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# The Symplectic Structure of N=2 Supergravity and its Central Extension *

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**Abstract**

We report on the formulation of $N = 2$ $D = 4$ supergravity coupled to $n_V$ abelian vector multiplets in presence of electric and magnetic charges. General formulae for the (moduli dependent) electric and magnetic charges for the $n_V + 1$ gauge fields are given which reflect the symplectic structure of the underlying special geometry. The specification to Type IIB strings compactified on Calabi-Yau manifolds, with gauge group $U(1)^{h_{21}+1}$ is given.

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# The Symplectic Structure of N=2 Supergravity and its Central Extension*

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We report on the formulation of $N=2$, $D=4$ supergravity coupled to $n_V$ abelian vector multiplets in presence of electric and magnetic charges. General formulae for the (moduli dependent) electric and magnetic charges for the $n_V+1$ gauge fields are given which reflect the symplectic structure of the underlying special geometry. Model independent sum rules obeyed by these charges are obtained. The specification to Type IIB strings compactified on Calabi-Yau manifolds, with gauge group $U(1)^{h_{21}+1}$, is given.

## 1. INTRODUCTION

Recent developments towards a deeper understanding of the exact quantum structure of supersymmetric non abelian gauge theories [1–3] and their extension to gravity couplings [4,5], including superstring theories[6], have revealed that many aspects of nonperturbative physics, in the infrared regime, can be exactly analysed in virtue of their supersymmetric structure.

In particular, the peculiar rôle played by the dilaton-axion complex scalar field $S$ in 4D heterotic strings, allows one to extend the powerful non-renormalization theorems of rigid renormalizable gauge theories to the string case [7,8] . $N=2$ theories are specially interesting examples because they exhibit (unlike $N\geq 3$ theories) non trivial dynamics, but nevertheless they may be exactly solved by making some hypothesis about their nonperturbative behaviour.

Exact solutions of the infrared regime of both $N=2$ super Yang-Mills theories [1,2] and their string analogue [9] have been the subject of intensive investigation over the last year [10] , and spectacular results [11–14] substantiating proposed assumptions about the exact solutions have been obtained.

For $N=2$ Yang-Mills theories coupled to gravity, which are supposed to be a low energy manifestation of $N=2$ 4D heterotic strings, the nonperturbative physics of the low energy effective action can be inferred by the assumption that a given non-abelian theory, with gauge group $G$ of rank $r$, in the particular phase where the gauge group is broken to $U(1)^r$ is equivalent, at the nonperturbative level, to a Type IIB supergravity[2] (which is the low energy theory for Type IIB strings) compactified on a Calabi–Yau manifold [4] (or its mirror [15]) with Hodge numbers $h_{11}=n_H-1$ , $h_{21}=n_V=r+1$. Here $(n_V,n_H)$ denote the number of vector multiplets and hypermultiplets, neutral with respect to the abelian gauge group $U(1)^r$ on the heterotic side. Since $n_V$ includes the (dual to the heterotic) dilaton-axion vector multiplet, it follows that $n_V\geq 1$ $(h_{21}\geq 1)$ and also $n_H\geq 1$, as on the Type II side there is always the universal hypermultiplet [16,17] which corresponds to the Type II string dilaton.

If one would like to compare string theories with rigid theories [18–21] with a rank $r$ gauge group, then $n_V=r+1$ (because the dilaton-axion degree of freedom is frozen in the rigid limit) and this means that $h_{21}=r+1\geq 2$. Since the overall gauge group is $U(1)^{r+2}$ (including the graviphoton), an example of a new nonperturbative stringy phenomenon is the possibility of having states which have in addition electric and magnetic charges with respect to all $r+2$ vector fields, as it occurs in higher N theories, under

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the assumption of S duality or more generally U duality [22,23].

In section 2 we will summarize some basic formulae for the underlying moduli space of $n_V$ complex scalar fields[24,25] of vector multiplets [26]. It is the geometrical properties of this space (special geometry) which determine the low energy couplings of vector fields as well as the central extension [27] of the $N = 2$ supersymmetry algebra. These quantities receive both perturbative and non perturbative corrections [7,8] on the heterotic side and are supposed to be exactly computed on the Type II side [9,11], at the classical level, under the hypothesis of string-string duality [28,6] and the identification [9,10] of the "dual" dilaton-axion degree of freedom with one particular element of the $H_{21}$ cohomology. Non perturbative phenomena must also occurr on the Type II side, since the central extension of the $N = 2$ algebra implies the existence of electric and magnetic charged states carrying Ramond-Ramond charges [23] together with the fact that the hypermultiplet moduli space should also undergo quantum corrections to match the classical quaternionic space on the heterotic side [11,29]. We will not discuss the hypermultiplet dynamics [30] in this report. Such corrections may be viewed as particular wrappings of 10D p-branes [29,31] ($p = 1, 3, 5$) around the Calabi-Yau manifold and therefore are non-perturbative in nature at the string level. In section 3 we will give general formulae for the moduli dependent charges of the $U(1)^{n_V+1}$ gauge fields and some general relations that such charges obey as a consequence of special geometry and its symplectic structure. Finally, the application of these results to the compactification of Type IIB strings on Calabi-Yau threefolds is given in section 4.

