

Chains of $N=2$, $D=4$ heterotic/type II duals

G. Aldazabal ^{*1}, A. Font ^{†2}, L.E. Ibáñez^{1,3} and F. Quevedo⁴

¹ Departamento de Física Teórica,
Universidad Autónoma de Madrid,
Cantoblanco, 28049 Madrid, Spain.

²Theory Group, Department of Physics,
The University of Texas,
Austin, TX 78712, USA.

³Department of Physics and Astronomy,
Rutgers University,
Piscataway, NJ 08855-0849, USA.

⁴ Theory Division, CERN, 1211 Geneva 23, Switzerland.

Abstract

We report on a search for $N = 2$ heterotic strings that are dual candidates of type II compactifications on Calabi-Yau threefolds described as $K3$ fibrations. We find many new heterotic duals by using standard orbifold techniques. The associated type II compactifications fall into chains in which the proposed duals are heterotic compactifications related one another by a sequential Higgs mechanism. This breaking in the heterotic side typically involves the sequence $SU(4) \rightarrow SU(3) \rightarrow SU(2) \rightarrow 0$, while in the type II side the weights of the complex hypersurfaces and the structure of the $K3$ quotient singularities also follow specific patterns. Some qualitative features of the relationship between each model and its dual can be understood by fiber-wise application of string-string duality.

*Permanent Institutions: CNEA, Centro Atómico Bariloche, 8400 S.C. de Bariloche, and CONICET, Argentina.

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1 Introduction

In the last few months, evidence has been found in favor of a strong-weak coupling duality between type II strings compactified on certain Calabi-Yau (CY) manifolds and certain $N = 2$ heterotic strings in four dimensions [1, 2, 3, 4, 5, 6, 7]. Highly non-trivial perturbative and non-perturbative checks have been performed for a few pairs of duals in which the type IIA strings are compactified on CYs with small number of Kähler deformations.

In spite of this recent progress, at the moment there is no general construction that produces the heterotic dual of a given type II compactification. In fact, the general idea [1] is not that any model has a dual description but rather that there are some string models that admit both type II and heterotic dual realizations whereas there will be models that admit only one or the other (or none). Of course, finding examples with both type II and heterotic dual interpretations is extremely interesting because using both descriptions simultaneously allows to extract non-perturbative information about the relevant $N = 2$ theory considered. In this context, it has been pointed out [3] that CY compactifications corresponding to $K3$ fibrations seem to play an important rôle in heterotic/type II duality.

Our understanding of $D = 4$, $N = 2$ type II/heterotic duality is also quite incomplete regarding the issue of how the spaces of models are connected. Indeed, it has been proposed [1] that stringy gauge symmetry enhancement may provide a way of continuously connecting many (or all) $N = 2$ heterotic vacua. It would be interesting to see explicitly in more detail how such a ‘web’ of heterotic vacua is actually formed. Obviously, there is also the question of how this web in the heterotic side maps into the type II side, yielding compactifications connected somehow. In fact, it is known that CY spaces are connected along paths where conifold singularities develop [8]. In the type IIB theory these singularities appear in the moduli space of vector multiplets and the transition to a different CY can be explained in terms of blackhole condensation [9, 10]. We would like to know how these transitions translate into the heterotic side. Moreover, we would also need to understand the problem of singularities in the moduli space of hypermultiplets [11].

It seems clear that in order to address some of the above issues, as well as to extend this duality to the $N = 1$ case, a better understanding of the space of $N = 2$ heterotic compactifications is needed. In comparison, the CY threefolds in the type II side are much better known and, due to its application to $N = 1$ heterotic compactifications, there exist long lists of models with different topological data.

Such a systematic study in the heterotic $N = 2$ case is lacking.

In the present note we begin a systematic exploration of $N = 2$ heterotic models. Most of our examples are obtained by compactifying the heterotic string on symmetric orbifolds $T^4/Z_M \times T^2$ and simultaneously embedding the Z_M symmetry in the gauge degrees of freedom. Besides the fact that the orbifold conformal field theory and partition function are well known, there is perhaps another naive motivation to use this kind of compactification towards constructing pairs of heterotic/type II duals. This duality is supposed to have its roots on an underlying string-string duality [12] between the type IIA string compactified on $K3$ and the heterotic string compactified on a 4-torus. Considering $K3$ fibrations on the type II side and using string-string duality fiber-wise, it has been argued [5] that on the heterotic side the $K3$ fibers should be replaced by T^4 fibers.

Starting with an specific $N = 2$ heterotic parent we derive chains of descendant models obtained by appropriate Higgsing, both using hypermultiplets and vector multiplets. In many examples we find that the last four elements of the chains, typically involving the sequential gauge breaking $SU(4) \rightarrow SU(3) \rightarrow SU(2) \rightarrow 0$, have candidate type II duals that appear in the lists of $K3$ fibrations in ref. [3]. Moreover, the weights of the corresponding weighted projective spaces follow specific sequential patterns. These patterns also reflect in a certain structure of the quotient singularities of the $K3$ fiber. For example, to the chain of heterotic breakings $SU(3) \rightarrow SU(2) \rightarrow 0$ there corresponds $K3$ singularities of type $A_i \rightarrow A_{i-1} \rightarrow A_{i-2}$. We also find that many of the $K3$ fibrations listed in ref. [3] can be put into similar sequences, although heterotic duals for all of them are not yet available. These sequences of CY models presumably correspond to manifolds connected through some sort of transition in the type II language.

This note is organized as follows. In section 2 we briefly review orbifold compactifications. In section 3 we construct chains of heterotic models and conjecture their type II duals. In section 4 we discuss the structure of the chains. Finally, in section 5 we present our conclusions and outlook.

2 Constructing $N = 2$ heterotic models

There are many possibilities available to build $N = 2$, $D = 4$ heterotic models with different gauge groups. They fall essentially into two classes: left-right symmetric and asymmetric. Examples of the first class are obtained by compactifying on $K3 \times T^2$ and simultaneously embedding the spin connection into the gauge degrees of

freedom in a modular invariant manner. There are many possible ways to construct the $K3$ and also many possible modular invariant gauge embeddings. An alternative to following the ‘Calabi-Yau’ approach of ref. [1], is to consider exact conformal field theory (CFT) constructions. During the past years several formalisms have been developed to construct $N = 1$ heterotic CFT’s based on free or coset theories (for a collection of some relevant papers see ref. [13]). Extending these techniques to $N = 2$, myriads of left-right symmetric models can be constructed for which the CFT is known and therefore the couplings can be explicitly computed.

