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Quantum Poisson-Lie T-duality and WZNW model

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Abstract

A pair of conformal σ-models related by Poisson-Lie T-duality is constructed by starting with the $O(2,2)$ Drinfeld double. The duality relates the standard $SL(2,R)$ WZNW model to a constrained σ-model defined on the $SL(2,R)$ group space. The quantum equivalence of the models is established by using a path integral argument.

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1 Introduction

Target space duality in string theory has attracted a considerable attention in recent years because it sheds some light on the geometry and symmetries of string theory. Much is known about the standard Abelian σ-model T-duality [1, 2, 3, 4, 5, 6, 7]. However, the structure and role of the non-Abelian duality still remains to be uncovered. The non-Abelian duality between the isometric σ-model on a group manifold G and the non-isometric σ-model on its Lie algebra $\mathcal{G}$ discussed in [8, 9, 10, 11] did miss a lot of features characteristic to the Abelian duality. One could even question if the term 'duality' was applicable since the original and the dual σ-models did not enter the picture symmetrically. Indeed, while the original model on G was isometric, which was believed to be an essential condition for performing a duality transformation, the 'dual' one did not possess the G-isometry. As a result, it was not known how to perform the inverse duality transformation to get back to the original model.

A solution to this problem was proposed recently in [12] where it was argued that the two theories are, indeed, *dual* to each other from the point of view of the so called Poisson-Lie T-duality (this term was introduced later in [13]). In [12] a large class of new dual pairs of σ-models associated with each Drinfeld double D [14] (or, more precisely, with each Manin triple $(\mathcal{G}, \tilde{\mathcal{G}}, \mathcal{D})$ corresponding to D) was constructed. The main idea of the approach is to replace the so far key feature of the T-duality - the requirement of isometry - by a weaker condition which is the Poisson-Lie symmetry of the theory. This generalized duality is associated with the two groups forming a Drinfeld double and the duality transformation exchanges their roles. We shall review some elements of the Poisson-Lie T-duality in section 2.

The discussion in [12] was purely classical. It is obviously of central importance to try to understand if there is a quantum analogue of the Poisson-Lie T-duality relating appropriate correlation functions in the two models. In particular, one would like to know if there are dual pairs of conformal σ-models. This is the question we address in the present paper. As we shall show in section 3, there exists a simple example of such dual pair associated with the $O(2,2)$ Drinfeld double. Here $G = SL(2,R)$ and the corresponding model is a constrained σ-model with a target space being $SL(2,R)$ group manifold. Its dual associated with $\tilde{G} = B_2$ (where B_2 is the Borel subgroup of $SL(2,C)$) is the $SL(2,R)$ WZNW model. In the section 4 we shall give a path integral argument demonstrating quantum equivalence of the two such dual models related to the groups G and $\tilde{G}$. A detailed account of our approach and its range of applicability will appear later.

2 Poisson-Lie T-duality

In this section we will describe the construction of a dual pair of σ-models which have equivalent field equations (and symplectic structure of their phase spaces [12]) in the sense that there exists a well defined (though possibly nonlocal) transformation which to every solution of the first model associates a solution of the dual one and vice versa. They are dual in a new 'Poisson-Lie' sense which generalizes the Abelian T-duality [1] and the non-Abelian T-duality between σ-models on a group and on its Lie algebra [8, 9, 10, 11].[4] For simplicity we shall consider only the case of the σ-models defined on the group space; generalization to the case when a group G acts freely on the target space (G-bundle), the Abelian analogue of which was discussed in [1, 4], was given in [12, 13].

For the description of the Poisson-Lie duality we need the crucial concept of the Drinfeld double which is simply a Lie group D such that its Lie algebra $\mathcal{D}$ can be decomposed into a pair of maximally isotropic subalgebras with respect to a non-degenerate invariant bilinear form on $\mathcal{D}$ [14]. An isotropic subspace of $\mathcal{D}$ is such that the value of the invariant form on any two vectors belonging to the subspace vanishes (maximally isotropic means that this subspace cannot be enlarged while preserving its isotropy). Any such decomposition of the double into the pair of maximally isotropic subalgebras $\mathcal{G} + \tilde{\mathcal{G}} = \mathcal{D}$ is usually referred to as the Manin triple.

Consider now an n-dimensional linear subspace $\mathcal{E}^+$ of the Lie algebra $\mathcal{D}$ and its orthogonal complement $\mathcal{E}^-$ such that $\mathcal{E}^+ + \mathcal{E}^-$ span the whole algebra $\mathcal{D}$. We shall show that these data determine a dual pair of σ-models with the targets being the groups G and $\tilde{G}$ respectively [12, 13]. Indeed, consider the following *field equations* for the mapping $l(\xi^+, \xi^-)$ from the two-dimensional spacetime with light-cone variables $\xi^\pm$ into the Drinfeld double group D

$$\langle \partial_\pm l \, l^{-1}, \mathcal{E}^\mp \rangle = 0. \tag{1}$$

[4] In [15] there was suggested a connection between non-Abelian axial-vector duality [16] and Poisson-Lie T-duality.

