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The BRST–antibracket cohomology of $2d$ gravity conformally coupled to scalar matter

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Abstract

We compute completely the BRST–antibracket cohomology on local functionals in two-dimensional Weyl invariant gravity for given classical field content (two dimensional metric and scalar matter fields) and gauge symmetries (two dimensional diffeomorphisms and local Weyl transformations). This covers the determination of all classical actions, of all their rigid symmetries, of all background charges and of all candidate gauge anomalies. In particular we show that the antifield dependence can be entirely removed from the anomalies and that, if the target space has isometries, the condition for the absence of matter field dependent anomalies is more general than the familiar 'dilaton equations'.

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1 Introduction

Wess and Zumino [1] have shown that anomalies satisfy consistency conditions. In turn, these consistency conditions can be used as a tool to classify possible anomalies. The solution of these conditions is one instance of a cohomology calculation: the cohomology of the BRST operator on local functionals. In ghost number one this cohomology indeed provides all solutions of the Wess–Zumino consistency conditions, i.e. determines the general form of possible anomalies. Other instances of cohomological analysis are equally important physically. For example, at ghost number zero, it yields the most general action that is compatible with a given symmetry, and in ghost number (-1) it provides all rigid symmetries of the action [2].

The ingredients needed to perform the cohomological analysis are: the field content, the gauge transformation laws of the fields, and the classical equations of motion (e.o.m.). The e.o.m. intervene in two places: on the one hand, the BRST operator may be nilpotent only on shell, and on the other hand, classical observables are physically equivalent if their difference is proportional to e.o.m.. The BRST cohomology modulo this equivalence is called the *weak* BRST cohomology.

For a class of theories which contain diffeomorphisms in the gauge group, a general method for the analysis of the cohomology was set up in [3]. This class of theories contains for instance Einstein gravity as well as supergravity theories, but it does not cover *all* diffeomorphism invariant theories. In particular it does not include Weyl invariant gravity theories in two dimensions, such as the standard bosonic string theory described at the classical level by the Polyakov action. The reason is the absence of an independent Weyl gauge field[1] in these theories whose presence would be a crucial prerequisite for applying the methods of [3]. As we shall see, this is responsible for considerable differences in the cohomological analysis and its results for these two dimensional models when compared with more "standard" gauge theories such as Yang–Mills theory or Einstein gravity.

In fact, in spite of its central importance to string theory, the BRST cohomology on local functionals has, to our knowledge, never been analysed exhaustively in the literature for the case of Weyl invariant $d = 2$ gravity theories. (For recent contributions, see [4, 5, 6, 7]). The filling of this gap is the purpose of this paper, for the case that all matter fields are scalar fields. The results have been announced already partly in [8, 9]. Some of them are of course common knowledge. In particular this holds for results on the strong cohomology, i.e. for the BRST cohomology which does not take the e.o.m. into account. We shall see however that many important aspects of the theory show up only in the weak cohomology, such as the rigid symmetries and the so-called background charges or the dilaton terms which can cancel Weyl anomalies and are well-known in string theory [10]. Moreover we will show in a companion paper [11] that the results on the weak cohomology allow one in particular cases to construct interesting generalizations of the theory (so-called consistent deformations [12]) which are reminiscent of non-critical string theories and possibly provide new models for the latter.

The necessity of re–analysing the cohomology appears clearly if one would blindly extend the results of [3] to the present situation: one would conclude, for example, that candidate anomalies can be assumed to depend only on the undifferentiated Weyl ghost c, the undifferentiated zweibeins and on tensor fields (the two dimensional Riemann curvature

[1] One could introduce the Weyl gauge field, but not without further ado. See section 5.

R, the matter fields and their covariant derivatives) but not, e.g., on the diffeomorphism ghosts. However, it is well known that this is incorrect: the important Weyl anomaly

$$\mathcal{A}_0 = \int d^2x \ c\sqrt{g}R \ , \tag{1.1}$$

can be split, after the addition of BRST–exact terms, in a left and right handed part, which separately solve the consistency equations [13, 14]. These two parts are cohomologically inequivalent, involve the diffeomorphism ghosts and cannot be written entirely in terms of the Weyl ghost and tensor fields up to BRST–exact terms. We will show, among other things, how the absence of the Weyl gauge field modifies the conclusions of [3] in a way that implies this result.

The starting point of our analysis will be the field content, and the symmetry transformations. These will include the diffeomorphisms, of course, and the Weyl transformations. We will realise them on the scalar matter fields, and on the two-dimensional metric[2]. The symmetries will entail the corresponding ghosts, in our case diffeomorphism ghosts and the Weyl ghost.

Although it may be customary, after Faddeev and Popov, to introduce also antighosts, this is in fact quite superfluous to investigate the classical cohomology. This is especially obvious in the Batalin–Vilkovisky (BV) framework [15, 16, 9, 17, 18] (also called the field–antifield formalism)[3]. The reason is that antighosts (as well as their antifields), and Lagrange multiplier fields that come with gauge fixing, are introduced as so–called *trivial systems*, implying that they leave the cohomology groups unchanged. Therefore, the antighosts will be absent from our analysis. A related feature is that no gauge fixing is needed: the formulation of the calculation, and its results, are made entirely without reference to any gauge fixing, and are therefore at every stage manifestly gauge independent.

The other side of the coin is that *antifields* are present. The BV cohomology will then have to be analysed in a space of functionals of fields and antifields. This has, for our purposes, the additional advantage that it automatically takes into account the *weak* nature of the relevant cohomology calculation since the antifields implement the e.o.m. in the cohomological analysis. To exploit this last feature, we have to know the classical action. This classical action itself need not be fixed on beforehand however: it will be determined, in an intermediate step, from the strong BRST cohomology in the space of integrated local functionals with ghost number zero depending on *fields only*, not on antifields.

The computation of the cohomology is carried out in three main steps. First we map the cohomological problem on integrated local functionals to the analogous problem on local functions of the fields and antifields. This map is quite standard and provided by the so-called descent equations. In the second step we isolate and eliminate successively trivial systems. This reduces the problem to a set of equations for "superfields" in the undifferentiated matter fields and first order derivatives of the diffeomorphism ghosts. The third step consists in solving these equations. Here we need the explicit form of

[2]Alternatively, one could introduce zweibeins and include Lorentz invariance: this amounts to a technical difference only. Since for scalars one does not need the zweibeins, we refrain from introducing them.

[3]We will (very summarily) introduce the necessary ingredients, and the relation with BRST cohomology, in section 2.

the action which is computed in an intermediate step by solving the "strong" superfield equations first.

Our analysis is local in two senses: on the one hand we work in the space of local functionals which are, by definition, polynomial in derivatives of all the fields and antifields, and on the other hand we ignore global aspects of the base and target manifold completely.

Let us now give an outline of the paper. In section 2 we will (very briefly) introduce the necessary elements from the BV framework, and write down the elements of the extended action that follow directly from our assumed symmetry transformations. We will also describe more accurately the cohomology calculation to be performed. In section 3 we make a first change of variables, showing how the determinant of the metric and the Weyl ghost occur as a trivial system when one introduces Beltrami variables to describe the metric. The resulting chiral splitting [13, 14] runs through the rest of the paper, and also, technically, it simplifies the calculations significantly. In section 4 we perform, following [19, 3], the above mentioned first step of the computation that takes us from local functionals to local functions via descent equations, and also give a short discussion of the type of global considerations that we will *not* take into account in the rest of the paper. In section 5 we prepare the second step by introducing *chiral tensor fields* and *covariant ghost variables*, the former being a generalisation of the usual tensor fields that we will explain and the latter forming a subset of the derivatives of ghosts. In section 6 we then conduct the second step which reduces the cohomological analysis to local functions generated by only a few chiral tensor fields and covariant ghost variables. There the above mentioned superfields show up. We then compute in section 7 the strong BRST cohomology by a first analysis of the equations these superfields have to satisfy. This provides in particular the most general classical action which we discuss in detail in section 8. We are then in the position to finish the calculation by solving the (weak) superfield equations completely. This is done in section 9 where we also enumerate all the resulting solutions on the level of local functions. In section 10 we spell out the corresponding local functionals for the most interesting cases (ghost numbers) and discuss their physical significance. Although cohomologically there is a complete chiral split, for example for the anomalies, but also for rigid symmetries and counterterms, in many cases of practical interest only the left–right symmetric combinations are relevant. In section 11 we therefore specialize our results to that case. This makes the connection with the case of primary importance for string theory, where an anomaly is tolerated in the Weyl symmetry only, and a dilaton field is introduced. We conclude with a discussion, including pointers to previously published partial results. Finally, in the appendices, we collected various formulas and technical results, but also a side result on the relation of target space reparametrisations with the cohomology.

2 Assumptions and definition of the problem

In the BV formalism, the fundamental object is the antibracket defined for two arbitrary functionals F and G of fields Φ^A and antifields Φ^*_A by

$$(F,G) = F\frac{\overleftarrow{\delta}}{\delta\Phi^A}\cdot\frac{\overrightarrow{\delta}}{\delta\Phi^*_A}G - F\frac{\overleftarrow{\delta}}{\delta\Phi^*_A}\cdot\frac{\overrightarrow{\delta}}{\delta\Phi^A}G\ . \tag{2.1}$$

The consistency equation for the anomaly $\mathcal{A}$ is then

$$\mathcal{S}\mathcal{A} \equiv (S,\mathcal{A}) = 0\ , \tag{2.2}$$

where S is the extended action (which itself satisfies the BV 'master' equation $(S,S)=0$). Two solutions $\mathcal{A}$ and $\mathcal{A}'$ of (2.2) are equivalent anomalies (related by field redefinitions, change of regularization, or local counterterms) iff

$$\mathcal{A}' - \mathcal{A} = \mathcal{S}M \tag{2.3}$$

where M is the integral of a local function (a 'local functional'). Now, anomalies normally have ghost number 1. Hence, what we have to solve for their classification is the cohomology of $\mathcal{S}$ with ghost number 1 on local functionals. As we mentioned already we will not restrict ourselves to this case but perform the analysis for all other ghost numbers as well. For ghost number 0 this is relevant e.g. for the renormalization problem.

If the gauge algebra is 'closed', antifields enter only linearly in S. That will be the case here. The 'Slavnov' operator $\mathcal{S}$ can then be split in a 'Koszul–Tate' operator and the remaining part which we call the 'BRST' operator s:

$$\mathcal{S} = \delta_{KT} + s\ . \tag{2.4}$$

This splitting is related to the antifield number. The latter is defined to be zero for fields (which have non–negative ghost numbers), and minus the ghost number for antifields. δ_{KT} is the part of $\mathcal{S}$ which lowers the antifield number (by 1), while s is the part which does not change the antifield number. For general gauge theories with an 'open' gauge algebra there are also terms which raise the antifield number, but not in the cases treated in this paper. Note that on the fields we thus have $\mathcal{S} = s$ whereas on the antifields both δ_{KT} and s are nonvanishing. The expansion of $\mathcal{S}^2 = 0$ in antifield number implies that δ_{KT} and s are separately nilpotent and anticommute:

$$\delta_{KT}^2 = s\delta_{KT} + \delta_{KT}s = s^2 = 0\ . \tag{2.5}$$

The equation $s^2 = 0$ holds only due to the lack of further terms in (2.4) and is not true for gauge theories with on open algebra where s^2 vanishes only weakly, i.e. 'up to field equations'.

We consider scalar fields X^μ, $\mu = 1,\dots,D$ in interaction with the $d = 2$ metric fields $g_{\alpha\beta} = g_{\beta\alpha}$ with $\alpha,\beta\in\{+,-\}$. The coupling of the X^μ and $g_{\alpha\beta}$ is assumed to be generally covariant and Weyl invariant at the classical level. More precisely we require the classical action, denoted by $S_{cl}(X^\mu, g_{\alpha\beta})$, to be invariant under two dimensional diffeomorphisms and local dilatations so that the extended action S reads

$$S = S_{cl}(X^\mu, g_{\alpha\beta}) - \int d^2x\left(\mathcal{S}\Phi^A\right)\Phi^*_A\ , \tag{2.6}$$

where $\mathcal{S}\Phi^A$ denotes the BRST–transformations of the fields corresponding to their transformations under two dimensional diffeomorphisms and local dilatations, [4]

$$
\begin{aligned}
\mathcal{S}g_{\alpha\beta} &= \xi^\gamma\partial_\gamma g_{\alpha\beta}+\partial_\alpha\xi^\gamma\cdot g_{\gamma\beta}+\partial_\beta\xi^\gamma\cdot g_{\alpha\gamma}+c\,g_{\alpha\beta}\ ;\\
\mathcal{S}X^\mu &= \xi^\alpha\partial_\alpha X^\mu\ ;\quad \mathcal{S}\xi^\beta=\xi^\alpha\partial_\alpha\xi^\beta\ ;\quad \mathcal{S}c=\xi^\alpha\partial_\alpha c\ .
\end{aligned}
\tag{2.7}
$$

Here ξ^α are the ghosts for general coordinate transformations and c is the ghost for local dilatations. The sets of fields and antifields are accordingly given by

$$
\{\Phi^A\}=\{X^\mu,g_{\alpha\beta},\xi^\alpha,c\}\ ;\qquad \{\Phi^*_A\}=\{X^*_\mu,g^{*\alpha\beta},\xi^*_\alpha,c^*\}\ .
$$

The ghosts and the antifields X^*_μ and $g^{*\alpha\beta}$ are odd–graded whereas X^μ, $g_{\alpha\beta}$, ξ^*_μ and c^* are even graded. With no loss of generality, $g^{*\alpha\beta}$ is taken to be symmetric since $g_{\alpha\beta}$ is symmetric too[5]. The ghost number is zero for X^μ and $g_{\alpha\beta}$, (-1) for their antifields, one for the ghosts ξ^α and c, and (-2) for the antifields of the ghosts.

