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DUALITY SYMMETRIES IN N = 2 HETEROTIC SUPERSTRING*

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Abstract

We review the derivation and the basic properties of the perturbative prepotential in N=2 compactifications of the heterotic superstring. We discuss the structure of the perturbative monodromy group and the embedding of rigidly supersymmetric monodromies associated with enhanced gauge groups, at both perturbative and non-perturbative level.

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1. Overview

Duality transformations seem to play an important role in understanding the non-perturbative dynamics of supersymmetric Yang-Mills theories. Several exact results have been obtained mainly in the cases of $N = 4$ and $N = 2$ extended supersymmetry due to the relation of the BPS mass formula with the central extension of the supersymmetry algebra [1]. Duality has been established as a non-perturbative symmetry in the ultraviolet finite $N = 4$ theories where, in particular, their partition function has been calculated [2]. As a first step towards non-trivial four-dimensional theories, $N = 2$ supersymmetric Yang-Mills provide simple examples of theories having non vanishing β-function and exhibiting interesting properties such as asymptotic freedom. Every gauge boson is in the same multiplet with one complex scalar and a Dirac spinor. The vacuum of the theory is then infinitely degenerate and is characterized by the expectation values (VEVs) of the Higgs fields which break the non-abelian group down to its maximal abelian subgroup. The effective two-derivative action describing the interactions of the massless abelian vector multiplets is completely determined by its analytic prepotential which was computed exactly [3]. It turns out that although perturbatively there is a non-abelian symmetry at the point where the Higgs expectation value vanishes giving rise as usually to a confining phase, the exact theory is always in the Higgs phase. Moreover, there are points in the moduli space of the Higgs expectation values where non-perturbative solitonic excitations corresponding to monopoles and dyons become massless.

Recently, there has been considerable progress in extending these results to supergravity in the context of superstring theory. The additional important ingredient is the conjecture of string-string duality according to which the heterotic and type II superstring is the same theory at the non-perturbative level [4]. Moreover for $N = 2$ supersymmetric compactifications the dilaton, S, whose expectation value plays the role of the string coupling constant, belongs to a hypermultiplet in type II string while in the heterotic case it belongs to a vector multiplet. Using the fact that vector multiplets and neutral hypermultiplets do not couple to each other in the low energy theory, this duality provides a very powerful method for extracting non-perturbative physics of one model from the perturbative computations in the dual model and vice versa [6].

Consider for instance the ten-dimensional heterotic $E_8 \otimes E_8$ superstring compactified on the six-dimensional manifold $\mathbf{T^2} \times \mathbf{K_3}$ with spin connection identified with the gauge connection which gives in four dimensions an $N = 2$ supersymmetry and a gauge group $E_7 \otimes E_8 \otimes U(1)^{2+2}$. One factor of $U(1)^2$ is associated to the universal dilaton vector multiplet and the graviphoton, while the second $U(1)^2$ is associated to the two-dimensional torus $\mathbf{T^2}$. At a generic point of the vector moduli space the gauge group is broken to $U(1)^{19}$ and there are no charged massless hypermultiplets. However there are special points where massless charged hypermultiplets appear and their VEVs reduce the rank of the gauge group. On the other hand in the moduli space of hypermultiplets, there are special points where additional vector multiplets become massless leading to an increase in the rank. As a result one can get any rank r starting from $r = 2$ up to a maximum of $2 + 22$ corresponding to the simple free fermionic constructions.

The classical moduli space of vector multiplets in these theories is

$$\left.\frac{SU(1,1)}{U(1)}\right|_{\text{dilaton}} \times \frac{O(2,r)}{O(2)\times O(r)}\Big/\Gamma \tag{1}$$

where Γ is the discrete T-duality group which in the simplest case is $O(2,r;\mathbf{Z})$ [7]. At the generic points in this moduli space the gauge group is abelian $U(1)^r$ and there are no charged massless states. However there are complex codimension 1 surfaces where one of the $U(1)$'s is enhanced to $SU(2)$, due to the appearance of two extra charged massless vector multiplets, or some

charged hypermultiplets become massless. The perturbative correction to the prepotential, which due to the $N = 2$ non-renormalization theorem occurs only at the one-loop level, develops a logarithmic singularity near these surfaces [8, 9]. As a result the classical duality group Γ gets modified at the perturbative level [8]. At the full non-perturbative level, from the analysis in the rigid case, this enhanced symmetry locus is expected to split into several branches where non-perturbative states corresponding to dyonic hypermultiplets become massless.

This class of heterotic theories is dual to the type II superstring compactified on Calabi-Yau threefolds [5, 6]. The latter is characterized by the two Betti numbers b_{11} and b_{12} which determine the number of massless vector multiplets and hypermultiplets to be b_{11} and $b_{12} + 1$, respectively, where the extra $+1$ is accounted for by the dilaton. At the perturbative level the gauge group is abelian $U(1)^r$ with $r = b_{11} + 1$ (the $+1$ accounts for the graviphoton) and there are no charged massless matter fields. Since the dilaton belongs to a hypermultiplet, the tree level prepotential is exact at the full quantum level. Moreover this tree level prepotential can be computed exactly, *i.e.* including the world-sheet instanton corrections, by using mirror symmetry [10]. A generic feature of the prepotential is that it has logarithmic singularities near the conifold locus in the moduli space of the Calabi-Yau manifold [11]. This singularity is due to the appearance of massless hypermultiplets, corresponding to charged black holes, at the conifold locus [12]. It can then be understood as a one-loop effect involving this massless black hole in the internal line.