A simple start is provided by symmetric toroidal orbifold compactifications on $T^4/Z_M \times T^2$. Acting on the (complex) bosonic transverse coordinates, the Z_M twist θ has eigenvalues $e^{2\pi i v_a}$, where v_a are the components of $v = (0, 0, \frac{1}{M}, -\frac{1}{M})$. Unbroken $N = 2$ SUSY requires $M = 2, 3, 4, 6$ [14]. The embedding of θ on the gauge degrees of freedom is usually realized by a shift V such that MV belongs to the $E_8 \times E_8$ or $Spin(32)/Z_2$ lattice. This shift is restricted by the modular invariant constraint $M(V^2 - v^2) = \text{even}$. All possible embeddings for each of the four allowed orbifolds can be easily found.

In the $E_8 \times E_8$ case, we find 2 inequivalent embeddings for Z_2 , 5 for Z_3 , 12 for Z_4 and 59 for Z_6 , leading to different patterns of $E_8 \times E_8$ symmetry breaking to rank-16 subgroups¹. A similar analysis can be carried out in the $Spin(32)/Z_2$ case. Each of these models is only the starting point for a big class of models generated by adding Wilson lines in the form of further shifts in the gauge lattice satisfying extra modular invariant constraints, permutations of gauge factors, etc.. The possibility of enhanced symmetry groups, at special points in the six-torus moduli space, can also be considered. We will not analyze those generalizations here. We will also restrict to left-right symmetric twists.

To find the spectrum for each model, we can easily adapt the analysis of the $N = 1$ compactifications [17] to the $N = 2$ case. There are M sectors twisted by θ^n , $n = 0, 1, \dots, M - 1$. Each particle state is created by a product of left and right vertex operators $L \otimes R$. At a generic point in the six-torus moduli space, the massless states follow from

$$m_R^2 = N_R + \frac{1}{2}(r + n v)^2 + E_n - \frac{1}{2} \quad ; \quad m_L^2 = N_L + \frac{1}{2}(P + n V)^2 + E_n - 1 \quad (1)$$

Here r is an $SO(8)$ weight with $\sum_{i=1}^4 r_i = \text{odd}$ and P a gauge lattice vector with $\sum_{I=1}^{16} P^I = \text{even}$. E_n is the twisted oscillator contribution to the zero point energy and it is given by $E_n = n(M - n)/M^2$.

¹The inequivalent Z_4 and Z_6 shifts for a single E_8 have been classified in refs. [15, 16]. Those results can be used to find the combinations satisfying modular invariance in the $N = 2$ case.

The multiplicity of states satisfying eq. (1) in a θ^n sector is given by:

$$D(\theta^n) = \frac{1}{M} \sum_{m=0}^{M-1} \chi(\theta^n, \theta^m) \Delta(n, m) \quad (2)$$

where

$$\Delta(n, m) = \exp\{2\pi i[(r + nv) \cdot mv - (P + nV) \cdot mV + \frac{1}{2}mn(V^2 - v^2) + m\rho]\} \quad (3)$$

In the above $\chi(\theta^n, \theta^m)$ is a numerical factor that takes into account the fixed point degeneracy. More precisely, $\chi(1, \theta^m) = 1$, implying that for the untwisted sector $D(1)$ only projects out the toroidal states which are not invariant under the twist. Otherwise, $\chi(\theta^n, \theta^m)$ is the number of simultaneous fixed points of θ^n and θ^m . Recall that θ acts on T^4 and always leaves fixed points, unlike the $N = 1$ case where there are some twists with fixed tori. The factor ρ only appears in the case of oscillator states ($N_L \neq 0$), the phase $e^{2\pi i\rho}$ indicates how the corresponding oscillator is rotated by θ .

Let us now consider the spectrum in the different sectors. In the untwisted sector, the right-moving massless states have $N_R = 0$ and $r^2 = 1$, the left-moving states come either from $N_L = 1$ or $P^2 = 2$. The $L \otimes R$ invariant states are selected by the condition $P \cdot V - r \cdot v - \rho = \text{int}$, with $\rho = 0, 1/M, -1/M$. The left-moving oscillators from the spacetime coordinates combine with the right-moving states with $r \cdot v = 0$ to generate the $N = 2$ supergravity multiplet, including the graviphoton and the vector-dilaton, giving rise to a model independent $U(1)^2$. The same right-moving states combined with the left-moving oscillators corresponding to the unrotated internal complex coordinate, generate two extra $U(1)$ vector multiplets. This $U(1)^2$ symmetry can be enhanced to a non-Abelian group for particular values of the moduli of the unrotated 2-torus. The model dependent non-Abelian vector multiplets are given by the $r \cdot v = 0$ states combined with the gauge lattice vectors satisfying $P \cdot V = \text{int}$ and with gauge oscillators. The states with $P \cdot V = 1/M + \text{int}$ combine with $r \cdot v = 1/M$ to generate charged matter hypermultiplets. The corresponding antiparticles come from $r \cdot v = -1/M$. Finally, the moduli hypermultiplets come from the same right-moving states combined with the oscillators corresponding to the two rotated coordinates ($\rho = \pm 1/M$). These are two model independent gauge singlets (they happen to be enhanced to four singlets for the Z_2 orbifold.)

The twisted sectors contain only matter hypermultiplets. For instance, the right-moving sector of the θ -twisted sector is always given by the two vector plus two spinorial weights of $SO(8)$ satisfying $r^2 = 1$, $r \cdot v = -1/M$. They correspond to the degrees of freedom of an $N = 2$ hypermultiplet whose gauge quantum numbers,

coming from the left-moving sector, depend on the model. The antiparticles of an n -twisted sector arise in the $(M - n)$ -twisted sector. For sectors of order two ($n = M/2$), particles and antiparticles are in the same sector, thus in order not to overcount the states the corresponding degeneracy is half of what is obtained in equation (2).

