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Dispersion Relations from the Hard Thermal Loop Effective Action in a Magnetic Field

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Abstract

Dispersion relations for fermions at high temperature and in a background magnetic field are calculated in two different ways. First from a straightforward one-loop calculation where, in the weak field limit, we find an expression closely related to the standard dispersion relations in the absence of the magnetic field. Secondly, we derive the dispersion relations directly from the Hard Thermal Loop effective action, which allows for an exact solution (i.e. to all orders in the external field), up to the last numerical integrals.

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1 Introduction

We know that thermal effects on dispersion relations are of extreme importance at high temperature, where the whole concept of propagating particles is drastically changed [1, 2]. Since these effects are large and influence the mass shell conditions, it is also important to include them consistently to all orders and not only as first-order corrections. This is consistent to leading order in $(eT)^2$. Both fermions and gauge bosons develop new branches and the zero-temperature masses become less important. New decay channels can open up because of the effective thermal masses of the propagating excitations. It is also interesting to study how strong external fields influence the physics of these effective modes. We have considered two ways of calculating the dispersion relations for fermions at high temperature in a constant background magnetic field. One method is to simply do the one-loop calculation at finite temperature, using exact propagators in the external field [3]. The other is to start from the already resummed effective action for Hard Thermal Loops [4, 5] (HTLs). It is an effective action that generates all HTLs at tree level and which contains the external fields to all orders.

2 One-loop self-energy

In Ref. [3] the fermionic dispersion relation at finite temperature in a constant magnetic field was calculated using the electron propagator in the Furry picture and Schwinger's proper-time method. In the high-temperature limit it turns out that Schwinger's proper-time formulation [6] of the exact propagator is the easiest to use for the loop calculation. The thermal propagator can be constructed as [7]

$$iS_{\rm vac}(p) - f_F(p_0)\Big(iS_{\rm vac}(p) - iS^*_{\rm vac}(p)\Big)\ , \tag{1}$$

where for a magnetic field parallel to the z-direction[1]

$$\begin{aligned} iS_{\rm vac}(p) &= \int_0^\infty ds \frac{e^{ieBs\sigma_z}}{\cos eBs} \exp\left[is\left(p_\parallel^2 - \frac{\tan eBs}{eBs}p_\perp^2 - m^2 + i\varepsilon\right)\right] \\ &\times \left\{\gamma p_\parallel - \frac{e^{-ieBs\sigma_z}}{\cos eBs}\gamma p_\perp + m\right\}\ . \end{aligned} \tag{2}$$

After performing the loop integrals in $\Sigma(x',x) = \langle x'|\hat{\Sigma}|x\rangle$ it is possible to extract the gauge-invariant operator $\hat{\Sigma}(p_0, p_z, \mathbf{\Pi}_\perp)$, where $\mathbf{\Pi}_\perp = (p_x - eA_x, p_y - eA_y)$, in a gauge with zero z-component. (We shall use the gauge $A_\mu = (0, 0, -B\,x, 0)$.) Then we take the weak field limit, but keep eB exactly wherever it is added linearly to the canonical momentum like $\mathbf{\Pi}_\perp^2 - eB\sigma_z$. The reason being that $\mathbf{\Pi}_\perp^2 - eB\sigma_z = \not{\mathbf{\Pi}}_\perp^2$, so eB can be reabsorbed in $\not{\mathbf{\Pi}}_\perp^2$, which

[1]The sign convention here, that the particle has a positive charge e, differs from Ref. [3].

need not be small. When the final expression for the self-energy is used in the effective Dirac equation, we obtain

$$[\not{\Pi} - m - \hat{\Sigma}(p_0, p_z, \mathbf{\Pi}_\perp)]\Psi =$$

$$\left[s(p_0, \mathbf{\Pi}^2)\gamma_0 p_0 - r(p_0, \mathbf{\Pi}^2)\gamma_z p_z - r(p_0, \mathbf{\Pi}^2 - eB\sigma_z)\not{\Pi}_\perp - m\right]\Psi = 0 \quad , \tag{3}$$