The equivalence between $N = 2$ heterotic compactifications on $\mathbf{T^2} \times \mathbf{K_3}$ and type II on Calabi-Yau has been checked explicitly in several examples involving rank 3 and 4 [13, 14, 15]. In particular it has been shown that after identifying the heterotic dilaton with a particular T-modulus of type II, the prepotential in the type II theory reproduces the perturbative prepotential of the corresponding heterotic model in the weak coupling limit $S \to \infty$. Moreover at finite values of S, the conifold singularity structure reproduces the exact results of the rigid supersymmetric Yang-Mills theory [14]. Finally, besides the comparison of the low energy theory, further non-trivial tests have been performed by analyzing the structure of higher dimensional interactions [15].

In this talk, we review the derivation and the basic properties of the perturbative prepotential in $N = 2$ compactifications of the heterotic superstring. We concentrate on the rank-four example which involves, besides the dilaton, two complex moduli, T and U, parameterizing the two-dimensional torus $\mathbf{T^2}$. The associated $U(1) \otimes U(1)$ gauge group becomes enhanced to $SU(2) \otimes U(1)$ along the $T = U$ line, and further enhanced to $SO(4)$ or to $SU(3)$ at $T = U = i$ and $T = U = \rho(=e^{2\pi i/3})$, respectively. At these particular surfaces the one-loop prepotential develops logarithmic singularities. We study the corresponding monodromies and exhibit the resulting modifications to the classical duality transformations. The perturbative monodromies of the rigid supersymmetric theories at the enhanced symmetric points with the maximum non-abelian gauge symmetry form an infinite dimensional subgroup of the full string (perturbative) monodromy group.

2. String computation of the prepotential

The simplest way to determine the prepotential is to reconstruct it from the Kähler metric of moduli fields. Indeed, the Kähler potential of the effective N=2 locally supersymmetric theory can be written as

$$K = -\ln(iY)\,, \qquad Y = 2F - 2\bar{F} - \sum_{Z=S,T,U} (Z - \bar{Z})(F_Z + \bar{F}_Z)\,, \tag{2}$$

where F is the analytic prepotential and $F_Z \equiv \partial_Z F$. Its general form is:

$$F \;=\; STU + f(T,U)\,, \tag{3}$$

where the first term proportional to the dilaton, is the tree-level contribution, and the one-loop correction is contained in a dilaton-independent analytic function $f(T,U)$. In our conventions S is defined such that $\langle S\rangle = \frac{\theta}{\pi} + i\frac{8\pi}{g^2}$ where g is the string coupling constant and θ the usual θ-angle. Higher loop corrections are forbidden from analyticity and the axionic shift,

$$D: \quad S \to S + \lambda \ , \tag{4}$$

which is an exact continuous symmetry in string perturbation theory.

The one loop moduli metric can be obtained by expanding (2),

$$K_{Z\bar{Z}} \ = \ K^{(0)}_{Z\bar{Z}}[1 + \frac{2i}{S-\bar{S}}\mathcal{I} + \cdots] \quad Z = T, U \ , \tag{5}$$

where the tree level metric $K^{(0)}_{Z\bar{Z}} = -(Z-\bar{Z})^{-2}$ and $\mathcal{I}$ is given by

$$\mathcal{I} \ = \ -\frac{i}{2}(\partial_T - \frac{2}{T-\overline{T}})(\partial_U - \frac{2}{U-\overline{U}})f \ + \ \text{c.c.} \tag{6}$$

$\mathcal{I}$ can be computed by a one loop string calculation of an amplitude involving the antisymmetric tensor using the method of ref.[16]. From the expression (6) one can easily deduce the third derivative of the one loop prepotential, $\partial_T^3 f$,

$$\partial_T^3 f = -i(U-\overline{U})^2 D_T \partial_T \partial_{\overline{U}} \mathcal{I} \ , \tag{7}$$

where the covariant derivative $D_T = \partial_T + \frac{2}{T-\overline{T}}$.

Since T, U belong to the coset $O(2,2)/O(2)^2$, the classical duality group is in general a subgroup of $O(2,2)$ restricted into the rational numbers and in the simplest case it is $O(2,2;\mathbf{Z}) \sim SL(2,\mathbf{Z})_T \otimes SL(2,\mathbf{Z})_U$. Under $SL(2,\mathbf{Z})_T$ transformations,

$$T \to T_g \equiv \frac{aT+b}{cT+d} \ , \tag{8}$$

with a, b, c, d integers satisfying $ad - bc = 1$, while the physical quantity $\mathcal{I}$ is modular invariant. It then follows from eq.(7) that $\partial_T^3 f$ is a modular function of weight 4 in T and -2 in U.