We do not impose any restriction on the classical action S_{cl}, except that it is a local and regular[6] functional of the fields X^μ and $g_{\alpha\beta}$ and that (2.6) extends it to a proper (minimal) solution of the BV master equation in the sense of [15]. This requires that

(i) the integrand of S_{cl} is regular and depends polynomially on the partial derivatives of X^μ and $g_{\alpha\beta}$;

(ii) S_{cl} is invariant under (2.7), i.e. it should just satisfy $\mathcal{S}S_{cl}=0$;

(iii) S_{cl} has no nontrivial local symmetries apart from those imposed by (ii).

An extension of the requirement imposed by (i) on the integrand of S_{cl} serves as definition of local functions throughout the paper and fixes thereby the space of functions and functionals on which we will perform the cohomological analysis. Namely a local function depends by definition polynomially on the derivatives of the X^μ and $g_{\alpha\beta}$ and on the (undifferentiated) ghosts, antifields and their partial derivatives, whereas we allow for nonpolynomial dependence on the undifferentiated X^μ and $g_{\alpha\beta}$. Furthermore we allow a local function to depend explicitly on the two dimensional coordinates x^α (see section 4 for remarks on this point). A local functional is by definition an integrated local function of the fields and antifields.

The condition (ii) just requires S_{cl} to be invariant under diffeomorphisms and local Weyl transformations.

(iii) guarantees the properness of $\mathcal{S}$, i.e. the completeness of our approach in the sense that the BRST operator encodes all (nontrivial) local symmetries of the classical theory

[4]Differentiations will always act on everything to their right, unless the scope is limited by the "$\cdot$" punctuation mark.

[5]When treating symmetric tensors and their antifields one may sum over $\alpha\geq\beta$, or one can work symmetrically—which we will do. Then we have to take $\left(g_{\alpha\beta},g^{*\gamma\delta}\right)=\frac{1}{2}\left(\delta^\gamma_\alpha\delta^\delta_\beta+\delta^\delta_\alpha\delta^\gamma_\beta\right)$.

[6]Regular dependence on the fields X^μ and $g_{\alpha\beta}$ requires the action to be well–defined within the allowed range of values these fields may take. In the case of the metric this range is restricted by $\det(g_{\alpha\beta})<0$; in the case of the matter fields we do not specify the range since anyhow we neglect topological aspects, i.e. actually the regularity requirement will not matter in the subsequent analysis (see section 4). Regularity also includes the requirement that there exist solutions to the field equations, differentiability of the action, and that in the set of local functions that we consider all functions that vanish when the field equations are satisfied are actually a linear combination of the field equations.

(additional local symmetries of S_{cl} would make the introduction of further ghost fields necessary).

Of course the requirements (i)–(iii) characterize the models to which our analysis apply only indirectly through the symmetries and field content of S_{cl} (and through the locality requirement). The derivation of its most general explicit form will in fact be part of our results, see section 8. A simple example for a functional satisfying (i)–(iii) is of course

$$S_{cl} = -\int d^2x\ \tfrac{1}{2}\sqrt{g}\, g^{\alpha\beta}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu \eta_{\mu\nu} \tag{2.8}$$

where $\eta_{\mu\nu}$ is a constant symmetric non-degenerate matrix.

Our aim is to compute the cohomology of the operator $\mathcal{S}$ in the space of local functionals.

$$\mathcal{S}W^g = 0. \tag{2.9}$$

Two solutions represent the same class and are called equivalent, $W^g \approx W'^g$, if they differ by an $\mathcal{S}$–exact functional, i.e. if $W^g - W'^g = \mathcal{S}M^{g-1}$ holds for for some local functional M^{g-1}. The $\mathcal{S}$-operation increases the ghost number by one and is nilpotent.

3 A simplifying canonical transformation

There is a field redefinition that simplifies our problem considerably. It will eliminate some fields from the cohomology, and cause a chiral split of the transformation laws [13, 14]. Expressed in BV language, we take as the generating fermionic functional

$$\begin{aligned} F \ &= \int d^2x \left[e^*\sqrt{g} + h^{++\,*}\frac{g_{++}}{g_{+-}+\sqrt{g}} + h^{--\,*}\frac{g_{--}}{g_{+-}+\sqrt{g}} + \tilde{c}^*\left(c\sqrt{g} + \partial_\alpha\xi^\alpha\sqrt{g}\right)\right. \\ &\left. +c_+^*\left(\xi^+ + \frac{g_{--}}{g_{+-}+\sqrt{g}}\xi^-\right) + c_-^*\left(\xi^- + \frac{g_{++}}{g_{+-}+\sqrt{g}}\xi^+\right) + \tilde{X}^*_\mu X^\mu\right] \ . \end{aligned} \tag{3.1}$$

where $g = |det\ g_{\alpha\beta}| = g_{+-}^2 - g_{++}g_{--}$. This generates a canonical transformation from fields and antifields $\{\Phi, \Phi^*\}$ to $\{\tilde{\Phi}, \tilde{\Phi}^*\}$ through

$$\tilde{\Phi}^A = \frac{\delta F(\Phi, \tilde{\Phi}^*)}{\delta\tilde{\Phi}^*_A} \qquad\qquad \Phi^*_A = \frac{\delta F(\Phi, \tilde{\Phi}^*)}{\delta\Phi^A} \ . \tag{3.2}$$

In our case, $\{\tilde{\Phi}\} = \{X^\mu, h_{++}, h_{--}, e, c^\alpha, \tilde{c}\}$. We have changed from the three fields g_{++}, g_{--} and g_{+-} to e, h_{++} and h_{--}, the last two being the 'Beltrami variables'. This transformation becomes singular for $g_{++}g_{--} = 0$ and $g_{+-} < 0$ (simultaneously). However, as we shall discuss in section 4, the singularity can become important at most for global considerations and is thus negligible for our purposes. At the same time we have introduced more convenient combinations of the ghost fields, but their explicit relation to the original diffeomorphism ghosts ξ will remain important in the sequel. Also, it should be noted that the transformation between the old set of fields and the new set does not involve the antifields, so that the $\mathcal{S}$–cohomology in the antifield independent sector does not change.

After this canonical transformation the extended action takes the form

$$S = S_{cl}(X^\mu, h_{++}, h_{--}) + \int d^2x \big(\tfrac{1}{1-y} X^*_\mu c^\alpha \nabla_\alpha X^\mu$$
$$+ h^{++\,*} \nabla_+ c^- + h^{--\,*} \nabla_- c^+ + e^* \tilde{c} - c^*_+ c^+ \partial_+ c^+ - c^*_- c^- \partial_- c^- \big) , \tag{3.3}$$

where the covariant derivative ∇ is defined in appendix A, and y is the abbreviation

$$y \equiv h_{++} h_{--} \ . \tag{3.4}$$

The inverse of the above field transformation is given in appendix A.

Note that we have claimed in (3.3) that S_{cl} does not depend on e when written in terms of the new fields. This can be checked in particular for (2.8) which would lead to

$$S_{cl}(X^\mu, h_{++}, h_{--}) = -\int d^2x \, \frac{1}{1-y} \nabla_+ X^\mu \cdot \nabla_- X^\nu \eta_{\mu\nu} \ . \tag{3.5}$$

Indeed the master equation requires in particular $\delta S_{cl}/\delta e = 0$ which means that the integrand of S_{cl} is independent of e (up to a total derivative which we neglect). This implies that e and $\tilde{c}$ become a so–called trivial system since they have the simple transformation property $\mathcal{S} e = \tilde{c}$ and do not occur in the $\mathcal{S}$–transformation of the other new fields and antifields. Hence, e and $\tilde{c}$ can be omitted for any cohomology considerations and with no loss of generality we can assume, whenever we work with the new fields and antifields, that the complete set of fields and antifields is

$$\{\Phi^A, \Phi^*_A\} \ ; \qquad \{\Phi^A\} = \{X^\mu, h_{++}, h_{--}, c^+, c^-\} \ . \tag{3.6}$$

We shall see that the use of the new variables has additional advantages. In particular, apart from eliminating e and c, we have obtained that $\mathcal{S} h_{++}$ and $\mathcal{S} c^-$ involve only c^- but not c^+, which is the chiral splitting announced before.

We note that a similar simplifying canonical transformation can be done in the zweibein formulation in order to eliminate the Weyl and local Lorentz ghosts present in the zweibein formulation. Then e_+^+ and e_-^- become trivial together with (appropriate redefinitions of) these ghosts, and one is left with the matter fields, two functions of the vielbein components given by h_{++} and h_{--}, the diffeomorphism ghosts and with the corresponding antifields.

4 Descent equations and their integration

The first step towards a solution of (2.9) consists in an analysis of the descent equations arising from it. This traces our problem back to the $\mathcal{S}$-cohomology on local functions rather than on local functionals (integrals of local functions). The analysis of the descent equations is independent of the form of S_{cl} and has been first performed in this form in [19] (see also [3])[7]. We can adopt it since we are not interested in global aspects of the target manifold and the two dimensional base manifold. What this means is spelled out in the following, together with a discussion of the singularity in the transformation to the Beltrami variables defined by (3.1) and (3.2).

[7]The antifields which are not considered in [19, 3] can be treated on an equal footing with the fields as far as the analysis of the descent equations is concerned because (4.5) holds on fields and antifields.

A crucial tool within the analysis of the descent equations performed in [19, 3] is the 'algebraic Poincaré lemma' describing the cohomology of the exterior derivative d in the space of *local differential forms*. The latter are by definition forms $\omega_p = dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_p} \omega_{\alpha_1 \ldots \alpha_p}$ where $\omega_{\alpha_1 \ldots \alpha_p}$ are local functions (see section 2). The lemma has been derived by various authors independently (cf. e.g. [20] and references in [2]). It states that the closed forms which are not locally exact are exhausted by the constant 0–forms and by volume forms which have non-vanishing Euler–Lagrange derivative with respect to at least one field or antifield. Here a form ω_p is called *locally exact* if it can be written as $d\eta_{p-1}$ for some local form η_{p-1} locally, i.e. in any (sufficiently small) local neighbourhood in $\mathcal{M} \times \mathcal{T}$ where $\mathcal{M}$ and $\mathcal{T}$ denote the base and target space manifold respectively. The latter is the space in which (all) the fields and antifields take their values. Of course, a locally exact form can fail to be globally exact in $\mathcal{M} \times \mathcal{T}$.

The general version of the algebraic Poincaré lemma, taking global properties of $\mathcal{M} \times \mathcal{T}$ into account, has been derived in [21]. Famous examples for locally but not globally exact local forms are the integrands of characteristic classes (of nontrivial bundles). In two dimensional gravity this is in particular the integrand $d^2x\sqrt{g}R$ of the two dimensional Einstein action. Other examples for closed but globally non–exact forms present in $d = 2k$ dimensional gravitational theories are $(2k-1)$–forms in the metric components and their first derivatives discussed in [22]. The latter stem from the nontrivial De Rham cohomology of the target space of the metric components which itself originates in the requirement that the metric has Minkowskian signature. In our case there exists therefore a closed 1–form which generically fails to be globally exact if $g_{\alpha\beta}$ has signature $(-,+)$. Further closed local forms which fail to be globally exact can of course arise from nontrivial De Rham cohomology of the target space of the matter fields X^μ. A refinement of the analysis of the descent equations which takes into account the global properties of $\mathcal{T}$ has been given recently in [23].

In this paper we will completely neglect global aspects of the two dimensional base manifold and of the target manifold. This means that whenever we call a functional, form or function $\mathcal{S}$– or d–exact ('trivial'), we have in mind that it is locally exact in $\mathcal{M} \times \mathcal{T}$ which does not necessarily imply that it is globally exact as well.

For our purposes the singularity in the canonical transformation performed in section 3 is therefore harmless since it occurs only on the 2-dimensional subspace $\mathcal{T}_s = \{(g_{++}, g_{--}, g_{+-}):\ g_{++}g_{--} = 0,\ g_{+-} < 0\}$ of the 3-dimensional target space of the metric components given by $\mathcal{T}_g = \{(g_{++}, g_{--}, g_{+-}):\ g_{++}g_{--} - (g_{+-})^2 < 0\}$ where we assumed $g_{\alpha\beta}$ to have signature $(-,+)$. When using Beltrami variables, one thus actually works in a target space of metric components given by $\mathcal{T}_g - \mathcal{T}_s$ rather than by $\mathcal{T}_g$. We note that $\mathcal{T}_g - \mathcal{T}_s$ and $\mathcal{T}_g$ indeed have different de Rham cohomology. Hence, if one wants to consider seriously global aspects of $\mathcal{T} \times \mathcal{M}$ using Beltrami variables, the singularity in the transformation to these variables has to be taken into account.

Let us now turn to the discussion of the descent equations. The analysis takes advantage of the fact that a necessary condition for a local functional W^g to be a solution of (2.9) is that the $\mathcal{S}$–transformation of its integrand is a total derivative. If one views the integrand as a local 2–form with ghost number g,

$$W^g = \int \omega_2^g\ , \tag{4.1}$$

this requires $\mathcal{S}\omega_2^g + d\omega_1^{g+1} = 0$ for some local form ω_1^{g+1}, where $d = dx^\alpha \partial_\alpha$ is the exterior derivative operator. Using now[8]

$$\mathcal{S}^2 = d^2 = \mathcal{S}d + d\mathcal{S} = 0 \tag{4.2}$$

one derives by means of the algebraic Poincaré lemma the descent equations

$$\mathcal{S}\omega_2^g + d\omega_1^{g+1} = 0 \ ; \qquad \mathcal{S}\omega_1^{g+1} + d\omega_0^{g+2} = 0 \ ; \qquad \mathcal{S}\omega_0^{g+2} = 0 \ . \tag{4.3}$$

The analysis of (4.3) performed in [19, 3] shows that the local function (zero–form) ω_0^{g+2} occurring here is nontrivial ($\omega_0^{g+2} \neq \mathcal{S}\omega_0^{g+1}$) and does not involve explicitly the coordinates x^α whenever ω_2^g is nontrivial ($\omega_2^g \neq \mathcal{S}\omega_2^{g-1} + d\omega_1^g$). Conversely, any nontrivial x–independent solution of the last equation (4.3) apart from the constant gives rise to a nontrivial solution of (2.9) whose integrand can be obtained from it according to

$$\omega_2^g = \tfrac{1}{2} dx^\alpha dx^\beta \frac{\partial}{\partial \xi^\beta} \frac{\partial}{\partial \xi^\alpha} \omega_0^{g+2} \tag{4.4}$$

where $\partial/\partial\xi^\alpha$ indicates an ordinary derivative with respect to *undifferentiated* ghosts ξ^α (*not* the functional or Euler–Lagrange derivative). It is important here to use the ghosts ξ^α and *not* the ghosts c^α arising from (3.1). Namely, (4.4) originates in the property of $\mathcal{S}$ that one can represent the exterior derivative on the fields and antifields (and their derivatives) by

$$d = b\mathcal{S} - \mathcal{S}b \quad ; \qquad b \equiv dx^\alpha \frac{\partial}{\partial \xi^\alpha} \ . \tag{4.5}$$

(4.5) simply reflects that diffeomorphisms are encoded in $\mathcal{S}$ and does not hold on the coordinates x^α themselves. It is therefore important that ω_0^{g+2} depends only on the fields and antifields and their derivatives but not explicitly on the coordinates, as shown in [19, 3]. Using (4.5) (and its consequence $bd - db = 0$), as well as (4.2), it is then straightforward to show that one can 'integrate' the descent equations (4.3) in the form $\omega_1^{g+1} = b\omega_0^{g+2}$ and $\omega_2^g = \frac{1}{2}b^2\omega_0^{g+2}$, the latter being just (4.4).