where $\mathbf{\Pi}^2 = \mathbf{\Pi}_\perp^2 + p_z^2$ and

$$s(p_0, \mathbf{\Pi}^2) \quad = \quad \left(1 - \frac{\mathcal{M}^2}{2p_0\,|\mathbf{\Pi}|} \ln\left|\frac{p_0 + |\mathbf{\Pi}|}{p_0 - |\mathbf{\Pi}|}\right|\right) \quad , \tag{4}$$

$$r(p_0, \mathbf{\Pi}^2) \quad = \quad \left(1 + \frac{\mathcal{M}^2}{\mathbf{\Pi}^2}\left(1 - \frac{p_0}{2\,|\mathbf{\Pi}|} \ln\left|\frac{p_0 + |\mathbf{\Pi}|}{p_0 - |\mathbf{\Pi}|}\right|\right)\right) \quad . \tag{5}$$

The temperature dependence enters only through the thermal mass $\mathcal{M} = e^2T^2/8$. It is *almost* possible to guess the expression in Eq. (3) from the standard expression for the HTL Dirac equation [1, 2]. The usual momentum p_μ should be replaced with the gauge-invariant momentum Π_μ, but there is an ambiguity in replacing p^2 by Π^2 or by $\not{\Pi}\,\not{\Pi}$. The correct way follows from the calculations in Ref. [3].

In the massless limit (i.e. zero vacuum mass) the dispersion relation divides into a left- and a right-handed part. With a wave function $\Psi = (L, R)^T$ in the chiral representation the two dispersion relations become

$$L: \quad (Es_n + p_z r_n)(Es_{n-1} - p_z r_{n-1}) - 2eBnr_{n-\frac{1}{2}}^2 = 0 \quad , \tag{6}$$

$$R: \quad (Es_n - p_z r_n)(Es_{n-1} + p_z r_{n-1}) - 2eBnr_{n-\frac{1}{2}}^2 = 0 \quad , \tag{7}$$

where $s_n = s(E, p_z^2 + eB(2n+1))$ and similarly for r_n. These relations are valid for all $n \geq 1$, but in the lowest Landau level there is only one non-zero component for each of L and R, so the dispersion relations reduce to

$$L: \quad Es(E, p_z^2 + eB) + p_z r(E, p_z^2 + eB) = 0 \ , \tag{8}$$

$$R: \quad Es(E, p_z^2 + eB) - p_z r(E, p_z^2 + eB) = 0 \ . \tag{9}$$

The numerical solution in the lowest Landau level is shown in Fig. 1 for the right-handed particle. There are eight different branches corresponding to the states $(L/R)\times$ (particle/hole) $\times$ (positive/negative energy), for each given value of $n \geq 1$ and p_y. For $n = 0$ there are only half as many states, since only one spin orientation is possible in the lowest Landau level. Figure 1 only shows the right-handed branch, and the left-handed branch is obtained by reflection in p_z. In fact, from Eqs. $(6, 8)$ we find the combined symmetries $(L \leftrightarrow R$, $p_z \leftrightarrow -p_z)$, $(L \leftrightarrow R, E \leftrightarrow -E)$ and $(E \leftrightarrow -E, p_z \leftrightarrow -p_z)$. In particular, right-handed particles in the lowest Landau level can only propagate along the magnetic field (which