Integrating equation (7) one can determine f up to a quadratic polynomial in T and U,

$$f(T,U) = \int_{(T_0,U_0)}^{(T,U)} \{dT' Q(U,U')(T-T')^2 \partial_{T'}^3 f(T',U') + (T \leftrightarrow U, T' \leftrightarrow U')\} \ , \tag{9}$$

where (T_0, U_0) is an arbitrary point, and $Q(x,x')$ is the second order differential operator,

$$Q(x,x') = \frac{1}{2}(x-x')^2 \partial_{x'}^2 + (x-x')\partial_{x'} + 1 \ . \tag{10}$$

The path of integration should not cross any singularity of $\partial_T^3 f$, while the result of the integral depends on the homology class of such paths. Different choices of homology classes of paths change f by quadratic polynomials in T, U. Moreover under a modular transformation (8), f does not transform covariantly. Using its integral representation (9), we see that it has a weight -2 up to an addition of a quadratic polynomial $\mathcal{P}^g$ in T, U,

$$f(T_g, U) = (cT+d)^{-2}[f(T,U) + \mathcal{P}^g(T,U)] \ . \tag{11}$$

The same transformation properties hold for the U variable, as well as for the $T \leftrightarrow U$ exchange. These transformations should leave the physical metric (5), (6) invariant. Hence, one must have

$$\text{Im}\{(\partial_T - \frac{2}{T-\overline{T}})(\partial_U - \frac{2}{U-\overline{U}})\mathcal{P}^g(T,U) = 0 \ , \tag{12}$$

which is satisfied only if $\mathcal{P}^g(T,U)$ is a quadratic polynomial with real coefficients. In fact, we will see below that this ambiguity is related to the non-trivial quantum monodromies.

Modular invariance of the full effective action implies that the dilaton should also transform. Imposing the requirement that duality transformations should be compensated by Kähler transformations one finds,

$$S \to S + c\frac{f_U + \mathcal{P}^g_U}{cT+d} + \lambda_g \,, \tag{13}$$

up to an arbitrary additive axionic shift, λ_g. It follows that in the presence of one loop corrections one can define an invariant dilaton $S_{\rm inv}$ [9],

$$S_{\rm inv} \equiv S + \frac{1}{2} f_{TU} \ , \tag{14}$$

which however is not a special coordinate of $N=2$ Kähler geometry.

From a direct string computation of the one-loop metric and using eq.(7), one finds the following world-sheet integral representation of $\partial^3_T f$,

$$\partial^3_T f = 4\pi^2 \frac{U-\overline{U}}{(T-\overline{T})^2} \int d^2\tau\, C(\bar\tau) \sum_{p_L,p_R} p_L \bar p_R^3\, e^{\pi i\tau\,|p_L|^2}\, e^{-\pi i\bar\tau\,|p_R|^2} \,, \tag{15}$$

where the integration extends over the fundamental domain of the modular parameter $\tau \equiv \tau_1 + i\tau_2$, and C is a T-independent modular function of weight -2 with a simple pole at infinity due to the tachyon of the bosonic sector. The summation inside the integral extends over the left- and right-moving momenta of $\mathbf{T^2}$,

$$p_L = \frac{1}{\sqrt{2\,{\rm Im}T\,{\rm Im}U}}\,(n_1 + m_1\overline{T} + m_2\overline{U} + n_2\overline{TU}) \tag{16}$$

$$p_R = \frac{1}{\sqrt{2\,{\rm Im}T\,{\rm Im}U}}\,(n_1 + m_1 T + m_2\overline{U} + n_2 T\overline{U}) \tag{17}$$

with m_1, m_2, n_1 and n_2 integer numbers. The r.h.s. of eq.(15) is indeed an analytic function of T and U, as can be verified by taking derivatives with respect to $\overline{T}$ or $\overline{U}$. Using eqs (16), (17), one can easily show that the resulting expressions are total derivatives in τ and vanish upon integration.

At the plane $T=U$ there are two additional massless gauge multiplets which enhance the gauge symmetry to $SU(2)\otimes U(1)$. They correspond to lattice momenta (16), (17) with $n_1 = n_2 = 0$ and $m_2 = -m_1 = \pm 1$, so that $p_L = 0$ and $p_R = \pm i\sqrt{2}$. The gauge group is further enhanced at the two special points $T=U=i$ and $T=U=\rho$ giving rise to $SO(4)$ and $SU(3)$, respectively. It follows that the one-loop metric (6) has a logarithmic singularity of the form $\mathcal{I} \sim \frac{2}{\pi}\ln|T-U_g|$ for T close to U_g ($\neq i,\rho$) where g is an $SL(2,\mathbf{Z})_U$ element (8). As a result, the one-loop prepotential behaves as

$$f(T,U) \to -\frac{i}{\pi}[(cU+d)T - (aU+b)]^2 \ln(T-U_g)\,. \tag{18}$$

One can now use duality symmetry to determine f. As mentioned above $\partial^3_T f$ is a modular function of weight 4 in T and -2 in U. Moreover from its integral representation (15), it has a simple pole at $T=U$ (modulo $SL(2,\mathbf{Z})_U$) with residue $(-2i/\pi)$ in accordance with (18), and vanishes as $T\to i\infty$. These properties fix $\partial^3_T f$ uniquely to:

$$\partial^3_T f = -\frac{2i}{\pi}\frac{j_T(T)}{j(T)-j(U)}\left\{\frac{j(U)}{j(T)}\right\}\left\{\frac{j_T(T)}{j_U(U)}\right\}\left\{\frac{j(U)-j(i)}{j(T)-j(i)}\right\} \ , \tag{19}$$

where $j(T)$ is the meromorphic function with a simple pole with residue 1 at infinity and a third order zero at $T = \rho$.

