CERN-TH/95-253
BI-TP 95/33
arch-ive/9510031

Improved Actions for QCD Thermodynamics on the Lattice

B. Beinlich[1], F. Karsch[1,2] and E. Laermann[1]

[1]Fakultät für Physik, Universität Bielefeld, D-33615 Bielefeld, Germany
[2]Theory Division, CERN, CH-1211, Geneva, Switzerland

ABSTRACT

Finite cut-off effects strongly influence the thermodynamics of lattice regularized QCD at high temperature in the standard Wilson formulation. We analyze the reduction of finite cut-off effects in formulations of the thermodynamics of $SU(N)$ gauge theories with three different $O(a^2)$ and $O(a^4)$ improved actions. We calculate the energy density and pressure on finite lattices in leading order weak coupling perturbation theory ($T \rightarrow \infty$) and perform Monte Carlo simulations with improved $SU(3)$ actions at non-zero g^2. Already on lattices with temporal extent $N_\tau = 4$ we find a strong reduction of finite cut-off effects in the high temperature limit, which persists also down to temperatures a few times the deconfinement transition temperature.

CERN-TH/95-253
BI-TP 95/33
October 1995

1 Introduction

With increasing accuracy of lattice calculations it also becomes increasingly important to control and, as far as possible, eliminate the systematic errors introduced through the finite lattice cut-off in numerical calculations. Various procedures to achieve this goal without increasing the necessary computational effort drastically have been suggested in the past, including the early *Symanzik improvement* programme [1], renormalization group improved actions [2] as well as the recently constructed perfect actions [3, 4]. The resulting actions are not designed to lead to a better infrared behaviour of lattice regularized field theories. It thus may not have come as a surprise, that the application of these actions in lattice calculations, which aimed at a determination of long-distance properties of asymptotically free field theories, did show little advantages over the standard one-plaquette action originally proposed by Wilson [5]. In particular, this is the case for studies of the finite temperature deconfinement transition in $SU(N)$ gauge theories with Symanzik-improved actions [6, 7]. On the other hand the improved actions are expected to lead to a better ultraviolet behaviour of the theory. This has, for instance, been observed in studies of topological properties [8] or the short distance part of the heavy quark potential [4]. In these cases the unwanted short distance lattice artifacts could successfully be eliminated.

The high temperature behaviour of QCD is close to that of an ideal gas. Bulk thermodynamic quantities are therefore dominated by contributions from large momenta. These, however, are most strongly influenced by finite cut-off effects. One thus may expect that improved actions will be particularly useful for the analysis of bulk thermodynamic quantities. In fact, it is well known that in the standard Wilson formulation of lattice QCD the cut-off effects lead to strong modifications of the high temperature limit of quantities like, e.g. the energy density and pressure [9, 10]. These cut-off effects are $O((aT)^2)$ in the pure gauge sector. Calculations on lattices with temporal extent $N_\tau = 1/aT$ have recently been performed for the $SU(2)$ [13] and $SU(3)$ [14] gauge theories. They clearly show the influence of a finite ultraviolet cut-off on the behaviour of bulk thermodynamic quantities and their extrapolation to the continuum limit. In the standard Wilson formulation [5] lattices with temporal extent $N_\tau \gtrsim 8$ are needed in order to reduce deviations of, eg. ϵ/T^4, from the continuum extrapolation below a few percent. The interesting bulk thermodynamic quantities like energy density and pressure all have dimension four. In lattice calculations they are determined from operators, whose expectation values are proportional to $N_\tau^{-4} = (aT)^4$. It thus rapidly becomes difficult to calculate these operators with reasonable statistical significance. In fact, the effort to calculate them with a given accuracy on lattices of size $N_\sigma^3 \times N_\tau$, increases at least like N_τ^{11}, if one also keeps the physical size of the thermodynamic system constant ($N_\sigma/N_\tau =$ const.). It therefore should be clear that a huge improvement is already achieved, if one can perform calculations on lattices with, say $N_\tau = 4$, with cut-off distortions similar in magnitude to what one reaches in calculations with the Wilson action on lattices with twice that temporal extent.

It is the purpose of this paper to quantify the ultraviolet cut-off effects that can be expected to be present in calculations of thermodynamic quantities with improved actions and to examine the relevance of improved actions for the calculation of the equation of state in QCD at high temperature. We will discuss various improved actions and calculate finite cut-off corrections to the high temperature limit of the energy density in leading order perturbation theory. We also will present some numerical results for the pressure of a $SU(3)$ gluon gas calculated with an $O((aT)^2)$ improved action. We are going to discuss tree level improved actions ($g^2 \equiv 0$). As will become clear in the following this leads already to a significant improvement of the high temperature behaviour of bulk thermodynamic quantities in $SU(N)$ gauge theories. There is, however, no fundamental problem which would prohibit the extension of the present considerations to $O(g^2)$ improved actions and we will present evidence that this is, in fact, needed for temperatures close to the deconfinement transition temperature.

We will present in the next section two extensions of the standard one-plaquette Wilson action leading to an $O((aT)^2)$ improvement of thermodynamic observables as well as a specific choice of an action that leads to an improvement at $O((aT)^4)$. In section 3 we will discuss the perturbative high temperature limit of these improved actions. A first exploratory numerical analysis of these actions and a comparison with the standard one-plaquette Wilson action is presented in section 4. In section 5 we give our conclusions. Various details of our perturbative calculations are given in an Appendix.

2 Improved SU(N) Actions

In order to eliminate the $O(a^2)$ and higher order corrections to the lattice version of the Euclidean action one can add appropriately chosen larger loops to the basic 4-link plaquette term appearing in the standard one-plaquette Wilson action. We will, in particular, discuss here simple extensions of the one-plaquette action, which only involve larger planar loops, *i.e.* we will consider the generalized Wilson actions,

$$\begin{aligned} S^I &= \sum_{x,\nu>\mu} \sum_{l=1}^{\infty} \sum_{k=1}^{l} a_{k,l}\, W^{k,l}_{\mu,\nu}(x) \\ &\equiv \sum_{x,\nu>\mu} S^I_{\mu,\nu}(x) \quad , \end{aligned} \tag{2.1}$$

where $W^{k,l}_{\mu,\nu}$ denotes a symmetrized combination of $k \times l$ Wilson loops in the (μ,ν)-plane of the lattice,

$$W^{k,l}_{\mu,\nu}(x) = 1 - \frac{1}{2N}\Big(\mathrm{Re\ Tr} L^{(k)}_{x,\mu} L^{(l)}_{x+k\hat{\mu},\nu} L^{(k)+}_{x+l\hat{\nu},\mu} L^{(l)+}_{x,\nu} + (k \leftrightarrow l)\Big) \quad . \tag{2.2}$$

Here we have introduced the short hand notation for *long* links, $L^{(k)}_{x,\mu} = \prod_{j=0}^{k-1} U_{x+j\hat{\mu},\mu}$ and $x = (n_1, n_2, n_3, n_4)$ denotes the sites on an asymmetric lattice of size $N_\sigma^3 \times N_\tau$.

