , \\ f_\mu &\equiv \frac{1}{2} k_{\mu\alpha\beta} B_{\alpha\beta} \ , \end{aligned} \tag{6.11}$$

and the coupling constants e^2 (dimensionless) and g^2 (with dimensions mass^2) are given by

$$e^2 = 2NlE_C \ , \qquad \kappa = N \ , \qquad g^2 = \frac{\pi^2}{Nl} E_J \ . \tag{6.12}$$

The plasma frequency is given again by a product of these coupling constants,

$$m = \frac{egN}{\pi} = \sqrt{2N^2 E_C E_J} \ , \tag{6.13}$$

and for $ml \leq O(1)$ we can choose $l_0 = l$. Also in three dimensions we have thus obtained a coupled gauge theory in the limit of infinite magnetic permeabilities, encoded in the absence of the time components of both $\tilde{F}_{\mu\nu}$ and f_μ in the kinetic terms. As in two dimensions we shall henceforth consider the relativistic limit of this gauge theory:

$$\begin{aligned} Z &= \sum_{\substack{\{Q_\mu\} \\ \{M_{\mu\nu}\}}} \int \mathcal{D}A_\mu \int \mathcal{D}B_\mu \, \exp(-S) \ , \\ S &= \sum_x \frac{l^4}{4e^2} \tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu} - i\frac{l^4\kappa}{2\pi} A_\mu k_{\mu\alpha\beta} B_{\alpha\beta} + \frac{l^4}{2g^2} f_\mu f_\mu + il\sqrt{\kappa} A_\mu Q_\mu + il^2\sqrt{\kappa} B_{\mu\nu} M_{\mu\nu} \ . \end{aligned} \tag{6.14}$$

This is a pure gauge theory representation of a model of interacting charges and closed vortex loops. The identification of the physical degrees of freedom is analogous to the two-dimensional case:

$$\begin{aligned} q_\mu^{\text{charge}} &\equiv \frac{\kappa^{\frac{3}{2}}}{2\pi}\ k_{\mu\alpha\beta}B_{\alpha\beta}\ , \\ \phi_{\mu\nu}^{\text{vortex}} &\equiv \frac{1}{2\pi\kappa^{\frac{1}{2}}}\ \hat{k}_{\mu\nu\alpha}A_\alpha\ . \end{aligned} \tag{6.15}$$

The model (6.14) is a (Euclidean) lattice version of the so called *BF gauge theory* [19], whose Lagrangian is given by

$$\begin{aligned} \mathcal{L}_{BF} &= -\frac{1}{12g^2}f_{\mu\nu\rho}f^{\mu\nu\rho} + \frac{\kappa}{4\pi}B_{\mu\nu}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho} - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}\ , \\ f_{\mu\nu\rho} &\equiv \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}\ , \\ f^\mu &\equiv \frac{1}{6}\epsilon^{\mu\nu\lambda\rho}f_{\nu\lambda\rho}\ . \end{aligned} \tag{6.16}$$

Here A_μ describes an ordinary (3+1)-dimensional photon with field strength given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and dual field strength $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$. This photon has a topological BF coupling to an antisymmetric Kalb-Ramond [18] gauge field $B_{\mu\nu}$, whose field strength is given by the three-form $F_{\mu\nu\rho}$. The first term in (6.16) represents the kinetic term for the Kalb-Ramond gauge field. In addition to the usual invariance under gauge transformations of A_μ (6.16) is also invariant under gauge transformations $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu$. The Kalb-Ramond gauge theory describes a single massless scalar degree of freedom since all the components B_{i0} are Lagrange multipliers and B_{ij} has only one transverse component. When coupled to the usual Maxwell theory via a BF term, the Kalb-Ramond sector induces a topological mass

$$m = \frac{eg\kappa}{\pi} \tag{6.17}$$

for the photon. As already pointed out in (6.13) this mass represents the plasma frequency of Josephson junction arrays. The BF system is thus the natural three-dimensional generalization of (1.1), as expected.

As in two dimensions, the integer-valued variables Q_μ and $M_{\mu\nu}$ appearing in (6.14) represent (Euclidean) topological excitations whose role is to make the charge-vortex BF coupling *periodic*. They satisfy the constraints

$$\begin{aligned} \hat{d}_\mu Q_\mu &= 0\ , \\ \hat{d}_\mu M_{\mu\nu} &= \hat{d}_\nu M_{\mu\nu} = 0\ . \end{aligned} \tag{6.18}$$

The "electric" topological excitations Q_μ are exactly as in two dimensions; the "magnetic" topological excitations, instead, describe *compact surfaces* on the lattice (in a dilute phase) or infinite surfaces (in a dense phase).