## 2. SPECIAL GEOMETRY AND ITS SYMPLECTIC STRUCTURE

Let us consider here the space of the $n_V$ complex scalar fields $z^i$ of vector multiplets coupling[26] to $N = 2$ supergravity[32] . The local structure of this space is that of a Kähler-Hodge manifold with a Kähler metric $G_{i\bar{j}}$ whose curvature satisfies the "special geometry" constraint

$$R_{i\bar{j}l\bar{m}} = G_{i\bar{j}}G_{l\bar{m}} + G_{i\bar{m}}G_{l\bar{j}} - C_{ilp}\bar{C}_{\bar{j}\bar{m}\bar{p}}G^{p\bar{p}} \ . \tag{1}$$

Eq. (1) is a consequence of the $N = 2$ local supersymmetry algebra, and can actually be derived from the symplectic structure dictated by the Bianchi identities. Define the (covariantly holomorphic) symplectic sections of the Hodge bundle $\mathcal{L}$

$$V = (L^\Lambda, M_\Lambda) \qquad\qquad \Lambda = 0, \ldots, n_V$$

$$D_{\bar{\imath}}V = (\partial_{\bar{\imath}} - \frac{1}{2}\partial_{\bar{\imath}}K)V = 0 \tag{2}$$

such that

$$i < V, \bar{V} >= i(\bar{L}^\Lambda M_\Lambda - \bar{M}_\Lambda L^\Lambda) = 1 \ . \tag{3}$$

Defining $U_i = D_iV = (\partial_i + \frac{1}{2}\partial_iK)V = (f_i^\Lambda, h_{i\Lambda})$, one finds

$$D_iU_j = iC_{ijk}G^{k\bar{k}}\bar{U}_{\bar{k}} \ , \tag{4}$$

where $C_{ijk}$ is a covariantly holomorphic section of $(T^*)^3 \otimes \mathcal{L}^2$, totally symmetric in its indices. It was shown in ref [25] that eqs. (1)-(4) define a flat symplectic connection. From the above relations it is possible to derive some holomorphic identities which are equivalent to Picard-Fuchs equations whenever $V$ is associated to the periods of some Calabi-Yau manifold [24,33] . Eq. (1) can be solved by setting [5]

$$M_\Lambda = \mathcal{N}_{\Lambda\Sigma}L^\Sigma \quad , \quad h_{i\Lambda} = \bar{\mathcal{N}}_{\Lambda\Sigma}f_i^\Sigma \ ,$$

$$\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}L^\Lambda\bar{L}^\Sigma = -\frac{1}{2}, \tag{5}$$

where $\mathcal{N}_{\Lambda\Sigma}$ is a complex symmetric matrix. Then one has

$$< V, U_i >=< V, U_{\bar{\imath}} >= 0 \ , \tag{6}$$

$$G_{i\bar{j}} = -i < U_i, U_{\bar{j}} > \ , \tag{7}$$

$$C_{ijk} =< D_iU_j, U_k > \ . \tag{8}$$

Moreover, $\mathcal{N}_{\Lambda\Sigma}$ can be obtained in terms of the two $(n_V + 1) \times (n_V + 1)$ matrices

$$f_I^\Lambda = (f_i^\Lambda, \bar{L}^\Lambda) \quad , \quad h_{I\Lambda} = (h_{i\Lambda}, \bar{M}_\Lambda) \tag{9}$$

from the relation

$$\bar{\mathcal{N}}_{\Lambda\Sigma} = h_{I\Lambda}(f^{-1})^I_\Lambda \ . \tag{10}$$

Another useful relation is

$$U^{\Lambda\Sigma} = f_i^\Lambda G^{i\bar{\jmath}} f_{\bar{\jmath}}^\Sigma = -\frac{1}{2}(\mathrm{Im}\,\mathcal{N})^{\Lambda\Sigma} - \bar{L}^\Lambda L^\Sigma \ . \quad (11)$$