With a suitable choice of the coefficients $a_{k,l}$ it can be achieved that the generalized Wilson actions reproduce the continuum Euclidean Yang-Mills Lagrangian, $\mathcal{L} = -\frac{1}{2} F_{\mu,\nu} F_{\mu,\nu}$, up to some order $O(a^{2n})$ [1]. The standard one-plaquette Wilson action, $S^{(1,1)}$, with $a_{1,1} = 1$ and $a_{k,l} = 0$ for all $(k,l) \neq (1,1)$ receives $O(a^2)$ corrections in the naive continuum limit. Expanding the link variables, $U_{x,\mu} = \exp(igaA_\mu(x))$, in powers of a one finds

$$\begin{aligned} S^{(1,1)}_{\mu,\nu}(x) &= 1 - \frac{1}{N}\mathrm{Re\ Tr} U_{x,\mu} U_{x+\hat{\mu},\nu} U^{+}_{x+\hat{\nu},\mu} U^{+}_{x,\nu} \\ &= -\frac{1}{2N} g^2 a^4 \Big(F_{\mu,\nu} F_{\mu,\nu} + \frac{1}{12} a^2 F_{\mu,\nu} (\partial_\mu^2 + \partial_\nu^2) F_{\mu,\nu} \\ &\qquad + O(a^4) \Big) \quad . \end{aligned} \tag{2.3}$$

By adding an additional Wilson loop to the action one can achieve that corrections start only at $O(a^4)$. In particular we will consider here actions obtained by adding a planar 6-link and 8-link loop, respectively. The non-vanishing coefficients in these cases are,

$$\begin{aligned} I \equiv (1,2): &\qquad a_{1,1} = \frac{5}{3} \quad , \quad a_{1,2} = -\frac{1}{6} \\ I \equiv (2,2): &\qquad a_{1,1} = \frac{4}{3} \quad , \quad a_{2,2} = -\frac{1}{48} \end{aligned} \tag{2.4}$$

The action $S^{(1,2)}$ is a specific choice of the 6-link improved actions originally proposed by Symanzik [1]. The action $S^{(2,2)}$ has recently also been discussed in the context of NRQCD calculations [15, 16]. Its generalization to larger quadratic loops gives a straightforward procedure to eliminate also higher order cut-off effects. Already in $O(a^4)$ there exist two independent operators of dimension eight, which contribute to the finite cut-off effects. In general, one thus needs three independent loops of length six and eight in order to eliminate all cut-off effects proportional to a^2 and a^4, respectively. In an action constructed only from quadratic loops the different operators contribute, however, with the same relative weight. In that case it thus is sufficient to add only two additional, quadratic loops to obtain an $O(a^4)$ improved action. We will consider here the case where we add 2×2 and 3×3 Wilson loops to the plaquette term[a]. In this case corrections to the continuum limit start only at $O(a^6)$. The non-vanishing coefficients for the action $S^{(3,3)}$ are given by,

$$I \equiv (3,3): \qquad a_{1,1} = \frac{3}{2} \quad , \quad a_{2,2} = -\frac{3}{80} \quad , \quad a_{3,3} = \frac{1}{810} \tag{2.5}$$

In the following we will discuss the behaviour of thermodynamic quantities in the high temperature limit, which are obtained from the partition function,

$$Z = \int \prod_{x,\mu} \mathrm{d}U_{x,\mu} \, e^{-\beta S^I} \quad , \tag{2.6}$$

defined on lattices of size $N_\sigma^3 \times N_\tau$. Here $\beta = 2N/g^2$ is the bare gauge coupling. Thermodynamic quantities like the energy density or the pressure are obtained as derivatives of the logarithm of the partition function with respect to the temperature, $T = 1/N_\tau a$, and the volume, $V = (N_\sigma a)^3$. They are thus represented by expectation values of certain parts of the action, *i.e.* the operators used to evaluate thermodynamic quantities are improved to the same order as the action. In the following we will show explicitly that the $O(a^n)$ improvement of the action will lead to an $O((aT)^n)$ improvement of thermodynamic observables.

3 Perturbative High Temperature Limit

3.1 The generalized SU(N) Wilson Actions

A suitable quantity for the study of the influence of finite cut-off effects on the thermodynamics of $SU(N)$ gauge theories is the leading high temperature behaviour of the energy density or equivalently the pressure. In the limit $T \to \infty$ the energy density approaches that of a massless, non-interacting gluon gas, $\epsilon/T^4 \to \epsilon_{\mathrm{SB}}/T^4 = (N^2-1)\pi^2/15$. Corrections to this are of $O(g^2)$. Numerical investigations of the high temperature phase of $SU(N)$ gauge theories show that the energy density rapidly approaches this high temperature limit. More precisely, on lattices with finite temporal extent N_τ, it approaches a limiting value, which differs from $\epsilon_{\mathrm{SB}}/T^4$ due

[a]This procedure can easily be extended to higher orders. The resulting actions are, however, non-local in the sense that the magnitude of the coefficients of a $k \times k$ loop asymptotically only decreases powerlike, $|a_{k,k}| \sim k^{-4}$.

to ultraviolet cut-off effects. At temperatures a few times the critical temperature deviations from this limiting value are generally of the same order of magnitude as the cut-off distortion effects. In order to quantify the approach to the high temperature ideal gas limit it thus is important to reach a good approximation of this limit already on finite lattices. The cut-off effects due to finite values of N_τ can be studied in leading order perturbation theory. In fact, in the case of the one-plaquette Wilson action this is known for quite some time [9, 10].