In order to be more concrete, we now discuss the spectrum of a particular model. We will consider the Z_4 orbifold with standard embedding $V = \frac{1}{4}(1, -1, 0, \dots, 0)$ in one single E_8 . The gauge group is given by the roots P satisfying $P \cdot V = 0$, breaking the group to $E_7 \times U(1) \times E_8 \times U(1)^4$. The untwisted matter is given by the roots satisfying $P \cdot V = 1/4$ leading to an $N = 2$ hypermultiplet transforming under $E_7 \times U(1)$ as a $(56, 1)$ ². Since for these states $P \cdot V - r \cdot v = 0$, they survive the orbifold projection. To these states we have to add the two model independent singlets $2(1, 0)$ mentioned above.

The sector twisted by θ has several states depending on the value of N_L . For $N_L = 0$ the massless states are given by the E_8 weights satisfying $(P + V)^2 = 13/8$ that produce a $(56, -1/2)$ representation. The degeneracy factor is 4, since the number of fixed points of a Z_4 twist in T^4 is 4, i.e. $\chi(\theta, \theta^m) = 4$. For $N_L = 1/4$, the weights satisfy $(P + V)^2 = 9/8$ and there are two left-moving oscillator states, corresponding to the two rotated coordinates, with $\rho = -1/4$. Again equation (2) gives an overall multiplicity number of 4 leaving then a total of 8 singlets $(1, -3/2)$. For $N_L = 3/4$ there are a total of 6 oscillators with $\rho = -3/4$ and $D(\theta) = 4$ for the solution of $(P + V)^2 = 1/8$, implying $24(1, 1/2)$.

For the θ^2 sector, the multiplicity factor takes the form $D(\theta^2) = \frac{1}{2}(4 + \Delta(2, 1))(1 + \Delta(2, 1)^2)$. For $N_L = 0$ there are 5 $(56, 0)$ from solving $(P + 2V)^2 = 3/2$ which has $\Delta(2, 1) = 1$. For $N_L = 1/2$ there are two solutions of $(P + 2V)^2 = 1/2$ for which $(P + 2V) \cdot V = \pm 1/4$. Each combines with two pairs of oscillators with $\rho = \pm 1/4$, giving $\Delta(2, 1) = \pm 1$ and $D = 5, 3$ and altogether making 16 copies of $(1, 1) + (1, -1)$.

The total matter spectrum is

$$\begin{aligned}
\theta^0 & : (56, 1) + 2(1, 0) \\
\theta^1 & : 4[(56, -\frac{1}{2}) + 2(1, -\frac{3}{2}) + 6(1, \frac{1}{2})] \\
\theta^2 & : 5(56, 0) + 16[(1, -1) + (1, 1)]
\end{aligned} \tag{4}$$

Notice that in total there are 66 singlets of E_7 as expected from the $K3$ moduli

² The $U(1)$ charge is computed by the scalar product $(P + nV) \cdot Q$, where $Q = (1, -1, 0, \dots, 0) \times (0, \dots, 0)$ is the (unnormalized) $U(1)$ generator. All matter is singlet under the unbroken $E_8 \times U(1)^4$ group.

space (since one is used to break $U(1)$). Notice also that we have to be very careful in finding the number of gauge singlets (which are neglected in many discussions) since they play a crucial role in the identification of heterotic/type II dual pairs as we will see next.

3 Chains of heterotic duals

In the construction of heterotic $N = 2$ models, our main interest will be in finding examples whose number of hypermultiplets (n_H) and vector multiplets (n_V) matches the number of such multiplets in a type IIA compactification on a CY threefold with Hodge numbers b_{11} and b_{21} . Since the dilaton lives in a vector multiplet in the heterotic side, but in a hypermultiplet in the type IIA side, it must be that $(n_H, n_V) = (b_{21} + 1, b_{11} + 1)$. Furthermore, due to its moduli structure, it is expected that the appropriate CY manifolds for type II compactification should be understandable as $K3$ fibers on \mathbb{P}_1 [3]. The conjectured underlying string-string duality in six dimensions supports this interpretation [5]. Indeed, all examples of heterotic/type II dual pairs analyzed up to now do correspond to CYs that are $K3$ fibrations. Two lists of such manifolds were provided in ref. [3]. The first list includes 31 simple hypersurfaces in weighted \mathbb{P}_4 in which the associated $K3$ fibers are a subset of the 95 $K3$ transversal families classified in ref. [18]. The second list gives 25 additional $K3$ fibrations which are complete intersection CY spaces in weighted \mathbb{P}_5 . As a first exercise we will try to find $N = 2$ heterotic compactifications that match the spectra of type IIA theories compactified on those CYs.

Let us now describe our strategy. We will start with an specific heterotic $N = 2$ orbifold and use the hypermultiplets to break the gauge symmetry by the Higgs mechanism step by step, decreasing the rank of the group in one unit in each step (more complicated possibilities will be mentioned below). In principle, care must be taken not to spoil the $N = 2$ symmetry by giving vevs along non-flat directions. However, that is specially easy in an $N = 2$ theory. Looking at it from the $N = 1$ point of view, it is enough to impose the usual $N = 1$ D-flatness conditions and in addition an F-flatness condition coming from the $N = 1$ Yukawa coupling between the adjoint chiral field inside the $N = 2$ vector multiplet and the chiral fields inside the hypermultiplets. Both conditions together may be seen as D-flatness conditions of the $N = 2$ theory. Flat directions respecting $N = 2$ do in general exist. After each step of symmetry breaking we give generic vevs to the adjoint scalars in the (unbroken) vector multiplets. This has two effects, namely it gives masses to any

hypermultiplet charged with respect to the rank-reduced group, and it breaks the rank-reduced gauge group to its maximal Abelian subgroup. The final result will be a model with gauge group $U(1)^n$, with $2 \leq n \leq 20$. As explained in the previous section, the lower limit comes from the multiplets containing the graviphoton and the dilaton, whereas the upper limit is the maximal rank achievable for generic values of the T^4/Z_M or $K3$ moduli. Notice that the breaking of the gauge group down to its Abelian subgroup is necessary to match the type II side whose (perturbative) gauge group is just $U(1)^{b_{11}+1}$. Furthermore, the hypermultiplets must be neutral with respect to the $U(1)$ s, which is guaranteed by our construction in the heterotic side.