We conclude that, neglecting global properties of the base and target space, the cohomology of $\mathcal{S}$ on local functionals with ghost number g is isomorphic to its cohomology on those local functions with ghost number $(g+2)$ which do not depend explicitly on the x^α. On the representatives $\int \omega_2^g$ resp. ω_0^{g+2} of the corresponding cohomology classes this isomorphism is explicitly established through the substitution $\xi^\alpha \to \xi^\alpha + dx^\alpha$ which converts ω_0^{g+2} to $\omega_0^{g+2} + \omega_1^{g+1} + \omega_2^g$, cf. (4.4). Since we will compute the cohomology of $\mathcal{S}$ using the variables introduced in section 3, we note that this substitution rule translates into

$$c^\pm \ \to \ c^\pm + dx^\pm + h_{\mp\mp} dx^\mp \ ; \qquad \partial_\alpha c^\pm \to \partial_\alpha c^\pm + \partial_\alpha h_{\mp\mp} \cdot dx^\mp \qquad \text{etc.,} \tag{4.6}$$

where we used $\partial_\alpha dx^\beta = 0$. Note that these results imply already that the integrands of the solutions of (2.9) do not depend explicitly on the x^α, up to trivial contributions of course. We stress however that for the validity of the final result it is nevertheless important to allow for the presence of local functionals whose integrands depend explicitly on the x^α since otherwise there would be *more* nontrivial solutions of (2.9). Indeed, one would find

[8] The differentials dx^α are treated as odd graded variables which implies $\mathcal{S}dx^\alpha = -dx^\alpha\mathcal{S}$ and $dx^\alpha dx^\beta = -dx^\beta dx^\alpha$. The latter allows to omit the wedge product symbol.

additional solutions whose integrands are x–independent and trivial in the space of local x–dependent forms but nontrivial in the space of local x–independent forms. A typical example for such integrands is

$$\xi^\alpha \mathcal{L} = \mathcal{S}(-x^\alpha \mathcal{L}) + \partial_\beta(x^\alpha \xi^\beta \mathcal{L}) \ , \tag{4.7}$$

where $\mathcal{L}$ denotes a Weyl–invariant density such as, e.g., $\mathcal{L} = \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \cdot \partial_\beta X^\nu \eta_{\mu\nu}$. The occurrence of these additional solutions originates in a seeming harmless change of the algebraic Poincaré lemma when one formulates it in the space of local x–independent forms: in that space the differentials dx^α are not exact and therefore the descent equations do not always terminate with a zero-form! For instance, the descent equations arising from (4.7) terminate with the one-form $dx^\alpha \xi^+ \xi^- \mathcal{L}$ which is trivial in the space of x-dependent forms but not necessarily in the space of x-independent forms.

We finally mention that there are in principle two modifications of the results if the investigation is restricted to the space of forms which are globally defined on $\mathcal{M} \times \mathcal{T}$ rather than only locally: (a) those solutions which are only locally but not globally defined, disappear from the list of solutions we will find; (b) globally defined solutions ω_2^g which can locally be written as $\mathcal{S}\omega_2^{g-1} + d\omega_1^g$ have to be added to that list if they fail to be globally of this form.

5 Chiral tensor fields

We have shown in the previous section that the $\mathcal{S}$–cohomology on local functionals with ghost number g can be obtained from the $\mathcal{S}$–cohomology on local x–independent functions with ghost number $(g+2)$. In the next section we show that the latter cohomology can be reduced to the $\mathcal{S}$–cohomology in a particular subspace of local functions generated by quantities which we will call *covariant ghost variables* and *chiral tensor fields*. This section is devoted to prepare this result by introducing these quantities.

Usually, tensor fields are defined by their transformation laws under the symmetries of interest. This can be expressed just as well with the help of their BRST transformation, which gives a more convenient formulation for the analysis of the BRST cohomology. In many cases one finds that (components of) the gauge fields occur in trivial pairs together with all the derivatives of the ghost fields. They can therefore be eliminated from the BRST cohomology on local functions. The gauge fields and their derivatives then only remain in restricted combinations which are 'tensor fields'. Their BRST transformation involves only the undifferentiated ghosts. As a result, the representatives of the cohomology classes (of the BRST cohomology on local functions) can be expressed entirely in terms of tensor fields and the undifferentiated ghosts [3]. Well–known examples for such theories are Yang–Mills theories [19], ordinary (non-Weyl invariant) gravity in the vielbein formulation [19, 23] and supergravity theories [24].

Let us clarify this feature with the simplest example, Maxwell theory [25]. Consider local functions of the gauge field A_μ, the ghost C and their derivatives. The BRST transformations read just $sA_\mu = \partial_\mu C$, $sC = 0$. A first set of trivial pairs is thus $(A_\mu, \partial_\mu C)$. With one derivative more, there are the trivial pairs $(\partial_{(\mu} A_{\nu)}, \partial_\mu \partial_\nu C)$. This leaves the combinations $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$ unpaired. Obviously one can continue this separation to higher order derivatives. One then changes variables from $\{C; A_\mu, \partial_\mu C; \partial_\mu A_\nu, \partial_\mu \partial_\nu C; \ldots\}$ to

$\{C; A_\mu, \partial_\mu C; \partial_{(\mu} A_{\nu)}, F_{\mu\nu}, \partial_\mu\partial_\nu C; \ldots\}$, subdivided in the trivial pairs $\{(A_\mu, \partial_\mu C), (\partial_{(\mu} A_{\nu)}, \partial_\mu\partial_\nu C), \ldots\}$ and the unpaired variables $\{C, F_{\mu\nu}, \ldots\}$. The choice of the remaining combinations like $F_{\mu\nu}$ is dictated by the requirement that only unpaired (undifferentiated) ghost variables may appear in their BRST transformation. These remaining combinations are the tensor fields, ($F_{\mu\nu}$, $\partial_{(\mu} F_{\nu)\rho}$, etc.) and the undifferentiated ghost C.

Of course one should not expect that one can eliminate *all* derivatives of the ghosts from the cohomology in any gauge theory. That can be done if all ghosts are independent (which is also true in our case), and if there is a gauge field for each symmetry (which is not).

A well–known counterexample is provided already by ordinary gravity in the metric formulation where one can eliminate all derivatives of the diffeomorphism ghosts of second and higher order but not all of their first order derivatives: e.g. in two dimensions, it is not possible to pair off the three components of the metric with the four components of the gradients of the diffeomorphism ghosts (the remaining first order derivatives then play a role analogous to the undifferentiated Lorentz ghosts in the vielbein formalism). For the case treated in this paper, the situation is even more subtle since, apart from using the metric formulation of gravity, we do not introduce a Weyl gauge field. As a consequence, there are *infinitely many* derivatives of the ghosts which do not occur in trivial pairs and thus cannot be eliminated through the procedure sketched above[9]. This is easily checked by the following simple counting argument. Analogously to the above example of Maxwell theory we consider the BRST transformations of the derivatives of $g_{\alpha\beta}$ of fixed order ("level") n. They contain as leading terms derivatives of order $(n+1)$ of the diffeomorphism ghosts ξ^α and nth order derivatives of the Weyl ghost c, cf. (2.7). This suggests to assign level (-1) to the two undifferentiated ghosts ξ^α which are clearly unpaired. At level 0 there are the three components of the undifferentiated metric $g_{\alpha\beta}$ but four components $\partial_\alpha \xi^\beta$ and the undifferentiated Weyl ghost, i.e. two ghost variables remain unpaired. Similarly, at level 1, the 6 algebraically independent first order derivatives $\partial_\gamma g_{\alpha\beta}$ cannot be paired with the 6 second order derivatives of the ξ^α together with the 2 first order derivatives of c. Analogously one easily verifies that at all higher levels precisely two derivatives of the ghosts remain unpaired.

One may also check that the same feature occurs in the zweibein formulation. In this formulation one introduces zweibeins, but also the Lorentz ghost apart from the diffeomorphism and dilatation ghosts. The zweibeins e_α^a transform into the gradients $\partial_\alpha c^a$ of the diffeomorphism ghosts, leaving at level 0 the undifferentiated ghosts c and c' of local dilatations and local Lorentz transformations. At level 1 one has 8 components $\partial_\alpha e_\beta^a$ versus the 6 components $\partial_\alpha \partial_\beta c^a$ plus the 4 components $\partial_\alpha c$ and $\partial_\alpha c'$ and so on. Therefore we cannot directly adopt the methods and results developed in [19, 3, 23] for non–Weyl invariant gravity, or Weyl invariant gravity without Weyl gauge field.

There seems to be a way around this mismatch of derivatives of the ghosts and the gauge fields: one could introduce an extra gauge field b_α for Weyl transformations. Indeed, in presence of b_α the mismatch disappears since all derivatives of the Weyl ghost can be paired with the b_α and their derivatives as in the above example of Maxwell theory. As a consequence the cohomology problem could be treated as in [3]. Then however, one would be computing a different cohomology, including b_α dependence in the functionals. One

[9]Nevertheless it will turn out in the end that again all derivatives of order > 2 disappear, but the argument is more sophisticated than that of eliminating trivial pairs.

could eliminate this dependence by requiring invariance under arbitrary shifts $\delta b_\alpha = \Lambda_\alpha$, which expresses the absence of b_α. The mismatch then remains. Alternatively, this new invariance brings in another gauge field, and so on. Continuing in this way, one would get an infinite set of gauge invariances and gauge fields. In fact, this would amount to gauging two copies of the subalgebra $\{L_n | n \geq -1\}$ of the Virasoro algebra, as in [26]. We will denote these two copies henceforth by $\{L_n^+\}$ and $\{L_n^-\}$ respectively. Wishing to avoid the approach with an infinite tower of symmetries and gauge fields, we will not introduce a gauge field for the Weyl transformations. Of course we will then have to adapt the methods of [19, 3, 23].

In our approach we only introduce $g_{\alpha\beta}$ as gauge fields. As a consequence one *cannot* reduce the cohomology to a problem involving only undifferentiated ghost fields, or derivatives of ghosts up to some finite order, by the standard argument sketched above. However, we can still use this argument to get rid of all derivatives of the ghosts except for two at every level, as the above counting suggests. In particular, all 'mixed' derivatives, namely $\partial_\pm c^\mp$ and their derivatives, can be eliminated by the standard argument. The remaining derivatives can be chosen to be $(\partial_+)^{m+1} c^+$ and $(\partial_-)^{m+1} c^-$, where m is the level used in the above counting and runs from (-1) to infinity. These derivatives of c^+ and c^- are called the *covariant ghost variables.*

The more difficult task is to construct the quantities which take over the role usually played by tensor fields. We call them *chiral tensor fields.* Their characteristic property is that their BRST transformation may contain the covariant ghost variables, but no other derivatives of the ghosts. The fact that in our case the set of covariant ghost variables is infinite corresponds to the infinite set of (undifferentiated) ghosts in the approach using an infinite tower of gauge symmetries.

In the remainder of this section we will explicitly construct an appropriate basis for the chiral tensor fields, denoted by $\{\mathcal{B}^i\}$. This construction is slightly involved but a crucial and necessary step within the computation of the $\mathcal{S}$–cohomology. It is also interesting in itself since it shows how the above mentioned subalgebras $\{L_m^+\}$ and $\{L_m^-\}$ $(m \geq -1)$ of the Virasoro algebra come into play and are represented on the $\mathcal{B}^i$. In particular it turns out that the $\mathcal{B}^i$ can be chosen as eigenfunctions of L_0^+ and L_0^-. This will be very useful in the next section, since it will eventually allow to reduce the cohomological analysis to a problem where only a small finite subset of $\{\mathcal{B}^i\}$ and those six covariant ghost variables enter which correspond to the $sl(2)$ subalgebras $\{L_{-1}^\pm, L_0^\pm, L_1^\pm\}$.

It is understood in the following that all functions that occur are functions of the fields introduced in section 3 and of their derivatives. Treating the ghosts separately, we will use for the remaining variables the collective notation

$$[Z] \equiv \Big((\partial_+)^m (\partial_-)^n Z^i : \ m, n = 0, 1, 2, \ldots\Big) \ , \quad \{Z^i\} = \{h_{\pm\pm},\, X^\mu,\, h^{*\pm\pm},\, X^*_\mu,\, c^*_\pm\} \ .$$

We introduce the following notation for the above mentioned covariant ghost variables:

$$c_\pm^m = \frac{1}{(m+1)!} \left(\partial_\pm\right)^{m+1} c^\pm \ ; \quad m = -1, 0, 1, \ldots \ . \tag{5.1}$$

Using (B.1) one easily verifies that their BRST transformations read

$$\mathcal{S} c_\pm^m = \tfrac{1}{2} f^m{}_{nk} c_\pm^k c_\pm^n \ , \tag{5.2}$$

where ${f^m}_{nk}$ are the structure constants of the Virasoro algebra:

$$ {f^m}_{nk} = (n-k)\delta^m_{n+k} \ . \tag{5.3}$$

Note that the sum in (5.2) is finite due to $m,n,k \geq -1$ and that the covariant ghost variables transform among themselves, i.e. that no other derivatives of the ghosts occur in $\mathcal{S}c^m_\pm$.