Here the brackets denote the invariant bilinear form. In the vicinity of the unit element of D, there exists the unique decomposition of an arbitrary element of D into the product of elements from G and $\tilde{G}$, i.e.

$$l(\xi^+, \xi^-) = g(\xi^+, \xi^-)\tilde{h}(\xi^+, \xi^-). \tag{2}$$

Inserting this ansatz into Eq. (1) we obtain

$$\langle g^{-1}\partial_\pm g + \partial_\pm \tilde{h}\tilde{h}^{-1}, g^{-1}\mathcal{E}^\mp g\rangle = 0. \tag{3}$$

It is convenient to introduce a pair of bases T^i and $\tilde{T}_i$ in the algebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ respectively, satisfying the duality condition

$$\langle T^i, \tilde{T}_j\rangle = \delta^i_j. \tag{4}$$

Suppose there exists a matrix $E^{ij}(g)$ such that $(i, j = 1, \ldots, n)$

$$g^{-1}\mathcal{E}^+ g = \mathrm{Span}(T^i + E^{ij}(g)\tilde{T}_j), \tag{5}$$

$$g^{-1}\mathcal{E}^- g = \mathrm{Span}(T^i - E^{ji}(g)\tilde{T}_j). \tag{6}$$

The explicit dependence of the matrix E on g is given by the matrices of the adjoint representation of D and is easily obtained as follows [12, 13]:

$$\begin{aligned} g^{-1}\mathcal{E}^+ g &= \mathrm{Span}\ g^{-1}(T^i + E^{ij}(e)\tilde{T}_j)g \\ &= \mathrm{Span}\Big[(a(g)^i{}_l + E^{ij}(e)b(g)_{jl})T^l + E^{ij}(e)d(g)_j{}^l\tilde{T}_l\Big], \end{aligned} \tag{7}$$

where

$$g^{-1}T^i g \equiv a(g)^i{}_l T^l, \qquad g^{-1}\tilde{T}_j g \equiv b(g)_{jl}T^l + d(g)_j{}^l\tilde{T}_l. \tag{8}$$

Hence the matrix $E(g)$ is given by

$$E(g) = [a(g) + E(e)b(g)]^{-1}E(e)d(g). \tag{9}$$

We can now rewrite the field equations (3) in the form

$$-(\partial_-\tilde{h}\tilde{h}^{-1})^i = E^{ij}(g)(g^{-1}\partial_- g)_j \equiv A^i_-(g), \tag{10}$$

$$-(\partial_+\tilde{h}\tilde{h}^{-1})^i = -E^{ji}(g)(g^{-1}\partial_+ g)_j \equiv A^i_+(g). \tag{11}$$

This implies the 'zero curvature' condition for $A^i(g)$:

$$\partial_- A^i_+(g) - \partial_+ A^i_-(g) - \tilde{c}_{kl}{}^i A^k_+(g)A^l_-(g) = 0, \tag{12}$$

where $\tilde{c}_{kl}{}^{i}$ are the structure constants of the Lie algebra $\tilde{\mathcal{G}}$. Remarkably, it can be checked directly that the equations (12) are just the field equations for the σ-model on the group space G with the Lagrangian

$$L = E^{ij}(g)(g^{-1}\partial_+ g)_i(g^{-1}\partial_- g)_j. \tag{13}$$

Let us suppose now that instead of (2) we use the decomposition

$$l(\xi^+,\xi^-) = \tilde{g}(\xi^+,\xi^-)h(\xi^+,\xi^-), \tag{14}$$

where $\tilde{g} \in \tilde{G}$ and $h \in G$. If we assume that the matrix $E^{ij}(g)$ is invertible (in the vicinity of the origin of the group $g = e$ this is implied by the invertibility of $E^{ij}(e)$) then all the steps of the previous construction can be repeated. We end up with the dual σ-model

$$\tilde{L} = \tilde{E}_{ij}(\tilde{g})(\tilde{g}^{-1}\partial_+\tilde{g})^i(\tilde{g}^{-1}\partial_-\tilde{g})^j, \tag{15}$$

where the matrix $\tilde{E}_{ij}(\tilde{g})$ is defined by $(i = 1,\ldots,n)$

$$\tilde{g}^{-1}\mathcal{E}^+\tilde{g} = \mathrm{Span}(\tilde{T}_i + \tilde{E}_{ij}(\tilde{g})T^j), \tag{16}$$

$$\tilde{g}^{-1}\mathcal{E}^-\tilde{g} = \mathrm{Span}(\tilde{T}_i - \tilde{E}_{ji}(\tilde{g})T^j), \tag{17}$$

and is given by the exact analogue of the formula (9). Clearly, at the origin of the group ($g = e$ and $\tilde{g} = \tilde{e}$ respectively) the matrices E and $\tilde{E}$ are related as follows

$$E(e)\tilde{E}(\tilde{e}) = \tilde{E}(\tilde{e})E(e) = 1. \tag{18}$$

This is an indication that we have indeed obtained a generalization of the standard Abelian '$R \to 1/R$' duality.