We have performed a systematic search of chains of $N = 2$ models following the above procedure and starting mostly with orbifold compactifications. We will spare the reader the details of all these models and show the most relevant heterotic examples found up to now in our search. We hope to report more complete results in a future publication. As we said above, we have obtained several new heterotic models that match the lists of $K3$ fibrations in ref. [3]. One of the most interesting results is the existence of five chains, each of four CY spaces, whose conjectured duals are given by heterotic $N = 2$ models in which a ‘cascade gauge symmetry breaking’ takes place. Each such chain has very much the same structure. For instance, the four models in the chain have a gauge group (before adjoint Higgsing) of the form $SU(m) \times G_r$, with $m = 4, 3, 2, 0$. G_r is an additional gauge factor of rank $r = 12, 10, 8, 4$ and 11 for each of the five chains respectively.

We now describe each chain labelled by the value of r .

1) $r = 12$ chain

This chain may be obtained by appropriate Higgsing of the Z_2 orbifold with standard embedding and gauge group $E_7 \times SU(2) \times E_8 \times (U(1)^4)$. The hypermultiplets, transforming only under $E_7 \times SU(2)$, are

$$\begin{aligned} \theta^0 & : (56, 2) + 4(1, 1) \\ \theta^1 & : 8[(56, 1) + 4(1, 2)] \end{aligned} \tag{5}$$

Higgsing away the $SU(2)$ we are left with 65 singlet hypermultiplets. Now we give a vev to the adjoint Higgses inside $E_7 \times E_8$ and break it down to $U(1)^{15}$, while all 56-plets get a mass.

We are left altogether with 65 singlet hypermultiplets and 19 $U(1)$ s, i.e. a model of type $(n_H, n_V) = (65, 19)$ in the notation of [1]. In fact, this is nothing but the first of a series of models with $(n_H, n_V) = (65, 19), (84, 18), (101, 17), (116, 16)$

already constructed by Kachru and Vafa. They are obtained by a ‘cascade breaking’ $E_7 \rightarrow E_6 \rightarrow SO(10) \rightarrow SU(5)$. None of these models matches the Hodge numbers given in the tables in ref. [3]. However, the interesting results are obtained by continuing the breaking through $SU(5) \rightarrow SU(4) \rightarrow SU(3) \rightarrow SU(2) \rightarrow 0$. In this case it can easily be checked that a chain of models with $(n_H, n_V) = (167, 15), (230, 14), (319, 13), (492, 12)$ are generated. All of them have dual candidates in the lists in ref. [3].

2) $r = 10$ chain

The starting point is one of the four possible non-standard $E_8 \times E_8$ embeddings of the Z_3 orbifold, with gauge shift $V = \frac{1}{3}(1, -1, 0, \dots, 0) \times \frac{1}{3}(1, 1, -2, 0, \dots, 0)$. The gauge group turns out to be $E_6 \times SU(3) \times E_7 \times U(1) \times (U(1)^4)$. The massless hypermultiplet spectrum is easily found along the lines discussed in the previous section. Explicitly,

$$\begin{aligned} \theta^0 & : (27, 3; 1, 0) + (1, 1; 56, 1) + (1, 1; 1, -2) + 2(1, 1; 1, 0) \\ \theta^1 & : 9[(27, 1; 1, \frac{2}{3}) + (1, \bar{3}; 1, -\frac{4}{3}) + 2(1, \bar{3}; 1, \frac{2}{3})] \end{aligned} \quad (6)$$

We now Higgs the group $E_7 \times SU(3) \times U(1)$ as much as possible. The two last factors can be Higgsed away completely whereas the E_7 can only be broken to a subgroup of rank 6 (e.g., E_6) since there is only one 56 available for Higgsing. Now, giving a generic vev to the adjoint of this rank-6 group we are just left with an unbroken $E_6 \times U(1)^6 \times (U(1)^4)$ group with 12 (27) + 76 (1) hypermultiplets. We now proceed as in the previous chain by sequential Higgsing $E_6 \rightarrow SO(10) \rightarrow SU(5) \rightarrow \dots \rightarrow SU(3) \rightarrow SU(2) \rightarrow 0$. In this way we obtain models with $(n_H, n_V) = (76, 16), (87, 15), (96, 14), (129, 13), (168, 12), (221, 11), (322, 10)$. Again, the last four steps, corresponding to the sequential breaking $SU(5) \rightarrow SU(4) \rightarrow \dots \rightarrow 0$ have counterparts in the lists in ref. [3].

3) $r = 8$ chain

We consider the $E_8 \times E_8$ embeddings of the Z_4 orbifold with gauge shift $V = \frac{1}{4}(1, 1, 1, -3, 0, \dots, 0) \times \frac{1}{4}(1, 1, -2, 0, \dots, 0)$. The gauge group in this example is $SO(10) \times SU(4) \times E_6 \times SU(2) \times U(1) \times (U(1)^4)$. The massless hypermultiplets content is given by

$$\begin{aligned} \theta^0 & : (16, 4; 1, 1, 0) + (1, 1; 27, 2, 1) + (1, 1; 1, 2, -3) + 2(1, 1; 1, 1, 0) \\ \theta^1 & : 4[(16, 1; 1, 1, \frac{3}{2}) + (1, \bar{4}; 1, 2, -\frac{3}{2}) + 2(1, \bar{4}; 1, 1, \frac{3}{2})] \\ \theta^2 & : 5(10, 1; 1, 2, 0) + 3(1, 6; 1, 2, 0) \end{aligned} \quad (7)$$

This model has already an $SU(4)$ group at the start, so that a possibility would be

to Higgs away as far as possible the rest of the gauge group and then start breaking $SU(4)$ step by step. How far down can one break the rest of the group? It is obvious that the $SO(10)$ group can be broken completely since there are enough 16-plets and 10-plets to do the job.

On the other hand, we cannot Higgs away completely the E_6 factor, since there are only 2 (27)s. After examining the possible Higgsings, we conclude that the maximal breaking is $E_6 \times SU(2) \times U(1) \rightarrow SO(8)$. Altogether, the model before starting cascade breaking has gauge group $SU(4) \times SO(8) \times (U(1)^4)$ and has the following hypermultiplet content : $32(4) + 6(6) + 123(1)$. Giving generic vevs to the adjoints and proceeding by cascade symmetry breaking leads to the following models: $(n_H, n_V) = (123, 11), (154, 10), (195, 9), (272, 8)$. Again, these four models admit a $K3$ fibration interpretation in the type II side.