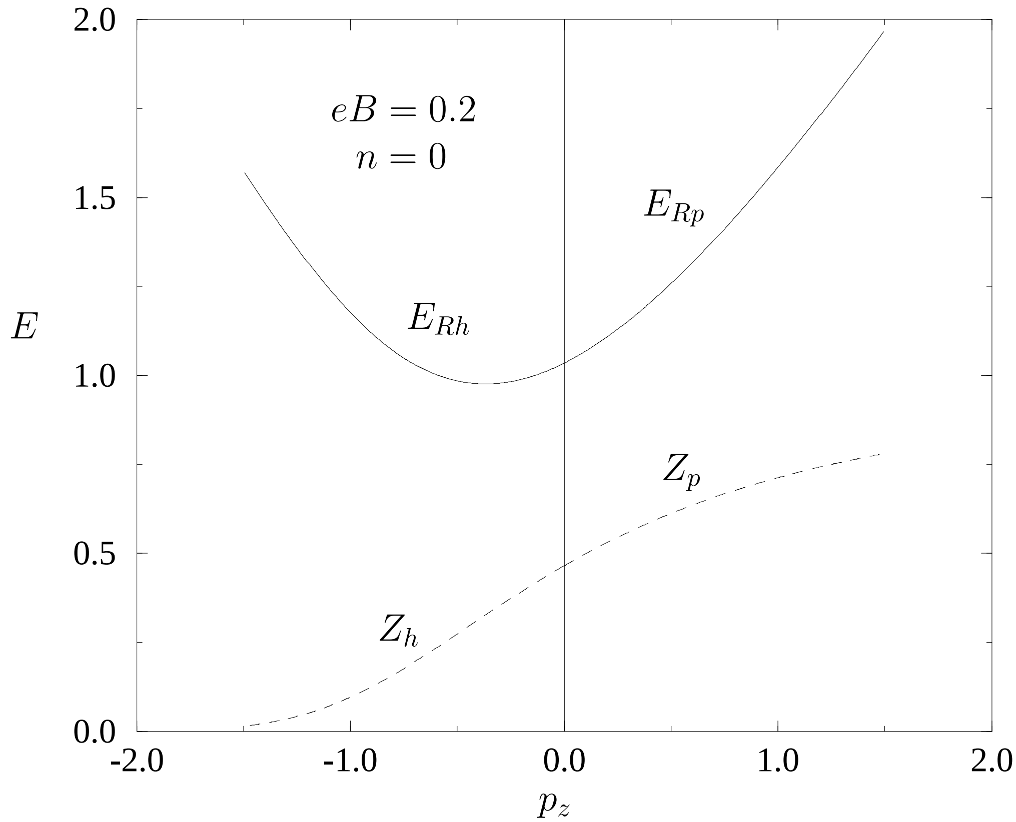

Figure 1: *Dispersion relation and spectral weight for the right-handed branch in the lowest Landau level. All dimensionful parameters are given in units of the thermal mass $\mathcal{M}$.*

points in the positive z-direction), while the left-handed particles and the right-handed holes propagate against the field.

The spectral weight presented in Fig. 1 is defined by

$$Z_i(p_z, n)^{-1} = \frac{d}{d\omega}\bigg|_{\omega=E_i(p_z,n)} \left(\mathrm{Tr}\left[(\mathcal{D}(\omega, p_z, n)\gamma_0)^{-1}\right]\right)^{-1} \quad , \tag{10}$$

where $\mathcal{D}$ is the 4×4 matrix in Eq. (3) that remains after acting on an off-shell Landau level. For a given chirality, $Z_i(p_z)$ is obviously very asymmetric in p_z since only particles propagate in one direction and the holes in the other.

3 HTL effective action in a background field

Instead of computing the one-loop self-energy as in Section 2, we can use the fact that the HTL effective action already contains the leading high-temperature contribution to all orders in the gauge field. The equation of motion that follows from a variation of the fermionic field should immediately give the dispersion relation. The HTL effective action for QED can be written as [8]:

$$
\mathcal{L}_{HTL} = -\frac{1}{4}F^2 + \frac{3}{4}\mathcal{M}_\gamma^2 F_{\mu\alpha}\left\langle \frac{u^\alpha u^\beta}{(\partial u)^2}\right\rangle F_\beta{}^\mu
$$

$$
+ \overline{\Psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\Psi - \mathcal{M}^2\overline{\Psi}\gamma_\mu\left\langle\frac{u^\mu}{u\cdot\Pi}\right\rangle\Psi \quad , \tag{11}
$$

where the average $\langle\cdot\rangle$ is defined by

$$
\langle f(u_0,\vec{u})\rangle = \int\frac{d\Omega}{4\pi}f(1,\vec{u}) \quad , \tag{12}
$$

where $\vec{u}$ is a spatial unit vector. The equation of motion for Ψ that follows is

$$
\left[\Pi\!\!\!/ - m - \mathcal{M}^2\gamma_\mu\left\langle\frac{u^\mu}{u\cdot\Pi}\right\rangle\right]\Psi = 0 \quad . \tag{13}
$$