By setting

$$L^\Lambda = e^{\frac{K(z,\bar{z})}{2}} X^\Lambda(z) \ , \ M_\Lambda = e^{\frac{K(z,\bar{z})}{2}} F_\Lambda(z) \quad (12)$$

it follows that $(X^\Lambda, F_\Lambda)$ are holomorphic sections of a line bundle, and all previous formulae can be written in terms of such sections. The Kähler potential is

$$K = -\ln i < \Omega, \bar{\Omega} > \ , \quad (13)$$

where $\Omega = (X^\Lambda, F_\Lambda) = e^{-K/2} V$. From eq. (6) one finds

$$X^\Lambda \partial_i F_\Lambda - \partial_i X^\Lambda F_\Lambda = 0 \ . \quad (14)$$

Note that under Kähler transformations $K \rightarrow K + f + \bar{f}$ and $\Omega \rightarrow \Omega e^{-f}$. Since $X^\Lambda \rightarrow X^\Lambda e^{-f}$, this means that we can regard, at least locally, the $X^\Lambda$ as homogeneous coordinates on the Kähler manifold[24,25], provided the matrix

$$e_i^a(z) = \partial_i (X^a / X^0) \quad a = 1, \ldots, n_V \quad (15)$$

is invertible. In this case, $F_\Lambda = F_\Lambda(X)$ and because of eq. (14) and the fact that $X^\Sigma \partial_\Sigma F_\Lambda = F_\Lambda$, one then has

$$F_\Lambda(X) = \partial_\Lambda F(X) \quad , \quad X^\Sigma \partial_\Sigma F = 2F \ . \quad (16)$$

$F(X(z))$ is the prepotential of $N = 2$ supergravity vector multiplet couplings [26] and "special coordinates" correspond to a coordinate choice for which [24,25,33]

$$e_i^a = \partial_i (X^\Lambda / X^0) = \delta_i^a \ . \quad (17)$$

This means $X^0 = 1$, $X^i = z^i$. Note that under coordinate transformations, the sections $\Omega$ transform as

$$\Omega' = e^{-f_{\mathcal{S}}(z)} \Omega \ , \quad (18)$$

where $\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is an element of $Sp(2n_V + 2, \mathbb{R})$,

$$\begin{aligned} A^T D - C^T B &= \mathbb{1} \ , \\ A^T C - C^T A &= B^T D - D^T B = 0 \ . \end{aligned} \quad (19)$$

Since $F = \frac{1}{2} X^\Lambda F_\Lambda$, this implies that

$$\begin{aligned} \tilde{F}(\tilde{X}) &= F(X) + X^\Lambda (C^T B)_\Sigma^\Lambda F_\Sigma \\ &+ \frac{1}{2} X^\Lambda (C^T A)_{\Lambda\Sigma} X^\Sigma + \frac{1}{2} F_\Lambda (D^T B)^{\Lambda\Sigma} F_\Sigma \end{aligned} \quad (20)$$

where

$$\tilde{X} = (A + B\mathcal{F}) X \quad , \quad \mathcal{F} = F_{\Lambda\Sigma} = \frac{\partial^2 F}{\partial X^\Lambda \partial X^\Sigma} \ . \quad (21)$$

We also note that

$$\tilde{\mathcal{N}}(\tilde{X}, \tilde{F}) = (C + D\mathcal{N})(A + B\mathcal{N})^{-1} \ , \quad (22)$$

a relation that simply derives from its definition, eq. (10) . It is useful to give formulae which relate the two symmetric matrices $\mathcal{N}_{\Lambda\Sigma}$ , $F_{\Lambda\Sigma}$ which exist whenever (15) is fullfilled. We first note that $\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}$, $\mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}$ are related to the kinetic and topological term $F^2$ and $F\tilde{F}$ of vector fields respectively in the low energy $N = 2$ supergravity lagrangian. Thus, a physical requirement is that

$$\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma} < 0 \ . \quad (23)$$

This is actually a consequence of the positivity of the Kähler metric $G_{i\bar{\jmath}}$ and eq. (7)

$$G_{i\bar{\jmath}} = -2\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma} f_i^\Lambda f_{\bar{\jmath}}^\Sigma \ . \quad (24)$$

Moreover, if $\mathcal{F}$ exists, then it follows that

$$\mathcal{N}_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} - 2i\ \bar{T}_\Lambda \bar{T}_\Sigma (L\,\mathrm{Im}\,\mathcal{F}\,L) \ , \quad (25)$$

where

$$\begin{aligned} T_\Lambda = -i \frac{(\mathrm{Im}\,\mathcal{F}\bar{L})_\Lambda}{\bar{L}\ \mathrm{Im}\,\mathcal{F}\ \bar{L}} &= 2i(\mathrm{Im}\,\mathcal{N}\ L)_\Lambda \ , \\ L\,\mathrm{Im}\,\mathcal{F}\,\bar{L} &= -\frac{1}{2} \\ T_\Lambda \bar{L}^\Lambda &= -i \\ 4\bar{L}\,\mathrm{Im}\,\mathcal{F}\,\bar{L} &= (L\,\mathrm{Im}\,\mathcal{N}\,L)^{-1} \ . \end{aligned} \quad (26)$$

