CERN-TH/95-248
hep-th/9509095

Poisson-Lie T-duality

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Abstract

A description of dual non-Abelian duality is given, based on the notion of the Drinfeld double. The presentation basically follows the original paper [1], written in collaboration with P. Ševera, but here the emphasis is put on the algebraic rather than the geometric aspect of the construction and a concrete example of the Borelian double is worked out in detail.

Lecture given at
Trieste Conference on S-Duality and Mirror Symmetry, June 1995.

CERN-TH/95-248
September 1995

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1 Introduction

Duality symmetries are expected to play an important role in string theory as tools for disentangling its full symmetry structure. They are specific to string physics and their study led to important insights in understanding the geometry of space-time from the string point of view. At present, much is known about the Abelian T-duality [2] and also the mirror symmetry [3]. In the contribution [4] (see also previous works [5, 6]) there appeared an interesting possibility of non-Abelian generalization of the standard Abelian T-duality. For a σ-model possessing a global non-Abelian isometry with respect to the group G, a non-Abelian dual was found which, however, turned out to lack the isometry that would make it possible to perform the duality transformation back to the original model. In the series of subsequent investigations [7] (for a review see [8, 9]) other relevant insights were obtained, but the understanding of the sense in which both models were 'dual' to each other still was basically missing. The latter problem was solved in [1] where it was shown that the models are indeed dual in the sense of the 'Poisson-Lie' T-duality.

In the present contribution, based on the paper with P. Ševera [1], I will describe the properties of the Poisson-Lie T-duality; in particular I will advocate the idea that the relevant structure underlying the (non-Abelian) T-duality is the Drinfeld double. In what follows, I shall construct mutually dual pairs of σ-models for an arbitrary Drinfeld double; the duality transformation will simply exchange the roles of the two groups forming the double. Then I shall consider the criteria under which a σ-model has its Poisson-Lie dual and, finally, I will present explicit forms of the dual pair of models for the Borelian double.

2 Manin triples and T-duality

In this section I will describe the construction of a dual pair of σ-models which are equivalent in the sense of having the same field equations and the symplectic structure of their phase spaces. They are dual in the new 'Poisson-Lie' sense which generalizes[2] the Abelian T-duality [2] and the non-

[2]Recently, a connection was also found in [10] between non-Abelian axial-vector duality [11] and Poisson-Lie T-duality.

Abelian T-duality between a group and its Lie algebra [4, 5, 6, 7]. For the description of the Poisson-Lie duality I shall need the crucial concept of the Drinfeld double, which is any Lie group D such that its Lie algebra $\mathcal{D}$ can be decomposed into a pair of maximally isotropic subalgebras with respect to a non-degenerate invariant bilinear form on $\mathcal{D}$ [12]. The isotropic subspace of $\mathcal{D}$ is such that the value of the form on two arbitrary vectors belonging to the subspace vanishes; maximally isotropic means that the subspace cannot be enlarged while preserving the property of isotropy. Any such decomposition of the double into the pair of maximally isotropic subalgebras $\mathcal{G} + \tilde{\mathcal{G}} = \mathcal{D}$ I shall refer to as the Manin triple.

Denote $M(\mathcal{D})$ the set of the Manin triples corresponding to a given Drinfeld double. It plays the role of the modular space of σ-models mutually connected by Poisson-Lie T-duality transformation. (In the Abelian case the Drinfeld double is $D = U(1)^{2d}$ and its modular space is nothing but $M(\mathcal{D}_{Abel}) = O(d, d, Z)$). Let us also remark that in general $M(\mathcal{D})$ has always at least two points, i.e. $\mathcal{G} + \tilde{\mathcal{G}} = \mathcal{D}$ and $\tilde{\mathcal{G}} + \mathcal{G} = \mathcal{D}$.