Having defined the covariant ghost variables, we are now in a position to give a precise definition of chiral tensor fields. The differential $\mathcal{S}$ decomposes into a 'Koszul–Tate part' δ_{KT} [27] and a 'BRST'–part s (see (2.4)). On the fields one has $\mathcal{S} = s$, as δ_{KT} has nonvanishing action only on the antifields. On the antifields δ_{KT} collects that part of the $\mathcal{S}$–transformation which does not involve the ghosts:

$$ \delta_{KT} c^*_- = -\nabla_+ h^{*++} + \hat{X}^*_\mu \nabla_- X^\mu \ ; \tag{5.4}$$

$$ \delta_{KT} h^{*++} = S_{cl} \frac{\overleftarrow{\delta}}{\delta h_{++}} \ ; \quad \delta_{KT} \hat{X}^*_\mu = S_{cl} \frac{\overleftarrow{\delta}}{\delta X^\mu} \frac{1}{1-y} \tag{5.5}$$

where, for reasons which will become clear soon, we have introduced

$$ \hat{X}^*_\mu = \frac{1}{1-y} X^*_\mu \ . \tag{5.6}$$

Note that the change from X^*_μ to $\hat{X}^*_\mu$ becomes singular for $y=1$. This singularity is actually the same that occurred already in section 3 in the field redefinitions leading to Beltrami variables since $y=1$ is equivalent to $\sqrt{g}(g_{+-}+\sqrt{g})=0$. Hence, we do not introduce further singularities here.

Explicit expressions for s are given in appendix B. Note that only (5.5) involves explicitly the classical action. The precise form of that action however does not matter in the following since chiral tensor fields are identified by their s–transformation. Now we give the definition: a chiral tensor field is a local function $T([Z])$ such that

$$ sT([Z]) = c^m_\alpha \, T^\alpha_m([Z]) \ . \tag{5.7}$$

The sum on the r.h.s. of (5.7) contains actually only finitely many nonvanishing T^α_m since s is a local operator and T is by definition a local function.

The nilpotency of s guarantees that the T^α_m in (5.7) are chiral tensor fields as well. To prove this, one applies s to (5.7) and concludes from $s^2=0$ that sT^α_m cannot involve mixed derivatives of the ghosts. Also the full operation $\mathcal{S}$ on T leads to chiral tensor fields, as we will now prove. This will be true if $T' := \delta_{KT}T$ is automatically a chiral tensor field when this holds already for T. To prove this, we have to show that sT' does not involve mixed derivatives of the ghosts. Now, $s\delta_{KT}+\delta_{KT}s=0$ and $\delta_{KT}c^m_\alpha=0$ imply $sT' = -\delta_{KT}sT = c^m_\alpha \delta_{KT} T^\alpha_m$, which evidently does not involve mixed derivatives of the ghosts. Therefore, $\mathcal{S}T$ depends only on tensor fields and ghosts $c^m_\pm$. This finishes the proof.

Note that the undifferentiated fields X^μ, $\hat{X}^*_\mu$, $h^{*\pm\pm}$ and $c^*_\pm$ are chiral tensor fields according to (B.3)–(B.6) (X^*_μ itself is not a chiral tensor field). The partial derivatives of a chiral tensor field however are in general not chiral tensor fields: we have to complete them to covariant ones. To that end we define 'Virasoro' operators L^α_m ($m \geq -1$, $\alpha = +,-$)

on chiral tensor fields (and, for later purpose, on the covariant ghost variables as well) through the anticommutators

$$L_m^\alpha := \frac{\partial}{\partial c_\alpha^m}\,\mathcal{S} + \mathcal{S}\,\frac{\partial}{\partial c_\alpha^m} = \frac{\partial}{\partial c_\alpha^m}\,s + s\,\frac{\partial}{\partial c_\alpha^m}\ , \qquad m = -1, 0, 1, \dots\ , \tag{5.8}$$

where one can use both s and $\mathcal{S}$ since δ_{KT} vanishes on c_α^m. Using the notation of (5.7), we obtain

$$L_m^\alpha T = T_m^\alpha\ ; \quad L_m^\pm c_\pm^n = f^n{}_{km} c_\pm^k\ ; \quad L_m^\pm c_\mp^n = 0\ .$$

Note that the L_m^α are derivations, i.e. they satisfy the product rule, since they are defined as anticommutators of two antiderivations. Furthermore their algebra closes on (functions of) chiral tensor fields and the covariant ghost variables and is isomorphic to the algebra of vector fields $z^{m+1}\frac{d}{dz}$ that are regular at $z=0$:

$$\left[L_m^\pm, L_n^\pm\right] = (m-n)L_{m+n}^\pm\ ; \qquad \left[L_m^+, L_n^-\right] = 0\ . \tag{5.9}$$

(5.9) is easily verified on the ghosts c_α^m using the Jacobi identity for the structure constants $f^k{}_{mn}$. One verifies it on any chiral tensor field T by evaluating $s^2T=0$ using (5.2). Indeed, since T_m^α is a chiral tensor field (see above), we have

$$sT_m^\alpha = s(L_m^\alpha T) = c_\beta^n L_n^\beta L_m^\alpha T$$

which requires (5.9) to hold on T in order to be consistent with $s^2T=0$.

We can now describe and construct the generators replacing in the new basis the X^μ, X_μ^*, $h^{*\pm\pm}$, $c_\pm^*$ and their partial derivatives. We denote the set of these new generators by $\{\mathcal{B}^i\}$ and require

(I). $\{\mathcal{B}^i\}$ consists of 'covariant derivatives' of the fields X^μ, $\hat{X}_\mu^*$, $h^{*\pm\pm}$, $c_\pm^*$ which complete (all) their partial derivatives to chiral tensor fields;

(II). each $\mathcal{B}^i$ is an eigenfunction of L_0^+ and L_0^-.

(II) is not really needed for the construction of a basis for the chiral tensor fields but can be imposed and will be useful later, as mentioned already above. It is indeed fulfilled for the undifferentiated fields X^μ, $\hat{X}_\mu^*$, $h^{*\pm\pm}$, $c_\pm^*$. This is evident from (B.3)–(B.6) which also yields the L_0^α–eigenvalues of these fields ('weights'), denoted by w_α:

Z^i	X^μ	$\hat{X}_\mu^*$	h^{*++}	h^{*--}	c_-^*	c_+^*
(w_+, w_-)	$(0,0)$	$(1,1)$	$(0,2)$	$(2,0)$	$(0,2)$	$(2,0)$

. (5.10)

We now observe that the operators L_{-1}^α already serve as covariant derivatives of the matter fields which we denote by $X_{m,n}^\mu$:

$$X_{m,n}^\mu = \left(L_{-1}^+\right)^m \left(L_{-1}^-\right)^n X^\mu = \left(\frac{\partial}{\partial c^+}\,\mathcal{S}\right)^m \left(\frac{\partial}{\partial c^-}\,\mathcal{S}\right)^n X^\mu\ , \quad m,n = 0,1,2,\dots \tag{5.11}$$

where the second equality holds due to $\mathcal{S}^2=0$. In order to see that the $X_{m,n}^\mu$ indeed complete the partial derivatives to covariant derivatives of the X^μ one verifies that

$$X_{m,n}^\mu = \frac{1}{(1-y)^{m+n}}\,(\nabla_+)^m\,(\nabla_-)^n\,X^\mu + \mathcal{O}(m+n-1)\ , \tag{5.12}$$

where $\mathcal{O}(m+n-1)$ denotes a complicated function of h_{++}, h_{--}, X^μ involving only their derivatives of $(m+n-1)$th and lower order. The first few $X^\mu_{m,n}$ are given in appendix A (in fact only $X^\mu_{0,0}$, $X^\mu_{1,0}$, $X^\mu_{0,1}$ and $X^\mu_{1,1}$ will ultimately be needed for the cohomology). The action of $L^\pm_m$ and s on $X^\mu_{m,n}$ can be obtained using the algebra (5.9) and the fact that $X^\mu_{0,0} = X^\mu$ has 'highest weight'

$$L^\pm_m X^\mu = 0 \qquad \forall m \geq 0 \ . \tag{5.13}$$

In particular, one easily verifies that $X^\mu_{m,n}$ is an eigenfunction of the L^α_0 with weights (m,n), using (5.9) and (5.10). In the next section we will show in detail that the change of generators from the $\partial^m_+\partial^n_- X^\mu$ to the $X^\mu_{m,n}$ is in fact local and invertible except where the transformation to the Beltrami variables itself becomes singular (cf. proof of lemma 6.1).

Analogously one checks that a basis $\{\hat{X}^*_{\mu\, m,n}\}$ for the covariant derivatives of the antifields $\hat{X}^*_\mu$ is given by

$$\hat{X}^*_{\mu\, m,n} = \left(L^+_{-1}\right)^m \left(L^-_{-1}\right)^n \hat{X}^*_\mu \ , \quad m,n = 0,1,2,\dots \ , \tag{5.14}$$

and that $\hat{X}^*_{\mu\, m,n}$ has weights $(m+1, n+1)$. Again, the change from X^*_μ and its partial derivatives to the $\hat{X}^*_{\mu\, m,n}$ is local and (locally) invertible, see next section. The action of $L^\pm_m$ and s on $\hat{X}^*_{\mu\, m,n}$ can be obtained using the algebra (5.9) and

$$L^\pm_m \hat{X}^*_\mu = 0 \qquad \forall m \geq 1 \ , \tag{5.15}$$

which follows from (B.4).

The construction of a complete basis for the covariant derivatives of the remaining antifields $h^{*\pm\pm}$ and $c^*_\pm$ is more subtle since (B.5) and (B.6) show that L^+_{-1} does not serve as an appropriate covariant derivative operator on h^{*++} or c^*_- due to $L^+_{-1}h^{*++} = \hat{X}^*_\mu X^\mu_{0,1}$ and $L^+_{-1}c^*_- = 0$. Analogous statements hold for L^-_{-1} on h^{*--} and c^*_+ of course. We therefore have to look for an alternative construction of covariant derivatives. It can be easily found. Namely the operators

$$D_\pm = \partial_\pm - \sum_{m\geq -1} \frac{1}{(m+1)!} (\partial_\mp)^{m+1} h_{\pm\pm} \cdot L^\mp_m \tag{5.16}$$

provide covariant derivatives $D_\alpha T$ of an arbitrary chiral tensor field T since they are constructed such that $sD_\alpha T$ does not contain $\partial_- c^+$, $\partial_+ c^-$ or any of their derivatives (again, the sum appearing in $D_\alpha T$ contains only finitely many nonvanishing terms since T is local by assumption). In fact we could have used the D_α to construct a basis for the covariant derivatives of all fields $Z \in \{X^\mu,\ \hat{X}^*_\mu,\ h^{*\pm\pm},\ c^*_\pm\}$ through $(D_+)^m(D_-)^n Z$. However that basis would not satisfy requirement (II) since the operators $D_\pm$ have the following commutation relations with the L^α_0:

$$[L^\pm_0, D_\pm] = L^\pm_{-1} \ ; \quad [L^\pm_0, D_\mp] = 0 \ . \tag{5.17}$$

On the other hand (5.17) and (5.9) imply

$$[L^\alpha_0, \tilde{D}_\beta] = 0 \tag{5.18}$$

where

$$\tilde{D}_\alpha = D_\alpha - L^\alpha_{-1} \ . \tag{5.19}$$

We note that the $\tilde{D}_\alpha$ vanish on X^μ and $\hat{X}^*_\mu$, i.e. on these fields one actually has $D_\alpha = L^\alpha_{-1}$. In contrast, $\tilde{D}_+ h^{*++}$ and $\tilde{D}_+ c^*_-$ do not vanish and complete $\partial_+ h^{*++}$ and $\partial_+ c^*_-$ to chiral tensor fields. In particular we have $\tilde{D}_+ c^*_- = D_+ c^*_-$ due to $L^+_{-1} c^*_- = 0$ and, more generally,

$$(\tilde{D}_\pm)^m c^*_\mp = (D_\pm)^m c^*_\mp \ . \tag{5.20}$$

It is now straightforward to construct a basis for the covariant derivatives of h^{*++} and c^*_- with definite weights through

$$c^*_{-\,m,n} = (L^-_{-1})^n (D_+)^m c^*_- \ ; \quad h^{*++}_{m,n} = (L^-_{-1})^n (\tilde{D}_+)^m h^{*++} \ ; \quad m,n = 0,1,2,\dots \ , \tag{5.21}$$

and analogously one constructs covariant derivatives $c^*_{+\,m,n}$ and $h^{*--}_{m,n}$ of h^{*++} and c^*_-. We note that one has

$$L^\alpha_m c^*_\pm = L^\alpha_m h^{*\pm\pm} = 0 \qquad \forall m \geq 1 \ . \tag{5.22}$$

One now checks again that all the derivatives of the $c^*_\pm$ and $h^{*\pm\pm}$ appear as leading terms (highest derivatives) in the new variables. Furthermore the change from the $c^*_\pm$, $h^{*\pm\pm}$ and their partial derivatives to the $c^*_{\pm\,m,n}$ and $h^{*\pm\pm}_{m,n}$ is local and invertible, see next section. E.g. we have, using (B.5) and (B.6),

$$\begin{aligned}
h^{*++}_{0,1} &= L^-_{-1} h^{*++} = \partial_- h^{*++} - h_{--} \hat{X}^*_\mu X^\mu_{0,1} \\
h^{*++}_{1,0} &= \tilde{D}_+ h^{*++} = \nabla_+ h^{*++} - (1-y) \hat{X}^*_\mu X^\mu_{0,1} \\
c^*_{-\,0,1} &= L^-_{-1} c^*_- = \partial_- c^*_- \\
c^*_{-\,1,0} &= D_+ c^*_- = (\partial_+ - h_{++}\partial_- - 2\partial_- h_{++}\cdot) c^*_- \ .
\end{aligned} \tag{5.23}$$

This completes the construction of $\{\mathcal{B}^i\}$. The complete list of the $\mathcal{B}^i$ and their weights is given by

$\mathcal{B}^i$	$X^\mu_{m,n}$	$\hat{X}^*_{\mu\,m,n}$	$h^{*++}_{m,n}$	$h^{*--}_{m,n}$	$c^*_{-\,m,n}$	$c^*_{+\,m,n}$
(w_+,w_-)	(m,n)	$(m+1,n+1)$	$(0,n+2)$	$(m+2,0)$	$(0,n+2)$	$(m+2,0)$

$(m,n = 0,1,2,\dots)$. (5.24)

We finally give the weights of the covariant ghost variables:

c^m_α	c^m_+	c^m_-
(w_+,w_-)	$(m,0)$	$(0,m)$

$(m = -1,0,1,\dots)$. (5.25)

As a side comment we remark that one has $h^{*\pm\pm}_{m+1,n} = -\delta_{KT} c^*_{\mp\,m,n}$ for $m,n \geq 0$ which can easily be checked explicitly for $m = n = 0$ and then extended to $m,n \geq 0$ using $[L^\alpha_m, \delta_{KT}] = 0$. This illustrates a general property of δ_{KT} explained above, namely that it maps chiral tensor fields to chiral tensor fields.