For the analysis of the high temperature ideal gas limit it is sufficient to evaluate perturbatively the difference between expectation values of space- and timelike parts of the action,

$$\begin{aligned}\frac{\epsilon}{T^4} \equiv 3\frac{p}{T^4} &= \frac{6N}{g^2} N_\tau^4 \Big(\langle S_\sigma^I \rangle - \langle S_\tau^I \rangle \Big) + O(g^2) \\ &= \frac{3}{2}(N^2-1) N_\tau^4 \Big(S_\sigma^{I,(2)} - S_\tau^{I,(2)} \Big) + O(g^2) \quad , \end{aligned} \tag{3.1}$$

where $S_\sigma^{I,(2)}$ and $S_\tau^{I,(2)}$ denote the $O(g^2)$ expansion coefficients of the expectation values of the space- and timelike parts of the action,

$$\begin{aligned}\langle S_\sigma^I \rangle &\equiv \frac{1}{3N_\sigma^3 N_\tau} \langle \sum_{x,\mu<\nu<4} S_{\mu,\nu}^I(x) \rangle = g^2 \frac{N^2-1}{4N} S_\sigma^{I,(2)} + O(g^4) \\ \langle S_\tau^I \rangle &\equiv \frac{1}{3N_\sigma^3 N_\tau} \langle \sum_{x,\mu=1,2,3} S_{\mu,4}^I(x) \rangle = g^2 \frac{N^2-1}{4N} S_\tau^{I,(2)} + O(g^4) \quad . \end{aligned} \tag{3.2}$$

All other terms appearing in the definition of the energy density are $O(g^2)$ and thus do not contribute in leading order perturbation theory [12].

In the following we will discuss the perturbative calculation of the expansion coefficients $S_\sigma^{I,(2)}$ and $S_\tau^{I,(2)}$ on finite lattices. Thereby we will also review some of the results obtained in perturbative calculations with the Wilson one-plaquette action on finite, asymmetric lattices [9, 10]. Our notation closely follows [10]. Perturbative results for the Symanzik action, $S^{(1,2)}$, on infinite lattices have also been obtained in [11].

The improved Wilson actions are defined in terms of the link variables[b], $U_{x,\mu} = \exp\left(igA_\mu(x)\right)$. They can be expanded in powers of g^2 and represented in momentum

[b] In the following the cut-off a has been set to unity.

space. Sums in Fourier space we denote by

$$\int_p = \frac{1}{N_\sigma^3 N_\tau} \sum_{(p \neq 0)} \quad , \tag{3.3}$$

where $p_\mu = (2\pi/N_\sigma) n_\mu$, $n_\mu = 0,\ 1, \ldots,\ N_\sigma - 1$ for $\mu = 1,\ 2,\ 3$ and $p_4 = (2\pi/N_\tau) n_4$, $n_4 = 0,\ 1, \ldots,\ N_\tau - 1$. The lattice gauge fields are defined on the middle of a link,

$$A_\mu(p) = \sum_x \mathrm{e}^{-ipx - ip_\mu/2} A_\mu(x + \hat{\mu}/2) \quad , \tag{3.4}$$

with $A_\mu \equiv A_\mu^a \lambda^a$, $a = 1, \ldots,\ (N^2-1)$, and the normalization $2\mathrm{Tr}\lambda^a\lambda^b = \delta_{ab}$. We also will use the short hand notation $s_\mu \equiv \sin(p_\mu/2)$.

For the evaluation of the leading order term in the energy density on finite lattices one only needs to keep the quadratic part in the perturbative expansion of the action, βS^I, and introduce a gauge fixing term. We combine both in the form

$$S_0 = -\frac{1}{2} \int_p \sum_{\mu,\nu} A_\mu^a(-p) \Delta_{\mu,\nu}(p) A_\nu^a(p) \quad , \tag{3.5}$$

with the inverse propagator

$$\Delta_{\mu,\nu}(p) = G_{\mu,\nu}(p) + \xi g_\mu(p) g_\nu(p) \quad . \tag{3.6}$$

The first term in Eq. (3.6) arises from the expansion of the action and the second term defines the covariant gauge fixing. It is convenient to separate in $G_{\mu,\nu}$ a diagonal part,

$$G_{\mu,\nu}(p) = D_\mu(p) \delta_{\mu,\nu} - E_{\mu,\nu}(p) \quad , \tag{3.7}$$

with

$$\begin{aligned} D_\mu(p) &= \sum_{l=1}^{\infty} \sum_{k=1}^{l} a_{k,l} \sum_{\nu=1}^{4} N_{\mu;\nu}^{k,l}(p) \quad , \\ E_{\mu,\nu}(p) &= \sum_{l=1}^{\infty} \sum_{k=1}^{l} a_{k,l} M_{\mu,\nu}^{k,l}(p) \quad , \end{aligned} \tag{3.8}$$

where $N^{k,l}_{\mu;\nu}(p)$ and $M^{k,l}_{\mu,\nu}(p)$ are obtained in $O(g^2)$ from the expansion of symmetrized $k \times l$ Wilson loops,

$$
\begin{aligned}
W^{k,l}_{\mu,\nu} &\equiv \frac{1}{N_\sigma^3 N_\tau} \sum_x W^{k,l}_{\mu,\nu}(x) \\
&= \frac{g^2}{4N} \sum_{p,a} \Big(N^{k,l}_{\mu;\nu}(p) A^a_\mu(p) A^a_\mu(-p) + N^{k,l}_{\nu;\mu}(p) A^a_\nu(p) A^a_\nu(-p) \\
&\qquad -2 M^{k,l}_{\mu,\nu}(p) A^a_\mu(p) A^a_\nu(-p) \Big) + O(g^4) \quad . \qquad (3.9)
\end{aligned}
$$

Explicit expressions for $N^{k,l}_{\mu;\nu}(p)$ and $M^{k,l}_{\mu,\nu}(p)$ are given in the Appendix.