A classification of various T-dualities is essentially given by the following types of underlying Drinfeld doubles:
i) Abelian doubles, which correspond to the standard Abelian T-duality [2];
ii) semi-Abelian doubles (there exists the decomposition $\mathcal{G} + \tilde{\mathcal{G}} = \mathcal{D}$ such that $\tilde{\mathcal{G}}$ is Abelian)[3], which correspond to the standard non-Abelian T-duality between a G-isometric σ-model with a G-target and a non-isometric σ-model with the target $\tilde{\mathcal{G}}$ viewed as the Abelian group [4, 5, 6];
iii) non-Abelian doubles (all the others). They correspond to the non-trivial Poisson-Lie T-duality described in [1] where none of the models from the dual pair is isometric with respect to the action of the group, which naturally acts on its target.

In what follows, I give a unique description of T-duality valid for every Drinfeld double; for the special cases of the types i) and ii) I recover the results known previously. I will call the group G with the algebra $\mathcal{G}$ the duality group and the group $\tilde{G}$ with the algebra $\tilde{\mathcal{G}}$ the coduality group, if the decomposition of the double is written as $\mathcal{D} = \mathcal{G} + \tilde{\mathcal{G}}$. We shall see in a while that for every such decomposition of the double there exist σ-models

[3] The semi-Abelian double can always be viewed as the cotangent bundle of the group manifold G with the group structure given by the semi-direct product of G and its coalgebra $\tilde{\mathcal{G}}$ on which G acts by the coadjoint action.

such that the duality group G acts freely on their targets and this action is Poisson-Lie symmetric with respect to the coduality group $\tilde{G}$; the notion of the Poisson-Lie symmetry, I will explain in section 3. Every such σ-model has its dual counterpart for which the role of the duality and coduality groups is interchanged. The dual model has the same field equations as the original one, of course, in appropriate variables. Moreover, the duality map is a symplectomorphism between the phase spaces of both models [1].

For the sake of clarity, I shall first consider the 'atomic duality' case in which the duality group acts on the resulting σ-model target not only freely but also transitively, which means that the target itself can be identified with the group manifold. If the duality group does not act transitively on the σ-model target, the free action means that the target is a principal G-bundle; I will refer to this situation as to the 'Buscher's duality' case and describe it afterwards. In the Abelian case with $U(1)$ duality group and another $U(1)$ coduality group, the atomic duality means that the target space is a one-dimensional circle; in other words, this is the standard $R \to 1/R$ duality. In the case of the semi-Abelian double, the most standard example [4, 5, 6] of atomic duality is provided by the principal chiral model on a simple group G

$$L(g) = Tr(g^{-1}\partial_- g)(g^{-1}\partial_+ g) \tag{1}$$

and its dual

$$\tilde{L}(\chi) = \tilde{E}_{ab}(\chi)\partial_-\chi^a\partial_+\chi^b. \tag{2}$$

Here χ^a are coordinates of the elements of coalgebra $\tilde{\mathcal{G}}$ and the matrix $E_{ab}(\chi)$ is given by

$$(\tilde{E}_\chi^{-1})^{ab} = \delta^{ab} + \chi^k c_k^{ab}, \tag{3}$$

c_k^{ab} being the structure constants of the Lie algebra $\mathcal{G}$ of G. The coduality group in this case is the Abelian group with the same number of generators as the dimension of $\mathcal{G}$. These results may be derived by choosing the Drinfeld doubles of types i) and ii), using the following method valid for a general double:

Consider an n-dimensional linear subspace $\mathcal{E}^+$ of the ($2n$-dimensional) Lie algebra $\mathcal{D}$ and its orthogonal complement $\mathcal{E}^-$ such that $\mathcal{E}^+ + \mathcal{E}^-$ span the whole algebra $\mathcal{D}$. I shall show that those data determine a dual pair of the σ-models with the targets being groups G and $\tilde{G}$ [1]. Indeed, consider the following field equations for the mapping $l(\xi^+, \xi^-)$ from the world-sheet of string into the Drinfeld double D considered as a group:

$$\langle \partial_\pm l l^{-1}, \mathcal{E}^\pm \rangle = 0. \tag{4}$$

Here obviously the bracket $\langle .,. \rangle$ means the invariant bilinear form on the double. According to Drinfeld, there exists the unique decomposition (at least in the vicinity of the unit element of D) of any arbitrary element of D as the product of elements from G and $\tilde{G}$, i.e.