Equation (13) is a non-local and non-linear differential equation, which is, in general, very difficult to deal with. What makes this equation much less tractable than the thermal Dirac equation in the absence of the B-field is that the average over $\vec{u}$ is difficult to perform because $[\Pi_\mu,\Pi_\nu] = -ieF_{\mu\nu} \neq 0$. Since the spatial symmetries of the system are unaltered by the thermal heat bath, we still expect the eigenfunctions to have the same spatial form as at zero temperature. In fact, after performing the u-integral in Eq. (13) the result can only be a function of the invariants $\mathbf{\Pi}_\perp^2$, p_0^2 and p_z, and the γ-structure has to be proportional to $\gamma\mathbf{\Pi}_\perp$, $\gamma_0 p_0$ and $\gamma_z p_z$. We shall therefore compute the matrix elements

$$
\langle\Phi_{\kappa'}|\left\langle\frac{u^\mu}{u\cdot\Pi}\right\rangle|\Phi_\kappa\rangle \quad , \tag{14}
$$

between the vacuum eigenstates

$$
\langle x|\Phi_\kappa\rangle = \exp[i(-p_0t+p_yy+p_zz)]I_{n;p_y}(x) \quad , \tag{15}
$$

$$
I_{n;p_y}(x) = \left(\frac{eB}{\pi}\right)^{1/4}\exp\left[-\frac{1}{2}eB\left(x-\frac{p_y}{eB}\right)^2\right] \times\frac{1}{\sqrt{n!}}H_n\left[\sqrt{2eB}\left(x-\frac{p_y}{eB}\right)\right] \quad , \tag{16}
$$

where $\kappa = \{p_0, n, p_y, p_z\}$ and $H_n[x]$ are Hermite polynomials. These states form a complete set of functions in four dimensions when the energy is off shell. In the chiral representation suitable spinors can be formed from Φ_κ as $\Psi_\kappa = \mathrm{diag}[\Phi_\kappa, \Phi_{\kappa-1}, \Phi_\kappa, \Phi_{\kappa-1}]\chi$ where χ is an undetermined space-time-independent spinor. The vacuum Dirac operator in Eq. (13) obviously gives an eigenvalue when acting on Ψ_κ, but it is more difficult to determine the action of the thermal part since Φ_κ cannot be an eigenfunction to $u\cdot\Pi$ for all u. One way to calculate the matrix element in Eq. (14) is to find a basis such that $v\cdot\Pi|v_p\rangle = v\cdot p|v_p\rangle$ and insert a unit operator $\int d^4p|v_p\rangle\langle v_p|$. This unit operator is, of course, independent of v after the p-integration, so in particular we can choose $v = u$ and change the order of integrations. After computing the matrix elements in Eq. (14) we find indeed that they are diagonal in κ for u_0 and u_z, and have a mixing with the first subdiagonals for u_x and u_y. Define $\langle u_{0,z,\pm}\rangle$ by

$$\langle\Phi_{\kappa'}|\left\langle\frac{u_{0,z}}{u\cdot\Pi}\right\rangle|\Phi_\kappa\rangle = (2\pi)^3\delta_{\kappa',\kappa}\langle u_{0,z}\rangle_\kappa \quad , \tag{17}$$

$$\langle\Phi_{\kappa'}|\left\langle\frac{u_x \pm iu_y}{u\cdot\Pi}\right\rangle|\Phi_\kappa\rangle = (2\pi)^3\delta_{\kappa',\kappa\mp1}\langle u_\pm\rangle_\kappa \quad , \tag{18}$$

and $\kappa\mp1 = \{p_0, n\mp1, p_y, p_z\}$. These are exactly the components that occur naturally when we include the γ-matrices in the chiral representation. The calculation of $\langle u_{0,z,\pm}\rangle$ is a bit lengthy but straightforward, and the result reads