6 Reduction to $H^*(\mathcal{S},\mathcal{C})$

In this section we prove that one can contract the $\mathcal{S}$–cohomology in the full space of local functions to a particular subspace which we will denote by $\mathcal{C}$. In the first step we will perform a reduction to the space of local functions $\omega(c^m_\alpha, \mathcal{B}^i)$ of the chiral tensor fields $\mathcal{B}^i$ and the covariant ghost variables c^m_α introduced in the previous section, and in the second step a reduction to the space of local functions $\omega(c^m_\alpha, \mathcal{B}^i)$ with vanishing L^+_0 and L^-_0 weights. The latter is the above mentioned subspace $\mathcal{C}$. The cohomology of $\mathcal{S}$ in $\mathcal{C}$ is denoted by $H^*(\mathcal{S},\mathcal{C})$. In the third step we will give a basis for the functions in $\mathcal{C}$, which will be described in terms of a finite number of 'superfields' in the undifferentiated matter fields X^μ and the ghosts c^0_α. The subsequent cohomological analysis will be in terms of those superfields. The final step, the computation of $H^*(\mathcal{S},\mathcal{C})$, can then be done by a direct calculation, which will be carried out in section 9.

In order to compute the $\mathcal{S}$-cohomology in the space of local x-independent functions, we have to solve

$$\mathcal{S}\omega^G([\Phi,\Phi^*]) = 0 \tag{6.1}$$

where ω^G has arbitrary ghost number G and its argument $[\Phi,\Phi^*]$ indicates the local[10] dependence on *all* fields, antifields and their partial derivatives collectively,

$$[\Phi,\Phi^*] \equiv \left\{(\partial_+)^m(\partial_-)^n\Phi^A,\, (\partial_+)^m(\partial_-)^n\Phi^*_A|\; m,n = 0,1,\ldots\right\} .$$

The set of fields Φ was given in (3.6). Two solutions of (6.1) are called equivalent if they differ by an $\mathcal{S}$–exact solution or a constant. The latter can occur only in the ghost number–zero section due to the absence of constant ghosts,

$$\omega^G([\Phi,\Phi^*]) \approx \omega'^G([\Phi,\Phi^*]) \;\Leftrightarrow\; \omega'^G - \omega^G = \mathcal{S}\eta^{G-1}([\Phi,\Phi^*]) + \lambda\,\delta^G_0 , \tag{6.2}$$

where η^{G-1} is a local function with ghost number $(G-1)$ and λ is a constant. This definition of equivalence is motivated by the fact that $\mathcal{S}$–exact solutions of (6.1), and the constants, correspond via the descent equations to (locally) trivial functionals, see section 4.

We will now isolate trivial pairs, which we can then remove from the computation of the cohomology. Trivial pairs are doublets of generators (U,V) satisfying

(a) U and V have the simple transformations $\mathcal{S}U = V$, $\mathcal{S}V = 0$;

(b) U and V do not occur in the $\mathcal{S}$-transformation of any other generator;

(c) U and V generate the algebra of functions of U and V *freely*, i.e. there are no extra[11] relations.

By a standard argument, using a contracting homotopy, one then easily shows that such trivial pairs of generators indeed do not contribute nontrivially to the S-cohomology (neglecting global properties of the target manifold). This reduces the computation of the $\mathcal{S}$-cohomology in the algebra of all generators to the same problem in the algebra of those

[10] A local function is polynomial in all these variables except for the undifferentiated $h_{\pm\pm}$ and X^μ.

[11] i.e. other than the Grassman algebra relations. This third condition is usually satisfied automatically, and will therefore be left out of focus. In section 9 we will meet an example where it is *not* valid.

generators which remain after the trivial pairs have been removed. The difficult part in this step is in fact not that of finding U's and V's satisfying (a) but the construction of a complete set of complementary generators, since they are conditioned by (b).

We used elimination of trivial pairs already before to remove the fields $(e, \tilde{c})$ and their antifields. Indeed, these pairs satisfy conditions (a)–(c) in the basis of generators introduced in section 3 (this was in fact one of the reasons for introducing that basis). Other trivial pairs of generators are the antighosts and corresponding Lagrange multiplier fields (and their derivatives) which one introduces for gauge fixing. They also satisfy evidently (a)–(c) and therefore have been omitted from the very beginning. In the cases just cited one can eliminate sets of fields completely from the cohomology since two *undifferentiated* fields (or antifields) group in trivial pairs respectively. Therefore all their derivatives group in trivial pairs as well and these fields disappear completely from the cohomology (both on local functions and on local functionals). This is different in the cases considered in the following since not all derivatives of the involved fields (ghosts) occur in trivial pairs.

Let us now derive the reduction to functions of the chiral tensor fields $\mathcal{B}^i$ and the covariant ghost variables c_α^m introduced in section 5. The transformation laws $\mathcal{S}h_{\pm\pm} = \nabla_\pm c^\mp$ suggest that trivial pairs are given by

$$(\mu_\ell, \mathcal{S}\mu_\ell) \quad \text{where} \quad \{\mu_\ell\} = \{(\partial_+)^m(\partial_-)^n h_{++}, (\partial_+)^m(\partial_-)^n h_{--} \,|\, m,n = 0,1,\ldots\} \ . \tag{6.3}$$

These pairs evidently satisfy property (a) but the fulfillment of (b) is not straightforward. Rather, we first have to complete $\{\mu_\ell, \mathcal{S}\mu_\ell\}$ to a new basis of generators satisfying (b) and replacing the old generators (field, antifields and their derivatives) in order to be able to remove the μ's and $(\mathcal{S}\mu)$'s. Of course we require the change of basis from the old to the new generators to be invertible and local.

The new generators $\mathcal{S}\mu_\ell$ replace the 'mixed' derivatives of the ghosts c^α, i.e. $\partial_+ c^-$, $\partial_- c^+$ and derivatives thereof, as one has $\mathcal{S}h_{\pm\pm} = \partial_\pm c^\mp + \ldots$. Hence, we can replace the ghosts c^α and all their derivatives by the $\mathcal{S}\mu_\ell$ and the $c_\pm^m$.

Now, a set of generators completing $\{\mu_\ell, \mathcal{S}\mu_\ell\}$ to a basis with the desired property (b) is given by $\{c_\alpha^m, \mathcal{B}^i\}$. This follows from the facts that by construction both $\mathcal{S}c_\alpha^m$ and $\mathcal{S}\mathcal{B}^i$ can be written entirely in terms of the c_α^m and $\mathcal{B}^i$ again, and that the change of basis is indeed local and (locally) invertible due to the following lemma:

Lemma 6.1 *Any local function of X^μ, $h_{\pm\pm}$, $c^\pm$, X^*_μ, $h^*_{\pm\pm}$, $c^*_\pm$ and their derivatives can be written as a local function of the $c_\pm^m$, $\mathcal{B}^i$, μ_ℓ and $\mathcal{S}\mu_\ell$ and vice versa*[12].

Proof: One easily verifies that the lemma is implied by the facts that (i) the undifferentiated fields X^μ, h_{++} and h_{--} are elements both of the old and of the new basis, and (ii) all other generators of the old basis can be written as local functions of the new generators and vice versa. (i) is relevant since X^μ, h_{++} and h_{--} can occur nonpolynomially in local functions, contrary to all other generators.

Hence, all we have to prove is (ii). To that end we assign a level to each generator given by the highest order of derivatives of fields or antifields occurring in them. The proof can then be performed inductively. First one verifies that (ii) holds at level 0. This is obvious

[12]A local function of the $c_\pm^m$, $\mathcal{B}^i$, μ_ℓ and $\mathcal{S}\mu_\ell$ depends polynomially on all these generators except possibly on the undifferentiated fields X^μ, h_{++} and h_{--}.

since the new generators with this level just agree with old ones (undifferentiated fields and antifields) up to the replacements $X^*_\mu \leftrightarrow \hat{X}^*_\mu$. In the second step of the induction one shows that (ii) holds at level n if it holds at all smaller levels.

For the derivatives of the X^μ that second step of the induction can be performed as follows. Consider the set of nth order derivatives of the X^μ, i.e. $\{(\partial_+)^{n-p}(\partial_-)^p X^\mu | p = 0,\ldots,n\}$. The corresponding set of new generators with the same level is $\{X^\mu_{n-p,p}|p = 0,\ldots,n\}$. Due to (5.12) (and (A.2)), one has

$$X^\mu_{n-p,p} = \sum_{q=0}^{n} M^{(n)}_{p,q}(h)\,(\partial_+)^{n-p}(\partial_-)^p X^\mu + \mathcal{O}(n-1)\ . \tag{6.4}$$

Here $\mathcal{O}(n-1)$ denotes a local function of generators with levels $k \le n-1$, cf. (5.12) and (A.2). Since (ii) is supposed to hold at all levels smaller than n we don't have to worry about this term. The question then is whether $M^{(n)}_{p,q}$ is invertible. This can be seen by considering the transformation of independent variables from $x^\pm$ to $y^\pm = (1-y)^{-1}(x^\pm + h_{\mp\mp}x^\mp)$. With constant $h_{\mp\mp}$ it is obvious that this leads to the same matrix for the transformation between $x^\pm$ derivatives and $y^\pm$ derivatives, since $\partial/\partial y^\pm = \nabla_\pm$. From this one easily sees that $\det M^{(n)} = (1-y)^{-n(n+1)/2}$. This proves that all $(\partial_+)^m(\partial_-)^n X^\mu$ are indeed local functions of the new generators and that the change from the $(\partial_+)^m(\partial_-)^n X^\mu$ to the $X^\mu_{m,n}$ is invertible except for $y = 1$. The latter is the same singularity that occurred already in the change to the Beltrami variables themselves, cf. remark after (5.6). Note that if we would have encountered here infinitely many further singularities (e.g. at any level a different one), then the change to the new generators would not have been allowed.

Analogously one checks that the change from the fields $X^*_\mu, h^{*\pm\pm}, c^*_\pm$ and their derivatives to the corresponding new generators is also local and invertible except for $y = 1$.

The $c^m_\pm$ contain the ghosts and their 'unmixed' derivatives. Using $\mathcal{S}h_{\pm\pm} = \nabla_\pm c^\mp = \partial_\pm c^\mp + \ldots$, cf. (B.2), we see that the mixed derivatives of the ghosts are the highest derivative parts of $\mathcal{S}\mu_\ell\}$. Therefore all the ghosts and their derivatives and all (derivatives of) $h_{\pm\pm}$ are replaced by $\{c^m_\pm, \mu_\ell, \mathcal{S}\mu_\ell\}$. ■

Since the new basis of generators has been constructed such that it satisfies requirements (a) and (b), we can now conclude that the trivial pairs of generators can be removed from the cohomology:

Lemma 6.2 *(i) Any solution of (6.1) can be expressed entirely in terms of the c^m_α and $\mathcal{B}^i$ modulo an $\mathcal{S}$−exact contribution,*

$$\mathcal{S}\omega^G([\Phi,\Phi^*]) = 0 \ \Rightarrow\ \omega^G([\Phi,\Phi^*]) = \hat{\omega}^G(c^m_\alpha,\mathcal{B}^i) + \mathcal{S}\eta^{G-1}([\Phi,\Phi^*])\ . \tag{6.5}$$

(ii) A function of the variables c^m_α and $\mathcal{B}^i$ is $\mathcal{S}$−exact iff it is the $\mathcal{S}$−transformation of a another function of these variables,

$$\hat{\omega}^G(c^m_\alpha,\mathcal{B}^i) = \mathcal{S}\eta^{G-1}([\Phi,\Phi^*]) \ \Leftrightarrow\ \hat{\omega}^G(c^m_\alpha,\mathcal{B}^i) = \mathcal{S}\hat{\eta}^{G-1}(c^m_\alpha,\mathcal{B}^i)\ . \tag{6.6}$$

Now we go to step 2: the reduction to the space $\mathcal{C}$, which includes only the zero eigenspaces of L^α_0. Since all variables c^m_α and $\mathcal{B}^i$ have by construction definite weights (cf.