With these definitions at hand, it is easy to evaluate the leading order expression for the energy density on finite lattices. Using the $O(g^2)$ expansion coefficients for $k \times l$ Wilson loops, given in the Appendix, one finds,

$$
S^{(k,l),(2)}_{\mu,\nu} = \int_p \Big(N^{k,l}_{\mu;\nu} \Delta^{-1}_{\mu,\mu} + N^{k,l}_{\nu;\mu} \Delta^{-1}_{\nu,\nu} - 2 M^{k,l}_{\mu,\nu} \Delta^{-1}_{\mu,\nu} \Big) \quad , \qquad (3.10)
$$

and finally

$$
\begin{aligned}
S^{I,(2)}_\sigma &= \frac{1}{3} \sum_{\mu<\nu<4} \sum_{l=1}^{\infty} \sum_{k=1}^{l} a_{k,l} S^{(k,l),(2)}_{\mu,\nu} \quad , \\
S^{I,(2)}_\tau &= \frac{1}{3} \sum_{\mu=1}^{3} \sum_{l=1}^{\infty} \sum_{k=1}^{l} a_{k,l} S^{(k,l),(2)}_{\mu,4} \quad . \qquad (3.11)
\end{aligned}
$$

The necessary inversion of the matrix $\Delta_{\mu,\nu}$ and the evaluation of the sums appearing in Eq. (3.11) have been performed using Mathematica. We have explicitly checked that all our results are independent of the choice of the gauge fixing function $g_\mu(p)$ and the gauge fixing parameter ξ. In particular we have verified that in the infinite volume limit the $O(g^2)$ expansion coefficients are identical for all improved Wilson actions,

$$
\lim_{N_\tau\to\infty} \lim_{N_\sigma\to\infty} S^{I,(2)}_\sigma \equiv \lim_{N_\tau\to\infty} \lim_{N_\sigma\to\infty} S^{I,(2)}_\tau \equiv \frac{1}{2} \quad . \qquad (3.12)
$$

We also have evaluated separately the $O(g^2)$ expansion coefficients for the expectation values of the different loops appearing in the improved actions. For $S^{(1,2)}$ our results on an infinite lattice agree with earlier calculations [11], while we disagree for

$S^{(2,2)}$ with results quoted in [16][c]. For the expansion of the plaquette expectation value we find

$$\langle 1 - \frac{1}{N}\mathrm{Re\ Tr} U_{x,\mu} U_{x+\hat{\mu},\nu} U^+_{x+\hat{\nu},\mu} U^+_{x,\nu} \rangle = g^2 \frac{N^2-1}{4N} \begin{cases} 0.5 & , \mathrm{I}=(1,1) \\ 0.366263 & , \mathrm{I}=(1,2) \\ 0.392641 & , \mathrm{I}=(2,2) \\ 0.358304 & , \mathrm{I}=(3,3) \end{cases} + O(g^4) \quad . \tag{3.13}$$

3.2 Cut-off Effects in the Wilson Action

For the one-plaquette action we use the gauge fixing function $g_\mu = 2s_\mu$, which, to lowest order in a, is the momentum representation of the lattice version of $\partial_\mu A_\mu$. Choosing the Feynman gauge, $\xi = 1$, we then obtain a diagonal propagator

$$\Delta_{\mu,\nu}(p) = D(p)\delta_{\mu,\nu} \quad \text{with} \quad D(p) = 4\sum_{i=1}^{4} s_i^2 \quad . \tag{3.14}$$

This leads to the familiar result for the energy density [10],

$$\frac{\epsilon}{T^4} = 6(N^2-1)N_\tau^4 \int_p \frac{s_1^2 - s_4^2}{D(p)} \quad . \tag{3.15}$$

Eq. (3.15) is the starting point for a discussion of the cut-off dependence of the high temperature limit of the energy density. We will consider here the cut-off dependence in the thermodynamic limit ($N_\sigma \rightarrow \infty$) and will comment on the influence of a finite spatial volume later. In the thermodynamic limit the sum over the spatial momenta can be replaced by an integral. The remaining sum over p_4 can be performed explicitly [19]. One then finds,

$$\frac{\epsilon}{T^4} = 4(N^2-1)N_\tau^4 \frac{1}{(2\pi)^3} \int_0^{2\pi} \mathrm{d}^3p \frac{\omega}{\sqrt{1+\omega^2}} \frac{1}{\exp(N_\tau x) - 1} \quad , \tag{3.16}$$

with $x = 2\ln(\omega + \sqrt{1+\omega^2})$ and $\omega^2 = s_1^2 + s_2^2 + s_3^2$. The right hand side of the above equation can be expanded in powers of $1/N_\tau \equiv aT$, which explicitly shows that the

[c]The gluon propagator given in Ref.[16] for the action $S^{(2,2)}$ corresponds to a gauge, in which $\Delta_{\mu,\nu} = D_\mu \delta_{\mu,\nu}$. In general, this cannot be achieved with the covariant gauge fixing term given in Eq. (3.6) and therefore does require non-trivial ghost contributions already in leading order perturbation theory, *i.e.* for the ideal gas [18]. The ghost contribution has not been calculated in Ref.[16]. For this reason the $O(g^2)$ result quoted there is not the complete contribution to the expansion coefficient for the plaquette expectation value at that order in g^2.

leading corrections to the continuum Stephan-Boltzmann law are indeed $O((aT)^2)$ [13]. This expansion is somewhat tedious but straightforward. In next-to-leading order we find[d]

$$\frac{\epsilon}{T^4} = 3\frac{p}{T^4} = (N^2-1)\frac{\pi^2}{15}\left[1 + \frac{30}{63}\cdot\frac{\pi^2}{N_\tau^2} + \frac{1}{3}\frac{\pi^4}{N_\tau^4} + O\left(\frac{1}{N_\tau^6}\right)\right] . \tag{3.17}$$

In Figure 1 we show the cut-off dependence of ϵ/T^4 obtained from Eq. (3.16) as well as the $O(N_\tau^{-2})$ and $O(N_\tau^{-4})$ results from Eq. (3.17). As can bee seen the leading

ideal11.ps

Figure 1: High temperature limit of the energy density of an $SU(N)$ gluon gas calculated with finite cut-off, $aT = 1/N_\tau$, in units of the continuum limit result (solid curve). The dashed and dashed-dotted curves show this ratio calculated up to $O(N_\tau^{-2})$ and $O(N_\tau^{-4})$, respectively.

finite size correction term given in Eq. (3.17) accounts for more than 70% of the total correction already for $N_\tau \geq 6$.

[d]Note that the natural expansion parameter is $\pi T a \equiv \pi/N_\tau$, *i.e.* half the smallest, non-vanishing Matsubara mode.

