Variational solution of the Gross-Neveu model: Finite N and renormalization[1]

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Abstract

We show how to perform systematically improvable variational calculations in the $O(2N)$ Gross-Neveu model for generic N, in such a way that all infinities usually plaguing such calculations are accounted for in a way compatible with the perturbative renormalization group . The final point is a general framework for the calculation of non-perturbative quantities like condensates, masses etc. . ., in an asymptotically free field theory. For the Gross-Neveu model, the numerical results obtained from a "2-loop" variational calculation are in a very good agreement with exact quantities down to low values of N.

1 Introduction.

It has been advocated for a long time ([1]-[10]) that the convergence of conventional perturbation theory may be improved by a variational procedure in which the separation of the action into "free" and "interaction" parts is made to depend on some set of auxiliary parameters. The results obtained by expanding to finite order in this "redefined" perturbation series are optimal in regions of the space of auxiliary parameters where they are least sensitive to these parameters. This is reasonable: such regions are those expected to resemble the most to the complete answer, where there should simply be *no* dependence on the auxiliary parameters. This intuitive argument can be made more precise, and there has recently appeared strong evidence that this optimized perturbation theory may indeed lead to a rigorously convergent series of approximations even in strong coupling cases. In particular, the convergence of this variational-like procedure has been rigorously established in the case of zero and one dimensional "field theories" [11].

Having these rigorous proofs, it is very tempting to try to apply these methods in renormalizable quantum field theory. The main obstacle which one should overcome in this case is the compatibility of the variational expansion with the usual renormalization program of the theory. Indeed there are very few field-theoretical cases where the method has been tested and gave non-trivial results. Most of these cases concern the ϕ^4 theory, which is unfortunately free in four dimensions, and the effective potential of the large-N Gross-Neveu (GN) model [6, 7, 9, 10].

In this paper, in the finite-N GN model, we shall show how one can build variational expansions which are compatible with perturbative renormalization and give non-trivial results even for low values of N. Our approach can be in principle extended to more general theories, like QCD, as a new non-perturbative approach to the calculation of (dynamical) chiral symmetry breaking parameters [12].

We shall first take a closer look at the large-N [13] limit and from there infer the procedure for the case of arbitrary N. Taking the fermion mass as a variational parameter we can obtain exact results in the large-N limit of the $O(2N)$ GN model, but as we shall see, in the finite-N case it seems necessary to go to infinitely high perturbative order just to obtain the right renormalization group (RG) behavior. We will see how to overcome this problem and interpret it in order to automatically include the correct renormalization group properties, obtaining finite answers as the space-time dimension goes to 2. This turns out to be a relatively simple exercise in renormalization theory, taken from an unusual point of view. At the end, we obtain a variational estimate of the vacuum energy and the mass gap, as a function of $\Lambda_{\overline{MS}}$, whose accuracy increases with the order of the perturbative expansion. Exact results for these two quantities are known [14, 15] and

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provide a test for the accuracy of the method. Note that our method also has definite connections with previous derivations of a mass gap in the framework of the (non-linear) sigma model [16, 17]. In particular our way of using the RG properties to relate the mass gap to the basic scale, $\Lambda_{\overline{MS}}$, is quite analogous with the one in the latter reference, although the subsequent method to extrapolate the pure RG results to the actual mass gap values is completely different. Moreover, as above mentioned the essential novelty in the present approach is to clarify the link between RG results and the variational expansion, allowing a treatment of the latter that is consistent with infinities and therefore much more adaptable to other theories.

In section 2, we explain the basic idea and ingredients of our method, using only the first coefficients of the renormalization group $\beta(g)$ and $\gamma(g)$ functions in order to keep the discussion clear. In section 3, we show how to generalize, formally, to arbitrary higher orders the previous construction, and derive explicit formulas for the fermion mass and the vacuum energy density using the presently known maximal perturbative order, i.e. the three-loop renormalization group functions. In section 4, we construct an alternative Ansatz, starting directly from renormalized quantities, which is shown to differ in general from the previous one by a specific renormalization scheme change, and lead to essentially similar results. It also gives a more transparent construction, in particular more appropriate to generalization to higher orders and other theories. Finally some technicalities are treated in two Appendices.

2   One-loop renormalization group properties

Let us start from the $O(2N)$ GN Lagrangian [18], modified with a non-zero fermion mass as follows

$$L_{GN}(m_0, g_0, x) = \bar{\Psi}(i\gamma_\mu \partial^\mu - m_0)\Psi + x\,(\frac{g_0^2}{2}(\bar{\Psi}\Psi)^2 + m_0\bar{\Psi}\Psi); \qquad (2.1)$$

where we introduced for convenience a parameter x, interpolating between the *free*, massive Lagrangian for $x = 0$, and the usual massless GN interaction Lagrangian, for $x = 1$ (summation over the $2N$ component of fermions Ψ is implicitly understood [1]). As is well known, one can solve the model exactly in the large N limit and obtain a non-trivial mass gap in the massless limit, i.e. for $x \to 1$ in the present context. Now how can we treat the theory as defined by (2.1) for arbitrary N? A direct application of the arguments developed in the introduction would consist of doing perturbation theory with respect

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[1]We consider the $O(2N)$ case only for convenience of comparison with the results of ref.[14], but in fact there are a priori no limitations in our method to consider N half-integer as well, i.e. corresponding to the usual $O(N)$ model.

to the reorganized Lagrangian above, i.e. expanding the above terms to some order in x. One would then look for extrema with respect to m, supposedly approximating the unknown exact result for $x \to 1$, which by definition does not depend on m. Lessons from the anharmonic oscillator case leads one to expect a systematic improvement when going to higher orders in the expansion in x, the better approximation being assumed for the flattest extrema structure.

Unfortunately, one immediately encounters a number of obstacles. Most importantly, before accessing to any physical quantity of interest for such an optimization the theory has to be renormalized, and there is an unavoidable mismatch between the expansion in x as introduced above and the ordinary perturbative expansion as dictated by the necessary counterterms, as we shall see explicitely in section 2.2. Independently of that, even in the most optimistic case where arbitrary orders of perturbation theory would be known (which is only true in the large N case), it is easily seen that at any *finite* order one only recovers a trivial result for $x \to 1$ (i.e. $m_0 \to 0$), which is the limit we are however interested in to identify a non trivial mass gap.

Actually these facts can be circumvented by advocating a certain resummation Ansatz, whose main ingredient exploits the renormalization group invariance of the theory and analytic continuation properties of the (arbitrary) mass parameter m [2], which at the end defines a mass gap as an integral transform as we shall derive below. This (renormalization group invariant) Ansatz is exact in the large N limit, where it can be shown to resum the x dependence exactly. In the finite N case it is only exact as far as the leading (or next-to-leading) renormalization group behavior is concerned. But the important point is that it also provides a non-trivial resummation of the perturbative serie in x, in the sense that it gives a finite and non-zero result in the $m \to 0$ limit. This is expected to be a sufficiently good "trial" Ansatz for a subsequent optimization with respect to m, as will be motivated below.

2.1 The mass gap in the large N limit

To be more concrete let us start with the expression of the one-loop mass and coupling counterterms of the model, expressed with the renormalized parameters m and g related to m_0 and g_0 by :

$$m_0 = m Z_m = \frac{m}{\left[1 - \frac{g^2(N-1)}{\pi \varepsilon}\right]^{\frac{N-1/2}{N-1}}} \quad ,$$

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[2]More precisely, of an arbitrary parameter related to the original Lagrangian mass m_0 in a way to be specified next.

$$g_0^2 = g^2 \mu^{-\varepsilon} Z_g = \frac{g^2 \mu^{-\varepsilon}}{1 - \frac{g^2(N-1)}{\pi\varepsilon}} \tag{2.2}$$

where μ is the arbitrary scale introduced by dimensional regularization. Note that those one-loop expressions are exact in the $N \to \infty$ limit, where it can be shown that there are no other leading N corrections. For finite N this one-loop RG dependence generates however only the leading terms in $1/\epsilon$, to all orders.

Consider now the following expression for a (bare) resummed mass, as inspired by the latter RG properties:

$$m_F(m_0) = \frac{m_0}{(1 - 2(N-1)g_0^2\tilde{\Gamma}\mu^{-\varepsilon}m_F^\varepsilon(m_0))^{\frac{N-1/2}{N-1}}} \tag{2.3}$$

(where $\tilde{\Gamma} \equiv \Gamma[-\varepsilon/2]/(4\pi)^{1+\varepsilon/2}$). Eq. (2.3) is obviously renormalization group invariant since expressed only in terms of the bare parameters m_0 and g_0, and can be expressed in terms of the renormalized parameters $m(\mu)$ and $g(\mu)$ with the help of eq. (2.2) above. When this is done, one obtains for m_F an expression which can be recursively expanded in powers of $g^2(\mu)$ and, as easily checked, is finite at each order as $\varepsilon \to 0$, at fixed $g^2(\mu)$ and $m(\mu)$, thanks to the recursive dependence in m_F in eq. (2.3). Explicitly one obtains

$$m_F(\bar{m}) = \bar{m} \left[1 + \frac{(N-1)}{\pi}\bar{g}^2 \ln\left[\frac{m_F(\bar{m})}{\bar{\mu}}\right]\right]^{-\frac{N-1/2}{N-1}}, \tag{2.4}$$

where the $\overline{MS}$ scheme scale, $\bar{\mu} = \mu e^{\frac{1}{2}(-\gamma_E + \ln 4\pi)}$ was introduced for convenience, and $\bar{g} \equiv g(\bar{\mu})$, $\bar{m} \equiv m(\bar{\mu})$. Eq. (2.4) resums the leading logarithmic dependence in $\bar{m}$, and in the $N \to \infty$ limit there are no other corrections. Using the recursivity one can rewrite (2.4) identically as

$$m_F(m'') = \Lambda_{\overline{MS}} \frac{m''}{f(m'')^{\frac{N-1/2}{N-1}}}; \tag{2.5}$$

where the scale-invariant, dimensionless mass parameter:
$m'' \equiv (\bar{m}/\Lambda_{\overline{MS}}) [(N-1)\bar{g}^2/\pi]^{-\frac{N-1/2}{N-1}}$ has been introduced for convenience and is related to f as

$$f(m'') + \frac{N-1/2}{N-1} \ln f(m'') \equiv \ln m'' \tag{2.6}$$

while $\Lambda_{\overline{MS}} \equiv \bar{\mu} \, exp[-\pi/((N-1)g^2(\bar{\mu}))]$ is the $\overline{MS}$ RG-invariant basic scale.
Now from (2.6), it is easily seen that for $m'' \to 0$ (i.e. $m \to 0$), $f(m'') \simeq (m'')^{\frac{N-1}{N-1/2}}$, so that one recovers the well known result,

$$m_F = \Lambda_{\overline{MS}} \tag{2.7}$$

for the mass-gap in the large N limit.