$$l(\xi^+, \xi^-) = g(\xi^+, \xi^-)\tilde{h}(\xi^+, \xi^-). \tag{5}$$

Inserting this ansatz into Eq. (1) we obtain:

$$\langle g^{-1}\partial_\pm g + \partial_\pm \tilde{h}\tilde{h}^{-1}, g^{-1}\mathcal{E}^\pm g \rangle = 0. \tag{6}$$

It is convenient to introduce mutually dual bases T^i and $\tilde{T}_i$ in both algebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ respectively, which means that

$$\langle T^i, \tilde{T}_j \rangle = \delta^i_j. \tag{7}$$

Then it is also convenient to write

$$g^{-1}\mathcal{E}^+ g = \mathrm{Span}(T^i + E^{ij}(g)\tilde{T}_j), i = 1, \ldots, n, \tag{8}$$

$$g^{-1}\mathcal{E}^- g = \mathrm{Span}(T^i - E^{ji}(g)\tilde{T}_j), i = 1, \ldots, n. \tag{9}$$

By inserting (8) and (9) into (6), it follows that

$$-(\partial_+ \tilde{h}\tilde{h}^{-1})^i = E^{ij}(g)(g^{-1}\partial_+ g)_j \equiv A^i_+(g), \tag{10}$$

$$-(\partial_- \tilde{h}\tilde{h}^{-1})^i = -E^{ji}(g)(g^{-1}\partial_- g)_j \equiv A^i_-(g). \tag{11}$$

Now $\tilde{h}$ can be easily eliminated, arriving at the final set of equations

$$\partial_+ A^i_-(g) - \partial_- A^i_+(g) - \tilde{c}_{kl}{}^{i} A^k_-(g) A^l_+(g) = 0, \tag{12}$$

where $\tilde{c}_{kl}{}^{i}$ are the structure constants of the Lie algebra $\tilde{\mathcal{G}}$. It can be directly checked, however, that the last equations are just the field equations of the σ-model with the Lagrangian

$$L = E^{ij}(g)(g^{-1}\partial_- g)_i (g^{-1}\partial_+ g)_j. \tag{13}$$

Equivalently, I may use the decomposition

$$l(\xi^+,\xi_-) = \tilde{g}(\xi^+,\xi^-)h(\xi^+,\xi^-), \tag{14}$$

where $\tilde{g} \in \tilde{G}$ and $h \in G$. All steps of the previous construction can be repeated, to end up with the dual σ-model

$$\tilde{L} = \tilde{E}_{ij}(\tilde{g})(\tilde{g}^{-1}\partial_-\tilde{g})^i(\tilde{g}^{-1}\partial_+\tilde{g})^j, \tag{15}$$

where the matrix $\tilde{E}_{ij}(\tilde{g})$ is defined as

$$\tilde{g}^{-1}\mathcal{E}^+\tilde{g} = \text{Span}(\tilde{T}_i + \tilde{E}_{ij}(\tilde{g})T^j), i = 1,\dots,n, \tag{16}$$

$$\tilde{g}^{-1}\mathcal{E}^-\tilde{g} = \text{Span}(\tilde{T}_i - \tilde{E}_{ji}(\tilde{g})T^j), i = 1,\dots,n. \tag{17}$$