From eqs. (4), (11) we get

$$\begin{aligned} G_{i\bar{\jmath}} &= 2\,\mathrm{Im}\,F_{\Lambda\Sigma}\, f_i^\Lambda f_{\bar{\jmath}}^\Sigma \\ U^{\Lambda\Sigma} &= \frac{1}{2}(\mathrm{Im}\,\mathcal{F}^{-1})^{\Lambda\Sigma} + L^\Lambda \bar{L}^\Sigma \\ &= \mathcal{T}_I^\Lambda G^{IJ} \bar{\mathcal{T}}_J^\Sigma \ , \end{aligned} \quad (27)$$

where $\mathcal{T}_I^\Lambda$, $G^{IJ}$ are $(n_V + 1) \times (n_V + 1)$ matrices

$$\begin{aligned} \mathcal{T}_I^\Lambda &= (\mathcal{T}_i^\Lambda, \mathcal{T}_0^\Lambda = L^\Lambda) \\ G^{IJ} &= (G^{ij} = G^{i\bar{\jmath}}, G^{i0} = 0, G^{00} = -1) \ . \end{aligned} \quad (28)$$

Eq. (27) implies, because of eq. (23), that $\mathrm{Im}\,\mathcal{F}$ is a matrix with $n_V$ positive and one negative eigenvalue. Note that $U^{\Lambda\Sigma}$ is a rank $n_V$ matrix since it annihilates the vector $T_\Lambda$, $\bar{T}_\Sigma$,

$$T_\Lambda U^{\Lambda\Sigma} = U^{\Lambda\Sigma}\bar{T}_\Sigma = 0 \ . \tag{29}$$

We will see in section 3 that $T_\Lambda$ is the graviphoton projector. From eq. (27) we can further compute

$$[\det 2\,\mathrm{Im}\,\mathcal{F}]^{-1} = \det(U^{\Lambda\Sigma} - L^\Lambda \bar{L}^\Sigma) \ , \tag{30}$$

and using (28) we find

$$\det 2\,\mathrm{Im}\,\mathcal{F} = -\det G_{i\bar{\jmath}} \mid \det \mathcal{T}_I^\Lambda \mid^{-2} \ . \tag{31}$$

In virtue of simple properties of determinants, it can be easily seen that

$$\det \mathcal{T}_I^\Lambda = e^{K/2\ (n_V+1)}(\det e)(X^0)^{n_V+1} \ , \tag{32}$$

with $e_i^a$ given by eq. (17), and hence

$$\mid \det \mathcal{T}_I^\Lambda \mid^2 = e^{K(n_V+1)}(X^0\bar{X}^0)^{n_V+1} \mid \det e \mid^2 \ . \tag{33}$$

It then follows that[26]

$$\partial_i \partial_{\bar{\jmath}} \ln\det \mathrm{Im}\,\mathcal{F} = \ln\det G_{i\bar{\jmath}} - (n_V+1)G_{i\bar{\jmath}} \ . \tag{34}$$

Since on a Kähler manifold the Ricci tensor can be written as

$$R_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} \ln\det G_{i\bar{\jmath}} \ , \tag{35}$$

one has

$$\partial_i \partial_{\bar{\jmath}} \ln\det \mathrm{Im}\,\mathcal{F} = -C_{ilp}\bar{C}_{\bar{\jmath}\bar{l}\bar{p}} G^{l\bar{l}} G^{p\bar{p}} \ . \tag{36}$$

Thus eq. (32) becomes

$$\det \mathrm{Im}\,2\mathcal{F}(t,\bar{t}) = -\det G_{a\bar{b}}(t,\bar{t}) e^{K(t,\bar{t})(n_V+1)} \ , \tag{37}$$

where $t^a = \frac{X^a}{X^0}$, $F(X^\Lambda) = (X^0)^2 f(t^a)$, $G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K(t,\bar{t})$, and

$$e^{-K(t,\bar{t})} = i\left[2f - 2\bar{f} + (\bar{t}^a - t^a)(f_a + \bar{f}_a)\right] \ . \tag{38}$$

## 3. ELECTRIC-MAGNETIC DUALITY, CENTRAL EXTENSION AND ELECTRIC-MAGNETIC CHARGE RELATIONS

In the previous section we have described the geometric structure of the metric of scalar fields. The metric $G_{i\bar{\jmath}}$ appears in the scalar kinetic term of the effective action

$$-G_{i\bar{\jmath}}\partial_\mu z^i \partial_\nu z^{\bar{\jmath}} g^{\mu\nu}\sqrt{-g} \ . \tag{39}$$