2.2 Usual plague of the x expansion and its cure

Up to now we have not made any use of the expansion in the "variational" parameter x and indeed, as expected, that was not necessary to obtain the correct result for the mass gap in the $N \to \infty$ limit. Since the large N result is a good consistency check, and the perturbation in x will play a definite rôle later on in the more complicated arbitrary N case, let us see what is happening when considering the x expansion.

Introducing the x expansion parameter in (2.1) formally amounts to substitute $m_0 \to m_0(1 - x)$, $g_0^2 \to g_0^2 x$ in any (bare) expression, in particular in (2.3). One however soon realizes that (2.3) is then no longer finite for $\epsilon \to 0$ at any order, apart for $x = 1$, in which case only the trivial result $m_F = 0$ is recovered, as announced. Indeed the resummed expression (2.4) resulted from the usual RG properties and has no reason to be compatible with the way the perturbative x expansion is introduced. The cure to that problem is to resum the series $m_F(x) \equiv \sum_{x=0}^{\infty} a_q x^q$, in a different way: this is made possible by analytic continuation in x, and an adequate contour integral can be shown [13] to resum exactly (in the large N limit) the above series, as explained in some details in Appendix A. Apart from technical details, the main effect of the resummation is to cancel the factor of $(1 - x)$ which arises from the substitution applied to m_0 and was responsible for the trivial result at any arbitrary order for $x \to 1$. The net result gives the mass gap as a specific contour integral expression:

$$
\left.
\begin{aligned}
& m_F(m_0') = \frac{m_0'}{2i\pi} \oint \frac{du\, e^u}{f_1^{\frac{N-1/2}{N-1}}(u)} \\
& \text{with} \\
& f_1(u) = 1 - 2(N-1)g_0^2 \tilde{\Gamma}\, (m_0' u)^\varepsilon \, [f_1(u)]^{-\frac{N-1/2}{N-1}\varepsilon}
\end{aligned}
\right\}
\tag{2.8}
$$

where $u \equiv q(1 - x)$ has been introduced as an appropriate change of variable to analyse the $x \to 1$, $q \to \infty$ limit of the qth-order expansion of $m_F(x)$, m_0' is obtained by the rescaling $m_0 \equiv q\, m_0'$, and the integration contour runs counterclockwise along the cut negative real axis.

The other nice thing about equation (2.8) is, that it has a smooth limit as $\varepsilon \to 0$, at fixed renormalized m' and g^2, where m' is obviously related to m_0' as in (2.2). Indeed in the limit $\varepsilon \to 0$, we have :

$$
\left.
\begin{aligned}
& m_F(m') = \frac{m'}{2i\pi} \oint \frac{du\, e^u}{[f_1'(u)]^{\frac{N-1/2}{N-1}}} \\
& \text{with:} \\
& f_1'(u) = 1 + \frac{g^2(N-1)}{\pi} \left(\ln \frac{m'u}{\overline{\mu}} - \frac{N-1/2}{N-1} \ln f_1' \right).
\end{aligned}
\right\}
\tag{2.9}
$$

As expected, this can be expressed in terms of the dimensionless parameter m'', given by

$$m'' = \frac{m'}{\Lambda_{\overline{MS}} \left[\frac{(N-1)g^2}{\pi}\right]^{\frac{N-1/2}{N-1}}} \quad , \tag{2.10}$$

which is invariant under the renormalization group of the *massive* theory i.e.

$$\left\{\mu\frac{\partial}{\partial\mu} - \frac{g^2(N-1)}{\pi}g\frac{\partial}{\partial g} - \frac{g^2(N-1/2)}{\pi}m'\frac{\partial}{\partial m'}\right\} m'' = 0 \quad . \tag{2.11}$$

After these rescalings one obtains:

$$m_F(m'') = \Lambda_{\overline{MS}}\,\frac{m''}{2i\pi}\oint\frac{du e^u}{f^{\frac{N-1/2}{N-1}}(u)} \quad , \tag{2.12}$$

with[3]

$$f(u) = \ln(m''u) - \frac{N-1/2}{N-1}\ln f(u) \quad . \tag{2.13}$$

This conceptually very simple expression defines a function $m_F(m'')$, which should be studied, looking for extrema, and/or its value at $m'' = 0$, as explained in references [13, 19]. In fact, as far as the large N limit is concerned the optimization is trivial, as one may expect: from (2.13) one can find the series expansion of $f(u)$ near the origin as $f(u) \simeq (m''u)^{\frac{N-1}{N-1/2}}$, from which it is immediate that the expression (2.12) has a simple pole at $u = 0$, whose residu simply gives $m_F = \Lambda_{\overline{MS}}$. Now however, the main interest of the above construction is that the resummation of the x expansion remains valid, when considering the generalization at higher RG orders of the different expressions, and when one includes as well the non-RG perturbative corrections which are present at the next order, as we shall see. Remark at this point that all the complexities of the renormalization procedure are hidden in the definition of the function $f(u)$ in (2.13). We note that once the pure number m'' is fixed, for example at an extremum of $m_F(m'')$, the corresponding value of m_F is an invariant of the renormalization group of the *massless* theory, as is evident from equation (2.12).

3 Higher order results

3.1 Generalized Ansatz

For finite N, and at a given order in g^2, one knows Z_m and Z_g from the knowledge of the renormalization group functions $\beta(g)$ and $\gamma(g)$ at the appropriate order in g^2. Hence we can ask the question :

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[3]Note that $f(u)$ is related to $f_1'(u)$ in (2.9) as $f(u) \equiv \frac{\pi}{(N-1)g^2}f_1'(u)$.

Can we guess for m_F (or any other physical quantity) an expansion in powers of $g_0^2 m_0^\varepsilon$ which is finite when $\varepsilon \to 0$ at fixed g^2 and m and has a power expansion in g^2 ?

The exact expression for m_F is of course an answer, but certainly not the only one, to this simple purely mathematical question. What we are after is a *class* of answers, and a general principle which will enable us to pick in this class the appropriate answer which reproduces the results of the calculation up to a given order in the loop expansion, *i.e.* in the expansion of m_F in powers of $g_0^2 m_0^\varepsilon$. The graphs of figure 1 give the perturbative expression for $m_F(m_0)$ to $\mathcal{O}(g_0^4)$, for arbitrary N:

$$
\begin{aligned}
m_F(m_0) \ = \ & m_0 + (2N - 1)\, g_0^2\, \tilde{\Gamma}\, m_0^{1+\varepsilon} \\
& + (2N - 1)\, g_0^4\, \tilde{\Gamma}^2\, m_0^{1+2\varepsilon}\, (1 + \varepsilon)\left\{ (2N - 1) - \frac{\Pi(\varepsilon)}{2} \right\} \\
& + O(g_0^6) \quad,
\end{aligned}
\tag{3.1}
$$

where we have used the following definitions:

$$
\begin{aligned}
\tilde{\Gamma} \ \equiv \ & \frac{\Gamma(-\frac{\varepsilon}{2})}{(4\pi)^{1+\varepsilon/2}} \quad, \\
\Pi(\varepsilon) \ \equiv \ & \frac{2^{-\varepsilon}}{\sqrt{\pi}} \frac{\Gamma(1/2 - \varepsilon/2)}{\Gamma(-\varepsilon/2)} \int_0^1 dx\, dy\, \frac{[x(1 - x)]^{-\varepsilon/2} y^{-1-\varepsilon/2}(1 + y)}{[(1 - y)^2 x(1 - x) + y]^{-\varepsilon}} \\
\underset{\varepsilon\to 0}{\simeq} \ & 1 - \frac{3\varepsilon}{2} + \Big[\frac{\pi^2}{24} - \frac{2.9022\ldots}{2}\Big]\varepsilon^2 + O(\varepsilon^3) \quad.
\end{aligned}
\tag{3.2}
$$

Now the previous expression (2.3) (from now on denoted as $\mathcal{F}$),

$$
\mathcal{F}(m_0) = \frac{m_0}{\left\{1 - 2(N - 1)\, g_0^2\, \tilde{\Gamma}\, \mu^{-\varepsilon}\, \mathcal{F}^\varepsilon\right\}^{\frac{N-1/2}{N-1}}}
\tag{3.3}
$$

reproduces eq. (3.1) only to order g_0^2 and the leading, $1/\varepsilon^2$ divergent terms of order g_0^4, while at the same time being finite, to all orders in g^2 fixed for $\varepsilon \to 0$, when the renormalized mass and coupling are substituted, as discussed in the previous section. It should be noted however that, even when restricting to the first RG order, m_F need not be identical to $\mathcal{F}$: at any order in g_0^2 there is room for a finite term, to be determined by an explicit calculation. This follows from the fact that

$$
\frac{g_0^2 \mathcal{F}^\varepsilon}{\left\{1 - 2(N - 1)g_0^2\tilde{\Gamma}\mathcal{F}^\varepsilon\right\}^{1+\varepsilon}}
\tag{3.4}
$$

is itself, as well as all its powers, in the class of renormalization group invariant functions of g^2 which are finite as $\varepsilon \to 0$, at fixed g^2. So it is clear that m_F can be expressed as a unique expansion in this expression. The above fact enables us to make compatible our variational procedure with renormalization theory at finite N as we shall see next. We must accordingly take into account the fact that equation (3.3) does not reproduce the

expansion of equation (3.1) at order g_0^4, since a finite term (as $\varepsilon \to 0$) survives, as well as a next-to-leading divergent term $\sim g_0^4/\varepsilon$, at that order [4].

To systematically correct these discrepancies, while keeping at all times smooth limits as $\varepsilon \to 0$ for fixed renormalized parameters, is achieved by noting that the renormalization group invariant:

$$\frac{g_0^2 m_0^\varepsilon}{\{1 - 2(N-1)g_0^2 \tilde{\Gamma} \, \mathcal{F}^\varepsilon\}^{1+\varepsilon \frac{N-1/2}{N-1}}} \tag{3.5}$$

is finite for $\varepsilon \to 0$ and has a power series expansion in both g_0^2 and g^2. With this invariant and all its positive powers, one will then reproduce the whole perturbative expansion (in powers of g_0^2) of equation (3.1).