Before proceeding further, note that the Poisson-Lie T-duality is a natural generalization of the standard Abelian $R \to 1/R$ symmetry. Indeed, at the group origin ($g = e$ and $\tilde{g} = \tilde{e}$) the matrices $E(e)$ and $\tilde{E}(\tilde{e})$ are related to each other as follows

$$E(e)\tilde{E}(\tilde{e}) = \tilde{E}(\tilde{e})E(e) = 1. \tag{18}$$

The explicit dependence of E on g and $\tilde{E}$ on $\tilde{g}$ is given by the matrices of the adjoint representation of $\mathcal{D}$:

$$g^{-1}\mathcal{E}^+g = \text{Span } g^{-1}(T^i + E^{ij}(e)\tilde{T}_j)g$$
$$= \text{Span}\Big[(a(g)^i{}_l + E^{ij}(e)b(g)_{jl})T^l + E^{ij}(e)d(g)_j{}^l\tilde{T}_l\Big], \tag{19}$$

where

$$g^{-1}T^ig \equiv a(g)^i{}_lT^l, \qquad g^{-1}\tilde{T}_jg \equiv b(g)_{jl}T^l + d(g)_j{}^l\tilde{T}_l. \tag{20}$$

Hence the σ-model matrix is given by

$$E(g) = (a(g) + E(e)b(g))^{-1}E(e)d(g). \tag{21}$$

The matrix of the dual σ-model can be obtained in a completely analogous way.

If there exists another maximally isotropic decomposition of the double $\mathcal{D}$ in two subalgebras $\mathcal{K}$ and $\tilde{\mathcal{K}}$ we obtain another pair of dual σ-models with some matrices $E'(k)$ and $\tilde{E}'(\tilde{k})$. This new dual pair again shares the equations of motions (4) with the old pair. Thus we indeed see that the set of the mutually equivalent σ-models is nothing but the set $M(\mathcal{D})$ introduced at the beginning of this section.

Note that the original field equations (4) have a huge symmetry $l \to lm$ where m is an arbitrary element of D. This symmetry is realized in a *non-local* way in the σ-model language, in particular the geometry of the target does not possess it as an isometry. Only in the specific cases of the Abelian and semi-Abelian duality is *part* of this symmetry realized locally and it does lead to the isometries of the target. This is the reason why T-duality has so far been considered to be connected with the isometries of the target.

3 Poisson-Lie symmetry

It is natural to pose a question when a σ-model with the free G action on its target admits the Poisson-Lie dual for some dual group $\tilde{G}$. As was mentioned in the previous section, the condition is the Poisson-Lie symmetry of the σ-model, defined as follows: Let

$$J_a = v_a^i(x)E_{ij}(x)\partial_+ x^j d\xi^+ - v_a^i(x)E_{ji}(x)\partial_- x^j d\xi^- \tag{22}$$

be the 'Noether' current 1-forms corresponding to the right action of the group G on the target $\mathcal{M}$ of the σ-model

$$S = \int d\xi^+ d\xi^- E_{ij}(x)\partial_- x^i \partial_+ x^j. \tag{23}$$

In these formulae $v_a^i(x)$ are the (left-invariant) vector fields corresponding to the right action of G on $\mathcal{M}$ and x^i are some coordinates on $\mathcal{M}$. If the action of G is isometry, then the Noether currents (22) are closed 1-forms on the world-sheets of extremal strings. If they are not closed but they rather obey on the extremal surfaces a condition

$$dJ_a = \frac{1}{2}\tilde{c}_a^{\ kl} J_k \wedge J_l. \tag{24}$$

(with $\tilde{c}_a^{\ kl}$ being the structure constants of some Lie algebra $\tilde{\mathcal{G}}$), we say that the σ-model has the G-Poisson-Lie symmetry with respect to the group $\tilde{G}$. One can interpret the conditions (24) also as (some of) the field equations of the model, if the group G acts transitively then they exhaust all equations of motion. Note also that the σ-models (13), defined in the previous section, are G-Poisson-Lie symmetric with respect to the group $\tilde{G}$, as can be directly seen from Eqs. (10)-(12).