The symmetric matrix $\mathcal{N}_{\Lambda\Sigma}$ appears in the vector part of the action (we set the fermions to zero)

$$\mathrm{Im}\,\mathcal{F}^{-\Lambda}\bar{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Sigma} = \mathrm{Im}\,\mathcal{F}^{-\Lambda}\mathcal{G}^-_\Lambda \ , \tag{40}$$

where $\mathcal{G}^-_\Lambda = \bar{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Sigma}$. The symplectic structure of the equations of motion comes by defining the $Sp(2n_V+2)$ symplectic (antiselfdual) vector field strength

$$\mathcal{Z}^- = (\mathcal{F}^{-\Lambda}, \mathcal{G}^-_\Lambda) \tag{41}$$

and writing, in form language

$$d\,\mathrm{Re}\,\mathcal{Z}^- = 0 \tag{42}$$

in a source free theory. In presence of electric and magnetic sources we can write

$$\int_{S_2} \mathcal{F}^\Lambda = n_m^\Lambda \ , \tag{43}$$

$$\int_{S_2} \mathcal{G}_\Lambda = n_\Lambda^e \ , \tag{44}$$

where the symplectic vector $\mathrm{Re}\,\mathcal{Z}^-$ is

$$\begin{aligned} \mathrm{Re}\,\mathcal{Z}^- &= (\mathcal{F}^\Lambda, \mathcal{G}_\Lambda) \\ \mathcal{G}_\Lambda &= \mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^\Sigma + \frac{1}{2}\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\tilde{\mathcal{F}}^\Sigma \ . \end{aligned} \tag{45}$$

Since $(n_m^\Lambda, n_\Lambda^e)$ are integers, eqs. (43), (44) are only covariant under $Sp(2n_V+2;\mathbb{Z})$ rotations. We can now define two symplectic invariant combinations[4,18] of the symplectic field strength vector $\mathcal{Z}^-$:

$$T^- = -<\mathcal{Z}^-, V> = T_\Lambda \mathcal{F}^{-\Lambda} \ , \tag{46}$$

with $V$ and $T_\Lambda$ defined by eqs (2), (26),

$$\mathcal{F}^{-i} = -<\mathcal{Z}^-, D_{\bar{\jmath}}\bar{V}> G^{i\bar{\jmath}} \ . \tag{47}$$

Because of eq. (5) we also have

$$<\mathcal{Z}^-, \bar{V}> = <\mathcal{Z}^-, D_j V> = 0 \ . \tag{48}$$

It then follows that[18,34]

$$-\frac{1}{2}\int_{S_2} T^- = Z = L^\Lambda n_\Lambda^e - M_\Lambda n_m^\Lambda \ , \tag{49}$$

$$-\frac{1}{2}\int_{S_2}\mathcal{F}^{+\bar{j}}G_{i\bar{j}} = D_i Z = Z_i \ . \tag{50}$$

The explicit expression for $T^-$, $\mathcal{F}^{-i}$ are

$$T^- = 2i\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}L^\Lambda\mathcal{F}^{-\Sigma} \ , \tag{51}$$

$$\mathcal{F}^{-i} = 2iG^{i\bar{j}}\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\bar{f}^\Lambda_{\bar{j}}\mathcal{F}^{-\Sigma} \ . \tag{52}$$

We may define the field strength $\hat{\mathcal{F}}^\Lambda$ orthogonal to the graviphoton,

$$\hat{\mathcal{F}}^- T_\Lambda = 0 \ \text{ i.e. } \ \hat{\mathcal{F}}^{-\Lambda} = \mathcal{F}^{-\Lambda} - i\,\bar{L}^\Lambda T_\Sigma \mathcal{F}^{-\Sigma} \tag{53}$$

and notice that $\mathcal{F}^{-i}$ is orthogonal to the graviphoton since in (52) $\mathcal{F}^{-\Lambda} \to \hat{\mathcal{F}}^{-\Lambda}$ in virtue of the fact that

$$\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\bar{f}^\Lambda_{\bar{j}}\,\bar{L}^\Sigma = 0 \ . \tag{54}$$