4) $r = 4$ chain

This chain can be obtained from the Z_6 orbifold with a $E_8 \times E_8$ embedding given by $V = \frac{1}{6}(1, 1, 1, 1, -4, 0, 0, 0) \times \frac{1}{6}(1, 1, 1, 1, 1, -5, 0, 0)$. The resulting model has gauge group $SU(5) \times SU(4) \times U(1) \times SU(6) \times SU(3) \times SU(2)$ and massless hypermultiplets

$$\begin{aligned}
\theta^0 & : (1, \bar{4}, -5; 1, 1, 1) + (10, 4, 1; 1, 1, 1) + (1, 1, 0; 6, 3, 2) + 2(1, 1, 0; 1, 1, 1) \\
\theta^1 & : (1, 1, \frac{10}{3}; 1, 3, 2) + (1, 4, -\frac{5}{3}; 6, 1, 1) + 2(1, 1, \frac{10}{3}; 6, 1, 1) \\
\theta^2 & : 5(1, \bar{4}, \frac{5}{3}; 1, \bar{3}, 1) + 4(\bar{5}, 1, -\frac{4}{3}; 1, \bar{3}, 1) \\
\theta^3 & : 3(1, 6, 0; 1, 1, 2) + 5(5, 1, -2; 1, 1, 2)
\end{aligned} \tag{8}$$

At this point we can Higgs away most of the symmetry to arrive at $SU(5) \times (U(1)^4)$ with essentially $4(10) + 22(5) + 118(1)$ hypermultiplets. Giving vevs to adjoint scalars and implementing cascade breaking leads to models with $(n_H, n_V) = (118, 8), (139, 7), (162, 6), (191, 5), (244, 4)$. The four last elements fall into the $K3$ fibrations classes of ref. [3].

Notice that the last model in this chain is identical to the rank four example discussed in detail in ref. [1]. In that reference the heterotic dual was obtained from compactification on $K3 \times T^2$ with a rank two bundle embedded in each E_8 . The existence of this chain suggests that this well studied model could also be continuously connected to the three CY compactifications with $(b_{21}, b_{11}) = (190, 4), (161, 5)$ and $(138, 6)$.

5) $r = 11$ chain

A simple way to construct this chain is to begin with example 7 in ref. [1] in which the $E_8 \times E_8$ heterotic string is compactified on $K3 \times T^2$ and the $U(1)^4$ generic symmetry coming from T^2 is enhanced to $SU(2) \times U(1)^3$ by choosing a modulus value $T = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. In addition, $SU(2)$ bundles are embedded in the first E_8 and in the enhanced $SU(2)$. The gauge group at this level is $E_7 \times E_8 \times U(1)^3$ with $8(56) + 65(1)$ hypermultiplets, as follows from the index theorem. Higgsing step by step we find the chain $(n_H, n_V) = (62, 18), (77, 17), (90, 16), (101, 15), (140, 14), (187, 13), (252, 12), (377, 11)$. The last four models again correspond to $K3$ fibrations in [3].

The five chains of models are collected in Table 1. To give an idea of the starting structure, the full gauge group before turning on generic vevs for the Cartan subalgebra is shown for each model as constructed above. The actual gauge group is purely Abelian. The CY threefolds ($K3$ fibrations) that match the numbers (n_H, n_V) are labelled by their weights in projective space. When the CYs are given by a hypersurface in $\mathbb{P}_4(1, 1, 2k_2, 2k_3, 2k_4)$, the $K3$ fiber is given by a hypersurface in $\mathbb{P}_3(1, k_2, k_3, k_4)$ [3]. In these cases, we have also recorded the polynomials of the $K3$ fibers and their quotient singularities.

The weights $(1, 1, 2k_2, 2k_3, \dots)$ of the CY compactifications follow an interesting pattern that repeats in all chains. The first element always correspond to one of the simple complete intersection CY spaces in weighted \mathbb{P}_5 given in ref. [3]. The remaining elements are simple hypersurfaces in weighted \mathbb{P}_4 . The second element is obtained by just deleting the last variable. The third member of the chain is obtained from the second by the replacement $(1, 1, 2k_2, 2k_3, 2k_4) \rightarrow (1, 1, 2k_2, 2k_3, 2k_4 + 2k_2)$. The fourth element is obtained by shifting the weights of the third element as $(1, 1, 2k_2, 2k_3, 2k_4) \rightarrow (1, 1, 2k_2, 2k_3 + 2k_2, 2k_4 + 2k_2)$. In the next section we will discuss how the grouping of each element in different chains is associated to families of quotient singularities in the $K3$ fibers.

It is natural to ask whether there are more CYs in the lists of ref. [3] whose weights follow the above pattern. Indeed, there are five additional candidate chains of just three elements whose weights are related as those of the first three elements in the other five chains. The elements of these chains are displayed in Table 2. For each chain there are two possible choices for the first element, the different b_{21} and last weight are indicated inside brackets. We have identified heterotic candidate duals for some (but not all) of them but we will spare the reader their construction. Finally, in addition to these 10 chains, there are pairs of models that also seem to be connected, one of the members is a complete intersection CY in \mathbb{P}_5 and the

other is a hypersurface in \mathbb{P}_4 . These pairs include the models with $(b_{21}, b_{11}) = (96, 12), (131, 11); (76[84], 10), (111, 9); (75[81], 9), (104, 8); (70[82], 6), (101, 5); (69, 3), (86, 2)$. Again, in some cases there are two possible choices for the first element as indicated by the different b_{21} inside brackets. From the CY models listed in the two tables of ref. [3], only those with $(b_{21}, b_{11}) = (143, 7), (68, 2)$ and $(76, 8)$ do not seem to fall into any of these chains.

Some comments are in order:

1) The various chains of models are not isolated from each other but rather seem to form a *web of heterotic $N = 2$ models* connected by different paths involving different directions of gauge symmetry breaking. Let us show as an example how the heterotic models with $(n_H, n_V) = (167, 15)$ and $(272, 8)$ (which belong to different chains) can be connected. We can start with an $SO(32)$ string compactified on any orbifold with standard embedding. This yields generically a gauge group $SO(28) \times SU(2) \times U(1) \times (U(1)^4)$ and $10(28, 2) + 66(1, 1)$ hypermultiplets. Higgsing down to $SO(22)$ using the hypermultiplets and then turning on vevs to the unbroken Cartan subalgebra we recover the model $(167, 15)$ belonging to the first chain. If we keep on Higgsing as much as possible we arrive at model $(272, 8)$ of the third chain. Thus, the heterotic $N = 2$ duals form a web of theories connected by different paths of moduli space, as expected.