The construction of our generalized Ansatz therefore goes as follows. We consider

$$\mathcal{F}(m_0) = \frac{m_0}{\{1 - 2(N-1)g_0^2 \tilde{\Gamma}(\frac{\mathcal{F}}{\mu})^\varepsilon\}^A} \quad , \tag{3.6}$$

entering the following renormalization group invariant expression for m_F,

$$\begin{aligned}
m_F(m_0) &= \frac{m_0}{\{1 - 2(N-1)g_0^2 \tilde{\Gamma}(\frac{\mathcal{F}}{\mu})^\varepsilon\}^B} \left[1 + \frac{\lambda \, g_0^2 m_0^\varepsilon}{\{1 - 2(N-1)g_0^2 \tilde{\Gamma}(\frac{\mathcal{F}}{\mu})^\varepsilon\}^D} + \right. \\
&\quad \left. \frac{\rho \, g_0^4 m_0^{2\varepsilon}}{\{1 - 2(N-1)g_0^2 \tilde{\Gamma}(\frac{\mathcal{F}}{\mu})^\varepsilon\}^{2D}} + \cdots \right] \quad ;
\end{aligned} \tag{3.7}$$

where the constants A, B and D are to be determined by *requiring* equation (3.7) to be finite as $\varepsilon \to 0$ when expressed in terms of m and g^2. A, B and D are therefore obtained as an expansion in powers of ε, where the different orders of the RG coefficients enter at the different order of ε. The coefficient λ, ρ, $\cdots$ are then given by matching the perturbative expression for m_F, equation (3.1). This is explained in more details in Appendix B.

With expression (3.7) and a straightforward generalization of the steps decribed in Appendix A, one can reach infinite order in the variational expansion in a very similar way as the one leading to equation (2.9), (2.12), and obtains the following generalized version of equation (2.12):

$$\begin{aligned}
m_F(m'') &= \left(\frac{e}{2}\right)^{-\frac{1}{2(N-1)}} \Lambda_{\overline{MS}} \times \\
&\quad \frac{m''}{2i\pi} \oint \frac{du e^u}{f_2^{\frac{N-1/2}{N-1}}(u)} \left\{ 1 + \frac{\mathcal{M}_1}{f_2} + \frac{\mathcal{M}_2}{f_2^2} + \cdots \right\} \quad ,
\end{aligned} \tag{3.8}$$

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[4]Indeed, even at order g_0^2, a finite term survives, if in equation (2.3) we replace $\Gamma(-\frac{\varepsilon}{2})$ for example by its leading behavior, $-2/\varepsilon$.

where now [5]

$$f_2(u) = \ln(m''u) - \frac{N}{N-1}\ln f_2(u) \quad,$$

$$\mathcal{M}_1 = \frac{3(N-1/2)}{4(N-1)^2} \quad,$$

$$\mathcal{M}_2 = -\frac{(N-1/2)(36N^2 - 62N + 17)}{64(N-1)^4}$$

$$-\frac{N-1/2}{2(N-1)^2}(-0.725551\ldots + \frac{\pi^2}{48}) \quad, \tag{3.9}$$

and in the overall factor in (3.8) we again identifed the usual renormalization group invariant scale at this order,

$$\Lambda_{\overline{MS}} = \overline{\mu}\exp\left[-\frac{\pi}{(N-1)g^2}\right]\left[\frac{(N-1)g^2}{2\pi}\right]^{\frac{1}{2(N-1)}}\left(1 - \frac{g^2}{2\pi}\right)^{-\frac{1}{2(N-1)}} \quad,$$

up to an overall factor independent of g^2 in (3.8).

We now discuss expression (3.8), which is the essential result of this paper. One can analyze the function $m_F(m'')$ in standard ways. One finds that for $m'' \to \infty$, its behavior is accurately described by the first term of equation (3.8)

$$m_F(m'') \sim \Lambda_{\overline{MS}}\left(\frac{e}{2}\right)^{-\frac{1}{2(N-1)}}\frac{m''}{(\ln m'')^{\frac{N-1/2}{N-1}}} \quad, \tag{3.10}$$

a typical renormalization group result, and that the $\mathcal{M}_n/f_2^n$ correction contributes a corrective term of order $(\ln m'')^{-n}$ to this, as well as $\ln\ln m''$ terms. Hence, by going to higher and higher orders of ordinary perturbation theory, one determines the function $m_F(m'')$ with an increasing accuracy for m'' large enough.

The same phenomenon had been noticed for the anharmonic oscillator case in references [19, 20]. Just as in that case, the hope is then that already with a low order calculation, one may reach an accurate value for, say, the physical mass gap by taking an extremum of $m_F(m'')$, the best one being presumably the one closest to $m'' = 0$. For the $N = \infty$ case, we have seen that all the coefficients $\mathcal{M}_n$ are zero, and the extremum at $m'' = 0$ is the exact answer. For finite N, the numerical convergence of the approximation can only be decided by an explicit calculation.

Notice that, when the order of the perturbative expansion increases, one can deduce from the behavior of $f(u)$ as u goes to zero, that $m_F(m'')$ becomes more and more singular as $m'' \to 0$. This same fact had been noticed on the case of the anharmonic oscillator.

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[5] Note the difference between the coefficient of the $\ln f_2$ term in (3.9) and the power of f_2 in (3.8). The previous equality of those quantities, in eq.(2.9) (and the consequent simple pole behavior for $m'' \to 0$) was an accident of the first RG order.

There, this singular behavior at $m'' = 0$ did not prevent the analogous function from *converging* to the right answer for a wider and wider range of the variational parameter when the order of the expansion increases. Thus, one could obtain very accurate results, in particular when resummation techniques are used to improve the behavior of the integrand of equation (3.8) for $u \to 0$. This will be discussed below.

3.2 Vacuum energy density

A similar treatment can be applied to the vacuum energy density $E_0^{(B)}$, with an appropriate modification to account for the structure of its divergences.

Indeed, in contrast with m_F, the vacuum energy, in the massive theory, is not multiplicatively renormalizable. Rather, $\frac{\partial^3 E_0^{(B)}}{\partial m_0^3}$ is, and consequently $\frac{\partial^3 E_0^{(B)}}{\partial m^3}$ is finite. Combining this fact with dimensional arguments, it follows that all infinities of $E_0^{(B)}$ which persist, at a given order of the perturbative expansion (after coupling constant, mass, and wave function renormalization), are proportional to $\frac{m_0^2}{2g_0^2}$. Hence, by subtracting from $E_0^{(B)}$ the quantity $\frac{m_0^2}{2g_0^2} h(\varepsilon)$ with a suitably chosen function $h(\varepsilon)$, $E_0^{(B)}$ can be made finite for $\varepsilon \to 0$ at fixed renormalized coupling constant g and mass m. It is crucial to remark that such a subtracted counterterm gives vanishing contribution to our procedure *for any* ε as soon as the order of the variational-perturbative expansion is larger than one[6]. Expanding now the regular function $h(\varepsilon)$ as

$$h(\varepsilon) = \sum_{n \geq 0} h_n \varepsilon^n \quad , \tag{3.11}$$

one can uniquely determine the coefficient of the ε^n term by a direct perturbative calculation to $n + 1$ loops. In contrast with the mass case, the fact that the $(n + 1)$ loop information is needed to determine unambiguously the $1/f^n$ perturbative corrections in our Ansatz, is of course a reminiscence of the original ambiguity due to the above mentioned additional divergences of the vacuum energy [7].

Once this is done, the finite terms that are left over can be treated in the same way as m_F was treated. We skip the details and only give the final answer, as a function of the variational parameter m'', at $\varepsilon = 0$

$$E_0(m'') = 2(N-1) \left(\frac{e}{2}\right)^{-\frac{1}{(N-1)}} \Lambda_{\overline{MS}}^2 \tag{3.12}$$

$$\times \frac{m''^2}{2i\pi} \oint \frac{u\,du\,e^u}{f_2^{\frac{N}{N-1}}(u)} \left\{ 1 + \frac{\mathcal{E}_1}{f_2} + \frac{\mathcal{E}_2}{f_2^2} + \ldots \right\} ,$$

---

[6]More precisely the subtraction procedure does not give any new contributions beyond the generic form in (3.12) below, but actually the specific values of the perturbative coefficients, $\mathcal{E}_i$ in (3.12), consistently include a dependence on the subtracted terms.

[7]Actually however, only the value of the simple pole in ε at order $n + 1$ is needed.

where

$$\mathcal{E}_1 = \frac{N(N^2 + N/2 - 3/4)}{2(N - 1/2)(N - 1)^2} \quad,$$

$$\mathcal{E}_2 = -\frac{N(N - 1/2)(N^2 - 6N + 7/2)}{8(N - 2/3)(N - 1)^4} \quad. \tag{3.13}$$

3.3 Refinements

While our main result formulas (3.8) and (3.12) contain in their derivation all the conceptual ingredients of the method, they leave room for some refinements, which are necessary before they can be of practical use for optimization.

The first one comes from realizing that, even at a given RG order, our different Ansätze are not unique, since there are infinitely many ways of introducing the purely perturbative terms, $1/f^n$ corrections. (Moreover beyond the first RG order even the resummation of the correct two-loop RG behaviour is not uniquely fixed by the function $\mathcal{F}$ as defined in eq. (3.9), as will be discussed in the next section). To take into account this freedom, let us introduce an arbitrary parameter a, from substituting $\bar\mu \to a\,\bar\mu$ in the different relevant expressions. This constant, which enlarges the class of our renormalization group invariant finite guesses for m_F, cannot be ruled out *a priori*, and parameterizes a renormalization scheme (RS) dependence. Of course, if one were able to work to all orders, the final answer would be independent on a. However, since we are truncating the expansion in equation (3.7), the choice of this scaling constant a turns out to be important in order to obtain a reliable numerical estimation for m_F. In the present model, the value of a which ensures a rapid convergence of the expansion in equation (3.7) is expected to be $a = 1 + O(1/N)$. We have however no *a priori* idea of the value to chose in more complicated theories as QCD. Hence we shall see in the following how to cope with the existence of the parameter a, intrinsic to renormalizable theories, in order to avoid too much additional arbitrariness in our results.