(5.24) and (5.25)), we can decompose any function $\hat{\omega}^G(c_\alpha^m, \mathcal{B}^i)$ into parts with definite weights,

$$\hat{\omega}^G(c_\alpha^m, \mathcal{B}^i) = \sum_{m,n} \hat{\omega}^G_{m,n}(c_\alpha^m, \mathcal{B}^i), \quad L_0^+ \hat{\omega}^G_{m,n} = m\hat{\omega}^G_{m,n}, \; L_0^- \hat{\omega}^G_{m,n} = n\hat{\omega}^G_{m,n} \; . \tag{6.7}$$

By their very definition (5.8), the L_0^α can be represented as anticommutators of $\mathcal{S}$ with other operators (namely with the derivatives with respect to c_α^0). Together with (6.7) this is already sufficient to conclude from $\mathcal{S}\hat{\omega}^G = 0$ the $\mathcal{S}$–exactness of all $\hat{\omega}^G_{m,n}$ apart from $\hat{\omega}^G_{0,0}$. Namely, (5.8) implies that $\mathcal{S}$ commutes with the L_0^α and therefore leaves their eigenspaces invariant. Hence, $\mathcal{S}\hat{\omega}^G = 0$ requires all $\hat{\omega}^G_{m,n}$ to be separately $\mathcal{S}$–invariant. This implies in turn that $\hat{\omega}^G_{m,n}$ is $\mathcal{S}$–exact unless both m and n vanish, cf. e.g. 'basic lemma' in [25]. Using the $\mathcal{S}$–invariance of the L_0^α–eigenspaces again, we can further conclude that $\hat{\omega}^G_{0,0}$ is $\mathcal{S}$–exact if and only if it is $\mathcal{S}$–exact in the space of functions with weights $(0,0)$. This is the space $\mathcal{C}$ of functions mentioned in the beginning of this section:

$$\mathcal{C} = \{\omega(c_\alpha^m, \mathcal{B}^i) : \; L_0^+\omega = L_0^-\omega = 0\} \; . \tag{6.8}$$

We have therefore shown that the computation of all solutions of (6.1) can be reduced to the computation of $H^*(\mathcal{S}, \mathcal{C})$:

Lemma 6.3 *(i) Any solution of (6.1) is a function in $\mathcal{C}$ up to an $\mathcal{S}$–exact part,*

$$\mathcal{S}\omega^G([\Phi, \Phi^*]) = 0 \;\Rightarrow\; \omega^G([\Phi, \Phi^*]) = \bar{\omega}^G + \mathcal{S}\eta^{G-1}([\Phi, \Phi^*]), \quad \bar{\omega}^G \in \mathcal{C} \; . \tag{6.9}$$

(ii) A function in $\mathcal{C}$ is $\mathcal{S}$–exact in the space of local functions iff it is $\mathcal{S}$–exact in $\mathcal{C}$,

$$\bar{\omega}^G = \mathcal{S}\eta^{G-1}([\Phi, \Phi^*]), \; \bar{\omega}^G \in \mathcal{C} \;\Leftrightarrow\; \bar{\omega}^G = \mathcal{S}\bar{\eta}^{G-1}, \; \bar{\eta}^{G-1} \in \mathcal{C} \; . \tag{6.10}$$

For later purposes, in step 3 we now characterize $\mathcal{C}$ more explicitly. Consider first the variables with weights $(0,0)$. They are collectively denoted by z^A:

$$\{z^A\} = \{X^\mu, \Theta^+, \Theta^-\} \quad \text{where} \quad \Theta^\pm \equiv c^0_\pm \; . \tag{6.11}$$

We interpret $\{z^A\}$ as coordinates of a superspace. A generic superfield $\mathcal{H}(z)$ is then a function of the form

$$\mathcal{H}(z) = H(X) + H^+(X)\Theta^+ + H^-(X)\Theta^- + H^{+-}(X)\Theta^+\Theta^- \; . \tag{6.12}$$

The functions $H(X), \ldots, H^{+-}(X)$ in the expansion (6.12) will be called the component fields of $\mathcal{H}(z)$.

Now, recall that among all generators c_α^m and $\mathcal{B}^i$ only the undifferentiated ghosts $c_+^{-1} = c^+$ and $c_-^{-1} = c^-$ have negative weights, given by $(-1,0)$ and $(0,-1)$ respectively, cf. (5.24) and (5.25). The nilpotency of c^+ and c^- implies that functions in $\mathcal{C}$ cannot involve generators whose L_0^+– or L_0^-–weight exceeds 1 (recall that we are dealing with local functions and therefore a function in $\mathcal{C}$ is polynomial in all variables c_α^m and $\mathcal{B}^i$ except for the $X^\mu_{0,0} = X^\mu$). Furthermore, whenever a variable with L_0^+–(L_0^-–) weight 1 occurs, it must appear necessarily together with c^+ (c^-). It is then easy to verify that any function in $\mathcal{C}$ can be expressed in terms of the z^A which have weights $(0,0)$ and in terms of the following zero-modes of the L_0^α:

$$\begin{array}{ll} T^\mu = c^+ X^\mu_{1,0} + c^- X^\mu_{0,1} \ ; & R^\mu = c^+ X^\mu_{1,0} - c^- X^\mu_{0,1} \ ; \\ T^+ = 2c_+^{-1} c_+^1 = c^+ \partial_+^2 c^+ \ ; & T^- = 2c_-^{-1} c_-^1 = c^- \partial_-^2 c^- \ ; \\ T^\mu_{+-} = c^+ c^- X^\mu_{1,1} \ ; & T^*_\mu = c^+ c^- \hat{X}^*_\mu \ . \end{array} \tag{6.13}$$

The motivation for introducing the linear combinations T^μ and R^μ of $c^+ X^\mu_{1,0}$ and $c^- X^\mu_{0,1}$ is that T^μ and $T^\pm$ group naturally in a "super-multiplet" corresponding to $\{z^A\}$ via the BRST operator (see below),

$$\{T^A\} = \{T^\mu, T^+, T^-\} \ . \tag{6.14}$$

Note that T^A is a vector of fermionic type, i.e. the first components T^μ are fermionic, while the others are bosonic. On the z's and the quantities (6.13), $\mathcal{S}$ takes the simple form

$$\begin{array}{ll} \mathcal{S} z^A = T^A \ ; & \mathcal{S} T^A = 0 \ ; \\ \mathcal{S} R^\mu = -2 T^\mu_{+-} \ ; & \mathcal{S} T^\mu_{+-} = 0 \ ; \\ \mathcal{S} T^*_\mu = \delta_{KT} T^*_\mu \ . & \end{array} \tag{6.15}$$

The explicit form of $\delta_{KT} T^*_\mu$ depends on the classical action. Therefore, we have to determine this action before we can completely compute $H^*(\mathcal{S}, \mathcal{C})$.

Due to the composite nature of the quantities (6.13), involving nilpotent ghosts, their algebra is not freely generated but subject to the following identities:

$$\begin{array}{l} T^\mu T^\nu = -T^\nu T^\mu = -R^\mu R^\nu \ ; \quad R^\mu T^\nu = R^\nu T^\mu \ ; \quad T^\mu T^\pm = \mp R^\mu T^\pm \ ; \\ T^\pm T^\pm = T^A T^B T^C = R^\mu T^A T^B = R^\mu R^\nu T^A = R^\mu R^\nu R^\rho = 0 \ ; \\ T^A T^\mu_{+-} = R^\nu T^\mu_{+-} = T^A T^*_\mu = R^\nu T^*_\mu = 0 \ ; \\ T^\mu_{+-} T^\nu_{+-} = T^\mu_{+-} T^*_\nu = T^*_\mu T^*_\nu = 0 \ . \end{array} \tag{6.16}$$

Taking these identities into account it is now straightforward to write down the most general function in $\mathcal{C}$. It can be parametrized by superfields multiplying the various non-vanishing monomials in the quantities (6.13). The parametrization we will use in the following sections is described by the following lemma.

Lemma 6.4 *Any function $\omega \in \mathcal{C}$ can be uniquely written in the form*

$$\begin{array}{rl} & \omega[\mathcal{A}, \mathcal{B}_A, \mathcal{F}_{AB}, \mathcal{C}_\mu, \mathcal{N}_{\mu\nu}, \mathcal{K}_\mu, \mathcal{H}_\mu] \\ = & \mathcal{A}(z) + T^A \mathcal{B}_A(z) - \tfrac{1}{2} T^A T^B \mathcal{F}_{BA}(z) (-)^A \\ & + R^\mu \mathcal{C}_\mu(z) + \tfrac{1}{2} R^\mu T^\nu \mathcal{N}_{\mu\nu}(z) + \left(T^\mu_{+-} + \tfrac{1}{2} R^\mu T^A \partial_A\right) \mathcal{K}_\mu(z) + T^*_\mu \mathcal{H}^\mu(z) \end{array} \tag{6.17}$$

where $\mathcal{A}(z), \ldots, \mathcal{H}^\mu(z)$ are superfields of the form (6.12) and $\mathcal{F}_{AB}$ and $\mathcal{N}_{\mu\nu}$ satisfy

$$\mathcal{F}_{++} = \mathcal{F}_{--} = 0 \ ; \quad \mathcal{F}_{AB} = -(-)^{AB} \mathcal{F}_{BA} \ ; \quad \mathcal{N}_{\mu\nu} = \mathcal{N}_{\nu\mu} \ . \tag{6.18}$$

Here $(-)^A$ denotes the grading of z^A, i.e. $(-)^\mu = 1$ and $(-)^\alpha = -1$.

That $\mathcal{N}_{\mu\nu}$ can be assumed to be symmetric follows from $R^\mu T^\nu = R^\nu T^\mu$, cf. (6.16). The graded antisymmetry of $\mathcal{F}_{AB}$ follows from the commutation relations of the T^A [13]. Note that two components, $\mathcal{F}_{++}$ and $\mathcal{F}_{--}$, are missing because of the identities $T^\pm T^\pm = 0$ occurring in (6.16).

The proof of the lemma is straightforward. We just note that the decomposition (6.17) is indeed unique since (6.18) implies that the $\mathcal{F}_{\mu\nu}$ are antisymmetric whereas the $\mathcal{N}_{\mu\nu}$ are symmetric under exchange of their indices, i.e. $T^\mu T^\nu \mathcal{F}_{\mu\nu}(z)$ and $R^\mu T^\nu \mathcal{N}_{\mu\nu}(z)$ are clearly independent functions in $\mathcal{C}$. ∎

Although it is not necessary to include the term $R^\mu T^A \partial_A \mathcal{K}_\mu(z)$ in (6.17) (it can be absorbed in the $\mathcal{F}_{A\mu}$–terms), we have introduced it for later convenience.

In the final step for the determination of the cohomology, we explicitly compute $\mathcal{S}$ on the function (6.17), and identify the kernel and the image of this operation. That is done first for the antifield–independent part in section 7, leading in section 8 to the classical action. From that action, we know also $\delta_{KT}T^*_\mu$ in (6.15), and can then make the analysis in full generality in section 9. $\mathcal{S}\omega = 0$ and $\omega \neq \mathcal{S}\eta$ will impose conditions on the superfields $\mathcal{A}(z), \ldots, \mathcal{H}^\mu(z)$ occurring in (6.17). In particular these conditions will involve derivatives of the superfields with respect to the z^A. Therefore it is convenient to introduce the following shorthand notations for these derivatives:[14]

$$\{\partial_A\} = \left\{\partial_\mu = \frac{\partial}{\partial X^\mu}, \, \partial_\alpha = \frac{\partial}{\partial \Theta^\alpha}\right\} . \tag{6.19}$$

This allows to express the $\mathcal{S}$–transformation of an arbitrary function of the z's through

$$\mathcal{S}\, F(z) = T^A \partial_A F(z) . \tag{6.20}$$

We end this section with three remarks.

1. Notice that all quantities (6.13) occur in pairs $(A, \mathcal{S}A)$ except for the T^*_μ. However this does not imply a trivial cohomology on functions of the A's and $(\mathcal{S}A)$'s since their algebra is not a free one due to (6.16). In particular z^A and T^A do not form a trivial pair, notwithstanding eq.(6.15): they do not obey the condition (c) (see the beginning of this section).

2. Note that functions in $\mathcal{C}$ involve only the six covariant ghost generators $c^{-1}_\pm$, $c^0_\pm$ and $c^1_\pm$. They correspond to the two $sl(2)$-copies $\{L^\pm_{-1}, L^\pm_0, L^\pm_1\}$. Furthermore, one easily checks that all $L^\pm_m$ with $m = 2, 3, \ldots$ vanish on the generators occurring in functions in $\mathcal{C}$. Hence, lemma 6.3 can be viewed as a reduction of the $\mathcal{S}$-cohomology in the space of local functions to the "weak $sl(2)$-Lie algebra cohomology" in $\mathcal{C}$. However, we cannot use the standard results on the Lie algebra cohomology here since the $sl(2)$-representations on the generators are not finite dimensional (recall that L^+_{-1} and L^-_{-1} act like derivatives on the generators which leads to infinite multiplets).[15]

[13] The sign factor for the corresponding term in ω is the natural one.

[14] Note the double use of the symbol ∂_α; we hope it is clear from the context whether this stands for $\partial/\partial\Theta^\alpha$ or $\partial/\partial x^\alpha$.

[15] Although only a finite number of generators contributes to $\mathcal{C}$, the usual results on the Lie algebra cohomology do not apply here since, by setting to zero the other generators, one would violate (5.9) (nevertheless $\mathcal{SC} \subset \mathcal{C}$ holds since the nilpotency of the ghosts prevents those generators which do not occur in functions $\omega \in \mathcal{C}$ from contributing to $\mathcal{S}\omega$).

3. Since $\mathcal{C}$ contains only functions with ghost numbers ranging from 0 to 6, we conclude that the cohomology of $\mathcal{S}$ on local functions is trivial for all other ghost numbers. According to section 4 this implies that the cohomology of $\mathcal{S}$ on local functionals can be nontrivial at most for ghost numbers ranging from -2 to 4 (in fact the value -2 does not occur since $H^0(\mathcal{S},\mathcal{C})$ is representated by a constant as one can easily check already at this stage).