If the matrix $E^{ij}(e)$ is not invertible, it may seem that the dual model does not exist because for the dual decomposition (14) we cannot find the dual σ-model matrix $\tilde{E}_{ij}(\tilde{g})$. However, we may proceed as follows. Let us represent the subspaces $\tilde{g}^{-1}\mathcal{E}^{\pm}\tilde{g}$ in (16), (17) as

$$\tilde{g}^{-1}\mathcal{E}^+\tilde{g} = \mathrm{Span}(\tilde{F}^{ij}(\tilde{g})\tilde{T}_j + T^i), \tag{19}$$

$$\tilde{g}^{-1}\mathcal{E}^-\tilde{g} = \mathrm{Span}(-\tilde{F}^{ji}(\tilde{g})\tilde{T}_j + T^i). \tag{20}$$

The existence of such a matrix $\tilde{F}^{ij}(\tilde{g})$ is correlated with the existence of the matrix $E^{ij}(g)$ in (5) or (6). In fact,

$$\tilde{F}(\tilde{e}) = E(e). \tag{21}$$

The explicit dependence of the matrix $\tilde{F}$ on $\tilde{g}$ is again given by the matrices of the adjoint representation of D. Here we have (cf. (7)–(9))

$$\begin{aligned} \tilde{g}^{-1}\mathcal{E}^{+}\tilde{g} = \text{Span}\ \tilde{g}^{-1}(T^i + \tilde{F}^{ij}(\tilde{e})\tilde{T}_j)\tilde{g} \\ = \text{Span}\Big[(\tilde{F}^{ij}(\tilde{e})\tilde{a}(\tilde{g})_j{}^l + \tilde{b}(\tilde{g})^{il})\tilde{T}_l + \tilde{d}(\tilde{g})^i{}_l T^l\Big], \end{aligned} \tag{22}$$

$$\tilde{g}^{-1}\tilde{T}_i\tilde{g} \equiv \tilde{a}(\tilde{g})_i{}^l\tilde{T}_l, \qquad \tilde{g}^{-1}T^i\tilde{g} \equiv \tilde{b}(\tilde{g})^{il}\tilde{T}_l + \tilde{d}(\tilde{g})^j{}_l T^l, \tag{23}$$

$$\tilde{F}(\tilde{g}) = \tilde{d}(\tilde{g})^{-1}(\tilde{F}(\tilde{e})\tilde{a}(\tilde{g}) + \tilde{b}(\tilde{g})). \tag{24}$$

For the dual decomposition (14), the field equations which follow from (1) are

$$\langle \tilde{g}^{-1}\partial_\pm\tilde{g} + \partial_\pm h h^{-1}, \tilde{g}^{-1}\mathcal{E}^{\mp}\tilde{g}\rangle = 0. \tag{25}$$

Using (19),(20) we obtain

$$(\tilde{g}^{-1}\partial_-\tilde{g})^i + \tilde{F}^{ij}(\tilde{g})(\partial_- h h^{-1})_j = 0, \tag{26}$$

$$(\tilde{g}^{-1}\partial_+\tilde{g})^i - \tilde{F}^{ji}(\tilde{g})(\partial_+ h h^{-1})_j = 0. \tag{27}$$

This set of equations can be represented in the following equivalent form

$$(\tilde{g}^{-1}\partial_-\tilde{g})^i + \tilde{F}^{ij}(\tilde{g})\lambda_{-j} = 0, \tag{28}$$

$$(\tilde{g}^{-1}\partial_+\tilde{g})^i - \tilde{F}^{ji}(\tilde{g})\lambda_{+j} = 0, \tag{29}$$

$$\partial_-\lambda_{+i} - \partial_+\lambda_{-i} + c^{kl}{}_i\lambda_{+k}\lambda_{-l} = 0, \tag{30}$$

where $c^{kl}{}_i$ are the structure constants of the Lie algebra $\mathcal{G}$. In (30) we recognize the zero curvature condition for the 'currents' in $\mathcal{G}$. The equations (28)-(30) are just the Euler-Lagrange equations corresponding to

$$\tilde{L} = -\lambda_{+i}\tilde{F}^{ij}(\tilde{g})\lambda_{-j} + \lambda_{+i}(\tilde{g}^{-1}\partial_-\tilde{g})^i + \lambda_{-i}(\tilde{g}^{-1}\partial_+\tilde{g})^i. \tag{31}$$

If the matrix $\tilde{F}(\tilde{g})$ were invertible, integrating out $\lambda_\pm$ from (31) we would get just the σ-model (15) with

$$\tilde{E}(\tilde{g}) = \tilde{F}^{-1}(\tilde{g}).$$

If $\tilde{F}(\tilde{g})$ does not have the inverse, we may integrate out only 'non-null' parts of λ's, while their 'null-vector' parts will play the role of the Lagrange multipliers, constraining the corresponding projections of the currents $\tilde{g}^{-1}\partial_\pm\tilde{g}$

to vanish. Hence we get a constrained (or 'singular') σ-model defined on the group space $\tilde{G}$. An example of such model will be discussed in the next section.[5]

To summarize, the Poisson-Lie duality is well defined also in the case when the σ-model matrix $E(g)$ of one of the models in the dual pair (with the group G as the target) is degenerate. Then the dual model action can written in the first-order form (31) and can be interpreted as that of the constrained σ-model with the target being the dual group $\tilde{G}$.[6] Similar situation happened in the context of the Abelian duality in a direction of a null isometry. It thus appears that the Poisson-Lie duality naturally acts on the set of σ-models enlarged by the constrained ones (for a discussion of such models see also [20, 21]). That means, in a sense, that also σ-model matrices with infinite eigenvalues are to be included into consideration.