Along the lines discussed above one can study a model with rank 3. The scalar components of the vector multiplets are the dilaton S and a modulus T which belongs to the coset $O(2,1)/O(2)$. The classical duality symmetry acting on T is $O(2,1;\mathbf{Z}) \equiv SL(2,\mathbf{Z})$. The prepotential is given by:

$$F = \frac{1}{2}ST^2 + f(T) \quad , \tag{20}$$

where $f(T)$ is the one loop correction to the classical prepotential $\frac{1}{2}ST^2$. At generic point in the T-moduli space, the gauge group is abelian, namely $U(1)^3$ including the vector partner of the dilaton and the graviphoton. However at $T = i \pmod{SL(2,\mathbf{Z})}$ two extra vector multiplets become massless, giving rise to an enhanced gauge group $U(1)^2 \times SU(2)$. Consequently the one-loop metric must have a singularity of the form, $\ln|T-i|$ for T close to i. This in turn implies that f must behave as $(T-i)^2 \ln(T-i)$.

As in the rank 4 case discussed above, one can construct the Kähler potential starting from the prepotential F and check that the requirement that the $SL(2,\mathbf{Z})$ transformations of T should be Kähler transformations implies that $f(T)$ transforms with weight -4, up to additive terms that are at most quartic in T.

Under $SL(2,\mathbf{Z})$ duality transformation the one-loop metric must transform covariantly. Using the expression for the metric in terms of f, one can easily deduce that $i\partial_T^5 f$ is a meromorphic form of weight 6 with respect to T, with a third order pole at $T = i$ and vanishing for $T \to \infty$. To fix completely $i\partial_T^5 f$ one uses the knowledge of the $SU(2)$ beta function coefficient and also the fact that the monodromy group of f must be embeddable in $Sp(6,\mathbf{Z})$ as dictated by $N = 2$ supergravity, as we will explain in the following. The result is:

$$\partial_T^5 f = -\frac{1}{18\pi i}\left\{\frac{j_T(T)}{j(T)-j(i)}\right\}^3 \left\{\frac{j(i)}{j(T)}\right\}^2 \left\{5 + 13\frac{j(T)}{j(i)}\right\} , \tag{21}$$

Expression (21) can be used to verify the heterotic-type II duality conjecture of ref.[6] in the weak coupling limit, that is the agreement with the prepotential of the type II string compactified on the Calabi-Yau threefold $X_{12}(1,1,2,2,6)$. Indeed, after identifying S and T with the two $(1,1)$ moduli of the type II side $t_{1,2}$ as $S = 2t_2$ and $T = t_1$, one can verify the agreement, in the large S limit, of the first few terms in the $q_1 \equiv \exp 2\pi i t_1$ expansion of the two expressions.

3. Monodromies of the one-loop prepotential

Let us now discuss the monodromy group acting on f in the rank 4 case [8]. At the classical level there is the usual action of the modular group acting on T and U upper half planes, namely $PSL(2,\mathbf{Z})_T \otimes PSL(2,\mathbf{Z})_U$. $PSL(2,\mathbf{Z})_T$ is generated by the two elements,

$$g_1 : \ T \to -1/T \qquad\qquad g_2 : \ T \to -1/(T+1) \ , \tag{22}$$

and similarly $PSL(2,\mathbf{Z})_U$ by the corresponding elements g_1', g_2'. These generators obey the $SL(2,\mathbf{Z})$ relations

$$(g_1)^2 = (g_1')^2 = (g_2)^3 = (g_2')^3 = 1 \ , \tag{23}$$

and the relations implied by the fact that the two $PSL(2,\mathbf{Z})$'s commute. There is also an exchange symmetry generator, namely:

$$\sigma : \ T \leftrightarrow U, \tag{24}$$

which satisfies $\sigma^2 = 1$. Moreover σ relates the two $PSL(2,\mathbf{Z})$'s via $g_1' = \sigma g_1 \sigma$ and $g_2' = \sigma g_2 \sigma$. The above relations can be thought of as the relations among the generators of the fundamental group of our classical moduli space in the following way: topologically each $PSL(2,\mathbf{Z})$ fundamental domain is a two-sphere S (S') with 3 distinguished points x_1 (x_1'), x_2 (x_2') and x_3 (x_3'), which can be taken to be the images of i, ρ and ∞ by the j-function. Associated with these three points we have generators g_i (g_i') of the fundamental group of orders 2, 3 and ∞ respectively, subject to the conditions $g_3 g_2 g_1 = 1$ and $g_3' g_2' g_1' = 1$. The total space is then the product of the two spheres S and S' minus $\{x_i\} \times S'$ and $S \times \{x_i'\}$, $i = 1, 2, 3$, and the fundamental group of the resulting 4-dimensional space is the product of the fundamental groups of the two punctured spheres. Including σ, we have the additional relations $g_i' = \sigma g_i \sigma$ and $\sigma^2 = 1$.