2) From the above comments it is clear that a given heterotic model can be obtained starting from compactifications that look very different from several points of view: different initial gauge group, different orbifolds, etc. Moreover, it must be pointed out that there is not a unique way to construct a full chain. For example, the $r = 12$ chain can be derived from any Z_M orbifold with standard embedding in $E_8 \times E_8$, or even, e.g., from Z_3 with a non-standard embedding. Also, the $r = 10$ chain can be obtained by first going to an enhanced $SU(3)$ symmetry point and then performing a Z_3 twist embedded partly in the enhanced $SU(3)$ and partly in one E_8 , leaving an unbroken $E_8 \times U(1)^2$, as in model 8 of [1]. In general there could be alternative starting models that yield the same chains but with different (same rank) G_r before adjoint Higgsing.

3) In the process of sequential gauge symmetry breaking leading to the five chains of models in Table 1, sometimes there may be bifurcations into other directions in Higgs space. For instance, there are examples in which cascade symmetry breaking can proceed through an alternative path containing the breakings $SU(4) \rightarrow SU(2) \times SU(2) \rightarrow SU(2) \rightarrow 0$, instead of proceeding through an $SU(3)$ intermediate step. This is for example the case of the second chain in Table 1. Proceeding through

$SU(2) \times SU(2)$ gives the model $(n_H, n_V) = (144, 12)$ instead of $(168, 12)$. The former corresponds to one of the models in Table 2, showing us another example of interconnection of different chains into a complicated web of heterotic models.

4) Not any cascade symmetry breaking of arbitrary $N = 2$ heterotic models leads to models with corresponding type II duals in the lists of [3]. It is not true either that any symmetry breaking chain ending by $SU(4) \rightarrow SU(3) \rightarrow \dots$, is going to give rise to heterotic models with type II duals, only some do. For example, we could have tried to construct the $r = 4$ chain by starting with the rank four example of [1]. This is an $E_7 \times E_7$ compactification with hypermultiplets $4(56, 1) + 4(1, 56) + 62(1, 1)$. Indeed, Higgsing completely the E_7^2 gives the last element of this chain with $(n_H, n_V) = (244, 4)$. However, trying to reproduce the preceding elements in the chain we find models $(215, 5)$, $(198, 6)$, $(183, 7)$, none of which have type II dual candidates in the lists of simple $K3$ fibrations.

5) There is the possibility that heterotic models, obtained in these symmetry-breaking chains, that do *not* have candidate duals in the lists of ref. [3] could correspond to a more general class of CY manifolds. It is thus sensible to look for candidate duals to the unmatched heterotic models in more general tables of CY spaces. We have done this check for several of the unpaired elements mentioned above and found that they sometimes (but not always) match CY compactifications classified in ref. [19]. In these cases, unlike in the chains reported in the present paper, the weights of the corresponding projective spaces do not seem to follow any obvious rule.

4 The structure of the chains of type II duals

At the moment we do not have a satisfactory understanding of which conditions heterotic models must fulfill in order to be dual to a type II compactified in one of the $K3$ fibrations listed in ref. [3]. It must be emphasized that those lists only include manifolds that are a simple generalization of the CYs with few moduli for which type II/heterotic duality has been tested, they are not supposed to be exhaustive compilations of $K3$ fibrations. Still, it would be interesting to understand the origin of all the properties of the dual chains of models described in this note.

It is natural to try to analyze our results in terms of the underlying 6-dimensional string-string duality [12] between type IIA compactifications on $K3$ and heterotic compactifications on T^4 . This duality maps the cohomology of $K3$ to the Narain lattice with signature $(20, 4)$. The idea is that if the type IIA theory is compactified

instead on a 6-dimensional Calabi-Yau which is a $K3$ fiber on \mathbb{P}_1 , the resulting $N = 2$ theory is expected to be dual to a heterotic compactified on a variety that looks like T^4 fibered over \mathbb{P}_1 . This would be the ‘adiabatic’ approximation suggested in ref. [5] in trying to explain the origin of type II/heterotic duality in four dimensions as a fiber-wise application of string-string duality. It turns out that there are certain singularities in the fibration that obstruct a direct application of this adiabatic argument. Nevertheless, the authors of ref. [5] were able to describe qualitatively certain features of type II/heterotic duality for the rank 3 and 4 examples of ref. [1]. We just briefly show here how these arguments generalize to our examples. The basic idea is to consider the monodromy of the cohomology of the $K3$ around the singularities in the fibration. String-string duality suggests that the sector of the $K3$ cohomology invariant under the monodromy is mapped to the invariant Narain lattice in the heterotic side.

To find the form of the $K3$ fibration, we first write the simplest transverse polynomial for the given CY in $\mathbb{P}_4(1, 1, 2k_2, 2k_3, 2k_4)$ and then set $X_0 = \lambda X_1$. After redefining $X_1 \rightarrow X_1^{1/2}$ we obtain an equation of the form

$$F(\lambda)X_1^d + X_2^{d/k_2} + \dots = 0 \tag{9}$$

where $d = 1 + k_2 + k_3 + k_4$ (for Fermat-type surfaces, such as those analyzed in [5], the mirror CY gives rise to the same $K3$ equation). For generic λ , this resulting equation describes a $K3$ in weighted \mathbb{P}_3 . The original CY is thus a $K3$ fibration over the \mathbb{P}_1 parametrized by λ . The fibration is singular when the coefficient of X_1 vanishes, otherwise it can be absorbed in X_1 . Near a simple zero we can write $F(\lambda) = \epsilon e^{i\theta}$ with $\epsilon \rightarrow 0$. To study the monodromy around this zero we make the transport $\theta \rightarrow \theta + 2\pi$. Now, since $F(\lambda)$ multiplies X_1^d , this is equivalent to the transformation $X_1 \rightarrow \zeta X_1$, where $\zeta = e^{2i\pi/d}$. In the following we will just consider $K3$ defining equations independent of λ and analyze the monodromy through the operation $X_1 \rightarrow \zeta X_1$.