The condition of the Poisson-Lie symmetry can be directly formulated at the level of the σ-model Lagrangian. It reads [1]

$$\mathcal{L}_{v_a}(E_{ij}) = \tilde{c}_a^{\ kl} v_k^m v_l^n E_{mj} E_{in}, \tag{25}$$

where $\mathcal{L}_{v_a}$ stands for the Lie derivative corresponding to the vector field v_a. The integrability condition for the Lie derivative gives the 'cocycle' condition of compatibility between the structure constants of Lie algebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ forming the double

$$\tilde{c}_k^{\ ac} c^l_{\ fa} - \tilde{c}_k^{\ al} c^c_{\ fa} - \tilde{c}_f^{\ ac} c^l_{\ ka} + \tilde{c}_f^{\ al} c^c_{\ ka} - \tilde{c}_a^{\ lc} c^a_{\ fk} = 0. \tag{26}$$

Needless to say, all formulae (22)-(25) have their dual counterparts, obtained just by exchanging the roles of $\mathcal{G}$ and $\tilde{\mathcal{G}}$.

Buscher's duality may be treated much in the same way as the atomic one; here we give just the final form of the dual pair of σ-models. The coordinates labelling the orbits of G in the target $\mathcal{M}$, we denote $y^\alpha(\alpha = 1, \ldots, k)$. The matrix of the σ-model E_{ij} has both types of indices corresponding to y^α and g. The Lagrangian reads

$$\begin{aligned} L = E_{\alpha\beta}(y)\partial_- y^\alpha \partial_+ y^\beta + E_{\alpha b}(y,g)\partial_- y^\alpha (g^{-1}\partial_+ g)^b + \\ + E_{a\beta}(y,g)(g^{-1}\partial_- g)^a \partial_+ y^\beta + E_{ab}(y,g)(g^{-1}\partial_- g)^a (g^{-1}\partial_+ g)^b. \end{aligned} \tag{27}$$

Note that the dependence of E_{ij} on g is fixed by condition (25). Explicitly

$$E(y,g) = (A(g) + E(y,e)B(g))^{-1} E(y,e) D(g), \tag{28}$$

where e is the unit element of G, $E(y,e)$ can be chosen arbitrarily and $A(g)$ is the $(k + dimG) \times (k + dimG)$ matrix

$$A(g) \equiv \begin{pmatrix} Id & 0 \\ 0 & a(g) \end{pmatrix}, \qquad B(g) \equiv \begin{pmatrix} 0 & 0 \\ 0 & b(g) \end{pmatrix} \tag{29}$$

and $D(g)$ is given in terms of $d(g)$ in the same way as $A(g)$ in terms of $a(g)$. Of course, $a(g), b(g)$ and $d(g)$ are the same as in (20). The σ-model matrix $E(y,g)$ given in (28) is in fact the most general solution of the condition (25) in the adapted coordinates (y,g). As far as the dual model $\tilde{E}$ is concerned:

$$\tilde{E}(y,\tilde{g}) = (\tilde{A}(\tilde{g}) + \tilde{E}(y,e)\tilde{B}(\tilde{g}))^{-1} \tilde{E}(y,e) \tilde{D}(\tilde{g}). \tag{30}$$

Here

$$\tilde{E}(y,\tilde{e}) = (A + E(y,e)B)^{-1}(C + E(y,e)D), \tag{31}$$

and

$$A = D = \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = C = \begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix}. \tag{32}$$

4 Borelian double

The simplest non-Abelian double is the $GL(2,R)$ group; its Lie algebra has the basis

$$T^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad T^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{33}$$

$$\tilde{T}^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{T}^2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{34}$$

Note that both sets of generators (33) and (34) span the Borelian subalgebras of the algebra $gl(2,R)$ (hence the name of the double) and they are maximally isotropic with respect to the non-degenerate invariant symmetric bilinear form defined by the brackets

$$\langle T^i, \tilde{T}_j \rangle = \delta^i_j, \qquad \langle T^i, T^j \rangle = \langle \tilde{T}_i, \tilde{T}_j \rangle = 0. \tag{35}$$