The objects defined by eqs. (46), (47) have the physical meaning of being the (moduli-dependent) vector combinations which appear in the gravitino and gaugino supersymmetry transformations respectively [26]. Indeed, setting to zero for simplicity the fermion bilinears and the $N = 2$ generalization of the Fayet-Iliopoulos term, we have

$$\begin{aligned} \delta\psi_{A\mu} &= \mathcal{D}_\mu\epsilon_A + \epsilon_{AB}T_\Lambda\mathcal{F}^{-\Lambda}_{\mu\nu}\gamma^\nu\epsilon^B \\ \delta\lambda^{iA} &= i\gamma^\mu\partial_\mu z^i\epsilon^A + \frac{i}{2}\mathcal{F}^{i-}_{\mu\nu}\gamma^{\mu\nu}\epsilon_B\epsilon^{AB} \ , \end{aligned} \tag{55}$$

where $\lambda^{iA}$, $\psi_{A\mu}$ are the chiral gaugino and gravitino fields, $\epsilon_A$ , $\epsilon^A$ are the chiral and antichiral supersymmetry parameters respectively, $\epsilon^{AB}$ is the $SO(2)$ Ricci tensor.

Eq. (49) define the central extension of the local supersymmetry algebra, $Z$ being the central charge. Eq. (50) defines in a geometrical way the charges of the other field strength vectors, orthogonal to the graviphoton. Note that the charges $(Z, Z_i)$ are in correspondence with the real charges $(n^\Lambda_m, n^e_\Lambda)$, but refer to the supermultiplet eigenstates, which are moduli dependent. The charges $(Z, Z_i)$ satisfy two model independent relations which follow from their definition and eqs. (11), (27)

$$\mid Z\mid^2 + \mid Z_i\mid^2 = -\frac{1}{2}P^t\mathcal{M}(\mathcal{N})P \ , \tag{56}$$

$$\mid Z_i\mid^2 - \mid Z\mid^2 = \frac{1}{2}P^t\mathcal{M}(\mathcal{F})P \ , \tag{57}$$

$$\begin{aligned} &P^t\mathcal{M}(\mathcal{N})P = \\ &(n^e_\Lambda - \bar{\mathcal{N}}_{\Lambda\Sigma}n^\Sigma_m)\mathrm{Im}\,\mathcal{N}^{-1\Lambda\Sigma}(n^e_\Delta - \mathcal{N}_{\Delta\Gamma}n^\Gamma_m) \end{aligned} \tag{58}$$

Here $P = (n^\Lambda_m, n^e_\Lambda)$ and $\mathcal{M}(\mathcal{N})$ is the real symplectic matrix

$$\begin{aligned} &\mathcal{M}(\mathrm{Re}\,\mathcal{N}, \mathrm{Im}\,\mathcal{N}) = \\ &\qquad \mathcal{R}^t(\mathrm{Re}\,\mathcal{N})\mathcal{D}(\mathrm{Im}\,\mathcal{N})\mathcal{R}(\mathrm{Re}\,\mathcal{N}), \end{aligned} \tag{59}$$

where $\mathcal{R}(\mathrm{Re}\,\mathcal{N}) = \begin{pmatrix} 1\!\!1 & 0 \\ -\mathrm{Re}\,\mathcal{N} & 1\!\!1 \end{pmatrix}$, $\mathcal{D}(\mathrm{Im}\,\mathcal{N}) = \begin{pmatrix} \mathrm{Im}\,\mathcal{N} & 0 \\ 0 & \mathrm{Im}\,\mathcal{N}^{-1} \end{pmatrix}$. The same matrix appears in (57) with $\mathcal{N} \to \mathcal{F} = F_{\Lambda\Sigma}$. It is evident that eqs. (56),(57) reflect the fact that $\mathrm{Im}\,\mathcal{N}$ is negative definite and that $\mathrm{Im}\,\mathcal{F}$ has $n_V$ positive and one negative eigenvalue. This comes from the signature of the quadratic form on the right-hand-side and further noticing that $\mathrm{Re}\,\mathcal{N}(\mathrm{Re}\,\mathcal{F})$ don't play any role because $\mathcal{R}$ can be included in the vector $P$

$$P_\mathcal{R} = \mathcal{R}P \ , \tag{60}$$

so that one gets $P^t\mathcal{M}P = P^t_\mathcal{R}\mathcal{D}P_\mathcal{R}$ which is manifestly a quadratic form with negative signature or $(n_V, 1)$ signature for the two cases at hand. As an illustrative example, let us compute some consequences of formulae (56),(57) for the homogeneous special manifold $SU(1,n)/SU(n)\times U(1)$ and $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2)\times SO(n)}$. If we take, in the first case

$$F(X) = -\frac{i}{2}(X_0^2 - X^2) = -\frac{i}{2}X^\Lambda\eta_{\Lambda\Sigma}X^\Sigma \ , \tag{61}$$

then we get

$$\mid Z_i\mid^2 - \mid Z\mid^2 = -\frac{1}{2}(n^e_\Lambda n^{e\Lambda} + n^\Lambda_m n_{m\Lambda}) \ . \tag{62}$$