The $K3$'s that appear in our models are defined by transverse polynomials in weighted \mathbb{P}_3 and have been studied in ref. [18]. The 20 (1,1) forms are the Kähler form, forms related to polynomial deformations of the hypersurface defining equation, and forms related to resolution of quotient singularities. In general, there are $(19 - s)$ polynomial deformations, where s is the total rank of the singularities due to fixed sets in the weighted \mathbb{P}_3 that intersect the hypersurface at isolated points. For example, in the model (492,12), there is a singular point at $(0, 0, -1, 1)$ associated with a Z_7 action. This singularity is of type A_6 , it is resolved by excising the point

and glueing in 6 copies of \mathbb{P}_1 that intersect according to the Cartan matrix of A_6 . This example has $s = 9$ since there are also A_1 and A_2 singular points at $(0, 1, -1, 0)$ and $(0, -1, 0, 1)$, associated with Z_2 and Z_3 actions. As another example, consider the model (162,6). In this case there is only an A_3 singularity at $(0, 0, 0, 1)$ associated to a Z_4 action.

Taking into account their dependence on X_1 , it is straightforward to show that all polynomial deformations transform under the monodromy. The Kähler form and the s $(1, 1)$ forms supported by the glued \mathbb{P}_1 s, are instead invariant. It can also be shown that the $(0, 2)$ and $(2, 0)$ forms are not invariant whereas the $(0, 0)$, and $(2, 2)$ forms are invariant [5]. Altogether, there are $s + 3$ invariant forms with signature $(s + 1, 2)$. This is expected to be mapped to a heterotic string with an $\Gamma(s + 1, 2)$ invariant lattice [5], in agreement with the structure of the heterotic compactifications considered. For instance, in the (492,12) model, the invariant Narain lattice is $\Gamma(10, 2)$, corresponding to the two unrotated left- and right-moving coordinates plus the eight-dimensional lattice of the unbroken gauge group.

The above discussion implies that the quotient singularities of the $K3$ are related to vector multiplets in the heterotic side and it can be seen as a complicated way of computing the number of unbroken $U(1)$ s as $n_V = s + 3$, or equivalently $b_{11} = s + 2$. However, there is more information regarding the structure of the quotient singularities. In Tables 1 and 2 these singularities are given when the CY is a hypersurface in \mathbb{P}_4 . It is evident that to each $SU(3) \rightarrow SU(2) \rightarrow 0$ heterotic cascade, there corresponds a ‘embedded’ singularity chain $A_i \rightarrow A_{i-1} \rightarrow A_{i-2}$ of the $K3$ in the type II side. This could be related to some physical process occurring in the type II string compactification.

5 Final comments and conclusions

We have identified a number of new candidates for $N = 2$, $D = 4$ type II/heterotic dual pairs. Besides the matching of the number of hypermultiplets and vector multiplets, we can claim further evidence from the fact that the heterotic duals come in symmetry-breaking chains that are mapped into type II compactifications on CY spaces that are $K3$ fibrations and also seem to be related to each other in a sequential manner.

This parallelism between transitions taking place on both the type II and the heterotic side is reminiscent of the ideas put forward in refs. [9, 10]. It is believed that the resolution of conifold singularities in type IIB CY compactifications, through the

appearance of massless blackhole hypermultiplets, is the dual of the Seiberg-Witten [20] mechanism in which massless monopoles appear at certain strong coupling points in the vector multiplet moduli space of the heterotic side. In the class of models described in this note the situation is not exactly the same. The transitions that we have in the heterotic side are very specific: they occur at *weak coupling* and correspond to sequential $SU(4) \rightarrow SU(3) \rightarrow SU(2) \rightarrow 0$ symmetry breaking through Higgsing of hypermultiplet scalars. Each $SU(n)$ group is then broken by Higgsing of vector multiplets. At points in the vector multiplet moduli space, where the non-Abelian groups are restored, they are very asymptotically non-free. It would be very interesting to understand the dual to these transitions in the type II side, probably in terms of new connections among different CY spaces.

There is an apparent lack of uniqueness in the mapping between heterotic and type II in the following sense. We can start with differently looking heterotic models before Higgsing (i.e., different Z_M orbifolds with different gauge embeddings) and obtain after Higgsing models that appear equally good candidates to be dual to a given type II model on a certain CY (i.e., same number of hypermultiplets and vector multiplets). Thus, these various heterotic constructions seem to have the same type II dual at strong coupling. An example of this is the first chain in Table 1 that may be obtained equally well from any of the Z_M orbifolds with standard embedding or even from non-standard embeddings. We could say that the dual loses information about at least some of the symmetries of the original heterotic model. This is perhaps not so surprising since in the Higgsing process we give vevs to hypermultiplets that carry e.g. Z_N charges, so that the original discrete symmetries of the orbifold are expected to be spontaneously broken.

Another point to remark is that the last elements of each heterotic chain are in some sense more generic (in the hypermultiplet and vector multiplet moduli) than the preceding elements in each chain. The gauge symmetry of these last elements in each chain cannot be further broken and we cannot continue increasing the number of massless hypermultiplets (and reducing the number of vector multiplets). In this sense, it is amusing to note that the heterotic $N = 2$ dual with the maximum number of massless hypermultiplets that can be constructed is the last element in the first chain with $(n_H, n_V) = (492, 12)$. This corresponds to the known CY compactification with maximum Euler characteristic $|\chi| = 960$, that was conjectured in ref. [19] to be the maximum achievable in CY compactifications.

There is also an interesting connection with cancellation of anomalies in $N = 1$ supergravity coupled to vector and hypermultiplets in 6 dimensions. Indeed, the

symmetric orbifold heterotic models described above can be thought of as compactifications taking place in a two step process, first to $D = 6$, $N = 1$ upon compactification on T^4/Z_M and then down to $D = 4$, $N = 2$ after further toroidal compactification. In $D = 6$ there are strong constraints, imposed by cancellation of gravitational anomalies, which require that the difference between the number of $N = 1$ hypermultiplets and vector multiplets be equal to 244 [21, 22, 23]³. After compactification on a generic T^2 , the massless states just arrange into the appropriate multiplets and the difference between the number of $N = 2$, $D = 4$ hypermultiplets and vector multiplets, without including the dilaton and toroidal $U(1)$ s, must still be equal to 244. This is certainly the case for all the symmetric orbifolds that we discussed. This argument applies only to the original (un-Higgsed) model, since in $D = 4$ there is not an analogous purely gravitational anomaly constraint. This explains why there are various heterotic $N = 2$ models leading upon Higgsing to $(n_H, n_V) = (244, 4)$. Any model in which it is possible to completely Higgs the rank 16 heterotic group is bound to yield 244 hypermultiplets due to the mentioned constraint.