$$\begin{aligned}\langle u_0\rangle_\kappa &= \frac{1}{n!\sqrt{2\pi}}\int_{-\infty}^{\infty} ds\, H_n^2(s)e^{-s^2/2} \\ &\times\left\{\frac{p_z}{2p^2}\ln\frac{p_0+p_z}{p_0-p_z} + \frac{Es\sqrt{2eB}}{2p^2\sqrt{p_0^2-p^2}}\arctan\frac{s\sqrt{2eB}}{2\sqrt{p_0^2-p^2}}\right\} \quad ,\end{aligned} \tag{19}$$

$$\begin{aligned}\langle u_z\rangle_\kappa &= \frac{1}{n!\sqrt{2\pi}}\int_{-\infty}^{\infty} ds\, H_n^2(s)e^{-s^2/2} \\ &\times\left\{-\frac{p_z}{p^2} + \frac{p_0(2p_z^2-eBs^2)}{4p^4}\ln\frac{p_0+p_z}{p_0-p_z}\right. \\ &\left.+\frac{p_z(2E^2-p^2)}{2p^4}\frac{s\sqrt{2eB}}{\sqrt{p_0^2-p^2}}\arctan\frac{s\sqrt{2eB}}{2\sqrt{p_0^2-p^2}}\right\} \quad ,\end{aligned} \tag{20}$$

$$\begin{aligned}\langle u_+\rangle_\kappa &= \frac{i}{\sqrt{2\pi n!(n-1)!}}\int_{-\infty}^{\infty} ds\, H_n(s)H_{n-1}(s)e^{-s^2/2} \\ &\times\left\{\frac{s\sqrt{2eB}}{2p^2} - \frac{Esp_z\sqrt{2eB}}{2p^4}\ln\frac{p_0+p_z}{p_0-p_z}\right. \\ &\left.+\frac{2p_z^2(p_0^2-p^2)-p_0^2eBs^2}{2p^4\sqrt{p_0^2-p^2}}\arctan\frac{s\sqrt{2eB}}{2\sqrt{p_0^2-p^2}}\right\} \quad ,\end{aligned} \tag{21}$$

$$\langle u_-\rangle_\kappa = -\langle u_+\rangle_{\kappa+1} \quad , \tag{22}$$

where $p^2 = p_z^2 + eBs^2/2$. With these definitions the Dirac equation effectively reduces to a 4×4 matrix in the spinor indices, since the other quantum numbers have been diagonalized. In the massless limit we take the determinant to find the dispersion relations. They factorize again as in Eqs. (6, 8) and the equations for the right-handed component are

$$\begin{aligned} n \geq 1: \qquad & \Big(p_0 - p_z - \mathcal{M}^2(\langle u_0\rangle_\kappa - \langle u_z\rangle_\kappa)\Big) \\ & \times\Big(p_0 + p_z - \mathcal{M}^2(\langle u_0\rangle_{\kappa-1} + \langle u_z\rangle_{\kappa-1})\Big) \\ & - \Big(\sqrt{2eBn} - i\mathcal{M}^2\langle u_+\rangle_\kappa\Big)^2 \;=\; 0 \;, \\ n = 0: \qquad & p_0 - p_z - \mathcal{M}^2(\langle u_0\rangle_\kappa - \langle u_z\rangle_\kappa) \;=\; 0 \;. \end{aligned} \tag{23}$$

In general there are imaginary parts in the functions $\langle u_{0,z,\pm}\rangle_\kappa$, which make the spectral