In the quantum case we have singularities at $T = U$ and consequently we must remove the diagonal in the product of the two punctured spheres and and the fundamental group of the resulting space is a braid group. One can adapt the results of ref.[17] to the present case, and obtain the following relations:

$$\begin{aligned}
&g_3 g_2 g_1 = \sigma^2, \qquad (g_1)^2 = (g_2)^3 = 1 \\
&g_i' = \sigma^{-1} g_i \sigma \\
&g_1 \sigma^{-1} g_2 \sigma = \sigma^{-1} g_2 \sigma g_1 \\
&\sigma g_i \sigma^{-1} g_i = g_i \sigma^{-1} g_i \sigma.
\end{aligned} \tag{25}$$

The full fundamental group is generated by three elements σ, g_1, g_2 subject to the above relations. In the quantum case σ^2 corresponds to moving a point around $T = U$ singularity, and it will not be equal to the identity. Notice that if one sets $\sigma^2 = 1$ one gets back the classical relations for the two commuting $PSL(2,\mathbf{Z})$'s. However in the quantum case $\sigma^2 \neq 1$ and the two $PSL(2,\mathbf{Z})$'s do not commute anymore. Under σ^2, f transforms as following:

$$Z_1 \equiv \sigma^2 : f(T,U) \to f(T,U) + 2(T-U)^2 \tag{26}$$

One can explicitly check the non commutativity of T and U duality transformations using the integral representation for f given in (9).

Having the generators and relations of the fundamental group, we will now determine the monodromy transformations of the prepotential f. As explained in the previous section, under the generators g_1, g_2 and σ, f transforms according to eq.(11) with three corresponding polynomials $\mathcal{P}^{g_1}$, $\mathcal{P}^{g_2}$ and $\mathcal{P}^{\sigma}$, quadratic in T, U with real coefficients. Imposing the group relations (25) and eq.(26), one can fix these polynomials in terms of 9 parameters which correspond to the freedom of adding to f a quadratic polynomial in T, U with real coefficients leaving the Kähler potential (2) invariant. In a particular base choice, one finds:

$$\begin{aligned}
\mathcal{P}^{g_1} &= 0 \qquad \mathcal{P}^{g_2} = 2(T^2 - 1) \\
\mathcal{P}^{\sigma} &= (T-U)^2 + (T-U)(-2UT + T + U + 2)\ .
\end{aligned} \tag{27}$$

The full monodromy group G contains a normal abelian subgroup H, which is generated by elements Z_g obtained by conjugating Z_1 by an element g which can be any word in the g_i's, g_i''s and their inverses. Z_g corresponds to moving a point around the singularity $T = U_g$, where the prepotential behaves as shown in (18). A general element of H is obtained by a sequence of such transformations and shifts f by:

$$f \to f + 2\sum_i N_i((c_i U + d_i)T - (a_i U + b_i))^2 \equiv f + \sum_{n,m=0}^{2} c_{nm} T^n U^m \quad N_i \in \mathbf{Z} \tag{28}$$

with a_i, b_i, c_i, d_i corresponding to some $SL(2,\mathbf{Z})$ elements for each i. Since the polynomial entering in (28) has 9 independent parameters c_{nm}, it follows that H is isomorphic to Z^9. The set of all conjugations of H by elements generated by g_i's and g_i''s defines a group of (outer) automorphisms of H which is isomorphic to $PSL(2,\mathbf{Z}) \times PSL(2,\mathbf{Z})$, under which c_{nm} transform as $(3,3)$ representation (in this notation the two $PSL(2,\mathbf{Z})$'s act on the index n, m respectively). Moreover, the conjugation by σ defines an automorphism which interchanges the indices n and m in c_{nm}. Thus the set of all conjugations of H is isomorphic to $O(2,2;\mathbf{Z})$, under which the c_{nm}'s transform as a second rank traceless symmetric tensor. Finally, the quotient group G/H is isomorphic to $O(2,2;\mathbf{Z})$, therefore G is a group involving 15 integer parameters. On the other hand, G is not the semidirect product of $O(2,2;\mathbf{Z})$ and H, since $O(2,2;\mathbf{Z})$ is not a subgroup of G, as it follows from the quantum relations (25). Of course for physical on-shell quantities the group H acts trivially and therefore one recovers the usual action of $O(2,2;\mathbf{Z})$.

In addition to the above monodromies there is also the axionic shift D, defined in eq.(4), and the full perturbative group of monodromies is the direct product of G with this abelian translation group.

The above monodromy group structure is best exploited in a field basis where all monodromies act linearly [5, 8]. To this end we use the formalism of the standard N=2 supergravity where the physical scalar fields Z^I of vector multiplets are expressed as $Z^I = X^I/X^0$, in terms of the constrained fields X^I and X^0. This is a way to include the extra $U(1)$ gauge boson associated with the graviphoton which has no physical scalar counterpart. In our case we have

$$S = \frac{X^s}{X^0} \quad T = \frac{X^2}{X^0} \quad U = \frac{X^3}{X^0} \tag{29}$$

and the prepotential (3) is a homogeneous polynomial of degree 2, $F(X^I) = (X^0)^2 F(S,T,U)$. The Kähler potential K is

$$K = -\log i(\bar{X}^I F_I - X^I \bar{F}_I) \ , \tag{30}$$

where F_I is the derivative of F with respect to X^I and $I = 0, s, 2, 3$. This has a generalization in basis where F_I is not the derivative of a function F [5].