The phase structure of three-dimensional Josephson junction arrays (in our approximation) is thus determined by the statistical mechanics of a coupled gas of lattice loops and surfaces. Its partition function can be easily obtained by a Gaussian integration over the gauge fields A_μ and $B_{\mu\nu}$ in (6.14):

$$\begin{aligned} Z_{\rm Top} &= \sum_{\substack{\{Q_\mu\} \\ \{M_{\mu\nu}\}}} \exp\left(-S_{\rm Top}\right) \ , \\ S_{\rm Top} &= \sum_x \frac{e^2\kappa}{2l^2}\ Q_\mu \frac{\delta_{\mu\nu}}{m^2-\nabla^2} Q_\nu + \frac{g^2\kappa}{2}\ M_{\mu\nu}\frac{\delta_{\mu\alpha}\delta_{\nu\beta}-\delta_{\mu\beta}\delta_{\nu\alpha}}{m^2-\nabla^2}M_{\alpha\beta} \\ &\quad + i\frac{\pi m^2}{l}\ Q_\mu \frac{k_{\mu\alpha\beta}}{\nabla^2\left(m^2-\nabla^2\right)} M_{\alpha\beta} \ , \end{aligned} \tag{6.19}$$

with m defined in (6.17). This partition function can be interpreted as the Euclidean partition function for a lattice model of interacting particles (whose world-lines are parametrized by the closed loops Q_μ) and closed Nielsen-Olesen type strings [21] (whose world-sheets are parametrized by the compact surfaces $M_{\mu\nu}$). In a derivative expansion the string action takes the form

$$S_{\rm strings} = \sum_x \frac{\pi^2}{\kappa e^2} M_{\mu\nu}M_{\mu\nu} + \dots \ . \tag{6.20}$$

In the dilute gas approximation, where $M_{\mu\nu}$ can take only the values $0, \pm 1$, this term measures the *area* of the world-sheet and is thus the standard Nambu-Goto term [8], with *string tension* $\pi^2/\kappa e^2 l^2$ (remember that $1/e^2l^2$ was the vortex mass in two dimensions). The parameter $1/l$ plays thus the role of the Higgs mass in our lattice model; higher order terms in (6.20) involve both the curvature and internal excitations of the string. Particle-string interactions are encoded in the topological Aharonov-Bohm term (third term in (6.19)) measuring the linking of the closed world-lines of particles and compact world-sheets of strings in four Euclidean dimensions. As in the two-dimensional case this term vanishes for loops and surfaces separated on distances much larger than $1/m$. In this case the denominator reduces to $m^2\nabla^2$ and, by using either one of the representations

$$\begin{aligned} Q_\mu &= l k_{\mu\alpha\beta} Y_{\alpha\beta} \ , \\ M_{\mu\nu} &= l \hat{k}_{\mu\nu\alpha} X_\alpha \ , \end{aligned} \tag{6.21}$$

and the equations (6.3) one recognizes that the whole term reduces to ($i2\pi$integer), which is equivalent to 0.

Unfortunately, the statistical mechanics of random surfaces [21] is much less understood than its random loop counterpart [20] and we cannot use self-duality arguments anymore. In order to proceed further we shall assume that the same three types of phases as in the two-dimensional case can exists and we will point out the differences that can nonetheless arise. First of all we can repeat the same argument as in the two-dimensional case to find that in the electric condensation phase the global gauge symmetry associated with A_μ is spontaneously broken $R_A \to Z_A$ while in the magnetic condensation phase it is the Kalb-Ramond global gauge symmetry which is spontaneously broken $R_B \to Z_B$. Secondly we can consider the expectation values of the Wilson loop (4.6) and the *'t Hooft surface*

$$S_H \equiv \exp\left(i\phi\kappa^{\frac{3}{2}} \sum_x l^2 \phi_{\mu\nu} B_{\mu\nu} \right) , \tag{6.22}$$

where $\phi_{\mu\nu}$ vanishes everywhere but on the plaquettes of a compact surface, where it takes the value 1.These expectation values are given by

$$\begin{aligned} \langle L_W \rangle &= \frac{Z_{\text{Top}}\left(Q_\mu + \frac{q}{\kappa} q_\mu, M_{\mu\nu}\right)}{Z_{\text{Top}}\left(Q_\mu, M_{\mu\nu}\right)} , \\ \langle S_H \rangle &= \frac{Z_{\text{Top}}\left(Q_\mu, M_{\mu\nu} + \phi\kappa\phi_{\mu\nu}\right)}{Z_{\text{Top}}\left(Q_\mu, M_{\mu\nu}\right)} . \end{aligned} \tag{6.23}$$