If we take now the other case, we cannot compute $F_{\Lambda\Sigma}$ since it does not exist in the $SO(2,n)$ covariant basis given by $(X^\Lambda, F_\Lambda = S\eta_{\Lambda\Sigma}X^\Sigma)$, with $\eta = \mathrm{diag}(++,---)$. By explicit knowledge of

$$\mathcal{N}_{\Lambda\Sigma} = (S - \bar{S})(\Phi_\Lambda\bar{\Phi}_\Sigma + \bar{\Phi}_\Lambda\Phi_\Sigma) + \bar{S}\eta_{\Lambda\Sigma} \ , \tag{63}$$

with $\Phi_\Lambda = X_\Lambda/(X^\Lambda \bar{X}_\Lambda)^{\frac{1}{2}}$, $(X^\Lambda X_\Lambda = 0)$ we get

$$\mathrm{Im}\mathcal{N}_{\Lambda\Sigma} = \frac{i}{2}(S - \bar{S})L_{\Lambda\Sigma} \ , \tag{64}$$

with $L_{\Lambda\Sigma} = \eta_{\Lambda\Sigma} - 2(\Phi_\Lambda\bar{\Phi}_\Sigma + \bar{\Phi}_\Lambda\Phi_\Sigma)$, $L_{\Lambda\Sigma}L^{\Sigma\Delta} = \delta_\Sigma^\Delta$ so we finally obtain

$$\mid Z_i \mid^2 + \mid Z \mid^2 = -n_\Lambda n_\Sigma L^{\Lambda\Sigma}\frac{\mid m_1 - m_2 S\mid^2}{i(S-\bar{S})} \tag{65}$$

and we have set

$$n_m^\Lambda = m_2 n^\Lambda \quad , \quad n_\Lambda^e = m_1 n_\Lambda \ . \tag{66}$$

Since we have that

$$\mid Z \mid^2 = \frac{1}{i(S-\bar{S})} n_\Lambda n_\Sigma \Phi^\Lambda \bar{\Phi}^\Sigma \mid m_1 - m_2 S \mid^2 \tag{67}$$

we get finally

$$\mid Z_i \mid^2 - 3 \mid Z \mid^2 = -n_\Sigma n^\Sigma \frac{\mid m_1 - m_2 S\mid^2}{i(S-\bar{S})} \ . \tag{68}$$

The factor 3 can be understood because $i$ runs over $n+1$ values and one has to match the $(n,2)$ signature of the right hand side.

## 4. TYPE IIB STRINGS ON CALABI-YAU THREEFOLDS

We can specify the previous formulae to the case of Type IIB 10D supergravity on Calabi-Yau manifolds. In this case $n_V = h_{21}$ and the holomorphic section [17,24,25]

$$(X^\Lambda, F_\Lambda) \tag{69}$$

is related to the $(3,0)$ form $\Omega$

$$\Omega = X^\Lambda \alpha_\Lambda - F_\Lambda \beta^\Lambda \ , \tag{70}$$

where $(\alpha,\beta)$ is a basis in $H^3$ with $\int \alpha_\Lambda \wedge \beta^\Sigma = \delta_\Lambda^\Sigma$, $\int \alpha_\Lambda \wedge \alpha_\Sigma = \int \beta^\Lambda \wedge \beta^\Sigma = 0$. The self-dual five form of the Type IIB supergravity can be written as follows

$$\mathcal{Z} = \mathcal{F}^\Lambda \alpha_\Lambda - \mathcal{G}_\Lambda \beta^\Lambda \tag{71}$$

with the symplectic (real) vector field strength as defined by eq. (45). It is easy to show that

$$\int_{S_2\times B} Z = n_m^\Lambda \quad , \quad \int_{S_2 \times A} Z = n_\Lambda^e \tag{72}$$

and, denoting by $*$ the Hodge dual,

$$\mathcal{Z}^* = \mathcal{Z} \tag{73}$$

due to the property[36]

$$\begin{aligned} \alpha^* &= A\alpha + B\beta \quad \alpha^{**} = -\alpha \\ \beta^* &= C\alpha + D\beta \quad \beta^{**} = -\beta \ , \end{aligned} \tag{74}$$

where

$$\mathcal{S} = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \tag{75}$$

is a symplectic matrix with entries

$$\begin{aligned} A &= -D^T = \mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1} \\ C &= (\mathrm{Im}\mathcal{N})^{-1} \\ B &= -\mathrm{Im}\mathcal{N} - \mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} \end{aligned} \tag{76}$$

and satisfying $\mathcal{S}^2 = -\mathbb{1}$. From eqs. (70) and (74)-( 76) it also follows that

$$\Omega^* = i\bar{\Omega} \quad , \quad D_i\Omega^* = -i\, D_{\bar{\imath}}\bar{\Omega} \ . \tag{77}$$