Next, let us comment on the field equations for the pair of dual models (13) and (15). They both have the form of the zero curvature conditions with respect to the algebras $\tilde{\mathcal{G}}$ and $\mathcal{G}$. Such σ-models were called Poisson-Lie symmetric in [12, 13] and were shown to be dualizable with respect to the Poisson-Lie duality. It is important to be able to express the corresponding flat connection in terms of the data defining the σ-model. As it is clear from Eq.(30), the currents are most easily identified in the first-order formalism (31). Their components are just the λ's in (31) and this is true no matter whether the matrix $\tilde{F}(\tilde{g})$ is regular or degenerate. In the degenerate case, however, we arrive at an interesting conclusion that some of the components of the non-Abelian connection (which is flat according to equations of motion) play the role of the Lagrange multipliers.

It may seem that the Poisson-Lie T-duality relates only σ-models with group targets G and $\tilde{G}$. However, it was shown in [12, 13] that this $G \leftrightarrow \tilde{G}$ duality is only the special case (referred to as 'atomic' duality in [13]) of the

[5]Let us remark that for $\tilde{F}^{ij}$ being an antisymmetric tensor $\tilde{F}^{ij} = -\tilde{F}^{ji}$, the σ-model (31) becomes topological. The consistency of equations of motion then implies the Jacobi identity for the tensor $\tilde{F}$ which may be interpreted as a Poisson bracket on the target group manifold. Topological σ-models of this type were introduced as Poisson σ-models in [17, 18]. In particular, the case of $\tilde{F}^{ij}(e) = 0$ leads to the gauged WZNW model [18]. We shall discuss topological σ-models in the context of Poisson-Lie T-duality elsewhere.

[6]Alternatively, by analogy with the gauged WZNW type models containing extra auxiliary vector fields (see [19] and refs. there) one may trade λ's for new fields and as a result get a model defined on a target of dimension $\dim\tilde{G}+2\times$ (number of null-vectors of the matrix $\tilde{F}$).

Poisson-Lie T-duality. When the double is Abelian, the atomic duality is just the '$R \leftrightarrow 1/R$' duality between the free scalar theories on the group spaces $U(1)$ and $\tilde{U}(1)$. In the Abelian case, however, the notion of T-duality is much broader: every σ-model such that its target is isometric with respect to the (free) action of the Abelian duality group, is dualizable. Similar generalization is possible for the Poisson-Lie T-duality [12, 13]: every σ-model such that a group G acts freely on its target space and its action is Poisson-Lie symmetric with respect to the dual group $\tilde{G}$ has dual counterpart such that $\tilde{G}$ acts freely on its target space and its action is Poisson-Lie symmetric with respect to G. A full classification of the target spaces admitting Poisson-Lie symmetry, their description in 'adapted' coordinates (in which the Poisson-Lie symmetry is explicit) and the corresponding form of the Poisson-Lie dual target space are given in [12, 13].

3 $O(2,2)$ double and $SL(2,R)$ WZNW model

In this section we shall describe an example of a pair of Posson-Lie dual σ-models which is associated with the $O(2,2)$ double. Consider the Lie algebra $sl(2,R)$ defined by

$$[H, E_\pm] = \pm 2E_\pm, \qquad [E_+, E_-] = H, \tag{32}$$

and equipped with the standard Killing-Cartan non-degenerate symmetric invariant bilinear form

$$\langle E_+, E_- \rangle = 1, \qquad \langle H, H \rangle = 2. \tag{33}$$

It can be checked that the direct sum of the two copies of $sl(2,R)$

$$\mathcal{D} = sl(2,R) \oplus sl(2,R) \tag{34}$$

with the bilinear form (also denoted by $\langle .,. \rangle$)

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle \tag{35}$$

is the algebra of the Drinfeld double which we shall refer to as the $O(2,2)$ double. The notation $(x_1, x_2) \in \mathcal{D}$ obviously means that x_1 (x_2) is from the first (second) copy of $sl(2,R)$ in (34). The decomposition of the double into the pair of maximally isotropic subalgebras is given as

$$\mathcal{D} = sl(2,R)_{diag} + b_2 \tag{36}$$

where $sl(2,R)_{diag}$ is generated by

$$\tilde{T}_0 = \frac{1}{2}(H,H), \quad \tilde{T}_+ = (E_+,E_+), \quad \tilde{T}_- = (E_-,E_-) \tag{37}$$

and b_2 (which is the Lie algebra of the Borel subgroup B_2 of $SL(2,C)$) by

$$T^0 = \frac{1}{2}(H,-H), \quad T^+ = (0,-E_-), \quad T^- = (E_+,0). \tag{38}$$

These two sets of generators are dual to each other in the sense of (4).