With exactly the same computation as in the two-dimensional case we find the dominant contributions

$$\begin{aligned} \langle L_W \rangle_{\text{mag. cond.}} &= \exp\left(\frac{-e^2 q^2}{2\kappa l^2} q_\mu \frac{\delta_{\mu\nu}}{-\nabla^2} q_\nu \right) , \\ \langle S_H \rangle_{\text{el. cond.}} &= \exp\left(-\frac{g^2\kappa^3\phi^2}{2} \phi_{\mu\nu} \frac{\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}}{-\nabla^2} \phi_{\alpha\beta} \right) . \end{aligned} \tag{6.24}$$

Small loop (surface) corrections can alter only the coefficients of the Coulomb potentials in these results. The long-range nature of the interaction kernels is associated with the Goldstone bosons due to the spontaneous symmetry breaking present in both the electric and the magnetic condensation phases. In four Euclidean dimensions the results (6.24) for the 't Hooft surface in the electric condensation phase implies that the self-energy of a circular vortex loop of radius R is proportional to $R\ln R$. As in two dimensions this is tantamount to *logarithmic confinement* of magnetic fluxes and we conclude thus

that the electric condensation phase is still a *superconducting phase*. The result (6.24) for the Wilson loop in the magnetic condensation phase, instead, represents a *perimeter law*, implying a $1/r$ Coulomb potential between charges. The amount of energy required to separate a charge-anticharge pair is finite, although the interaction potential is long-range. We identify this as an *insulating phase* (as opposed to the superinsulator in two dimensions). Clearly, the dilute phase for both topological excitations corresponds again to a *metallic phase*. As already mentioned, it is harder to estimate the position of the phase transitions in three dimensions due to the lack of self-duality.

Figure Captions

Fig. 1. The function $\frac{mlG(ml)\pi}{\mu}$ for $\mu = \ln 5$.

Fig. 2. Phase diagram of fabricated Josephson junction arrays (adapted from the last paper in [5]). Solid squares denote a transition from metallic behaviour to superconducting behaviour when the temperature is lowered; open squares denote a corresponding transition from metallic to insulating behaviour.

References

[1] J. Fröhlich and U. Studer, Rev. Mod. Phys. 65 (1993) 733, J. Fröhlich, R. Götschmann and P. A. Marchetti, J. Phys. A: Math. Gen. 28 (1995) 1169.

[2] X.-G. Wen, Int. Jour. Mod. Phys. B6 (1992) 1711, "Topological Orders and Edge Excitations in FQH states", MIT preprint 95-148; A. Zee, "From Semionics to Topological Fluids", in "Cosmology and Elementary Particles", Proceedings of the 1991 Rio Piedras Winter School, D. Altschuler, J. F. Nieves, J. P. de Leon and M. Ubriaco eds., World Scientific, Singapore (1991).

[3] For a general review see: E. Fradkin, "Field Theories of Condensed Matter Systems", Addison-Wesley, Redwood City (1991).

[4] B. Blok and X.-G. Wen, Phys. Rev. B42 (1990) 8133, Phys. Rev. B42 (1990) 8145; J. Fröhlich and T. Kerler, Nucl. Phys. B354 (1991) 369; J. Fröhlich and A. Zee, Nucl. Phys. B364 (1991) 517, X.-G. Wen and A. Zee, Phys. Rev. B46 (1992) 2290.

[5] U. Eckern and A. Schmid, Phys. Rev. B39 (1989) 6441; R. Fazio and G. Schön, Phys. Rev. B43 (1991) 5307; R. Fazio, A. van Otterlo, G. Schön, H. S. J. van der Zant and J. E. Mooij, Helv. Phys. Acta 65 (1992) 228.

[6] M. C. Diamantini, P. Sodano and C. A. Trugenberger, Nucl. Phys. B448 (1995) 505.

[7] A. Kovner and B. Rosenstein, Phys. Rev. B42 (1990) 4748; G. W. Semenoff and N. Weiss, Phys. Lett. B250 (1990) 117; N.Dorey and N. E. Mavromatos, Phys. Lett. B250 (1990) 107, Nucl. Phys. B386 (1992) 614.