Clearly the symplectic transformations acting on (X^I, F_I) leave the Kähler potential invariant. Since the monodromy group leaves K invariant, we expect it to be a subgroup of the symplectic group $Sp(8)$. A general symplectic transformation is

$$\begin{pmatrix} X^I \\ F_I \end{pmatrix} \to \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \begin{pmatrix} X^I \\ F_I \end{pmatrix} \tag{31}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are 4×4 matrices satisfying the defining relations of the symplectic group, $\mathbf{a}^t\mathbf{c} - \mathbf{c}^t\mathbf{a} = 0$, $\mathbf{b}^t\mathbf{d} - \mathbf{d}^t\mathbf{b} = 0$ and $\mathbf{a}^t\mathbf{d} - \mathbf{c}^t\mathbf{b} = \mathbf{1}$. Under this transformation, the vector kinetic term $\mathrm{Im}\mathcal{F}^I_{\mu\nu}\overline{N}_{IJ}\mathcal{F}^{J\mu\nu}$ transforms as $N \to (\mathbf{c} + \mathbf{d}N)(\mathbf{a} + \mathbf{b}N)^{-1}$. It follows that for $\mathbf{b} \neq 0$ the gauge coupling gets inverted and therefore in a suitable basis the perturbative transformations must have $\mathbf{b} = 0$. When $\mathbf{b} = 0$ the symplectic constraints become $\mathbf{d}^t = \mathbf{a}^{-1}$ and $\mathbf{c} = \mathbf{a}^{t^{-1}}\tilde{\mathbf{c}}$ with $\tilde{\mathbf{c}}$ an arbitrary symmetric matrix. In this case, the vector kinetic term changes by $\tilde{\mathbf{c}}_{IJ}\mathrm{Im}\mathcal{F}^I\mathcal{F}^J$ which, being a total derivative, is irrelevant at the perturbative level. However at the non-perturbative level, due to the presence of monopoles, the matrix $\tilde{\mathbf{c}}$ must have integer entries.

At the classical level one can easily verify that the $PSL(2,\mathbf{Z})_T$ transformation (8) (and similarly $PSL(2,\mathbf{Z})_U$ transformation given by interchanging X^2 with X^3 and F_2 with F_3) acts on X^I and F_I by a symplectic matrix, whose entry $\mathbf{b}$ is however different from zero. It is therefore convenient to perform a change of basis into (X^I, F_I), where $I = 0, 1, 2, 3$ and $X^1 = F_s$ and $F_1 = -X^s$. In the new basis the tree level $O(2,2;\mathbf{Z})$ transformations are block diagonal, i.e. $\mathbf{b} = \mathbf{c} = 0$ and $\mathbf{d} = \mathbf{a}^{t^{-1}}$.

Having found a basis which is appropriate for the perturbative monodromies one can proceed to recast the previous discussion about the monodromy transformations of f in a linear, symplectic form and get the symplectic matrices corresponding to the braid group generators. We will not give their explicit form here, but just mention their basic features. They have $\mathbf{b} = 0$, that is they are of the form:

$$\begin{pmatrix} \mathbf{a} & 0 \\ \mathbf{a}^{t-1}\tilde{\mathbf{c}} & \mathbf{a}^{t-1} \end{pmatrix} . \tag{32}$$

The matrices $\tilde{\mathbf{c}}$ turn out to be symmetric and satisfy $\mathrm{Tr}\eta\tilde{\mathbf{c}} = 0$, where η is the $O(2,2)$ metric $\eta = \mathrm{diag}(\sigma_1, -\sigma_1)$.

The abelian group H introduced in (28) is generated by symplectic matrices (32) with $\mathbf{a} = \mathbf{1}$, and

$$\tilde{\mathbf{c}} = \sum_i 2N_i g_i^t \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 - 2 \end{pmatrix} g_i \ , \tag{33}$$

where g_i can be chosen for instance as $PSL(2,\mathbf{Z})_T$ matrices. These matrices are traceless with respect to η, and depend on 9 integer parameters. They form the 9-dimensional representation of $O(2,2;\mathbf{Z})$ corresponding to the second rank symmetric traceless tensors, as explained previously.

Finally the axionic shift in the above symplectic basis corresponds to $\mathbf{a} = 1$ and $\mathbf{c} = -\lambda\eta$, which commutes with the above matrices of G, as expected. The parameter λ should also be quantized at the non-perturbative level. In this way one generates all possible symmetric 4×4 lower off-diagonal matrices depending on 10 integer parameters, the trace part being generated by η. The full monodromy group is generated by 4 generators: g_1, g_2, σ and the axionic shift.

4. Subgroups of Rigid Monodromies

At the semiclassical level, one $O(2,2)$ model embeds four distinct gauge groups: $U(1) \otimes U(1)$ at a generic point of the moduli space, $SU(2) \otimes U(1)$ along the $T = U$ line, $SO(4)$ at $T = U = i$, and $SU(3)$ at $T = U = \rho$. We will discuss now the relation between the monodromies of the respective N=2 globally supersymmetric (rigid) Yang-Mills theories and the superstring monodromy group.