Starting with the double as the group $D = SL(2,R) \times SL(2,R)$ we can follow the procedure of section 2 and choose two mutually orthogonal subspaces $\mathcal{E}^{\pm}$

$$\mathcal{E}^+ = (sl(2,R),0), \qquad \mathcal{E}^- = (0, sl(2,R)). \tag{39}$$

The reason for such a choice is that the corresponding field equations (1) are manifestly chiral,

$$\partial_\pm g_\mp = 0, \qquad l \equiv g_+ \times g_-. \tag{40}$$

Here g_+ and g_- are the elements from the first and the second copy of $SL(2,R)$ from the decomposition of the double and $\times$ means the direct product of two $SL(2,R)$ elements. The general solution of (40) is thus

$$l(\xi^+,\xi^-) = g_+(\xi^+) \times g_-(\xi^-)\ . \tag{41}$$

The appearence of the non-Abelian chiral bosons suggests a close relation to WZNW model and thus a possibility of conformal invariance of the quantum theory.

To obtain the dual pair of σ-model corresponding to the choice (39) we are to find the explicit form of Eqs. (5) and (19) in this case. According to (2), (36) we may write

$$g_+ \times g_- = (b_+ \times b_-)(\zeta \times \zeta) \tag{42}$$

where $b_+ \times b_- \equiv b \in B_2$ and $\zeta \times \zeta \in SL(2,R)_{diag}$. As follows from (38), the ordinary (not direct) product of the diagonal parts of the $SL(2,R)$ elements

$b_\pm$ is the identity element in $SL(2,R)$. The corresponding σ-model with the Borel group B_2 as the target space is

$$L = E^{ij} J_{+i}(b) J_{-j}(b), \tag{43}$$

where $J_{\pm i}(b)T^i$ are defined by

$$b^{-1}\partial_\pm b \equiv J_{\pm i}(b)T^i \tag{44}$$

and the matrix E^{ij} has the following explicit form

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{45}$$

The reason why the matrix E^{ij} does not depend on the group variable b is that the chosen subspaces $\mathcal{E}^\pm$ are invariant with respect to the adjoint action on the double.

Since E is *singular*, the dual model should be described by the first-order Lagrangian (31). Indeed, using the dual parametrization of the double (cf. (2),(14),(42))

$$g_+ \times g_- = (\eta \times \eta)(c_+ \times c_-), \tag{46}$$

where $\eta \times \eta \in SL(2,R)_{diag}$ and $c_+ \times c_- \in B_2$, and the relation (21) we find that (31) takes the form

$$\tilde{L} = -\lambda_{+i} E^{ij} \lambda_{-j} + \lambda_{+i} J^i_-(\eta) + \lambda_{-i} J^i_+(\eta). \tag{47}$$

Here $J^i_\pm(\eta)$ are defined by

$$\eta^{-1}\partial_\pm \eta \equiv J_\pm(\eta) = \frac{1}{2} J^0_\pm(\eta) H + J^+_\pm(\eta) E_+ + J^-_\pm(\eta) E_-. \tag{48}$$

Let us now look more closely at the pair of the mutually dual models (43) and (47). The first Lagrangian (43) can be rewritten in terms of the $sl(2,R)$-currents $J^i_\pm(b_\pm)$ defined as in (48)

$$L(b) = -J^0_+(b_+) J^0_-(b_-) - J^-_+(b_-) J^+_-(b_+). \tag{49}$$

This follows from the obvious relations

$$J_0(b) = J^0(b_+) = -J^0(b_-), \quad J_+(b) = -J^-(b_-), \quad J_-(b) = J^+(b_+), \tag{50}$$

where $J_i(b)$ were defined in (44) (we suppress the 2d space-time indices of these currents).

The action corresponding to (49) is nothing but the $SL(2,R)$ WZNW action I for the argument $b_-b_+^{-1}$ (we shall not explicitly indicate the standard measure of integration $d\xi^+d\xi^-$)

$$\int L(b) = -4I(b_-b_+^{-1}), \tag{51}$$

where

$$I(u) \equiv \frac{1}{8\pi}\int_{\partial M} \mathrm{Tr}\ (\partial_+ u \partial_- u^{-1}) + \frac{1}{12\pi}\int_M \mathrm{Tr}\ (u^{-1}du)^{\wedge 3}. \tag{52}$$

One can easily check this using the Polyakov-Wiegmann relation [22]

$$I(ff') = I(f) + I(f') - \frac{1}{4\pi}\int \mathrm{Tr}(f^{-1}\partial_+ f \partial_- f' f'^{-1}). \tag{53}$$

The conclusion is that the first (43) in the pair of dual models is actually the WZNW model on the group manifold $SL(2,R)$. Indeed, the combination $b_-b_+^{-1}$ can be interpreted as the Gauss decomposition of a group element u parametrizing the $SL(2,R)$ group space. The Gauss decomposition $u = b_-b_+^{-1}$ was used [23] in representing the WZNW theory in terms of free fields. We have gone in the opposite direction, starting from the simple action (49) on the Borel group B_2 and recovering the WZNW action at the end.