Applying now the variational procedure, we finally obtain a last version of equation (3.8), expressed in terms of the usual renormalization group invariant parameter $\Lambda_{\overline{MS}}$,

$$\frac{m_F}{\Lambda_{\overline{MS}}} = \left(\frac{\mathrm{e}}{2}\right)^{-\frac{1}{2(N-1)}} \frac{m'' a}{2i\pi} \oint \frac{du\,\mathrm{e}^u}{f_2^{\frac{N-1/2}{N-1}}(u)} \left\{1 + \frac{\mathcal{M}_1(a)}{f_2} + \frac{\mathcal{M}_2(a)}{f_2^2} + \ldots\right\} \quad, \tag{3.14}$$

where the recursively defined function f_2 is given by

$$f_2 + \frac{N}{N-1} \ln f_2 = \ln\left(\frac{m'' u}{a}\right) \quad. \tag{3.15}$$

The coefficients in the expansion are now given by

$$
\begin{aligned}
\mathcal{M}_1(a) &= \mathcal{M}_1(1) - \frac{(N - 1/2)}{(N - 1)} \ln a \quad, \\
\mathcal{M}_2(a) &= \mathcal{M}_2(1) + \frac{(N - 3/4)(N - 1/2)}{(N - 1)^2} \ln^2 a + \\
& \quad \frac{(N - 1/2)(N^2 - 5/2N + 9/8)}{(N - 1)^3} \ln a \quad,
\end{aligned}
\tag{3.16}
$$

where $\mathcal{M}_1(1)$ and $\mathcal{M}_2(1)$ are defined in equation (3.9).

Similarly, we obtain for the energy

$$
\frac{E_0}{2(N - 1)\Lambda_{\overline{MS}}^2} = \left(\frac{e}{2}\right)^{-\frac{1}{(N-1)}} \frac{(m'')^2 \, a^2}{2i\pi} \oint \frac{u du e^u}{f_2^{\frac{N}{N-1}}(u)} \left\{ 1 + \frac{\mathcal{E}_1(a)}{f_2} + \frac{\mathcal{E}_2(a)}{f_2^2} + \dots \right\}
\tag{3.17}
$$

with

$$
\begin{aligned}
\mathcal{E}_1(a) &= \mathcal{E}_1(1) - \frac{N}{(N - 1)} \ln a \quad, \\
\mathcal{E}_2(a) &= \mathcal{E}_2(1) + \frac{N(N - 1/2)}{(N - 1)^2} \ln^2 a - \frac{3N(N - 1/2)}{2(N - 1)^2} \ln a \quad,
\end{aligned}
\tag{3.18}
$$

$\mathcal{E}_1(1)$ and $\mathcal{E}_2(1)$ being given in equation (3.13).

We are now asked to find a stationary value for these expressions in term of m'' :

$$
\frac{\partial m_F}{\partial m''} = 0 \quad \text{and} \quad \frac{\partial E_0}{\partial m''} = 0 \quad .
\tag{3.19}
$$

In order to get a feeling of what we have achieved, we have plotted in figure 3 the values of the mass gap $m_F/\Lambda_{\overline{MS}}$ versus the variational parameter m'', for different values of the scaling constant a ranging from $a = 1.1$ to $a = 1.2$. Similarly, we have plotted in figure 4 the values of the vacuum energy $E_0/\Lambda_{\overline{MS}}^2$ versus m'', for a ranging from $a = 0.9$ to $a = 1.1$. In both cases, we have taken a reasonably small number of "flavors", $N = 3$. We have limited the graph to the vicinity of the origin in m'' since it is the region of interest (for large m'' the curves are following the expected perturbative behavior). The dashed horizontals represent the known exact values for the mass [8] [14], obtained from the Bethe Ansatz:

$$
\frac{m_F}{\Lambda_{\overline{MS}}} = \frac{(4e)^{\frac{1}{2(N-1)}}}{\Gamma\left(1 - \frac{1}{2(N-1)}\right)} \quad,
\tag{3.20}
$$

---

[8]In these figures, we have used an expansion for $m'' \to 0$ in order to produce the curves in a reasonable computing time. The corresponding slight change in the shapes of the displayed curves is irrelevant for the discussion.

and for the vacuum energy[9] [15]

$$E_0 = -\frac{m_F^2}{8} \cot \frac{\pi}{2(N-1)} \quad .\tag{3.21}$$

Many comments are now in order:

First of all, we have succeeded in producing finite values for the mass gap m_F and the vacuum energy E_0 at 2-loops order, including a variational mass m'', in a way consistent with the renormalization group . The extension of the procedure to higher orders in perturbation is only a matter of computational effort. Moreover, the method can be applied, at least in principle, to some of the order parameters governing the chiral symmetry breakdown in the realistic case of QCD, as we shall see in a subsequent paper.

Secondly, these expressions, equations (3.14) and (3.17), exhibit stationary values with respect to the parameter m'', as can be seen on figures 3 and 4 . We can then proceed and look at these stationary values.

We immediately meet an obvious problem: due to the large dispersion of the curves when the scaling constant a is varied, it seems difficult to predict a reliable numerical value for the ratios $m_F/\Lambda_{\overline{MS}}$ and $E_0/\Lambda_{\overline{MS}}^2$. Actually, the stationary values for the ratio $m_F/\Lambda_{\overline{MS}}$ are seen to lie in the range 1 to 3 when a is varied from 1.1 to 1.2, and one cannot hope to extract from that more than a vague estimate of the mass gap.

However, one can see in figure 3 that the flattest curve, whose stability plateau gets closest to $m'' = 0$, lies quite near to the expected exact value. Moreover, the divergence of the curves near $m'' = 0$, where we would expect to obtain the exact answer, as in the $N \to \infty$ case, is only due to the breakdown of perturbation theory as $m'' \to 0$. This, and previous experience with the anharmonic oscillator case [19], suggest us to try an extrapolation towards $m'' = 0$ in order to improve the numerical precision of our results. This will be done now.

3.4 Optimization results

The strategy is to turn the expansion in powers of $1/f_2$ of equation (3.14) into an expansion around $m'' = \infty$, whose first term appears in equation (3.10), later to be extrapolated by some variant of Padé techniques to $m'' = 0$. This however produces an expansion containing $\ln \ln \ldots \ln m''$, which is not suitable for an extrapolation towards $m'' = 0$. In

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[9]We are grateful to Prof. Al. B. Zamolodchikov for communicating his result to us before publication.

fact, equation (3.15) suggests that instead of m'' one should use the variable η defined by

$$\eta + \frac{N}{N-1} \ln \eta = \ln\left(\frac{m''}{a}\right) \quad . \tag{3.22}$$

In terms of η, the $\ln \ln \ldots \ln m''$ disentangle, and expression (3.14) now admits a very nice asymptotic expansion for $\eta \to \infty$, which is readily seen to give[10]

$$
\begin{aligned}
\frac{m_F}{\Lambda_{\overline{MS}}} &= \left(\frac{e}{2}\right)^{-\frac{1}{2(N-1)}} \frac{N-1/2}{N-1} \, a e^\eta \eta^{-\frac{N-3/2}{N-1}} \left\{ 1 + \frac{1}{\eta} \left(2 \frac{N-3/2}{N-1/2} \mathcal{M}_1(a) \right. \right. \\
&\quad + \left. \left. \frac{(2N-3/2)}{N-1} \gamma_E - \frac{N}{N-1} \right) + O(1/\eta^2) \right\} \quad .
\end{aligned}
\tag{3.23}
$$

Taking now the derivative of the logarithm of this expression yields a power series in $1/\eta$, which is suitable for a Padé approximant analysis:

$$
\begin{aligned}
\frac{\partial \ln\left(\frac{m_F}{\Lambda_{\overline{MS}}}\right)}{\partial \eta} &\simeq 1 - \frac{1}{\eta} \frac{N-3/2}{N-1} - \frac{1}{\eta^2}\left(2 \frac{N-3/2}{N-1/2} \mathcal{M}_1(a) + \right. \\
&\quad \left. \frac{(2N-3/2)}{N-1} \gamma_E - \frac{N}{N-1} \right) + \ldots \quad .
\end{aligned}
\tag{3.24}
$$

From the large-N case studied in a preceding paper, we can make the guess that, were it not for the fact that perturbation theory breaks down at $m'' = 0$, $m_F(m'')$ would approach the exact mass gap as

$$m_F(m'') \underset{m'' \to 0}{\simeq} m_F(0) + O(e^{-\text{Const}/m''}) \quad . \tag{3.25}$$

Therefore we can try a Padé approximant of the form

$$\frac{\partial \ln\left(\frac{m_F}{\Lambda_{\overline{MS}}}\right)}{\partial \eta} \simeq e^{C_1/\eta} \frac{1 + C_2/\eta + C_4/\eta^2 + \ldots}{1 + C_3/\eta + C_5/\eta^2 + \ldots} \quad , \tag{3.26}$$

which ensures an exponential convergence of m_F near $m'' = 0$, *provided* that $C_1 < 0$. However this is by no way limitative, in particular for *finite N* there are no guarantees that the exponential behavior persists, therefore we shall also use simple Padé approximants (i.e similar to (3.26) without exponential factor), and compare different Padé results. For any Padé approximant type we can now determine the coefficients $\{ C_1 , C_2 , C_3 \ldots \}$ by matching the large-η expansion (3.24), and integrating equation (3.26) from $\eta \to +\infty$,

---

[10]In (3.23) the coefficients of η^{-n} involve integrals of the type $I(p) = \frac{1}{2i\pi} \oint dy \, e^y \ln^p y$, which are easily evaluated as a (finite) expansion of real integrals expressable in terms of Euler Gamma functions.

where m_F is given by equation (3.24), to $\eta = 0$, will produce an extrapolated value at $m'' = 0$ for the mass gap $m_F/\Lambda_{\overline{MS}}$ [11].

Very likeky, it should exist an optimal Padé order, which may however vary as N varies, and is difficult to guess a priori. One may naively assume at first that the best Padé approximant should be of the same order of the known perturbative terms. Actually the situation is slighlty more complicated, since the RG coefficients, which is a genuine information independent of the purely perturbative terms, also enter Padé approximant expansions. Moreover, the larger N is, the more perturbative information may be considered available, since *all* the purely perturbative coefficents $\mathcal{O}(1/f^n)$ in m_F are vanishing for $N \to \infty$ (although we do not know *how fast* these are vanishing as functions of N). Hence, one can test the convergence properties of the increasing order Padé approximants (3.26) in the large-N case, and infer from it the accuracy of the lower Padés at finite N. Without much loose of generality in our numerical study we compare Padé approximants of different orders, from (1,1) to (2,3).