In the case of gauge group of rank 2, the rigid monodromy group is a subgroup of $Sp(4,\mathbf{Z})$. It is generated by Weyl reflections

$$\mathbf{a}_k \ = \ \mathbb{1} - 2\frac{\alpha_k \otimes \alpha_k}{\alpha_k^2} \ , \tag{34}$$

where $\alpha_k,\ k = 1,2$ are simple roots of the gauge group. The corresponding monodromy matrices can be written as

$$M_k \ = \ \begin{pmatrix} \mathbf{a}_k & 0 \\ \mathbf{a}_k^{t\,-1}\tilde{\mathbf{c}}_k & \mathbf{a}_k^{t\,-1} \end{pmatrix}, \qquad \tilde{\mathbf{c}}_k = \alpha_k \otimes \alpha_k \ . \tag{35}$$

In the field basis in which the classical duality $O(2,2;\mathbf{Z})$ group is realized as block diagonal $Sp(8,\mathbf{Z})$ matrices discussed in the previous section, the surfaces of enhanced symmetries in the $\Gamma_{2,2}$ lattice (16-17), $p_L = 0$ and $p_R^2 = 2$, are defined by the equation

$$p_L \sim \frac{n_I^{\alpha_i} X^I}{X^0} = n_1^{\alpha_i} + n_2^{\alpha_i} TU + m_1^{\alpha_i} T + m_2^{\alpha_i} U = 0 \ , \tag{36}$$

for the particular choice of $n_I^{\alpha_i} = (n_1, n_2, m_1, m_2)^{\alpha_i} \in \Gamma_{2,2}$ obeying $n_I\, n^I = 2(n_1 n_2 - m_1 m_2) = 2$. These $\Gamma_{2,2}$ vectors represent the root vectors α_i of enhanced symmetries corresponding to gauge

multiplets that become massless on the surface. It is well known that Weyl reflections associated with these roots can be represented as $O(2,2;\mathbf{Z})$ duality transformations [7]; they correspond to Weyl reflections $(\mathbf{a}^{\alpha_i})^I{}_J = \delta^I_J - n^{\alpha_i I} n^{\alpha_i}{}_J$ in the surfaces (36). In the Table below, we give a list of duality transformations and the respective $O(2,2)$ matrices,[a] for Weyl reflections associated with the simple roots.

O(2, 2) embeddings of Weyl reflections

$T=U$ $SU(2)$	$n_I^{\alpha_1}=(0,0,1,-1)$ $\sigma: \quad T\to U$ $\mathbf{a}^{\alpha_1}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$		
$T=U=i$ $SO(4)$	same as above	$n_I^{\alpha_2}=(1,1,0,0)$ $g_1\sigma g_1^{-1}: \quad T\to -1/U$ $\mathbf{a}^{\alpha_2}=\begin{pmatrix}0&-1&0&0\\-1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	
$T=U=\rho$ $SU(3)$	same as above	$n_I^{\alpha_3}=(1,1,0,1)$ $g_2^{-1}\sigma g_2: \quad T\to -(U+1)/U$ $\mathbf{a}^{\alpha_3}=\begin{pmatrix}0&-1&0&-1\\-1&0&0&-1\\1&1&1&1\\0&0&0&1\end{pmatrix}$	$n_I^{\alpha_4}=(1,1,1,0)$ $g_2\sigma g_2^{-1}: \quad T\to -1/(U+1)$ $\mathbf{a}^{\alpha_4}=\begin{pmatrix}0&-1&-1&0\\-1&0&-1&0\\0&0&1&0\\1&1&1&1\end{pmatrix}$

The appearance of massless states at the enhanced symmetry surfaces gives rise to the logarithmic singularities which modify the classical monodromies. Near the surface (36), the singular part of f_I is

$$f_I = -\frac{2i}{\pi} n_I^{\alpha_i} n^{\alpha_i}{}_J X^J \log\left(\frac{n_K^{\alpha_i} X^K}{X^0}\right) . \tag{37}$$

The identification of Weyl reflections as duality transformations is very useful if one wants to study the embedding of rigid monodromies $M_k \in Sp(4,\mathbf{Z})$ of eq.(35) into $Sp(8,\mathbf{Z})$, as a subgroup of the superstring monodromy group [18]. The explicit form of the respective matrices depends however on the choice of the base point which amounts to the freedom of adding to f a polynomial quadratic in T and U with real coefficients or equivalently, of conjugating the monodromy matrices by the elements of the normal abelian subgroup H. One can ask the

[a] As usual, these matrices are defined up to an overall multiplication by $-\mathbb{1}$.

question whether a base exists such that the monodromies associated with Weyl reflections in singular surfaces have the form of rigid monodromies of eq.(35). The answer is affirmative. Indeed one can change the base from the one used in the previous section and in ref.[8] to the new one which corresponds to

$$\begin{aligned} f &\rightarrow f + 1 + 2T + T^2 - 2TU - 2TU^2 - 4T^2U + \epsilon(TU-1)^2 \ , \\ S &\rightarrow S + 2U + 4T - 2\epsilon TU \ , \end{aligned} \tag{38}$$

where ϵ is still a free parameter. It is easy to verify that with an additional dilaton shift accompanying σ transformation, $\lambda_\sigma = 1$, the $Sp(8,\mathbf{Z})$ monodromies associated with Weyl reflections σ, $g_1\sigma g_1^{-1}$ and $g_2\sigma g_2^{-1}$ become precisely of the form of eq.(35), with the $O(2,2)$ matrices $\mathbf{a}$ as given in the Table and

$$(\tilde{\mathbf{c}}_k)_{IJ} = 2n_I^{\alpha_k} n^{\alpha_k}{}_J \ . \tag{39}$$

Notice that the third $SU(3)$ Weyl reflection $g_2^{-1}\sigma g_2$ is not of the above form, similarly to the rigid case, where only the Weyl reflections corresponding to the simple roots are of this form [19]. Finally the three Weyl reflections are related by conjugation with the Coxeter element g_2. Hence the rigid perturbative monodromies associated with all possible enhanced gauge symmetries are glued together in superstring theory, embedded in one monodromy group in a very natural way. From the above discussion it is clear that this conclusion applies also to arbitrary (4,4) compactifications.