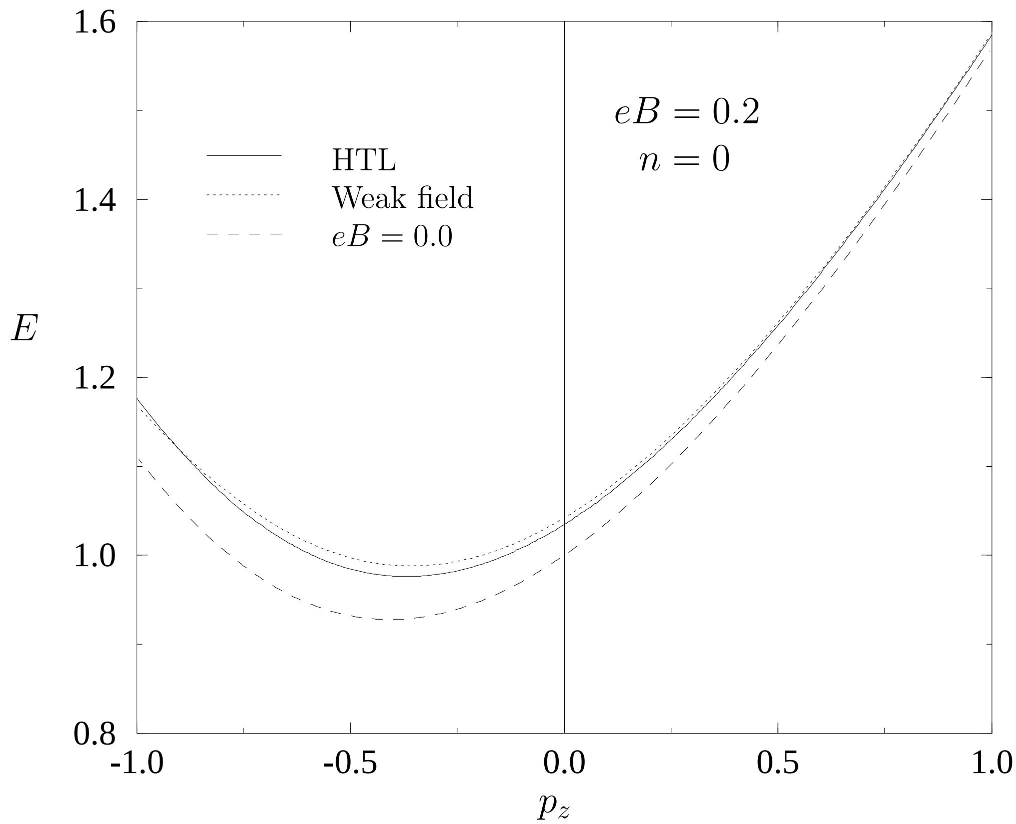

Figure 2: *Comparison of the dispersion relation from the HTL effective action and the weak field approximation. All dimensionful parameters are given in units of the thermal mass $\mathcal{M}$.*

functions more complicated. The imaginary parts of Eqs. (19) to (22) are determined by the analytic continuation $p_0 \to p_0 + i\epsilon$ for positive p_0; this amounts to the replacement

$$\begin{aligned} & \frac{1}{\sqrt{p_0^2 - p^2}} \arctan \frac{s\sqrt{2eB}}{2\sqrt{p_0^2 - p^2}} \\ \to & -\frac{1}{2\sqrt{p^2 - p_0^2}} \ln \frac{s\sqrt{2eB} + 2\sqrt{p^2 - p_0^2}}{s\sqrt{2eB} - 2\sqrt{p^2 - p_0^2}} - \frac{i\pi}{\sqrt{p^2 - p_0^2}} \;, \end{aligned} \tag{24}$$

for $p^2 > p_0^2$. It is anyway useful to solve Eq. (23), ignoring the imaginary part, since the real part indicates where the spectral functions are peaked, at least if the imaginary parts are small enough. This can conveniently be done numerically as all the integrals in Eqs. (19) to (22) are well convergent. The result for the lowest Landau level in a weak magnetic field ($eB = 0.2\,\mathcal{M}^2$) is shown in Fig. 2, and it agrees rather well with the weak-field result from Eq. (8).