The fact that the σ-model on the Borel group has an interpretation in terms of the $SL(2,R)$ WZNW model is quite interesting. By construction, the dual model also has the $SL(2,R)$ group as a target space. Thus the duality relates two different σ-models (unconstrained and constrained one) defined on the $SL(2,R)$ space. Under an appropriate choice of the currents, the σ-model matrix of one model is the inverse (in a loose sense, since E is degenerate) of the matrix of the other. This is true globally, i.e. not only at the group's origin as in the generic Poisson-Lie duality case (18).

The Lagrangian (47) of the second model can be put in a simpler form by integrating out all λ's except the Lagrange multipliers (those which drop out from the first term in (47)). The result is the Lagrangian of the constrained σ-model[7]

$$\tilde{L}(\eta) = J_+^0(\eta)J_-^0(\eta) + J_-^+(\eta)J_+^-(\eta) + \lambda_-^+ J_+^+(\eta) + \lambda_+^- J_-^-(\eta). \tag{54}$$

[7] As remarked at the end of section 2, the Lagrange multipliers are the components of the (flat) connection so it is convenient to write them also with the Lie algebra indices.

We shall give the explicit form of (54) (in a particular parametrisation of η) in the next section.

4 Duality and path integral

So far our discussion was purely classical – we have demonstrated the duality between the pair of the σ-models with the targets G and $\tilde{G}$ at the level of equations of motions. It would be highly desirable to find a path integral formulation of the Poisson-Lie T-duality and establish a quantum equivalence of the models. This, indeed, is possible to do in the example discussed in the previous section.

Consider the following action [3, 19]

$$S(g_+, g_-) = k[I_+(g_+) + I_-(g_-)], \tag{55}$$

where k=const and $I_\pm(g_\pm)$ are defined by ($\partial_{0,1} \equiv \frac{1}{2}(\partial_+ \pm \partial_-)$)

$$I_\pm(g_\pm) = \pm\frac{1}{4\pi}\int_{\partial M} \mathrm{Tr}\ (\partial_1 g_\pm^{-1}\partial_\mp g_\pm) + \frac{1}{12\pi}\int_M \mathrm{Tr}\ (g_\pm^{-1} dg_\pm)^{\wedge 3}, \tag{56}$$

or, equivalently, by (I is the WZNW action (52))

$$I_-(g_-) = I(g_-) - \frac{1}{8\pi}\int \mathrm{Tr}(\partial_+ g_- \partial_+ g_-^{-1}), \tag{57}$$

$$I_+(g_+) = I(g_+) - \frac{1}{8\pi}\int \mathrm{Tr}(\partial_- g_+ \partial_- g_+^{-1}). \tag{58}$$

The actions $I_\pm$ [24, 3] describe the non-Abelian chiral scalars. The corresponding field equations $\partial_1(g_\mp^{-1}\partial_\pm g_\mp) = 0$ imply

$$\partial_\pm g_\mp = 0, \tag{59}$$

provided the fields $g_\pm(\xi^+, \xi^-)$ are subject to appropriate boundary conditions [25]. The latter should be such that the equation $\partial_1 f = 0$ should have the unique solution $f = 0$, i.e.

$$\partial_1 f(g_+, g_-) = 0 \quad \rightarrow \quad f(g_+, g_-) = 0. \tag{60}$$

The actions (55)–(58) are of first-order type, i.e. are linear in the time derivatives of the fields. It is therefore natural to try to integrate one 'half' of the field variables in (55) to end up with a second-order and Lorentz invariant action. In fact, starting with (55) it is possible to integrate out the 'ratio' of g_- and g_+ explicitly, ending up with the standard WZNW action for g_-g_+ [3].

As we shall see below, there are two dual choices of what should be the 'half' of the variables to be integrated away. In the case of the double $O(2,2)$ these choices lead precisely to the dual pair of actions (43) and (47) considered in the previous section.

The equations (59) are identical to the basic equations (40) on the double. Thus we may attempt to use the action $S(g_+^{-1}, g_-)$ on the double as the one which 'interpolates' between the two dual σ-models. This basic idea can be illustrated on the simplest $U(1) \times U(1)$ free-theory case ($g_\pm = e^{ix_\pm}$) [3]

$$S = k[I_+(x_+) + I_-(x_-)], \quad I_\pm(x_\pm) = \pm\frac{1}{4\pi}\int \partial_1 x_\pm \partial_\mp x_\pm. \qquad (61)$$

Introducing the new fields $x, \tilde{x}$

$$x = \frac{1}{\sqrt{2}R}(x_+ + x_-), \qquad \tilde{x} = \frac{R}{\sqrt{2}}(x_+ - x_-) \qquad (62)$$

we get the action

$$S = \frac{k}{4\pi}\int (\partial_0 x \partial_1 \tilde{x} + \partial_1 x \partial_0 \tilde{x} - R^2 \partial_1 x \partial_1 x - R^{-2}\partial_1 \tilde{x}\partial_1 \tilde{x}), \qquad (63)$$

which is invariant under $x \to \tilde{x}, \tilde{x} \to x,\ R \to 1/R$. Integrating away in the path integral the variable x or $\tilde{x}$, one obtains the pair of free scalar actions related by the standard Abelian duality.