3.3 Improved Actions

Generalizing the discussion of the perturbative calculations with the one-plaquette Wilson action to the case of the improved actions is now straightforward, although the actual calculations turn out to be quite tedious because we did not succeed in finding a gauge fixing condition, which would lead to a simple diagonal propagator. Details of the perturbative calculations for the various improved actions considered by us are given in the Appendix.

The generalization of the leading order perturbative expansion for the energy density, given in Eq. (3.15) for the one-plaquette Wilson action, now reads

$$\frac{\epsilon}{T^4} = \frac{3}{2}(N^2-1)N_\tau^4 \int_p \frac{R_\sigma(p) - R_\tau(p)}{R^d(p)} \quad , \tag{3.18}$$

with functions R_σ, R_τ and R^d defined in the Appendix for the different improved actions.

We have evaluated the energy density from Eq. (3.18) numerically. Results for small values of N_τ obtained in the thermodynamic limit ($N_\sigma \to \infty$) are summarized in Table 1. We note that for all improved actions the deviation from the continuum results, $\epsilon_{\rm SB}/T^4$, is less than 1.2% already for $N_\tau = 6$ which is reached with the Wilson action only for $N_\tau \simeq 20$. At this level of accuracy it is, in fact, important to consider also the influence of a finite spatial extent of the lattice. A calculation of $\epsilon(N_\tau, N_\sigma)/\epsilon_{\rm SB}$ for finite N_σ indeed shows that the resulting infrared finite-size effects also are of the order of 1% for $N_\sigma/N_\tau \simeq 4$. Some results for finite spatial lattices are summarized in Table 2.

The energy density calculated perturbatively from the improved actions will show a cut-off dependence starting at $O((aT)^\alpha)$, *i.e.* corrections on lattices with temporal extent N_τ will asymptotically be proportional to $N_\tau^{-\alpha}$ with $\alpha = 4$ for the cases $I = (1,2)$ an $I = (2,2)$ and $\alpha = 6$ for $I = (3,3)$,

$$\frac{\epsilon(N_\tau)}{T^4} = (N^2-1)\frac{\pi^2}{15}\left[1 + c^I \left(\frac{\pi}{N_\tau}\right)^\alpha + O(N_\tau^{-(\alpha+2)})\right] \quad . \tag{3.19}$$

We have evaluated ϵ/T^4 also for large values of N_τ, up to $N_\tau \le 32$, in order to verify this asymptotic behaviour. In Figure 2 we show estimates for the expansion coefficients, $c^I(N_\tau)$, obtained at fixed values of N_τ by solving Eq. (3.19) for c^I. As can be seen, the cut-off dependence indeed shows the anticipated scaling behaviour. The coefficients c^I have then been determined from an extrapolation to $N_\tau \to \infty$, taking into account the subleading correction term proportional to $N_\tau^{-(\alpha+2)}$. These numbers are given in the last row of Table 1. The quoted errors reflect the difference

Table 1: Deviations from the continuum ideal gas behaviour on spatially infinite lattices with temporal extent N_τ for the standard one-plaquette Wilson action and various improved actions. In the last row we give the coefficients of the leading cut-off correction term as defined in Eq. (3.19). The number in brackets gives the difference between the estimate for c^I obtained for $N_\tau = 32$ and the extrapolated value.

	$\epsilon(N_\tau)/\epsilon_{\rm SB}$			
N_τ	I=(1,1)	I=(1,2)	I=(2,2)	I=(3,3)
4	1.495186	0.986568	1.087709	1.044357
6	1.181566	0.997528	1.011994	1.008095
8	1.086700	1.000309	1.003752	1.001199
10	1.051708	1.000253	1.001582	1.000320
12	1.034756	1.000150	1.000780	1.000110
c^I	0.4762	0.044 (2)	0.178 (2)	0.394 (19)

between the extrapolated value and the last calculated approximant for $N_\tau = 32$. We note that in the case of the improved actions $S^{(1,2)}$ and $S^{(2,2)}$ not only the $O(N_\tau^{-2})$ corrections have been eliminated, but also the magnitude of the $O(N_\tau^{-4})$ coefficient has been reduced strongly when compared to the one-plaquette action.

The improved actions lead to a drastic reduction of finite cut-off effects already for small values of N_τ, where in the case of the one-plaquette Wilson actions the cut-off effects were not at all dominated by the leading $O(N_\tau^{-2})$ contribution, which has been eliminated in the improved actions. In fact, in the case of the improved action $S^{(1,2)}$, deviations from the continuum ideal gas behaviour are always less than 1.5% already for $N_\tau \geq 4$, which is to be compared with the nearly 50% deviations at $N_\tau = 4$ for the one-plaquette Wilson action. We note, however, that also in this case the leading N_τ^{-4} dependence of the cut-off effects starts dominating only for $N_\tau \geq 10$. This is different for the actions $S^{(2,2)}$ and $S^{(3,3)}$, where the leading correction dominates already for $N_\tau \geq 6$.

4 Monte Carlo Results for Improved Actions

The perturbative calculations have shown that already the $O(a^2)$ improved actions $I = (2,2)$ and in particular $I = (1,2)$ provide a large reduction of finite cut-off effects in the high temperature ($T \to \infty$) limit even for small values of N_τ. We have used these actions as well as the $O(a^4)$ improved action, $I = (3,3)$, to analyze the cut-off dependence at non-vanishing values of g^2, *i.e.* for $T < \infty$. The recently performed

Table 2: Deviations from the continuum ideal gas behaviour on finite spatial lattices with temporal extent N_τ for the standard one-plaquette Wilson action and various improved actions.

	$\epsilon(N_\tau, N_\sigma)/\epsilon_{\rm SB}$							
	$N_\sigma = 4N_\tau$				$N_\sigma = 6N_\tau$			
N_τ	I=(1,1)	I=(1,2)	I=(2,2)	I=(3,3)	I=(1,1)	I=(1,2)	I=(2,2)	I=(3,3)
4	1.4833	0.9747	1.0758	1.0325	1.4917	0.9830	1.0842	1.0408
6	1.1697	0.9857	1.0001	0.9962	1.1780	0.9940	1.0085	1.0046
8	1.0748	0.9884	0.9919	0.9893	1.0832	0.9968	1.0002	0.9977
10	1.0398	0.9884	0.9897	0.9884	1.0482	0.9967	0.9981	0.9968
12	1.0229	0.9883	0.9889	0.9882	1.0312	0.9966	0.9973	0.9966

simulations with the one-plaquette Wilson action on lattices with $N_\tau = 4$, 6 and 8 [14] do provide here a good basis for a comparison.