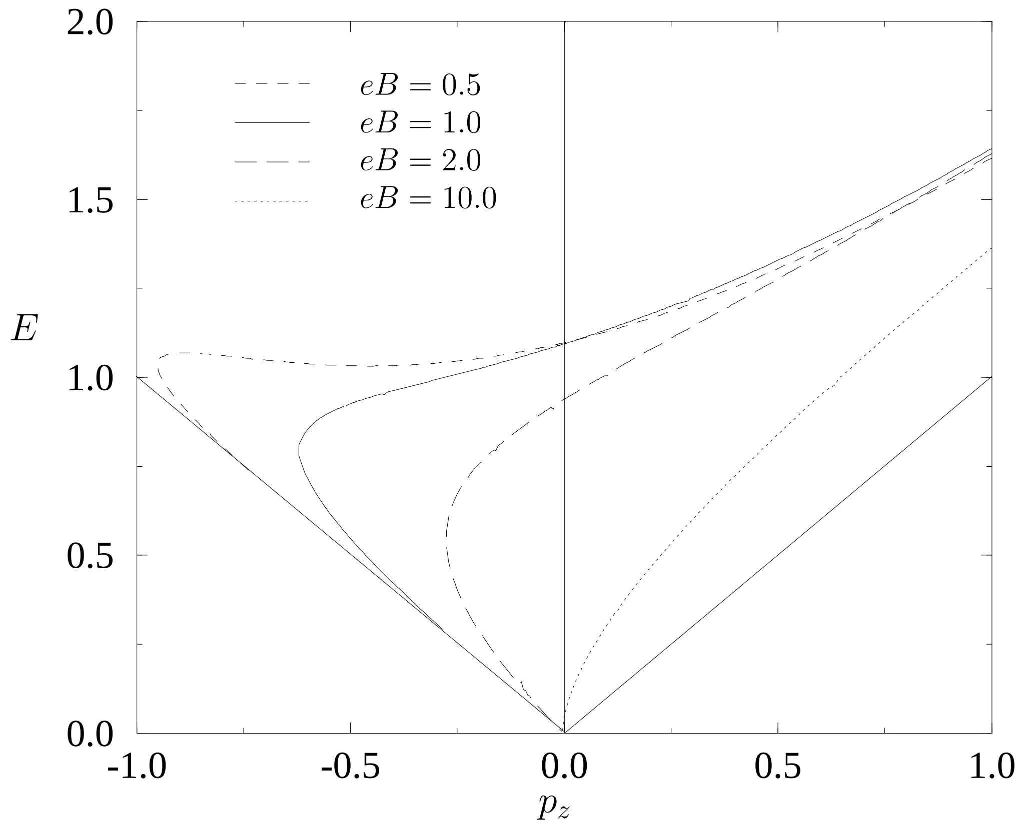

Figure 3: *Dispersion relation for the right-handed branch in the lowest Landau level ($n = 0$). As the B-field increases thermal effects become less important and the dispersion approaches the light cone, which is indicated by solid lines. All dimensionful parameters are given in units of the thermal mass $\mathcal{M}$.*

It is, of course, more interesting to see what happens at larger field strength, which cannot be treated by Eq. (8). The dispersion relations for several field strengths are shown in Fig. 3. Apart from the changes in the particle branch there is a new branch coming from the light cone, which eventually joins the hole branch and disappears. This new branch can be understood mathematically, from Eq. (23), by studying it close to the light cone. In the absence of the B-field, the hole branch exists because the logarithm in Eq. (4) becomes dominant close to the light cone, and the sign is such that it allows for a positive energy solution to the part of that Dirac equation which normally only gives the antiparticle solution. In the present case, there is a compensation from the new terms in Eq. (23), which change the behaviour close to the light cone and a new branch can exist. As the B-field increases the hole branch and its partner become less extended, and the particle branch approaches a massless mode. This is physically very reasonable since for very strong field strengths the thermal effects should disappear.

At this point it is worth discussing the approximations involved. The HTL effective action is derived under the assumption that the temperature is much larger than the external momenta. Here, the momentum is at least $\Pi \sim \sqrt{eB}$, which should be kept smaller than T. But, if we consider the coupling constant e to be very small this approximation should be valid even for $eB \sim \mathcal{M}^2$. On the other hand, when $\Pi \gg T$ the vacuum part is dominant and the dispersion relation is still approximately valid. The other approximation was to neglect the imaginary part. This approximation becomes worse for increasing eB and cannot be motivated for large eB. A new massless excitation, such as the new hole partner, could have important physical consequences only if its spectral weight is non-negligible. This remains to be studied and it can only be done correctly using the full spectral function.

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