We shall argue that the non-Abelian generalizations of the relations (63) are given by the products (2) and (14) in the Drinfeld double. Note that the treatment of the non-Abelian case requires the introduction of the four variables $g, \tilde{g}$ and $h, \tilde{h}$ as opposed to the two variables x and $\tilde{x}$ in (63). The reason for that is that g ($\tilde{g}$) does not commute with $\tilde{h}$ (h).

Consider now the path integral for the non-Abelian action $S(g_+^{-1}, g_-)$ (55)

$$Z = \int [dg_+][dg_-] \exp\left[iS(g_+^{-1}, g_-)\right] \qquad (64)$$

and assume that the first parametrization (42) is used. Using the Polyakov-Wiegmann relation (53) we get

$$Z = \int [db][d\zeta] \exp ik[I(b_-b_+^{-1}) - \frac{1}{8\pi}\int \mathrm{Tr}(2\partial_1\zeta\zeta^{-1} - J_-(b_+) + J_+(b_-))^2]. \tag{65}$$

The integration over ζ gives a trivial contribution [3] since one can replace the integral over ζ by the integral over $B = \partial_1\zeta\zeta^{-1}$ (the resulting Jacobian is equal to one under the choice of the boundary conditions (60)).[8] As a result,

$$Z = \int [db] \exp [ikI(b_-b_+^{-1})], \tag{66}$$

i.e. we get precisely the first model (51) of our dual pair.

To obtain the dual model we have to start with the parametrization (46) and integrate away the fields $c_\pm$. Choosing the parametrization of $c_\pm$ such that

$$g_- = \eta c_- = \eta \begin{pmatrix} e^\chi & 0 \\ 0 & e^{-\chi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \theta & 1 \end{pmatrix} \equiv \eta_- \begin{pmatrix} 1 & 0 \\ \theta & 1 \end{pmatrix}, \tag{67}$$

$$g_+ = \eta c_+ = \eta \begin{pmatrix} e^{-\chi} & 0 \\ 0 & e^\chi \end{pmatrix} \begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix} \equiv \eta_+ \begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix}, \tag{68}$$

we get

$$I_-(g_-) = I_-(\eta_-) + \frac{1}{2\pi}\int J_+^+(\eta_-)\partial_1\theta, \tag{69}$$

$$I_+(g_+^{-1}) = I_+(\eta_+^{-1}) - \frac{1}{2\pi}\int J_-^-(\eta_+)\partial_1\rho, \tag{70}$$

where the current components were defined in (48). Thus

$$\begin{aligned} Z = \int & [d\eta][d\chi][d\theta][d\rho] \exp ik[I_-(\eta_-) + I_+(\eta_+^{-1})] \\ & \times \exp (\frac{1}{2\pi}ik\int [J_+^+(\eta_-)\partial_1\theta - J_-^-(\eta_+)\partial_1\rho]). \end{aligned} \tag{71}$$

Using that

$$I_-(\eta_-) = I_-(\eta) + \frac{1}{2\pi}\int (\partial_+\chi\partial_1\chi + J_+^0(\eta)\partial_1\chi), \tag{72}$$

[8] The determinant of the operator $\partial_1 - B$ should be computed using the Green fuction $\theta(x_1 - x_1')$ of ∂_1. This choice makes the problem essentially a one-dimensional one and ensures the absence of anomaly.

$$I_+(\eta_+^{-1}) = I_+(\eta^{-1}) - \frac{1}{2\pi}\int(\partial_-\chi\partial_1\chi - J_-^0(\eta)\partial_1\chi), \tag{73}$$

and the Polyakov-Wiegmann relation (53), the total action can be rewritten as

$$S(g_+^{-1}, g_-) = \frac{1}{4\pi}k\int\Big[-(J_+^0(\eta)J_-^0(\eta) + J_+^-(\eta)J_-^+(\eta) + J_+^+(\eta)J_-^-(\eta)) \tag{74}$$
$$+\frac{1}{4}(4\partial_1\chi + J_+^0(\eta) + J_-^0(\eta))^2$$
$$+(J_+^-(\eta) + 2e^{-2\chi}\partial_1\theta)J_+^+(\eta) + (J_-^+(\eta) - 2e^{-2\chi}\partial_1\rho)J_-^-(\eta)\Big].$$