In a first step we have evaluated the difference of action densities, studied perturbatively in the previous sections, at finite values of the gauge coupling β on lattices of size $16^3 \times 4$ ($24^3 \times 4$ in the case of the action $S^{(2,2)}$). We have performed calculations of the action differences for the $SU(3)$ gauge theory,

$$\left(\frac{\epsilon}{T^4}\right)_0 = 3\beta N_\tau^4 \Big(\langle S_\sigma^I \rangle - \langle S_\tau^I \rangle \Big) \quad , \tag{4.1}$$

at values of $\beta = 6/g^2$ well above the critical value for the deconfinement transition. Some results are given in Table 3. As can be seen, also at finite values of β the results for the different actions show a cut-off dependence, which is close to the result calculated in the limit $\beta \to \infty$. In all cases the asymptotic value is approached from above. For the one-plaquette Wilson action this has been shown to be in accordance with the perturbatively calculated $O(g^2)$ correction to $(\epsilon/T^4)_0$ [10]. This behaviour should not be confused with the correction to the complete energy density, ϵ/T^4, which is negative.

A more detailed comparison with the one-plaquette Wilson action is possible through a complete calculation of the temperature dependence of bulk thermodynamic quantities along the line presented in [14]. This also requires the calculation of the action densities on large, zero temperature lattices. As this is quite time consuming we have performed such a complete analysis only for the action $I = (2, 2)$. On a lattice with four sites in the temporal direction this action leads, in the limit $T \to \infty$, to finite cut-off effects which are of similar size as those found for the one-plaquette Wilson action on lattices with eight sites in the temporal direction. A direct comparison of results obtained with both action thus is possible.

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Figure 2: Estimate of the coefficient c^I of the leading cut-off dependence, obtained by solving Eq. (3.19) for c^I at fixed N_τ, with $\alpha = 2$ for the one-plaquette Wilson action (triangle), $\alpha = 4$ for $I = (1,2)$ (crosses), $I = (2,2)$ (circle) and $\alpha = 6$ for $I = (3,3)$ (squares). In the limit $N_\tau \to \infty$ this quantity yields the coefficient c_I given in Table 1.

We have calculated the action densities $\langle S^{(2,2)} \rangle_T$ and $\langle S^{(2,2)} \rangle_0$ on lattices of size $24^3 \times 4$ and 24^4, respectively. An integration of their differences calculated at various values of β yields the pressure[e]

$$\frac{p}{T^4}\Big|_{\beta_0}^{\beta} = N_\tau^4 \int_{\beta_0}^{\beta} \mathrm{d}\beta' (\langle S^{(2,2)} \rangle_0 - \langle S^{(2,2)} \rangle_T) \quad , \tag{4.2}$$

where β_0 is a value of the coupling constant below the phase transition point at which the pressure can savely be approximated by zero. In order to compare this calculation with corresponding results for the one-plaquette Wilson action, we have to define a common temperature scale. For this purpose we determine the relation between β and the cut-off via the 2-loop renormalization group equation, $a\Lambda = R(\beta_{\mathrm{eff}})$ using there an effective coupling constant obtained in terms of the action

[e] We refer to Refs. [13, 14] for more details on the formalism.

Table 3: Differences of space- and timelike action densities times N_τ^4 calculated on lattices of size $16^3 \times 4$ for $I = (1,1)$, (1,2) and (3,3) and $24^3 \times 4$ in the case of $I = (2,2)$. The last row gives the perturbative result in the limit $\beta \to \infty$ (the corresponding value in the continuum is $\epsilon_{\rm SB}/T^4 = 5.2638$). Couplings have been chosen well above the critical coupling for the deconfinement transition, which occurs at $\beta_c = 5.6908$ (2) ($I = (1,1)$, [20]), 4.0752 (13) ($I = (1,2)$, [7]), 4.3995 (2) ($I = (2,2)$, [this study]) and $3.7 < \beta_c < 4.2$ ($I = (3,3)$, [this study]).

	$(\epsilon/T^4)_0$			
β	(1,1)	(1,2)	(2,2)	(3,3)
6.0	7.359 (12)	5.706(57)	6.513 (74)	5.88 (12)
10.0	8.046 (102)	5.42 (10)	6.203 (90)	5.79 (16)
15.0	7.837 (63)	5.445(84)	5.982 (79)	5.58 (16)
20.0	7.806 (48)	5.18 (10)	5.860 (94)	5.56 (13)
∞	7.808	5.131	5.707	5.435

density

$$\beta_{\rm eff} = \frac{N^2-1}{4\langle S^I\rangle_0} \quad . \tag{4.3}$$

In the case of the one-plaquette Wilson action, this has been found to be a reasonable parametrization of the temperature scale [14]. Combined with a determination of the critical coupling for the deconfinement transition on the $24^3 \times 4$ lattice,

$$\beta_c(N_\tau = 4) = 4.3995 \pm 0.0002 \quad , \tag{4.4}$$

the temperature can be expressed in units of the critical temperature $T/T_c = R(\beta_{{\rm eff},c})/R(\beta_{\rm eff})$.

In Figure 3 we show the pressure calculated with the action $S^{(2,2)}$ on a $24^3 \times 4$ lattice and compare this with the calculations performed with the one-plaquette Wilson action on lattices of size $32^3 \times 4$, 6 and 8 [14].

We note that at high temperatures, $T \gtrsim 4T_c$, the pressure calculated from the improved action, $S^{(2,2)}$, on a lattice with $N_\tau = 4$ indeed is in good agreement with results obtained for the one-plaquette Wilson action on a lattice with $N_\tau = 8$. This confirms that the improvement of the actions at $g^2 = 0$ persists also at non-vanishing values of the gauge coupling. For temperatures closer to T_c we find, however, results, which are compatible with those obtained with the one plaquette Wilson action. This is not too surprising. Already the analysis of the cut-off effects in calculations druckimp.ps

Figure 3: The pressure in units of T^4 versus T/T_c calculated with the improved action $I = (2,2)$ on a $24^3 \times 4$ lattice. This is compared with calculations using the standard one-plaquette Wilson action on lattices of size $32^3 \times 4$, 6 and 8 [14]. The dashed horizontal lines give the corresponding ideal gas limits for the (1×1) Wilson action and the solid line corresponds to the ideal gas limit for the $(2,2)$-improved action (see Table 2). The arrow points at the continuum result for an ideal gas.

with the Wilson action has shown that the magnitude of the cut-off corrections does vary with temperature.