We see that the Poisson-Lie duality will relate the σ-models with the same target, namely the group manifold of the Borel group B_2 whose Lie algebra is generated by (33) or (34). The following parametrization of the first copy (33) of B_2 is convenient

$$g = \begin{pmatrix} e^{\chi} & \theta \\ 0 & 1 \end{pmatrix}. \tag{36}$$

The matrices $a(g), b(g)$ and $d(g)$ from Eq. (20) read

$$a(g) = \begin{pmatrix} 1 & e^{-\chi}\theta \\ 0 & e^{-\chi} \end{pmatrix}, \tag{37}$$

$$b(g) = \begin{pmatrix} 0 & -e^{-\chi}\theta \\ \theta & e^{-\chi}\theta^2 \end{pmatrix}, \tag{38}$$

$$d(g) = \begin{pmatrix} 1 & 0 \\ -\theta & e^{\chi} \end{pmatrix}, \tag{39}$$

and the inverse σ-model matrix $E^{-1}(e)$ at the unit element of B_2 is

$$E^{-1}(e) = \begin{pmatrix} x & y \\ u & v \end{pmatrix}. \tag{40}$$

By a direct application of formula (21), the σ-model Lagrangian (13) is worked out as follows:

$$L = \frac{x\theta^2 + \theta(u+y) + v}{\theta^2 + \theta(u-y) + (xv-uy)} \partial_-\chi\partial_+\chi + \frac{x}{\theta^2 + \theta(u-y) + (xv-uy)} \partial_-\theta\partial_+\theta$$

$$-\frac{y + (x-1)\theta}{\theta^2 + \theta(u-y) + (xv-uy)} \partial_-\chi\partial_+\theta - \frac{u + (x+1)\theta}{\theta^2 + \theta(u-y) + (xv-uy)} \partial_-\theta\partial_+\chi.$$

With a slight abuse of the notation, the convenient parametrization of the second copy (34) of B_2 is

$$\tilde{g} = \begin{pmatrix} 1 & 0 \\ -\theta & e^{\chi} \end{pmatrix} \tag{41}$$

and the dual σ-model Lagrangian (15) has the same structure

$$\tilde{L} = \frac{\tilde{x}\theta^2 + \theta(\tilde{u}+\tilde{y}) + \tilde{v}}{\theta^2 + \theta(\tilde{u}-\tilde{y}) + (\tilde{x}\tilde{v}-\tilde{u}\tilde{y})} \partial_-\chi\partial_+\chi + \frac{\tilde{x}}{\theta^2 + \theta(\tilde{u}-\tilde{y}) + (\tilde{x}\tilde{v}-\tilde{u}\tilde{y})} \partial_-\theta\partial_+\theta$$

$$-\frac{\tilde{y} + (\tilde{x}-1)\theta}{\theta^2 + \theta(\tilde{u}-\tilde{y}) + (\tilde{x}\tilde{v}-\tilde{u}\tilde{y})} \partial_-\chi\partial_+\theta - \frac{\tilde{u} + (\tilde{x}+1)\theta}{\theta^2 + \theta(\tilde{u}-\tilde{y}) + (\tilde{x}\tilde{v}-\tilde{u}\tilde{y})} \partial_-\theta\partial_+\chi.$$

Here the parameters $\tilde{x}, \tilde{y}, \tilde{u}$ and $\tilde{v}$ are related to the original set x, y, u and v as

$$\begin{pmatrix} x & y \\ u & v \end{pmatrix} \begin{pmatrix} \tilde{x} & \tilde{y} \\ \tilde{u} & \tilde{v} \end{pmatrix} = \begin{pmatrix} \tilde{x} & \tilde{y} \\ \tilde{u} & \tilde{v} \end{pmatrix} \begin{pmatrix} x & y \\ u & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{42}$$

which are nothing but relations (18) in the context of this concrete example.

5 Acknowledgement

I thank A. Alekseev, K. Gawędzki, E. Kiritsis, S. Shatashvili, P. Ševera and A. Tseytlin for useful discussions.

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