The results of the procedure for $N \to \infty$ are presented in table 1, where we have given for different values of the scaling constant a, the value of the extrapolated ratio, using a Padé 1-1, a Padé 2-2 or a Padé 3-3 approximant[12]. The exact value is $m_F/\Lambda_{\overline{MS}} = 1$. We see that the stationary (in a) values of the successive Padés do indeed converge, albeit rather slowly, as the order increases, and with what may be considered relatively large errors (20% for the 1-1 Padé). This indicates that the extrapolation procedure is probably far from optimal, and could be improved by using information on the asymptotic behavior of the large-η expansion for example.

For finite N, different Padé approximants give quite different answers, however it turns out that on the average the results are not worst than in the large N case. Results for the extrapolation of $m_F/\Lambda_{\overline{MS}}$ to $m'' = 0$ are presented in table 2 for different values of N and different Padé types, where we show the optimal values and the exact values from reference [14]. In addition we illustrate in figure 5 a comparison of the a dependence for the different Padé types, for the lowest value of $N = 2$, the behavior for other values being very similar.

We note the great stability as a function of a when a is varied in a wide range, in contrast with the dispersion of the curves in figure 3. We see that in some cases the optimal values are quite remarkably close to the exact ones, even for very low values of

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[11]Note that the re-integration of $m_F/\Lambda_{\overline{MS}}$ is unique, as the integration constant is fixed by the asymptotic $m'' \to \infty$ behavior.

[12]The blank entries correspond to values of a where the coefficient C_1 is positive. C_1 is a complicated function of a, and there may be more than one range of values of a where it is negative, which happens for the first time for the 3-3 Padé.

a	Padé 1-1	Padé 2-2	Padé 3-3
0.1		0.9643	
0.18	1.2115	0.9913	1.0031
0.2	1.2030	0.9965	1.0044
0.25	1.2041	1.0080	1.0093
0.5	1.3115	1.0463	
0.7	1.4379	1.0623	0.9526
0.8	1.5105	1.0668	0.9594
0.9	1.5896	1.0696	0.9602
1.0	1.6760	1.07095	0.9582
1.1	1.7711	1.07096	0.9544
1.2	1.8768	1.0698	0.9491
1.3	2.0025	1.0676	0.9422
1.5	2.2636	1.0602	0.9231
opt.	1.203	1.071	0.9602
			1.002
error	+20%	+7%	-4%, +0.2%

Table 1: extrapolation of $m_F/\Lambda_{\overline{MS}}$ towards $m'' = 0$ using Padé approximants in the case $N = \infty$.

N. This is encouraging especially in view of the fact that the (only known) first two non-RG perturbative corrections were used. Actually however, the extremely good agreement for specific Padé types, for instance at $N = 2$ and $N = 3$, should be attributed to numerical occasion. What is certainly more significant is the *average* error over different values of N, for a given Padé order and type. The overall best result for arbitrary N are obtained with the simple Padé (2,3) and the exponential Padé (2,2) which we note are both using expansion to the fifth order in $1/\eta$ in (3.24). This is most probably due to the fact that, at the order where we are working, for a fixed N and a, expression (3.14) actually depends on five *independent* parameters: three perturbative terms, namely the "zero order" overall coefficient, $\mathcal{M}_1$, and $\mathcal{M}_2$; *plus* two independent combinations of RG coefficients, namely the coefficient of $\ln f_2$ in (3.9) and the power of f_2 in (3.15). However it is not excluded that higher Padé orders would give even better results. Due to the rather complicated dependence of a given Padé upon the above parameters, we have not tried to systematically study this issue. It would certainly be of interest to further optimize the choice of the Padé approximants, but this would go beyond the scope of the present paper. At any rate, given the very different Padé approximant types confronted here, we can be confident that we have found a useful convergent variational scheme to compute non-perturbative quantities.

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	$N = 2$	$N = 3$	$N = 5$	$N = 8$
exact result	1.8604	1.4819	1.2367	1.133
Padé type				
$p_{1,1}(u) \times e^{(c1/u)}$ opt. result (error)	2.742 (32%)	1.83375 (19%)	1.48725 (17%)	1.3554 (16%)
$p_{1,2}(u)$ opt. result (error)	1.9278 (3.5%)	1.3079 (11.7%)	1.08105 (12.6%)	0.9961 (12%)
$p_{2,2}(u) \times e^{(c1/u)}$ opt. result (error)	1.758 (5.5%)	1.47498 (0.46%)	1.2843 (3.7%)	1.19626 (5.3%)
$p_{2,3}(u)$ opt. result (error)	1.6206 (12.8%)	1.3456 (9.2%)	1.1917 (3.6%)	1.1205 (1.1%)
$p'_{2,3}(u)$ opt. result (error)	1.8749 (0.77%)	1.4864 (0.3%)	1.2654 (2.3%)	1.1628 (2.6%)

Table 2: Extrapolation of $m_F/\Lambda_{\overline{MS}}$ towards $m'' = 0$ using different Padé approximants for finite N. The last two lines provide in addition a comparison of different RS for a same Padé approximant order: accordingly the results given in the last line were obtained using the alternative Ansatz form (4.14) derived in section 4.

A particular feature of renormalizable theories might however spoil this optimism: it is the dependence of perturbative computations on the renormalization scheme (RS). As is well known, apart from the arbitrariness of the renormalization scale, parametrized with a, already at the second RG order there is more arbitrariness in the renormalization prescriptions. Typically, only the first two perturbative coefficients of the beta function and the first coefficient of the anomalous mass dimension are scheme independent, so that higher coefficients can be set to arbitrary values by perturbative redefinitions of the coupling constant and finite renormalizations. In our case, this translates into the fact that in principle the higher coefficients $\mathcal{M}_i$ for i larger than one could be set to arbitrary values with an *ad hoc* renormalization scheme. This problem is well known in perturbative QCD. The only assurance is that any *reasonable* renormalization scheme should give similar results at a given order, and that the accuracy of the results should improve as the order increases. What reasonable means exactly is unknown, but dimensional regularization with minimal subtraction is generally believed to be such a reasonable scheme. To test convergence and accuracy, we do not have higher order calculations at our disposal, but we can use the other method: do the same computation at the same order with a different renormalization scheme (RS) which can be considered *a priori* as reasonable, and compare the results. This is what we now do.

4 Alternative Scheme Ansatz

The renormalization group invariant expression (3.7) has been reconciled in section 3 with a variational mass expansion, using renormalization group invariance properties of appropriate combinations of *bare* parameters, m_0, g_0 like in equation (3.6), and the resummation properties of the x series, resulting in the integral result (3.8). However, once this construction is understood, and a finite, renormalized mass gap (3.14) emerges, one can alternatively try to construct an Ansatz starting directly from *renormalized* quantities[13]. This will turn out to provide a more transparent intepretation of the different quantities involved in (3.8)–(3.14), which also turns out to be more appropriate for a generalization to other theories, like QCD typically.

Let us thus start from the renormalization group invariance properties of the renormalized mass and coupling constant, and examine what kind of non-perturbative information it may contain. Integrating the renormalization group equations for the running mass $m(\bar{\mu})$ in terms of the running coupling constant, $g(\bar{\mu})$, to two-loop renormalization group order exactly[14], with the "fixed point" boundary condition

$$M_F^{RG} \equiv m(M_F^{RG}),$$

one obtains after some algebra the expression

$$M_F^{RG} = m(\bar{\mu}) \ f^{-\frac{\gamma_0}{2b_0}} \ \left[\frac{1 + \frac{b_1}{b_0}g^2(\bar{\mu})f^{-1}}{1 + \frac{b_1}{b_0}g^2(\bar{\mu})}\right]^{-\frac{\gamma_1}{2b_1} + \frac{\gamma_0}{2b_0}} \tag{4.1}$$

where $f \equiv g^2(\bar{\mu})/g^2(M_F^{RG})$ satisfies the recursive relation

$$f = 1 + 2b_0 g^2(\bar{\mu}) \ln \frac{M_F^{RG}}{\bar{\mu}} + \frac{b_1}{b_0}g^2(\bar{\mu}) \ln\left[\frac{1 + \frac{b_1}{b_0}g^2(\bar{\mu})f^{-1}}{1 + \frac{b_1}{b_0}g^2(\bar{\mu})} \ f\right] ; \tag{4.2}$$

and the renormalization group coefficients b_i and γ_i are given in Appendix B. M_F^{RG} designates the part of the mass which is entirely determined from renormalization group properties, *i.e.* it does not include the purely perturbative non-logarithmic finite parts. The latter are consistently included as follows:

$$M_F \equiv M_F^{RG} \left(1 + m_{11}\frac{g^2(\bar{\mu})}{f} + m_{22}\frac{g^4(\bar{\mu})}{f^2} + \dots\right) \tag{4.3}$$

with m_{11} and m_{22} simply given by the non-logarithmic parts of the perturbative expression for the renormalized mass, which is evidently obtained from equation (3.1) by substituting

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[13]This procedure will also be treated in more details in the QCD context [12].

[14]*i.e.* keeping the exact dependence on g as given by the renormalization group but of course the renormalization group functions themselves being truncated at a given order.

$m_0 = Z_m m$, $g_0^2 = (\bar{\mu})^{-\varepsilon} Z_g g^2$. Using Z_m, Z_g at second order, one finds (in the $\overline{MS}$ scheme)

$$
\begin{aligned}
M_F^{pert} =~& m(\bar{\mu})\Big(1 - g^2(\bar{\mu})\frac{N-1/2}{\pi}\ln\frac{m(\bar{\mu})}{\bar{\mu}} \\
& + g^4(\bar{\mu})\Big[\frac{(N-1/2)(N-3/4)}{\pi^2}\ln^2\frac{m(\bar{\mu})}{\bar{\mu}} \\
& + \frac{(N-1/2)(N-1/4))}{\pi^2}\ln\frac{m(\bar{\mu})}{\bar{\mu}} \\
& + \frac{N-1/2}{\pi^2}(0.737775 - \frac{\pi^2}{96})\Big]\Big)
\end{aligned}
\tag{4.4}
$$

from which we immediately obtain

$$
m_{11} = 0 , \quad m_{22} = \frac{N-1/2}{\pi^2}(0.737775 - \frac{\pi^2}{96}) .
\tag{4.5}
$$

One can check that expansion (4.3) then reproduces the pertubative result (4.4), and in fact generates correctly, by construction, the leading and next-to-leading logarithms, $g^{2n}\ln^n[m(\bar{\mu})/\bar{\mu}]$ and $g^{2n}\ln^{n-1}[m(\bar{\mu})/\bar{\mu}]$ respectively, to all orders.