[8] For a review see: A. M. Polyakov, "Gauge Fields and Strings", Harwood Academic Press, Chur (1987).

[9] We thank G. W. Semenoff for pointing this out to us.

[10] A. A. Odintsov and Y. V. Nazarov, Phys. Rev. B51 (1995) 1133.

[11] M. Y. Choi, Phys. Rev. B50 (1994) 10088.

[12] A. Stern, Phys. Rev. B50 (1994) 10092.

[13] For a review see R. B. Laughlin, "The Incompressible Quantum Fluid", in "The Quantum Hall Effect", R. E. Prange and S. M. Girvin eds., Springer Verlag, New York (1990).

[14] For a review see: F. Wilczek, "Fractional Statistics and Anyon Superconductivity", World Scientific, Singapore (1990); A. Lerda, "Anyons", Springer-Verlag, Berlin (1992).

[15] G. 't Hooft, Nucl. Phys. B190 [FS3](1981) 455.

[16] J. L. Cardy and E. Rabinovici, Nucl. Phys. B205 [FS5] (1982) 1; J. L. Cardy, Nucl. Phys. B205 [FS5] (1982) 17.

[17] R. B. Laughlin, Phys. Rev. Lett. 60 (1988) 2677; see also [14].

[18] M. Kalb and P. Ramond, Phys. Rev. D9 (1974) 2273; F. Lund and T. Regge, Phys. Rev. D14 (1976) 1524.

[19] A. P. Balachandran, V. P. Nair, B.-S. Skagerstam and A. Stern, Phys. Rev. D26 (1982) 1443; T. J. Allen, M. Bowick and A. Lahiri, Mod. Phys. Lett. A6 (1991) 559; A. P. Balachandran and P. Teotonio-Sobrinho, Int. Jour. Mod. Phys. A9 (1994) 1569.

[20] See for example: C. Itzykson and J.-M. Drouffe, "Statistical Field Theory", Cambridge University Press, Cambridge (1989).

[21] Excellent reviews are: J. Fröhlich, "The Statistical Mechanics of Surfaces", in "Applications of Field Theory to Statistical Mechanics", L. Garrido ed., Lecture Notes in Physics Vol. 216, Springer-Verlag, Berlin (1985); "Statistical Mechanics of Membranes and Surfaces", Jerusalem Winter School for Theoretical Physics, Vol. 5, D. Nelson, T. Piran and S. Weinberg eds., World Scientific, Singapore (1989), F. David, "Introduction to the Statistical Mechanics of Random Surfaces and Membranes", in "Two-Dimensional Quantum Gravity and Random Surfaces", Jerusalem Winter School for Theoretical Physics, Vol. 8, D. J. Gross, T. Piran and S. Weinberg eds., World Scientific, Singapore (1992).

[22] R. Jackiw and S. Templeton, Phys. Rev. D23 (1981) 2291; J. Schonfeld, Nucl. Phys. B185 (1981) 157; S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. 48 (1982) 975, Ann. Phys. (N.Y.) 140 (1982) 372.

[23] J. Fröhlich and P. A. Marchetti, Comm. Math. Phys. 121 (1989) 177; P. Lüscher, Nucl. Phys. B326 (1989) 557; V. F. Müller, Z. Phys. C47 (1990) 301; D. Eliezer and G. W. Semenoff, Ann. Phys. (N.Y.) 217 (1992) 66.

[24] For a review see: J, W. Negele and H. Orland, "Quantum Many-Particle Systems", Addison-Wesley, Redwood City (1988).

[25] For a review see: H. Kleinert, "Gauge Fields in Condensed Matter", World Scientific, Singapore (1989).

[26] See for example: V. L. Streeter, "Fluid Dynamics", McGraw-Hill, New York (1948); P. D. McCormack and L. Crane, "Physical Fluid Dynamics", Academic Press, New York (1973).

[27] T. Banks, R. Myerson and J. Kogut, Nucl. Phys. B129 (1977) 493; see also B. Svetitski, Phys. Rep. 132 (1986) 1.

[28] See for example J. Kogut, Rev. Mod. Phys. 55 (1983) 775.

[29] A. S. Goldhaber, R. Mackenzie and F. Wilczek, Mod. Phys. Lett. A4 (1989) 21.

[30] N. Read, Phys. Rev. Lett. 65 (1990) 1502.

This figure "fig1-1.png" is available in "png" format from:

http://arXiv.org/ps/hep-th/9511168

This figure "fig1-2.png" is available in "png" format from:

http://arXiv.org/ps/hep-th/9511168