7 Strong BRST cohomology on antifield independent functions

We have shown in sections 5 and 6 that the computation of the $\mathcal{S}$-cohomology on local functions can be reduced to the computation of $H^*(\mathcal{S},\mathcal{C})$ which is the $\mathcal{S}$-cohomology in the subspace of local functions described by lemma 6.4. As a first step towards the computation of this cohomology we will now compute the $\mathcal{S}$-cohomology in a subspace $\mathcal{C}^\Phi$ of $\mathcal{C}$ given by the antifield independent functions. (We denote this cohomology by $H^*(\mathcal{S},\mathcal{C}^\Phi)$.) This can be done consistently, since the closure of the algebra (absence of quadratic terms in antifields in the extended action) implies that the $\mathcal{S}$-transformation of any function in $\mathcal{C}^\Phi$ is again contained in $\mathcal{C}^\Phi$. Note that the resulting cohomology classes are not a subset of the cohomology classes of $\mathcal{S}$ in the space of local functions of fields *and* antifields: the image of $\mathcal{S}$ acting on that space contains functions in $\mathcal{C}^\Phi$. Therefore it can happen that an $\mathcal{S}$-invariant function in $\mathcal{C}^\Phi$ is trivial in $H^*(\mathcal{S},\mathcal{C})$ although it is nontrivial $H^*(\mathcal{S},\mathcal{C}^\Phi)$. Functions with this property always contain the field equations. Whereas $H^*(\mathcal{S},\mathcal{C})$, to be computed in section 9, is a *weak* cohomology, $H^*(\mathcal{S},\mathcal{C}^\Phi)$ is the strong BRST cohomology, since on fields the operation $\mathcal{S}$ is the BRST operator.

The main reason for computing $H^*(\mathcal{S},\mathcal{C}^\Phi)$ first is that it provides, for ghost number 2, the general classical action S_{cl} described in section 2. The latter has to be determined before we can compute $H^*(\mathcal{S},\mathcal{C})$ completely, since it fixes the $\mathcal{S}$-transformation of the quantities T^*_μ, cf. (6.15).

Now, any function in $\mathcal{C}^\Phi$ takes the form (6.17) with $\mathcal{H}^\mu = 0$. Using (6.15) and the identities (6.16) we obtain for the $\mathcal{S}$-transformation of a generic element of $\mathcal{C}^\Phi$:

$$\mathcal{S}\omega[\mathcal{A},\mathcal{B}_A,\mathcal{F}_{AB},\mathcal{C}_\mu,\mathcal{N}_{\mu\nu},\mathcal{K}_\mu,0] = \omega[0,\tilde{\mathcal{B}}_A,\tilde{\mathcal{F}}_{AB},0,0,\tilde{\mathcal{K}}_\mu,0] \tag{7.1}$$

where

$$\begin{aligned}
\tilde{\mathcal{B}}_A(z) &= \partial_A\mathcal{A}(z)\;;\\
\tilde{\mathcal{F}}_{AB}(z) &= \partial_A\mathcal{B}_B(z) - (-)^{AB}\partial_B\mathcal{B}_A(z) \quad \text{for} \quad [AB]\neq[++] \text{ or } [--]\;;\\
\tilde{\mathcal{K}}_\mu(z) &= -2\mathcal{C}_\mu(z)\;.
\end{aligned} \tag{7.2}$$

Recall that $\tilde{\mathcal{F}}_{++}$ and $\tilde{\mathcal{F}}_{--}$ do not occur in ω, cf. (6.18).

We now analyse the implications for the cohomology. $\mathcal{S}\omega = 0$ requires all superfields (7.2) to vanish. The last equation in (7.2) shows (i) that $\mathcal{S}\omega = 0$ requires $\mathcal{C}_\mu = 0$, and (ii) that the superfield $\mathcal{K}_\mu$ can be always removed from ω by subtracting an $\mathcal{S}$-exact function. Next we observe that $\mathcal{N}_{\mu\nu}$ is not restricted by (7.2) and does not occur in the image of $\mathcal{C}^\Phi$ under $\mathcal{S}$. Hence, the terms in ω involving the superfields $\mathcal{N}_{\mu\nu}$ clearly represent nontrivial cohomology classes of $H^*(\mathcal{S},\mathcal{C}^\Phi)$.

The remaining functions are the (graded) antisymmetric $\mathcal{F}_{AB}(z)$, $\mathcal{B}_A(z)$ and $\mathcal{A}(z)$. They form a super 2–form, 1–form and 0–form, on which $\mathcal{S}$ acts as a superderivative. The first two equations (7.2) show that $\mathcal{S}\omega = 0$ restricts the superfields $\mathcal{A}$ and $\mathcal{B}_A$, and that $\mathcal{F}_{AB}(z)$ is trivial if it is of the form $\partial_A \mathcal{B}'_B - (-)^{AB}\partial_B \mathcal{B}'_A$ for some superfields $\mathcal{B}'_A$. The condition on $\mathcal{A}$, namely $\partial_A \mathcal{A} = 0$, clearly implies that $\mathcal{A}$ is constant. The same equation implies that a super–one–form $\mathcal{B}_A$ which is exact is in the image of $\mathcal{S}$, while $\mathcal{S}\omega = 0$ requires $\mathcal{B}_A$ to be "almost closed" in superspace. It would be closed if also the conditions $\tilde{\mathcal{F}}_{\pm\pm} = 0$ were present, but this is not the case, as stated already. This is the reason why the super-one-form defined through the $\mathcal{B}_A(z)$ is not exact in superspace[16]. The extent to which this super–one–form is only "almost exact" is described in the following "super–Poincaré lemma for almost closed super–one–forms":

Lemma 7.1 *The general solution of*

$$\partial_A \mathcal{B}_B(z) - (-)^{AB}\partial_B \mathcal{B}_A(z) = 0 \quad \textit{for} \quad [AB] \neq [++] \ \textit{or} \ [--], \tag{7.3}$$

is given by $\partial_A \mathcal{A}'(z) + \delta_A^+ a_{++}\Theta^+ + \delta_A^- a_{--}\Theta^-$*, i.e.*

$$\mathcal{B}_\mu(z) = \partial_\mu \mathcal{A}'(z), \quad \mathcal{B}_\pm(z) = \partial_\pm \mathcal{A}'(z) + a_{\pm\pm}\Theta^\pm \ . \tag{7.4}$$

where a_{++} *and* a_{--} *are arbitrary (X–independent) constants.*

Proof: Explicitly, the equations (7.3) read

$$\partial_{[\mu}\mathcal{B}_{\nu]}(z) = 0 \ ; \quad \partial_\mu \mathcal{B}_\alpha(z) - \partial_\alpha \mathcal{B}_\mu(z) = 0 \ ; \quad \partial_+\mathcal{B}_-(z) + \partial_-\mathcal{B}_+(z) = 0 \ , \tag{7.5}$$

and we have to prove that this implies (7.4) for some $\mathcal{A}'(z)$ and $a_{\pm\pm}$. From $\partial_{[\mu}\mathcal{B}_{\nu]}(z) = 0$ we conclude $\mathcal{B}_\mu = \partial_\mu \mathcal{B}(z)$ for some superfield $\mathcal{B}(z)$, using the usual Poincaré lemma. The second set of equations (7.5) then yields $\partial_\mu(\mathcal{B}_\alpha - \partial_\alpha \mathcal{B}) = 0$. Using the usual Poincaré lemma again (this time for zero-forms), we conclude $\mathcal{B}_\alpha = \partial_\alpha \mathcal{B} + \rho_\alpha + a_{\alpha\beta}\Theta^\beta + d_\alpha \Theta^+\Theta^-$ where ρ_α, $a_{\alpha\beta}$ and d_α are constants and summation over β is understood. This implies

$$\partial_\alpha \mathcal{B}_\beta + \partial_\beta \mathcal{B}_\alpha = 2a_{(\alpha\beta)} + 2d_{(\beta}\epsilon_{\alpha)\gamma}\Theta^\gamma \ , \tag{7.6}$$

and the last equation (7.5) leads to $a_{(+-)} = 0$ and $d_\alpha = 0$. One now easily verifies that this implies (7.4) by setting $\mathcal{A}'(z) = \mathcal{B}(z) + \rho_\alpha \Theta^\alpha + a_{+-}\Theta^+\Theta^-$. ■

The fact that some nontrivial solutions remain is due to the *absence* of the equations $\tilde{\mathcal{F}}_{++} = 0$ and $\tilde{\mathcal{F}}_{--} = 0$: adding these also would kill the solutions.

Therefore we conclude:

Lemma 7.2 *The BRST-cohomology in* $\mathcal{C}^\Phi$ *is given by*

$$\begin{aligned} \mathcal{S}\omega = 0, \quad \omega \in \mathcal{C}^\Phi \quad &\Leftrightarrow \quad \omega = \omega^0 + a_{++}T^+\Theta^+ + a_{--}T^-\Theta^- \\ &-\tfrac{1}{2}T^A T^B \mathcal{F}_{BA}(z)(-)^A + \tfrac{1}{2}R^\mu T^\nu \mathcal{N}_{\mu\nu}(z) + \mathcal{S}\eta, \quad \eta \in \mathcal{C}^\Phi \end{aligned} \tag{7.7}$$

where ω^0, a_{++} *and* a_{--} *are constants. The functions* $\mathcal{F}_{AB}(z)$ *that give non-vanishing contributions are defined only up to "super-curls", i.e. up to*

$$\mathcal{F}_{AB}(z) \to \mathcal{F}_{AB}(z) + \partial_A \mathcal{B}'_B(z) - (-)^{AB}\partial_B \mathcal{B}'_A(z) \ . \tag{7.8}$$

[16] If it were really closed then it would be exact as well—this is easily proved, for all non-vanishing super-form degrees, just like the usual Poincaré lemma, using that the superspace coordinates z^A and the corresponding superspace differentials group in trivial pairs.

8 General classical action

We are now in the position to determine the general classical action S_{cl} described in section 2. Indeed, according to section 4, S_{cl} can be obtained from the most general $\mathcal{S}$-invariant antifield independent function with ghost number 2. Hence, it is provided by $H^2(\mathcal{S},\mathcal{C}^\Phi)$, i.e. by lemma 7.2 for ghost number 2.

Now, up to trivial solutions, the only parts with ghost number 2 contained in (7.7) are given by $\frac{1}{2}T^\mu T^\nu B_{\mu\nu}(X)$ and $\frac{1}{2}R^\mu T^\nu G_{\mu\nu}(X)$ where $B_{\mu\nu}(X)$ and $G_{\mu\nu}(X)$ are the lowest component fields of the superfields $\mathcal{F}_{\mu\nu}(z)$ and $\mathcal{N}_{\mu\nu}(z)$ respectively. Note that they are antisymmetric and symmetric respectively due to (6.18). Using (6.13) we obtain

$$\begin{aligned} \omega_0^2 &= \tfrac{1}{2}T^\mu T^\nu B_{\mu\nu}(X) + \tfrac{1}{2}R^\mu T^\nu G_{\mu\nu}(X) \\ &= c^+c^- X^\mu_{1,0}X^\nu_{0,1}[G_{\mu\nu}(X) + B_{\mu\nu}(X)] \end{aligned} \tag{8.1}$$

where we have specified the form degree and ghost number of ω again in order to make contact with the notation used in section 4. The remaining arbitrariness given by (7.8) affects only $B_{\mu\nu}(X)$ and reads

$$B_{\mu\nu}(X) \;\to\; B_{\mu\nu}(X) + 2\partial_{[\mu}B'_{\nu]}(X) \tag{8.2}$$

where $B'_\mu(X)$ is the lowest component field of the superfield $\mathcal{B}'_\mu(z)$ occurring in (7.8).

It is now straightforward to evaluate S_{cl} from (8.1) using the prescription given in section 4 which converts invariant functions to invariant functionals. Applying (4.4) resp. the substitution rule (4.6) to (8.1) results in the 2–form

$$\omega_2^0 = dx^+dx^-(1-y)X^\mu_{1,0}X^\nu_{0,1}[G_{\mu\nu}(X) + B_{\mu\nu}(X)] \ . \tag{8.3}$$

This is the integrand of the most general classical action. Using (A.15) it can be cast in a more familiar form:

$$S_{cl} = \int d^2x \left(\tfrac{1}{2}\sqrt{g}\, g^{\alpha\beta}G_{\mu\nu}(X)\partial_\alpha X^\mu\cdot\partial_\beta X^\nu + B_{\mu\nu}(X)\partial_+X^\mu\cdot\partial_-X^\nu\right) \ . \tag{8.4}$$

Notice that adding trivial contributions $\mathcal{S}\eta_0^1$ with $\eta_0^1 \in \mathcal{C}^\Phi$ to (8.1) results in adding (locally) exact forms to (8.3), i.e. total derivatives to the integrand of (8.4). Indeed, since $\eta_0^1 \in \mathcal{C}^\Phi$ does not involve antifields, the application of (4.4) to $\mathcal{S}\eta_0^1$ cannot give rise to a 2–form $\mathcal{S}\eta_2^{-1}$ but only to $d\eta_1^0$. Since we neglect total derivatives whether or not they are total differentials globally, this does not change (8.4). In particular, a change of $B_{\mu\nu}$ as in (8.2) gives indeed rise to a total derivative term as

$$2\int d^2x\, \partial_+X^\mu\cdot\partial_-X^\nu\cdot\partial_{[\mu}B'_{\nu]}(X) = \int d^2x \left[\partial_+\left(B'_\mu\partial_-X^\mu\right) - \partial_-\left(B'_\mu\partial_+X^\mu\right)\right] \ . \tag{8.5}$$

We obtain thus the well-known actions of the non–linear σ–models. Examples are the WZNW–models where the X^μ parametrize some Lie group manifold with group elements $G = \exp(X^\mu T_\mu)$, where T_μ is a suitable matrix representation of the Lie algebra. Then $g^{\alpha\beta}\partial_\alpha X^\mu\cdot\partial_\beta X^\nu\cdot G_{\mu\nu}$ equals $g^{\alpha\beta}Tr\,(G^{-1}\partial_\alpha G\cdot G^{-1}\partial_\beta G)$. Similarly the second contribution to the classical action can then be written as a topological term in 3 dimensions.