It is easy to see that the matrix $\mathcal{S}$ is related to the symplectic matrix $\mathcal{M}$ defined by eq. (59) by the relation

$$\mathcal{M} = \mathcal{I}\mathcal{S} \quad , \quad \mathcal{I} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \ . \tag{78}$$

The real cohomology basis in $H^3$ can be decomposed in the Dolbeault cohomology basis

$$H^3 = H^{(3,0)} + H^{(2,1)} + H^{(1,2)} + H^{(0,3)} \ . \tag{79}$$

This corresponds to introducing

$$\Omega \quad , \quad \bar{\Omega} \quad , \quad D_i\Omega \quad , \quad D_{\bar{\imath}}\bar{\Omega} \tag{80}$$

with the following vanishing intersections

$$\int \Omega \wedge D_i \Omega = \int \Omega \wedge D_{\bar{\imath}}\bar{\Omega} = 0 \ , \tag{81}$$

where $D_i\Omega = \partial_i\Omega - \frac{<\partial_i\Omega,\bar{\Omega}>}{<\Omega,\bar{\Omega}>}$, $i < \Omega, \bar{\Omega} >= e^{-K}$, $G_{i\bar{\jmath}} = -i < D_i\Omega, D_{\bar{\jmath}}\bar{\Omega} > e^K$. It is easy to see that $e^{K/2}\Omega \to V$, $e^{K/2}D_i\Omega \to U_i$ as defined in section 2 by eqs. (2), (3). If one defines the auxiliary five-form

$$\begin{aligned} &\mathcal{Z}^- = \mathcal{F}^{-\Lambda}\alpha_\Lambda - \mathcal{G}_\Lambda^- \beta^\Lambda = \\ &i\, e^{K/2}(T^-\bar{\Omega} + \mathcal{F}^{-i} D_i \Omega) \end{aligned} \tag{82}$$

(the above identity follows from (46)-(48) ), then [18,35]

$$\mathcal{Z} = \frac{i}{2}e^{K/2}(T^-\bar\Omega - T^+\Omega + \mathcal{F}^{-i}D_i\Omega - \mathcal{F}^{+\bar\imath}D_{\bar\imath}\bar\Omega) = \mathrm{Re}\,\mathcal{Z}^- \ . \qquad (83)$$

We then see that from eq. (83) and (49), (50) it follows that

$$\int_{S_2\times C} Z\wedge\Omega = -\frac{1}{2}e^{-K/2}\int_{S_2} T^- = X^\Lambda n^e_\Lambda - F_\Lambda n^\Lambda_m = Z^{(3,0)} \ , \qquad (84)$$

$$\int_{S_2\times C} Z\wedge D_i\Omega = -\frac{1}{2}e^{-K/2}\int_{S_2} \mathcal{F}^{+\bar\jmath}G_{i\bar\jmath} = D_iX^\Lambda n^e_\Lambda - D_iF_\Lambda n^\Lambda_m = Z^{(2,1)}_i \ . \qquad (85)$$

The charges defined by eqs. (84), (85) $(Z^{(3,0)}, Z^{(2,1)}_i)$ define the physical charges with respect to the vector field strengths which are supermultiplet eigenstates. In fact, $(Z^{(3,0)}, Z^{(0,3)}, Z^{(2,1)}_i, Z^{(1,2)}_{\bar\imath})$ correspond to electric and magnetic charges of the $U(1)^{h_{21}+1}$ gauge group of the complex cohomology. We also note that the 4D antiselfdual part of the five-form $\mathcal{Z}$ decomposes under $H^{(0,3)}+H^{(2,1)}$, due to the helicity properties of the gravitino and gaugino.

Note that in the Type II string theories these charges are nonperturbative in nature[23,11] since all the gauge fields are Ramond-Ramond vectors (coming from the five-form). The charges $(Z^{(3,0)}, Z^{(2,1)}_i)$ are then due to the solitonic three-brane configurations. The BPS states have masses given by

$$M = e^{K/2} \mid Z^{(3,0)} \mid = \mid Z \mid \ . \qquad (86)$$

The charges $(Z^{(3,0)}, Z^{(2,1)}_i)$ satisfy sum rules given by eqs. (56), (57). These charges are the generalization to Calabi-Yau manifolds of particular combinations $Q^2_R$ and $Q^2_L$ of string theories. Black hole level matching[9] should be compared to a quantization condition between these quantities, as shown in the examples at the end of section 3.

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