Next, we introduce the variational mass expansion in analogy with the procedure derived in section 2, by simply *assuming* the result to be given from performing the substitution

$$
m(\bar{\mu}) \rightarrow m(\bar{\mu})\, u
\tag{4.6}
$$

everywhere in expressions (4.1), (4.2), (4.3), and integrating the resulting expressions along an appropriate contour in the complex u plane, with the weight $du\, u^{-1}\, e^u$, similarly to e.g expression (3.8). This leads to the final formula,

$$
\frac{m_F(a)}{\Lambda_{\overline{MS}}} = \frac{2^{-C} a\, m'''}{2i\pi}\oint_C du\frac{e^u}{f_2'^A[C + f_2']^B}\Big(1 + \frac{\mathcal{M}_1'(a)}{f_2'} + \frac{\mathcal{M}_2'(a)}{f_2'^2} + \ldots\Big),
\tag{4.7}
$$

where we defined $f_2' \equiv f/(2b_0 g^2(\bar{\mu})) = [2b_0 g^2(M_F^{RG})]^{-1}$ which satisfies

$$
f_2'(u) = \ln[m''' u] - A\ln f_2' - (B - C)\ln[C + f_2'] .
\tag{4.8}
$$

In eqs. (4.7) and (4.8), A, B, C are defined as

$$
\begin{aligned}
A &= \frac{\gamma_1}{2b_1} = \frac{N-1/2}{2(N-1)} , \\
B &= \frac{\gamma_0}{2b_0} - \frac{\gamma_1}{2b_1} = \frac{N-1/2}{2(N-1)} , \\
C &= \frac{b_1}{2b_0^2} = -\frac{1}{2(N-1)} .
\end{aligned}
\tag{4.9}
$$

and

$$
m''' \equiv \frac{m(\bar{\mu})}{\Lambda_{\overline{MS}}\, 2^{-C}\, [2b_0 g^2(\bar{\mu})]^{A+B}\, [1 + \frac{b_1}{b_0}g^2(\bar{\mu})]^{-B}} .
\tag{4.10}
$$

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In equation (4.7), we also included the RS dependence through the parameter a, in a way similar to (3.14). Its explicit dependence, dictated by the renormalization group, reads:

$$\mathcal{M}_1'(a) = -\frac{(N-1/2)}{(N-1)} \ln a , \tag{4.11}$$

$$\mathcal{M}_2'(a) = \frac{(N-1/2)}{(N-1)^2}(0.737775 - \frac{\pi^2}{96}) + \frac{(N-1/2)(N-3/4)}{(N-1)^2} \ln^2 a$$
$$+ \frac{(N-1/2)(N-1/4)}{(N-1)^2} \ln a .$$

It is instructive to examine more closely the differences between the two forms (3.14) and (4.7): at first sight they look very different, as not only the functional form of M_F, the power coefficients, and the perturbative coefficients $\mathcal{M}_i$, $\mathcal{M}_i'$ are different, but the form of the recursive functions, f_2 in equation (3.15) and f_2' in equation (4.8) are also different.

Actually, we shall see next that this is in fact nothing but a RS difference, although in a not too conventional form. Since, at the second RG order, the RG properties allows such RS changes, this proves *a posteriori* that the construction when substitution (4.6) is made in the relevant renormalized expressions, followed by the appropriate contour integration, gives an equally acceptable Ansatz. In particular it corresponds, up to RS change, to an expression resumming infinite orders of the x series expansion, as is originally aimed.

To make the precise connection between the two different RS choices, note first that the definition of the variational parameter m''' in (4.10) and m'' in equation (3.14) indeed differs (except for $N \to \infty$):

$$m''' = exp\Big[\frac{-1}{2(N-1)}\Big] m'' . \tag{4.12}$$

The latter relation, by also comparing [15] the form of (4.8) with f_2 in (3.15) allows to express perturbatively the relation between the two schemes, for f_2' and f_2 sufficiently large:

$$f_2' \simeq f_2 \Big[1 - \frac{1}{2(N-1)} \frac{1}{f_2} + \frac{(3N+1/2)}{4(N-1)^2} \frac{1}{f_2^2} + O\Big(\frac{1}{f_2^3}\Big)\Big] . \tag{4.13}$$

One can easily check, by expanding in equation (4.7) in powers of $1/f_2$ using (4.13), that one recovers, up to higher order $O(1/f_2^3)$ terms, the previous form (3.14) [16]. Consequently the two schemes are *perturbatively* equivalent, as they should. For $f_2 \to 0$ (equivalently $f_2' \to 0$), *i.e.* $m'' \to 0$ (equivalently $m''' \to 0$) the two different forms have however no a

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[15]From (4.12) one may think at first that this RS change is simply given as a specific choice of a. This is not so, since the different form of f_2' in (4.8) with respect to (3.15) is also essential, especially for $m''' \to 0$, and cannot be obtained from the alternative form f_2 by simply changing a.

[16]But with $b_2 = \gamma_2 = 0$, because in the alternative Ansatz these third order RG coefficients do not appear, while in the previous Ansatz those are needed at intermediate stage for consistency, see Appendix B for more details.

priori reason to give the same results, due to a different behavior of f_2 and f_2' close to the origin.

To proceed, we perform the same type of Padé approximants, as described in section 3, with the new form of the mass gap (4.7). Actually formula (4.7) as it stands is not directly convenient, due to the presence of an extra branch cut starting at $f_2' = -b_1/(2b_0^2)$ *i.e.* on the positive real axis ($b_1 < 0$), therefore preventing a continuation down to $m''' \to 0$ like in the calculations of section 3. One may possibly infer that $b_1 < 0$ is an artifact of the low orders of perturbation theory (*i.e.* the Gross-Neveu model is asymptotically free to all orders in the large-N limit [18]). But in any event the analyticity structure is not uniquely determined, since it clearly depends on the scheme and precise form of the defining relation for f_2, as illustrated here from the two different expressions (3.15) and (4.8). It is in fact possible to recover a structure of singularities similar to the previous case (while still using the different information provided from the alternative scheme) by simply expanding (4.7) for large f_2' (namely in the perturbative range above the extra cut at $f_2' = 1/(2(N-1))$), but keeping the *same* definition of the variational parameter m''' in equation (4.10)[17]. This replaces expansion (4.7) by

$$\frac{m_F}{\Lambda_{\overline{MS}}} = 2^{\frac{1}{2(N-1)}} \frac{m''' a}{2i\pi} \oint \frac{du \, e^u}{f_2^{\frac{N-1/2}{N-1}}(u)} \left\{ 1 + \frac{\mathcal{M}''_1(a)}{f_2} + \frac{\mathcal{M}''_2(a)}{f_2^2} + \ldots \right\} , \qquad (4.14)$$

where f_2 verifies the very same relation as in equation (3.15) (but now with $m'' \to m'''$), and the cut again starts at $f_2 = 0$ towards the real negative axis. The perturbative coefficients are of course different:

$$\mathcal{M}''_1(1) = \frac{N-1/2}{4(N-1)^2} ,$$

$$\mathcal{M}''_2(1) = \mathcal{M}'_2(1) - (N-1/2)\frac{(16N^2 - 14N - 3)}{64(N-1)^4} . \qquad (4.15)$$

One can now directly apply the Padé techniques as described in section 3, to the form (4.14). The most interesting results are illustrated with the Padé (2,3) case in the last line of table 2, and are substantially better than the corresponding order results in the previous scheme. In particular the error is much more stable when varying N. Optimal values in this alternative scheme for Padé approximants of lower orders are only slightly different from those of the original scheme as given in table 2. For larger N, the two schemes give more and more similar results, as expected since for $N \to \infty$ they are *strictly* equivalent: $m''' \to m''$, and all perturbative coefficients $\mathcal{M}_i(1)$ go to 0. These different facts provide a good check of the stability of our results with respect to changes in the renormalization scheme. Indeed it strongly indicates that it should also be possible

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[17]Of course the larger N is, the closer one can safely approach the origin for f_2' (equivalently m'''), as expected.

to optimize with respect to the RS scheme (in addition to the optimization with respect to a). The variation of the scheme through the parameter a (whose dependence does differ in the alternative Ansatz at the next-to-leading order, $(\ln a)/f_2^2$), provides an extra illustration of the consistency of our results. Besides, these two different renormalization schemes are important to clarify the variational expansion procedure when applied either to bare or renormalized parameters. To illustrate a more general RS dependence one could further exploit the RS dependence of the renormalization group coefficients themselves, *e.g.* starting with γ_1. Although we have not tried explicitely this in the present case, we see no compelling reasons to study a more general case within the framework of the GN model, where we know the exact results anyway. The preceding facts should already be considered a strong indication that the variational mass method does lead to non-trivial results, at least in appropriate renormalization schemes.

The vacuum energy density can be treated along the same lines. However, in that case, the change of sign of the energy when m'' approaches 0 complicates matters in a way which we do not report here, as it is a separate issue.

5   Conclusion

In this paper, we have shown how to introduce a variational procedure in an asymptotically free field theory, in a way which is compatible with the renormalization group: we have found how to rescale the variational parameter in order to take account of all infinities (at least in the framework of dimensional regularization) so that the physics of the variational approximation is smooth when the cut-off is removed (in our case, when the dimension is varied around its critical value). We have further shown that the numerical results obtained with an extra very simple extrapolation procedure are in good to remarkable agreement with exact values in the case of the GN model. This establishes the theoretical possibility and potential usefulness of perturbative calculations for computing non-perturbative quantities. This framework is in principle directly applicable to the QCD case, apart form possible complications due to the *a priori* different analyticity structure of m of the relevant QCD expressions. In a subsequent paper [12], using a bare chiral symmetry breaking fermion mass as variational parameter, we shall obtain values for some parameters of spontaneous chiral symmetry breaking, m_Q (the part of the constituent quark mass due to chiral symmetry breaking), the pion decay constant f_π and the condensate $< \bar{\psi}\psi >$.