Many questions concerning this class of new type II/heterotic dual pairs still remain. In particular, it would be interesting to fully understand the physical process occurring in the type II side corresponding to these heterotic chains. One would expect that the CY manifolds in a chain are somehow connected. Perhaps the best framework to address this question is that of toric geometry in which a criterion for singularity transitions can be formulated [24, 25]. Eventually, one would like to have a direct method to obtain dual pairs in a systematic way. We hope that the new examples and regularities discussed in this note will shed some light in these issues.

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Group	(n_H, n_V)	CY weights	$K3$ fiber	$K3$ Singularity
$SU(4) \times E_8 \times U(1)^4$	(167, 15)	(1, 1, 12, 16, 18, 20)		
$SU(3) \times E_8 \times U(1)^4$	(230, 14)	(1, 1, 12, 16, 18)	$X_1^{24} + X_2^4 + X_3^3 + X_4^2 X_2 = 0$	$A_1 + A_2 + A_8$
$SU(2) \times E_8 \times U(1)^4$	(319, 13)	(1, 1, 12, 16, 30)	$X_1^{30} + X_2^5 + X_3^3 X_2 + X_4^2 = 0$	$A_1 + A_2 + A_7$
$E_8 \times U(1)^4$	(492, 12)	(1, 1, 12, 28, 42)	$X_1^{42} + X_2^7 + X_3^3 + X_4^2 = 0$	$A_1 + A_2 + A_6$
$SU(4) \times E_6 \times U(1)^4$	(129, 13)	(1, 1, 6, 10, 12, 14)		
$SU(3) \times E_6 \times U(1)^4$	(168, 12)	(1, 1, 6, 10, 12)	$X_1^{15} + X_2^5 + X_3^3 + X_4^2 X_2 = 0$	$2A_2 + A_5$
$SU(2) \times E_6 \times U(1)^4$	(221, 11)	(1, 1, 6, 10, 18)	$X_1^{18} + X_2^6 + X_3^3 X_2 + X_4^2 = 0$	$2A_2 + A_4$
$E_6 \times U(1)^4$	(322, 10)	(1, 1, 6, 16, 24)	$X_1^{24} + X_2^8 + X_3^3 + X_4^2 = 0$	$2A_2 + A_3$
$SU(4) \times SO(8) \times U(1)^4$	(123, 11)	(1, 1, 4, 8, 10, 12)		
$SU(3) \times SO(8) \times U(1)^4$	(154, 10)	(1, 1, 4, 8, 10)	$X_1^{12} + X_2^6 + X_3^3 + X_4^2 X_2 = 0$	$3A_1 + A_4$
$SU(2) \times SO(8) \times U(1)^4$	(195, 9)	(1, 1, 4, 8, 14)	$X_1^{14} + X_2^7 + X_3^3 X_2 + X_4^2 = 0$	$3A_1 + A_3$
$SO(8) \times U(1)^4$	(272, 8)	(1, 1, 4, 12, 18)	$X_1^{18} + X_2^9 + X_3^3 + X_4^2 = 0$	$3A_1 + A_2$
$SU(4) \times U(1)^4$	(139, 7)	(1, 1, 2, 6, 8, 10)		
$SU(3) \times U(1)^4$	(162, 6)	(1, 1, 2, 6, 8)	$X_1^9 + X_2^9 + X_3^3 + X_4^2 X_2 = 0$	A_3
$SU(2) \times U(1)^4$	(191, 5)	(1, 1, 2, 6, 10)	$X_1^{10} + X_2^{10} + X_3^3 X_2 + X_4^2 = 0$	A_2
$U(1)^4$	(244, 4)	(1, 1, 2, 8, 12)	$X_1^{12} + X_2^{12} + X_3^3 + X_4^2 = 0$	A_1
$SU(4) \times E_8 \times U(1)^3$	(140, 14)	(1, 1, 8, 12, 14, 16)		
$SU(3) \times E_8 \times U(1)^3$	(187, 13)	(1, 1, 8, 12, 14)	$X_1^{18} + X_2^3 + X_3^3 X_2 + X_4^2 X_3 = 0$	$A_1 + A_3 + A_6$
$SU(2) \times E_8 \times U(1)^3$	(252, 12)	(1, 1, 8, 12, 22)	$X_1^{22} + X_2^2 + X_3^4 X_4 + X_4^3 X_3 = 0$	$A_1 + A_3 + A_5$
$E_8 \times U(1)^3$	(377, 11)	(1, 1, 8, 20, 30)	$X_1^{30} + X_2^3 + X_3^5 X_2 + X_4^2 = 0$	$A_1 + A_3 + A_4$

Table 1: Chains of type IIA/heterotic duals denoted by (n_H, n_V) . The corresponding $K3$ fibers and their singularities are also shown for the examples which are hypersurfaces in \mathbb{P}_4 .

(b_{21}, b_{11})	CY weights	$K3$ singularity
(117[121], 13)	(1, 1, 8, 10, 12, 12[14])	
(164, 12)	(1, 1, 8, 10, 12)	$A_1 + A_4 + A_5$
(227, 11)	(1, 1, 8, 10, 20)	$A_1 + 2A_4$
(104[108], 12)	(1, 1, 6, 8, 10, 10[12])	
(143, 11)	(1, 1, 6, 8, 10)	$A_2 + A_3 + A_4$
(194, 10)	(1, 1, 6, 8, 16)	$A_2 + 2A_3$
(94[98], 10)	(1, 1, 4, 6, 8, 8[10])	
(125, 9)	(1, 1, 4, 6, 8)	$2A_1 + A_2 + A_3$
(164, 8)	(1, 1, 4, 6, 12)	$2A_1 + 2A_2$
(98[102], 6)	(1, 1, 2, 4, 6, 6)	
(121, 5)	(1, 1, 2, 4, 6)	$A_1 + A_2$
(148, 4)	(1, 1, 2, 4, 8)	$2A_1$
(76[84], 4)	(1, 1, 2, 2, 4, 4)	
(99, 3)	(1, 1, 2, 2, 4)	A_1
(128, 2)	(1, 1, 2, 2, 6)	0

Table 2: Additional candidate chains of $K3$ fibrations denoted by (b_{12}, b_{11}) .