Let us now turn to the question of non-perturbative monodromies. As in the rigid case, we expect that in the quantum moduli space, the singularities corresponding to enhanced gauge symmetries disappear and are replaced by the ones associated to monopole or dyonic states that become massless. If we label the dyonic charge by $(q_I; g^J)$ where q and g are the electric and magnetic charges respectively, then the monodromy due to this state becoming massless is given by the following symplectic matrix:

$$M_{(q,g)} = \begin{pmatrix} \mathbb{1} - g\otimes q & -\frac{1}{2} g\otimes g \\ 2q\otimes q & \mathbb{1} + q\otimes g \end{pmatrix}. \tag{40}$$

The factors of 2 in the above equation are due to our normalization of the charge vectors. The three singularities associated with the Weyl reflections σ, $g_1\sigma g_1^{-1}$ and $g_2\sigma g_2^{-1}$ each split into two singularities; one corresponding to a pure monopole state and the second that to a dyonic state. The pure monopole state carries the magnetic charge $g^I = \eta^{IJ} n_J^{\alpha_k}$ with $n_J^{\alpha_k}$ defined in the Table. The monodromy matrix associated to this monopole is then obtained by substituting the magnetic charge in (40). The monodromy corresponding to the dyonic state can be obtained by using the fact that the product of the monodromies associated to the pure monopole and the dyonic state equals the perturbative monodromy corresponding to α_k. More explicitly the dyonic charges (q,g) are given by $q_I = n_I^{\alpha_k}$ and $g^I = -\eta^{IJ} n_J^{\alpha_k}$. Finally the monodromies for the two dyonic states associated with the third $SU(3)$ root α_3 are obtained by the action of the Coxeter element g_2.

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References

[1] For recent reviews, see: A. Ceresole, R. D'Auria and S. Ferrara, hep-th/9509160; M.J. Duff, hep-th/9509106; J.H. Schwarz, hep-th/9509148.

[2] A. Sen, Phys. Lett. B 329 (1994) 217; C. Vafa and E. Witten, Nucl. Phys. B 431 (1994) 3; for a review see D. Olive, hep-th/9508089.

[3] N. Seiberg and E. Witten, Nucl. Phys. B 431 (1994) 484; A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, Phys. Lett. B 344 (1995) 169; P.C. Argyres and A. Faraggi, Phys. Rev. Lett. 74 (1995) 3931.

[4] C.M. Hull and P. Townsend, Nucl. Phys. B 438 (1995) 109; M.J. Duff, Nucl. Phys. B 442 (1995) 47; E. Witten, Nucl. Phys. B 443 (1995) 85.

[5] A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, Nucl. Phys. B 444 (1995) 92.

[6] S. Kachru and C. Vafa, hep-th/9505105; S. Ferrara, J.A. Harvey, A. Strominger and C. Vafa, hep-th/9505162.

[7] For a review, see A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. 244 (1994) 77.

[8] I. Antoniadis, S. Ferrara, E. Gava, K.S. Narain and T.R. Taylor, Nucl. Phys. B 447 (1995) 35.

[9] B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, Nucl. Phys. B 451 (1995) 53.

[10] P. Candelas, X. De la Ossa, P. Green and L. Parkes, Nucl. Phys. B 359 (1991) 21; P. Candelas, X. De la Ossa, A. Font, S. Katz and D. Morrison, Nucl. Phys. B 416 (1994) 481; S. Hosono, A. Klemm, S. Theisen and S.T. Yau, Comm. Math. Phys. 167 (1995) 301.

[11] P. Candelas, P. Green and T. Hübsch, Nucl. Phys. B 330 (1990) 49; P. Candelas and X. De la Ossa, Nucl. Phys. B 342 (1990) 246.

[12] A. Strominger, hep-th/9504090.

[13] V. Kaplunovsky, J. Louis and S. Theisen, hep-th/9506110; A. Klemm, W. Lerche and P. Mayr, Phys. Lett. B 357 (1995) 313; I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, as reported by K.S. Narain at the *Trieste Conference on Mirror Symmetry and S-Duality*, 5-9 June 1995.

[14] M. Billó, A. Ceresole, R. D'Auria, S. Ferrara, P. Frè, T. Regge, P. Soriani and A. Van Proeyen, hep-th/9506075; S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, hep-th/9508155; I. Antoniadis and H. Partouche, hep-th/9509009.

[15] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, hep-th/9507115.

[16] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, Nucl. Phys. B 407 (1993) 706.

[17] G.P. Scott, Proc. Camb. Phil. Soc. 68 (1970) 605.

[18] G.L. Cardoso, D. Lüst and T. Mohaupt, hep-th/9507113.

[19] A. Klemm, W. Lerche and S. Theisen, hep-th/9505150.