5 Conclusions

In the high temperature limit the thermodynamics of $SU(N)$ gauge theories is dominated by high momentum modes. In lattice calculations these modes are most strongly affected by the finite ultraviolet cut-off. We have shown here that the use of tree level improved actions for the gauge fields indeed leads to a strong reduction of finite cut-off effects in thermodynamic observables like the energy density or the pressure. The perturbative analysis of the high temperature limit of the energy density has shown that already on lattices with a temporal extent as small as $N_\tau = 4$

the cut-off effects can easily be reduced to a few percent.

A first Monte Carlo calculation of the pressure of an $SU(3)$ gauge theory using an improved action has shown that this also removes the dominant cut-off effects at non-zero values of the gauge coupling. At high temperatures the calculations performed with an improved action on a lattice with $N_\tau = 4$ lead to results which are compatible with those obtained with the standard one-plaquette Wilson action on a lattice with $N_\tau = 8$, where this action leads to similarly small cut-off effects. However, closer to T_c we also find that the inherently non-perturbative features of the deconfinement phase transition play a much more important role. Here the results obtained with a tree level improved action coincide with those obtained with the standard Wilson action on the same size lattice. Clearly in this temperature region there is need for further improvement of the lattice regularized Yang-Mills action. This can, for instance, be achieved by including one-loop corrections in the coefficients $a_{k,l}$ of the improved action or by determining them through a Monte Carlo renormalization group approach at various values of g^2. Currently we also investigate the properties of tadpole improved actions [21].

Of course, considerations similar to those made here for $SU(N)$ gauge theories can also be applied to the fermion sector of QCD. In fact, in this case the influence of finite cut-off effects on the high temperature limit of thermodynamic observables are known to be even larger on lattices with small temporal extent. The calculation of the equation of state of QCD thus should profit even more from the use of improved actions. An investigation of this claim is under way.

Acknowledgements:

The numerical calculations presented here have been performed on a QUADRICS parallel computer funded by the Deutsche Forschungsgemeinschaft (DFG) under grant Pe 340/6-1. It also has been supported by the DFG through the grants Pe 340/3-2 and Pe 340/3-3. F.K. thanks the theory division of CERN for the kind hospitality extended to him during the summer 1995.

A Appendix

We give here explicit expressions for the leading order weak coupling expansion of the action densities on finite lattices for the improved Wilson actions defined in Eqs. (2.1 - 2.4), respectively. For completeness we also repeat the corresponding results for the standard one-plaquette Wilson action. In all cases we have checked that the results are gauge invariant.

In addition to the short hand notation $s_\mu \equiv \sin(p_\mu/2)$ we introduce the abbreviations

$$\begin{aligned} \Delta_\mu^-(p) &= e^{ip_\mu} - 1 \quad , \\ \Delta_\mu^{(k)}(p) &= \sum_{l=0}^{k-1} e^{ilp_\mu} \quad . \end{aligned} \tag{A.1}$$

The expansion of a symmetrized $k \times l$ Wilson loop defined in Eq. (2.2) to $O(g^2)$ can then be written as

$$\begin{aligned} W_{\mu,\nu}^{k,l} &\equiv \frac{1}{N_\sigma^3 N_\tau} \sum_x W_{\mu,\nu}^{k,l}(x) \\ &= \frac{g^2}{4N} \sum_{p,a} \Big(N_{\mu;\nu}^{k,l}(p) A_\mu^a(p) A_\mu^a(-p) + N_{\nu;\mu}^{k,l}(p) A_\nu^a(p) A_\nu^a(-p) \\ &\qquad -2 M_{\mu,\nu}^{k,l}(p) A_\mu^a(p) A_\nu^a(-p) \Big) + O(g^4) \end{aligned} \tag{A.2}$$

with

$$N_{\mu;\nu}^{k,l}(p) = \frac{1}{2} \Big(\Delta_\mu^{(k)}(p) \Delta_\mu^{(k)}(-p) \Delta_\nu^-(-lp) \Delta_\nu^-(lp) + (k \leftrightarrow l) \Big) \tag{A.3}$$

and

$$M_{\mu,\nu}^{k,l}(p) = \frac{1}{2} \Big(e^{i(p_\mu - p_\nu)/2} \Delta_\mu^{(k)}(p) \Delta_\nu^-(lp) \Delta_\nu^{(l)}(-p) \Delta_\mu^-(-kp) + (k \leftrightarrow l) \Big) \quad . \tag{A.4}$$

We note that $M_{\mu,\nu}^{k,l}$ has non-vanishing matrix elements also for $(\mu = \nu)$. In fact, due to the identity $\Delta_\mu^{(k)}(p)\Delta_\mu^-(lp) = \Delta_\mu^{(l)}(p)\Delta_\mu^-(kp)$ one finds that $M_{\mu,\mu}^{k,l} = N_{\mu;\mu}^{k,l}$. This term does not contribute in the expansion of Wilson loops. It is included in the definition of $E_{\mu,\nu}$ and cancels against the corresponding term in the diagonal part of the matrix $G_{\mu,\nu} = D_\mu(p)\delta_{\mu,\nu} - E_{\mu,\nu}(p)$, which has been introduced in Eq. (3.8). With the above relations we obtain for $D_\mu(p)$ and $E_{\mu,\nu}(p)$,

$$\begin{aligned} D_\mu(p) &= \sum_{l=1}^{\infty} \sum_{k=1}^{l} a_{k,l} \sum_{\nu=1}^{4} N_{\mu;\nu}^{k,l}(p) \\ &= 2 \sum_{l=1}^{n} \sum_{k=1}^{l} a_{k,l} \Big(\Big[k + 2 \sum_{j=1}^{k-1} (k-j) \cos(jp_\mu) \Big] \sum_{\nu=1}^{4} \sin^2(lp_\nu/2) \\ &\qquad + (k \leftrightarrow l) \Big) \end{aligned}$$

$$E_{\mu,\nu}(p) = \sum_{l=1}^{n}\sum_{k=1}^{l} a_{k,l} M^{k,l}_{\mu,\nu}(p) \tag{A.5}$$
$$= 2\sum_{l=1}^{n}\sum_{k=1}^{l} a_{k,l}\Big(\sum_{i=0}^{k-1}\sin((2i+1)p_\mu/2)\sum_{j=0}^{l-1}\sin((2j+1)p_\nu/2) + (k\leftrightarrow l)\Big)$$

Adding the gauge fixing functions g_μ one can construct the propagator matrix $\Delta_{\mu,\nu}$ (Eq. (3.6)), which we invert using Mathematica.