Acknowledgments

A A contour integral resumming the x dependence

We shall explain here in some details how to resum the x series generated from the perturbative expansion of (2.1). Consider the one-loop RG invariant (bare) expression for the mass m_F, as given in eq. (2.3):

$$m_F = \frac{m_0}{(1 - 4\pi b_0 g_0^2 \tilde{\Gamma} \mu^{-\varepsilon} m_F^\varepsilon(m_0))^{\frac{\gamma_0}{2b_0}}} \tag{A.1}$$

where the substitution

$$m_0 \rightarrow m_0(1 - x); \qquad g_0^2 \rightarrow g_0^2 x, \tag{A.2}$$

provides a new quantity $m_F(x)$. To pick up the x^q order term in $m_F(x) \equiv \sum_{q=0}^\infty a_q x^q$ (having in mind that we are actually interested in the limit $x \rightarrow 1$), a convenient trick is by contour integration:

$$m_F^{(q)}(x \rightarrow 1) \rightarrow \sum_{k=0}^q a_k = \frac{1}{2i\pi} \oint dx (\frac{1}{x} + \cdots + \frac{1}{x^{q+1}}) \, m_F(x). \tag{A.3}$$

Now performing the sum in (A.3) exhibits a $(1 - x)^{-1}$ factor which cancels the $(1 - x)$ from (A.2). This results in the expression [18]:

$$m_F^{(q)}(x \rightarrow 1) \rightarrow \frac{1}{2i\pi} \oint dx\, x^{-(q+1)} \, m_0 [f_0(x)]^{-\frac{\gamma_0}{2b_0}} , \tag{A.4}$$

where the contour is counterclockwise around the origin, and for convenience we defined the (recursive) function

$$f_0(x) \equiv 1 - 4\pi b_0 \, x \, g_0^2 \tilde{\Gamma} \, m_0^\epsilon (1 - x)^\epsilon \, (f_0)^{-\epsilon \frac{\gamma_0}{2b_0}} , \tag{A.5}$$

---

[18]In (A.4) there appeared in fact a factor of $1 - x^{-(q+1)}$, from which only the last term contributes to the integral due to the analyticity of $f_0(x)$ defined in (A.5).

directly dictated from eq. (A.1). $f_0(x)$ has (evidently) a power series expansion in x, but admits also an expansion in $(1-x)$, as noted by inverting its defining relation (A.5). This implies, in particular, that $x = 1$ is an (isolated) pole of m_F.

Provided that no extra singularities lie in the way, one may distort the integration contour in (A.4) to go around the cut lying along the real positive axis and starting at $x = 1$. Actually one can go a step further and reach the $q \to \infty$ limit: after distorsion of the contour only the vicinity of $x = 1$ survives for $q \to \infty$, that one can analyse by changing variable to

$$1 - x \equiv \frac{v}{q} \tag{A.6}$$

and rescaling m_0 by introducing $m_0 = m'_0 q$ (keeping m'_0 fixed as q goes to infinity). One finds in place of (A.4)

$$m_F^{(q)}(q \to \infty) \to \frac{1}{2i\pi} \oint \frac{dv}{v} e^v \frac{v\, m'_0}{f_0(v)^{\frac{\gamma_0}{2b_0}}} \tag{A.7}$$

where now $f_0(v) \equiv 1 - 4\pi b_0\, g0^2 \tilde{\Gamma}(m_0 v)^\epsilon (f_0)^{-\epsilon \frac{\gamma_0}{2b_0}}$. The crucial point in (A.7) is, that once performing renormalization via $m_0 = \bar{m} Z_m$, $g_0^2 = \bar{\mu}^{-\epsilon} Z_g g^2$, m_F is finite to all orders:

$$m_F = \frac{1}{2i\pi} \oint dv e^v \frac{\bar{m}}{f^{\frac{\gamma_0}{2b_0}}} \tag{A.8}$$

where the renormalized function $f = Z_g f_0 = 1 + 2b_0 \bar{g}^2 \ln[(\bar{m}v/\bar{\mu})\, f^{-(\gamma_0/2b_0)}]$.

We have thus recovered finite quantities with a non-trivial x expansion. In the latter derivation we only included the one-loop RG dependence, which is the exact result in the large N limit only. For arbitrary N one should include in the derivation the non-logarithmic perturbative terms, present e.g at the two loop order in (4.4). This can be done without affecting the contour integration properties, except that the resulting expression of m_F has a more complicated structure around $v \simeq 0$, but can be systematically expanded around the origin in the way discussed in section 2. Generalization of the previous construction to the next RG order is straightforward, since the recursive function f has a very similar form as in (A.5) above, where only the power coefficients are changed.

B Bare RG Ansatz

We give here some useful expressions needed for the construction of the generalized Ansatz using third order RG functions, as introduced in section 2.3, and leading to our main results (3.8), (3.9) and (3.14) for the mass gap. Derivation of the vacuum energy (3.17) is very similar, apart from the slight complication due to the subtracted terms as discussed at the end of section 2.

We start from expression (3.7), where the constants A, B and D are first to be determined by requiring equation (3.7) to be finite as $\epsilon \to 0$, when expressed in terms of m and g^2 with the help of RG mass and coupling counterterms at a given order (and in a specific renormalization scheme). The coefficient λ, $\rho, \ldots$ are then obtained by matching the perturbative expansion of (3.7) to the perturbative (bare) expression for m_F, eq. (3.1).

Defining the renormalization group coefficients governing the coupling constant and mass evolution, respectively, as

$$\beta(g) \equiv \mu \frac{dg}{d\mu} = \frac{\varepsilon}{2}g - b_0 \, g^3 - b_1 \, g^5 - b_2 \, g^7 - \ldots \tag{B.1}$$

and

$$\gamma_m(g) \equiv -\frac{\mu}{m}\frac{dm}{d\mu} = \gamma_0 \, g^2 + \gamma_1 \, g^4 + \gamma_2 \, g^6 + \ldots \tag{B.2}$$

the coefficients b_i and γ_i, known in the $\overline{MS}$ scheme up to three loop order [21] in the Gross-Neveu model, are expressed as

$$b_0 = \frac{N-1}{2\pi}, \; b_1 = -\frac{N-1}{4\pi^2}, \; b_2 = -\frac{(N-1)(2N-7)}{32\pi^3}; \tag{B.3}$$

and

$$\gamma_0 = \frac{N-1/2}{\pi}, \; \gamma_1 = -\frac{N-1/2}{4\pi^2}, \; \gamma_2 = -\frac{(N-1/2)(4N-3)}{16\pi^3}. \tag{B.4}$$

From these one obtains explicit expressions of the counterterms, Z_g and Z_m, as expansions both in g^2 and ε to the required order. We need a counterterm for $\mathcal{F}$ in (3.7) as well, that we define according to $\mathcal{F}_0 \equiv Z_F \mathcal{F}$, where Z_F can be determined consistently together with A, B and D for given Z_m and Z_g. In terms of Z_g and Z_m, the finiteness of eq.(3.7) requirement explicitely reads

$$B \, \ln z + \ln Z_m \equiv \text{ finite}; \tag{B.5}$$

$$(1 + \varepsilon A) \, \ln z + \varepsilon \, \ln Z_m + \ln Z_g \equiv \text{ finite};$$

$$D \equiv 1 + \varepsilon A \; ;$$

where we exhibited the singular part of Z_F as

$$\ln z \equiv \ln\left[1 + \frac{g^2}{2\pi} - \frac{(N-1)g^2}{\pi\varepsilon}\right]. \tag{B.6}$$

From inspection of the formal expansion of (3.7) it turns out that actually A is needed to $O(\varepsilon)$ and B to $O(\varepsilon^2)$, to fix unambiguously the finite coefficients λ and ρ in (3.3) by matching with (3.1). Accordingly from (B.5) one needs to know Z_m, Z_g to order $O(1/\varepsilon^3)$: i.e the third order RG coefficients b_2, γ_2 in (B.3), (B.4) should be included for consistency.

The required expansion in powers of $1/\varepsilon$ of $\ln Z_g$, $\ln Z_m$ and $\ln z$ is completely determined by the expansion in ε of the perturbative zero, g_P, of the beta function, which is easily derived as

$$g_P^2 = \frac{\varepsilon}{2b_0} - \frac{b_1}{4b_0^3}\varepsilon^2 + \left(\frac{b_1^2}{4b_0^5} - \frac{b_2}{8b_0^4}\right)\varepsilon^3 + O(\varepsilon^4) \; . \tag{B.7}$$

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In this way we obtain after straightforward algebra,

$$
A = \frac{N}{N-1} + \frac{N-1/2}{(N-1)^2}\,\varepsilon + O(\varepsilon^2)\,, \tag{B.8}
$$

$$
B = \frac{N-1/2}{N-1} + \frac{3}{4}\frac{N-1/2}{(N-1)^2}\,\varepsilon + \frac{N(N-1/2)}{8(N-1)^3}\,\varepsilon^2 + O(\varepsilon^3)\,,
$$

and [19]

$$
\lambda = \frac{3}{4\pi}\frac{N-1/2}{N-1} \tag{B.9}
$$

$$
+ \frac{N-1/2}{8\pi(N-1)}\left[\frac{N}{N-1} + 3(\gamma_E - \ln 4\pi)\right]\varepsilon + O(\varepsilon^2)\,,
$$

$$
\rho = -\frac{(N-1/2)(36N^2 - 62N + 17)}{64\pi^2(N-1)^2}
$$

$$
- \frac{N-1/2}{2\pi^2}(-0.725551\ldots + \frac{\pi^2}{48}) + O(\varepsilon)\,,
$$

which directly leads to expressions (3.9) for $\varepsilon \to 0$. Let us finally remark that (B.5) leaves over non-zero finite perturbative terms, which are however completely determined order by order. This is inherent to our scheme here where the counterterm for $\mathcal{F}$ could not be put into a form where only the singularities appear (like is the case, by definition, for $\ln Z_m$ and $\ln Z_g$ in the $\overline{MS}$ scheme). Explicitely after using (B.8) we obtain

$$
B\,\ln z + \ln Z_m = -\frac{(N-1/2)}{(N-1)}\frac{g^2}{4\pi} + \frac{(N-1/2)(N+1)}{(N-1)}\frac{g^4}{16\pi^2} + O(g^6)\,, \tag{B.10}
$$

which obviously gives perturbative corrections when re-expanding expression (3.8), and which are indeed necessary to recover consistently the perturbative expression for the renormalized mass (4.4) [20].