We can now perform the change of variables in the path integral from θ and ρ to λ_-^+ and λ_+^- defined by

$$\lambda_-^+ = -J_+^-(\eta) - 2e^{-2\chi}\partial_1\theta, \tag{75}$$

$$\lambda_+^- = -J_-^+(\eta) + 2e^{-2\chi}\partial_1\rho. \tag{76}$$

The corresponding Jacobian is trivial (under the assumption of the boundary conditions (60) as in (65),(66), implying the Lorentz invariance of the total path integral). Then we are able to integrate away the χ-dependence. After a shift of λ_+^- we finally obtain

$$Z = \int[d\eta][d\lambda_+^-][d\lambda_-^+]\exp\Big(-\frac{1}{4\pi}ik\int[J_+^0(\eta)J_-^0(\eta) \tag{77}$$
$$+J_-^+(\eta)J_+^-(\eta) + \lambda_-^+J_+^+(\eta) + \lambda_+^-J_-^-(\eta)]\Big),$$

which is precisely the path integral corresponding to our dual model (54).[9]

The original action S (55) is conformal, being the sum of two chiral WZNW actions (in particular, Z in (64) does not depend on the conformal factor of the 2-metric apart from the overall 'central charge' term). This means that the actions we got from (64) by partial integrating out the subsets of variables are also conformal.

[9]Note that the overall constant k was not changed to $1/k$. This does not happen also in the case of the Abelian T-duality (cf. (62) and (63)). It is true that the Abelian duality (and in general Poisson-Lie duality, see (18)) change 'part' of the coupling matrix into its inverse, but not the overall coefficient (we consider the 'local' duality transformations, i.e. ignore the issue of zero modes).

The action (54) and the corresponding path integral (77) can be put in more explicit form by using the following parametrization of $\eta \in SL(2,R)$

$$\eta = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} 1 & v^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\phi} & 0 \\ 0 & e^{-\phi} \end{pmatrix}, \tag{78}$$

$$J_-^-(\eta) = e^{2\phi}\partial_- u, \qquad J_+^+(\eta) = -e^{-2\phi}v^{-2}\partial_+(u+v). \tag{79}$$

The classical constraints $J_-^- = 0,\ J_+^+ = 0$ then take the chiral form

$$\partial_- u = 0, \qquad \partial_+(u+v) = 0, \qquad u = u_+(\xi^+), \quad v = v_-(\xi^-) - u_+(\xi^+). \tag{80}$$

Under these constraints, the remaining components of the currents are

$$J_-^+ = e^{-2\phi}\partial_- v^{-1}, \quad J_+^- = -e^{2\phi}\partial_+ v, \quad J_-^0 = 2\partial_-\phi, \quad J_+^0 = 2\partial_+(\phi + \ln v), \tag{81}$$

so that the Lagrangian in (54) becomes

$$L' = J_+^0 J_-^0 + J_-^+ J_+^- = 4\partial_+(\phi + \ln v)\ \partial_-\phi + \partial_+ \ln v\ \partial_- \ln v, \tag{82}$$

and thus describes one off-shell scalar degree of freedom ϕ coupled to u_+ and v_-.

After λ's and u, v are integrated over in the path integral (77), one gets also the contributions of the determinants of the operators $e^{\pm 2\phi}\partial_\mp$ which lead (as in the last reference in [23]) to the shift of the coefficient of the $\partial_+\phi\partial_-\phi$ term and to the 2d curvature coupling $R^{(2)}\phi$ which combine to reproduce the central charge of the $SL(2,R)$ WZNW model.[10]

It should be noted that this reduction does not imply, however, that the $SL(2,R)$ WZNW model is equivalent to that of the free ϕ-theory. Indeed, (77) is just the vacuum partition function, while to have a complete picture of duality at the quantum level one is to compare the generating functionals for appropriate correlation functions. The correlators may contain the fields which are simply integrated out in the vacuum case (in particular, they are likely to depend on the 'Lagrange multipliers' λ's in (54),(77)).

[10]Let us note that the Lagrangian of the $SL(2,R)$ WZNW model in the standard Gauss decomposition parametrisation takes the form $L = k(\partial_+\psi\partial_-\psi + e^{-2\psi}\partial_+ u'\partial_- v')$, so that integrating out u' and v' one gets the free action for ψ with shifted $k' = k-2$ and linear 2d curvature coupling term (see Gerasimov et al, ref. [23]). Similar 'reduction' is possible for the chiral actions $I_\pm$ in (56). One finds that similar effective actions for the Cartan variables have the structure $\int[(k-2)\pm\partial_1\psi_\pm\partial_\mp\psi_\pm + \sqrt{g}R^{(2)}\psi_\pm]$. Starting from the sum of these actions (55) and integrating out the combination $\psi_+ - \psi_-$ one finds the corresponding action for the Cartan part $\psi = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-)$ of the WZNW model, cf. (61)–(66).

5 Outlook

It is very likely that the method we have described above can be used for the construction of Poisson-Lie T-dual of any maximally noncompact WZNW model (see [20] and refs. there). Another problem is to find a dictionary between correlators of local operators in the two dual models. This would reveal a nontrivial quantum content of the duality symmetry. In order to complete the full analogy between the standard Abelian and Poisson-Lie duality it would be also important to understand the issue of the zero modes which was not addressed here.

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