The explicit results for the standard one-plaquette Wilson action have been discussed in Section 3.1. We give here the result for the expansion coefficient, $S^{I,(2)}_{\mu,\nu}$, for improved actions, which we write in the form

$$S^{I,(2)}_{\mu,\nu} = \int_p \frac{R_{\mu,\nu}(p)}{R^d(p)} \quad . \tag{A.6}$$

In Eq. (3.18) we have introduced the functions R_σ and R_τ, which are defined as

$$R_\sigma = \frac{1}{3}(R_{1,2} + R_{1,3} + R_{2,3}) \quad ,$$
$$R_\tau = \frac{1}{3}(R_{1,4} + R_{2,4} + R_{3,4}) \quad . \tag{A.7}$$

In order to present the results in a convenient form, we have chosen a specific form of the gauge fixing function

$$I = (1,1) \quad : \quad g_\mu = 2\ s_\mu$$
$$I = (1,2) \quad : \quad g_\mu = 2\ (s_\mu + \frac{2}{3}\ s^3_\mu)$$
$$I = (2,2) \quad : \quad g_\mu = 2\ (s_\mu + s^3_\mu)$$
$$I = (3,3) \quad : \quad g_\mu = 2\ (s_\mu + \frac{1}{3}s^3_\mu + \frac{16}{45}s^5_\mu) \quad . \tag{A.8}$$

In the following we use the convention that all indices specified are mutually different. Permutations of indices that lead to identical expressions are counted only once. Whenever summations over permutations of indices are needed the relevant sums are specified explicitly. We present the result for the (μ,ν) component of the action. The remaining two directions of the 4-dimensional lattice are denoted by

(i,j). In those cases where indices can take values from the set (μ,ν) as well as (i,j) we denote them by $k,\ l,\ m$ and n. With these conventions we find

$I = (1,1)$:

$$\begin{aligned} R_{\mu,\nu} &= s_\mu^2 + s_\nu^2 \\ R^d &= \sum_{k=1}^{4} s_k^2 \end{aligned} \tag{A.9}$$

$I = (1,2)$:

$$\begin{aligned} R_{\mu,\nu} &= \frac{4}{3}(3 + s_\mu^2 + s_\nu^2) \\ &\quad \Big(D_i D_j (s_\mu g_\mu + s_\nu g_\nu)^2 + (D_\mu s_\mu^2 + D_\nu s_\nu^2)(D_i g_j^2 + D_j g_i^2) \Big) \\ R^d &= \sum_{k=1}^{4} g_k^2 D_l D_m D_n \end{aligned} \tag{A.10}$$

$I = (2,2)$:

$$\begin{aligned} R_{\mu,\nu} &= \frac{4}{3}(3 + s_\mu^2 + s_\nu^2 - s_\mu^2 s_\nu^2) \\ &\quad \Big(D_i D_j (s_\mu g_\mu + s_\nu g_\nu)^2 + (D_\mu s_\mu^2 + D_\nu s_\nu^2)(D_i g_j^2 + D_j g_i^2) \Big) \\ R^d &= \sum_{k=1}^{4} g_k^2 D_l D_m D_n \end{aligned} \tag{A.11}$$

$I = (3,3)$:

$$\begin{aligned} R_{\mu,\nu} &= 4\Big(1 + \frac{1}{3}(s_\mu^2 + s_\nu^2) + \frac{8}{45}(s_\mu^4 + s_\nu^4) + \frac{1}{9} s_\mu^2 s_\nu^2 - \frac{64}{135} s_\mu^2 s_\nu^2 (s_\mu^2 + s_\nu^2) + \frac{128}{405} s_\mu^4 s_\nu^4 \Big) \\ &\quad \Big[D_i D_j (s_\mu g_\mu + s_\nu g_\nu)^2 + (D_\mu s_\mu^2 + D_\nu s_\nu^2)(D_i g_j^2 + D_j g_i^2) \\ &\quad + \frac{256}{18225}(D_\mu s_\mu^2 + D_\nu s_\nu^2) A_{i,j} B_{j,i} \\ &\quad - \frac{256}{18225}\Big(D_i s_j^2 (A_{\mu,j} + A_{\nu,j})(B_{\mu,j} + B_{\nu,j}) + D_j s_i^2 (A_{\mu,i} + A_{\nu,i})(B_{\mu,i} + B_{\nu,i}) \Big) \\ &\quad - \frac{4096}{225} s_i^2 s_j^2 (s_j^2 - s_i^2)^2 \Big(s_\mu^2 (s_\mu^2 - s_i^2)(s_\mu^2 - s_j^2) + s_\nu^2 (s_\nu^2 - s_i^2)(s_\nu^2 - s_j^2) \Big)^2 \Big] \\ R^d &= \sum_{k=1}^{4} g_k^2 D_l D_m D_n + \frac{256}{18225} \sum_{k=1}^{3} \sum_{l=k+1}^{4} D_k D_l A_{n,m} B_{m,n} \\ &\quad - \frac{4096}{225} \sum_{k=1}^{4} D_k s_l^2 s_m^2 s_n^2 (s_m^2 - s_l^2)^2 (s_m^2 - s_n^2)^2 (s_l^2 - s_n^2)^2 \end{aligned} \tag{A.12}$$

Here we also have used the abbreviations

$$\begin{aligned} A_{k,l} &= s_k^2 (s_k^2 - s_l^2)(s_k^2 s_l^2 + 3(s_k^2 + s_l^2)) \\ B_{k,l} &= s_k^2 (s_k^2 - s_l^2)(-405 + 120(s_k^2 + s_l^2) + 184 s_k^2 s_l^2) \end{aligned} \tag{A.13}$$

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