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[19]Note that the order ε term in λ is required to determine ρ.

[20]The third order renormalization group dependence, via b_2 and γ_2 in (B.8), (B.9) and (B.10) actually cancels in the (two-loop) perturbative renormalized mass expression (4.4).

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Figure Captions.

Figure 1. Mass gap graphs at order 2.

Figure 2. Vacuum energy graphs at order 2.

Figure 3. Mass gap ratio $m_F/\Lambda_{\overline{MS}}$ versus variational parameter m'', for values of the scaling parameter a ranging from $a = 1.1$ (lowest curve) to $a = 1.2$ (highest curve). The horizontal dashed curve represents the exact value. We have taken $N = 3$ for the number of flavors.

Figure 4. Vacuum energy $E_0/\Lambda_{\overline{MS}}^2$ versus m'', for values of a ranging from $a = 0.9$ (lowest curve) to $a = 1.1$ (highest curve). The horizontal dashed curve gives the exact value. The number of flavors is $N = 3$.

Figure 5. Comparison of different Padé approximant types and orders, versus exact $m_F/\Lambda_{\overline{MS}}$, for $N = 2$. The curves showing extrema at 2.27, 1.62, 1.76, 1.93 and 1.87, have been obtained from the $p_{1,1}(u)$, $p_{2,3}(u)$, $p_{2,2}(u) \times e^{(c1/u)}$, $p_{1,2}(u)$ and $p'_{2,3}(u)$ Padé approximants, respectively.
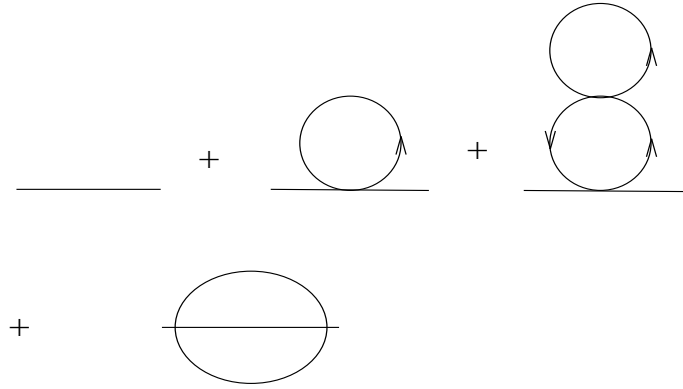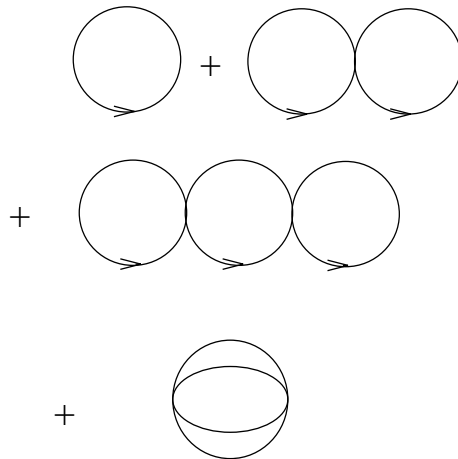
Figure 1: Mass gap graphs at order 2.

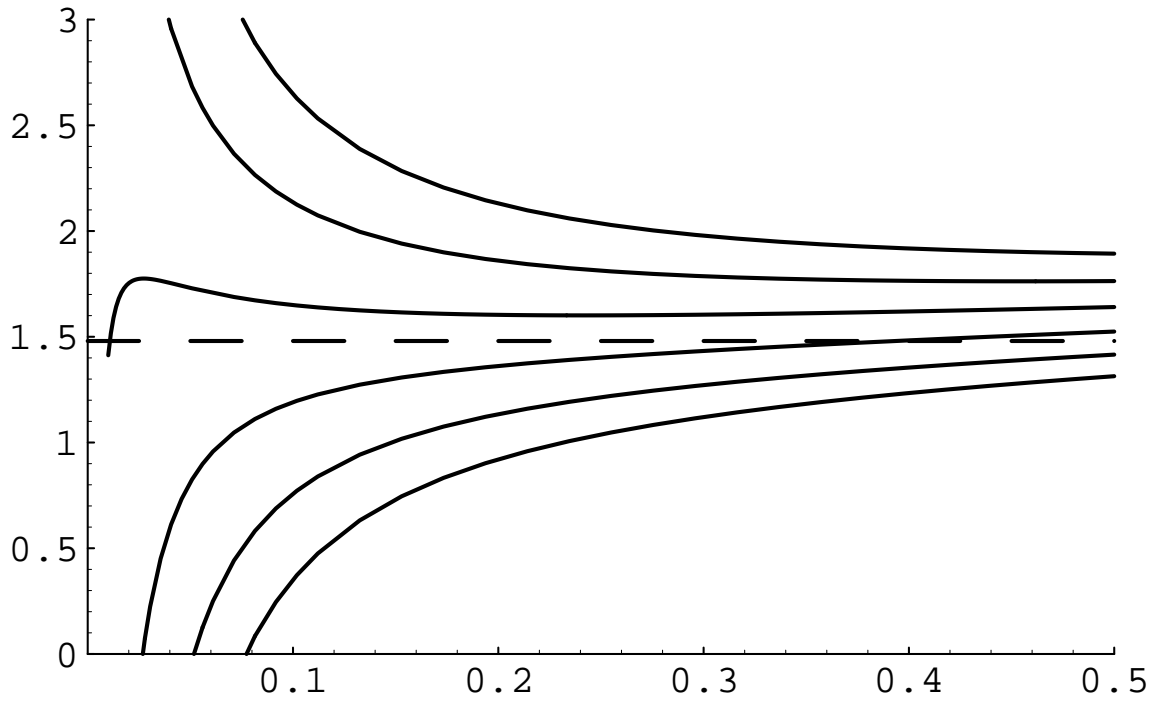Figure 2: Vacuum energy graphs at order 2.

Figure 3: Mass gap $m_F/\Lambda_{\overline{MS}}$ versus variational parameter m'' .
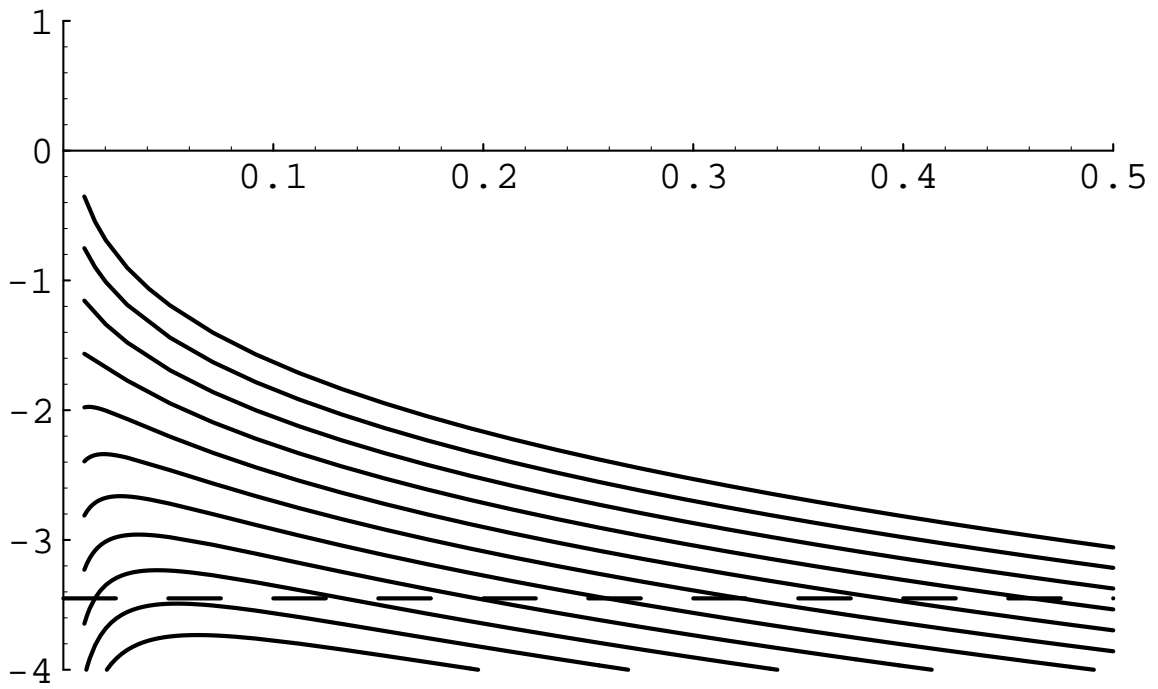
Figure 4: Vacuum energy $E_0/\Lambda_{\overline{MS}}{}^2$ versus variational parameter m''.
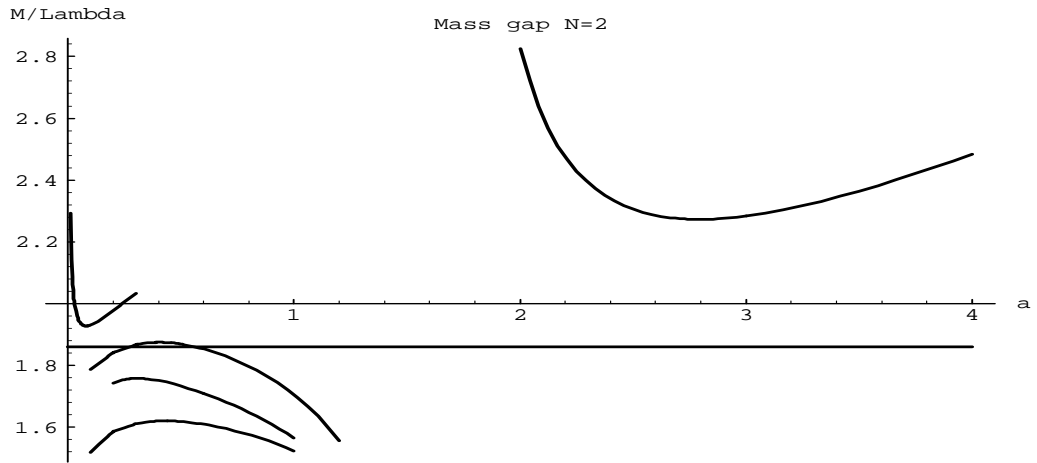
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