With formulas of appendix A, the general classical action (8.4) can also be written as

$$\begin{aligned} S_{cl} = \int d^2x\, \Big\{ &\tfrac{1}{1-y}G_{\mu\nu}(X)[(1+y)\partial_+X^\mu\cdot\partial_-X^\nu - h_{++}\partial_-X^\mu\cdot\partial_-X^\nu \\ &- h_{--}\partial_+X^\mu\cdot\partial_+X^\nu] + B_{\mu\nu}(X)\partial_+X^\mu\cdot\partial_-X^\nu\Big\} \ , \end{aligned} \tag{8.6}$$

which gives a suitable form for the equations of motion for X^μ, providing the δ_{KT}–transformation of X^*_μ given in (B.10). This results in the following $\mathcal{S}$–transformation of the quantity T^*_μ defined in (6.13):

$$\begin{aligned}\mathcal{S}T^*_\mu &= -2G_{\mu\nu}T^\nu_{+-} - 2\Gamma^-_{\rho\sigma,\mu}c^+c^-X^\rho_{1,0}X^\sigma_{0,1}\\ &= -2G_{\mu\nu}T^\nu_{+-} - 2\Gamma^+_{\rho\sigma,\mu}c^+c^-X^\sigma_{1,0}X^\rho_{0,1}\\ &= -2G_{\mu\nu}T^\nu_{+-} - \Gamma_{\rho\sigma,\mu}R^\rho T^\sigma + \tfrac{1}{2}H_{\rho\sigma\mu}T^\rho T^\sigma \qquad (8.7)\end{aligned}$$

where the notations of (A.16)–(A.17) have been used. Note that (8.7) would not change even if we took into account global properties of the base or target manifold since the equations of motion for X^μ remain the same whether or not the total derivative terms one adds to S_{cl} are globally exact.

(8.4) is the most general functional satisfying requirements (i) and (ii) imposed on the classical action in section 2. We shall carry out the analysis in the following for this general form of the classical action. That means that we will not assume any particular properties of the functions $G_{\mu\nu}(X)$ and $B_{\mu\nu}(X)$, not even invertibility of $G_{\mu\nu}(X)$ (This is also the reason why we use the 'Levi–Civita connection' in the form with all indices down.). The only non-degeneracy restriction we impose is the implication

$$G_{\mu\nu}(X)\,h^\nu(X) = \Gamma_{\mu\nu,\rho}(X)\,h^\rho(X) = H_{\mu\nu\rho}(X)\,h^\rho(X) = 0 \quad \Rightarrow \quad h^\mu = 0 \qquad (8.8)$$

since otherwise requirement (iii) imposed on S_{cl} in section 2 would be violated. Indeed, the presence of a non–vanishing solution h^μ of $G_{\mu\nu}h^\nu = \Gamma_{\mu\nu,\rho}h^\rho = H_{\mu\nu\rho}h^\rho = 0$ would give rise to an additional gauge symmetry of S_{cl} generated by $\delta_\epsilon X^\mu = \epsilon\, h^\mu(X)$ and $\delta_\epsilon\, g_{\alpha\beta} = 0$ where ϵ is an arbitrary function (on the two dimensional base manifold).

As already mentioned, some of the solutions (8.4) can still be cohomologically trivial when they can be written as

$$\int(\mathcal{S}\eta_2^{-1} + d\eta_1^0) \qquad (8.9)$$

for some 2–forms η_2^{-1} and η_1^0 involving nontrivially the antifields. This may look strange at first since (8.4) itself is needed to define the $\mathcal{S}$–transformation of the antifields. Nevertheless some functionals (8.9) connect two different twodimensional actions, which are then physically equivalent. These connections have a natural interpretation in terms of (infinitesimal) target space reparametrizations. They have been called sigma model symmetries or pseudo–symmetries [28], and occur naturally in the cohomological analysis which we perform. A generalisation of this statement, concerning field redefinitions in general can be found in appendix C.

9 Complete computation of $H^*(\mathcal{S},\mathcal{C})$

After this intermezzo, which was necessary to determine the full $\mathcal{S}$ transformation law (8.7) of T^*_μ, we come back to the computation of $H^*(\mathcal{S},\mathcal{C})$. We will compute the most general $\mathcal{S}$–invariant function (6.17) modulo trivial solutions. We work henceforth with a given action, i.e. for given functions $G_{\mu\nu}(X)$ and $B_{\mu\nu}(X)$. Fixing these functions is needed, because $\mathcal{S}T^*_\mu$ depends on them. Nevertheless we will not have to impose restrictions on these functions, i.e. we will compute $H^*(\mathcal{S},\mathcal{C})$ completely for *any* given choice of them. In particular we do not assume $G_{\mu\nu}(X)$ to be invertible.

We present the result of the calculation of $\mathcal{S}$ in the form of (6.17). It is more convenient however to express it in terms of

$$\hat{\mathcal{C}}_\mu = \mathcal{C}_\mu + \mathcal{H}_\mu \ , \tag{9.1}$$

where $\mathcal{H}_\mu$ is obtained from the superfield $\mathcal{H}^\mu$ occurring in (6.17) by lowering its index with $G_{\mu\nu}$

$$\mathcal{H}_\mu(z) = G_{\mu\nu}(X)\mathcal{H}^\nu(z) \ . \tag{9.2}$$

Hence, the general expression will contain $\hat{\mathcal{C}}_\mu$ instead of $\mathcal{C}_\mu$, i.e. the terms containing $\hat{\mathcal{C}}_\mu$ and $\mathcal{H}^\mu$ are given by

$$R^\mu \hat{\mathcal{C}}_\mu(z) - R^\mu \mathcal{H}_\mu(z) + T^*_\mu \mathcal{H}^\mu(z) \ . \tag{9.3}$$

With this choice of basis for the superfields, closed and exact functions can be easily identified. Using (6.15) and (8.7) one easily verifies that the result of $\mathcal{S}\omega$, gets modified from (7.2) to

$$\mathcal{S}\omega[\mathcal{A}, \mathcal{B}_A, \mathcal{F}_{AB}, \mathcal{C}_\mu, \mathcal{N}_{\mu\nu}, \mathcal{K}_\mu, \mathcal{H}^\mu] = \omega[0, \tilde{\mathcal{B}}_A, \tilde{\mathcal{F}}_{AB}, 0, \tilde{\mathcal{N}}_{\mu\nu}, \tilde{\mathcal{K}}_\mu, 0] \tag{9.4}$$

with

$$\begin{aligned} &\tilde{\mathcal{B}}_A(z) = \partial_A \mathcal{A}(z) \ ; \quad \tilde{\mathcal{N}}_{\mu\nu} = \mathcal{F}'_{(\mu\nu)} \ ; \quad \tilde{\mathcal{K}}_\mu(z) = -2\hat{\mathcal{C}}_\mu(z) \ ; \\ &\tilde{\mathcal{F}}_{AB}(z) = \partial_A \mathcal{B}_B(z) - (-)^{AB} \partial_B \mathcal{B}_A(z) + \tfrac{1}{2}(\mathcal{F}'_{AB} - (-)^{AB} \mathcal{F}'_{BA}) \ , \end{aligned} \tag{9.5}$$

where $\mathcal{F}'_{AB}$ are auxiliary quantities defined by

$$\mathcal{F}'_{\mu\nu} = D^-_\mu \mathcal{H}_\nu + D^+_\nu \mathcal{H}_\mu \ ; \quad \mathcal{F}'_{\mu\pm} = -\mathcal{F}'_{\pm\mu} = \mp\partial_\pm \mathcal{H}_\mu \ ; \quad \mathcal{F}'_{+-} = \mathcal{F}'_{-+} = 0 \ . \tag{9.6}$$

Here we have used the covariant derivatives (A.18). Note that only the symmetric (anti-symmetric) part of $\mathcal{F}'_{\mu\nu}$ enters in $\tilde{\mathcal{N}}_{\mu\nu}$ ($\tilde{\mathcal{F}}_{\mu\nu}$) and that one has

$$\mathcal{F}'_{(\mu\nu)} = \partial_\mu \mathcal{H}_\nu + \partial_\nu \mathcal{H}_\mu - 2\Gamma_{\mu\nu,\rho}\mathcal{H}^\rho \ ; \qquad \mathcal{F}'_{[\mu\nu]} = H_{\mu\nu\rho}\mathcal{H}^\rho \ . \tag{9.7}$$

The functions $\mathcal{K}_\mu$ and $\hat{\mathcal{C}}_\mu$ disappear from the cohomology, just as $\mathcal{K}_\mu$ and $\mathcal{C}_\mu$ in section 7. We now address the changes in the analysis of that section. The inclusion of the antifield dependent terms, i.e. the presence of the superfields $\mathcal{H}^\mu(z)$, modifies the result of section 7 in two ways:

1. New solutions of $\mathcal{S}\omega = 0$ involving non–vanishing $\mathcal{H}^\mu(z)$ may exist. As one has $\tilde{\mathcal{H}}^\mu = 0$ in (9.4), any $\mathcal{S}$-invariant function of this type gives a new solution of the cohomology problem.

2. Some of the solutions provided by lemma 7.2 become trivial.

We see immediately from (9.5) that the second modification applies only to solutions involving the superfields $\mathcal{F}_{AB}(z)$ and the $\mathcal{N}_{\mu\nu}(z)$, whereas the constant solutions and the two solutions $T^+\Theta^+$ and $T^-\Theta^-$ occurring in (7.7) remain nontrivial. We postpone a further specification until we analyse the cohomology at specific ghost numbers, and now elaborate on the first modification.

We now investigate whether there are 'new' solutions involving $\mathcal{H}^\mu$. The equation $\tilde{\mathcal{F}}_{\mu\nu} + \tilde{\mathcal{N}}_{\mu\nu} = 0$ takes the form

$$0 = D_\mu^- \mathcal{H}_\nu + D_\nu^+ \mathcal{H}_\mu + \partial_\mu \mathcal{B}_\nu - \partial_\nu \mathcal{B}_\mu \ , \tag{9.8}$$

which may be decomposed into a symmetric and an antisymmetric part, corresponding to $\tilde{\mathcal{N}}_{\mu\nu} = 0$ and $\tilde{\mathcal{F}}_{\mu\nu} = 0$ respectively. (9.8) is the Killing equation for $\mathcal{H}^\mu$. The new solutions therefore correspond to isometries of the target space.[17] We will see that they also encode the rigid symmetries of the sigma model. Apart from solving the Killing equation, there are no more conditions for the part of $\mathcal{H}^\mu$ which is independent of $\Theta^\pm$ and denoted[18] by H^μ, and thus has to satisfy

$$D_\mu^- H_\nu + D_\nu^+ H_\mu + \partial_\mu B_\nu - \partial_\nu B_\mu = 0 \ . \tag{9.9}$$

We call the non–vanishing H^μ solving these equations *Killing vectors*, and denote a basis for them by $\{\zeta_a^\mu(X)\}$. The corresponding vectors B_μ in the Killing equation (9.9) are denoted by $b_{\mu a}(X)$,

$$\{\zeta_a^\mu(X), b_{\mu a}(X) : \ a = 1, \ldots, N\} \ . \tag{9.10}$$

The conditions $\tilde{\mathcal{F}}_{\mu\pm} = 0$ require

$$0 = \partial_\mu \mathcal{B}_\pm - \partial_\pm (\mathcal{B}_\mu \pm \mathcal{H}_\mu) \ . \tag{9.11}$$

We can solve these equations for $\partial_+ \mathcal{B}_\mu$ and $\partial_- \mathcal{B}_\mu$ and insert the result in (9.8) after applying ∂_+ or ∂_- to the latter. Using (A.19), this results in

$$D_\mu^\pm (\partial_\pm \mathcal{H}_\nu) = 0 \ . \tag{9.12}$$

Hence, $\partial_\pm \mathcal{H}_\nu$ should be "covariantly constant" vectors. Such vectors are analysed in section D.3, where it is shown that they are related to the chiral symmetries, which for the example of WZW models are the Kač–Moody symmetries. In particular (9.12) requires the component fields $H^{\mu\pm}(X)$ of $\mathcal{H}^\mu(z)$ to solve

$$D_\mu^+ H_\nu^+ = 0 \ ; \qquad D_\mu^- H_\nu^- = 0 \ . \tag{9.13}$$

We denote a basis for these special Killing vectors by

$$\{\zeta_{a^+}^\mu(X) : \ a^+ = 1, \ldots, N^+\}; \quad \{\zeta_{a^-}^\mu(X) : \ a^- = 1, \ldots, N^-\} \ . \tag{9.14}$$

The numbers N^+ and N^- of ζ_{a^+}'s and ζ_{a^-}'s are in general different. As shown in appendix D.3, (9.13) implies that they satisfy (9.8) with $B_\mu^\pm = \mp H_\mu^\pm$. Therefore (9.14) are subsets of (9.10). This implies $N^c \equiv N^+ + N^- \leq N \leq D(D+1)/2$, since the latter is the maximal value of linearly independent Killing vectors (D being the range of μ).

The final equation $\tilde{\mathcal{F}}_{+-} = 0$ gives restrictions on the possible new solutions only through the component field $H_\mu^{+-}(X)$ of $\mathcal{H}_\mu(z)$. The conditions (9.12) imply that H_μ^{+-} should be covariantly constant for both signs of the torsion, i.e. $D_\mu^+ H_\nu^{+-} = D_\mu^- H_\nu^{+-} = 0$. Such vectors are considered in section D.4, where we find that $H_\mu^{+-} = \partial_\mu \Lambda$ for some

[17] In appendix D some properties of Killing vectors and Lie–derivatives are given, always allowing a degenerate metric.

[18] We use the notation introduced in (6.12).

"scalar" $\Lambda(X)$, see (D.35). However, (9.11) and $\tilde{\mathcal{F}}_{+-}=0$ imply $\Lambda = \textit{constant}$ and thus $H_\mu^{+-}=0$. As argued in appendix D.4, this is only possible if $H^{\mu\,+-}$ generates an extra gauge symmetry distinct from diffeomorphisms and Weyl transformations. (We obtain the equations (8.8)). We exclude this possibility using assumption (iii) of section 2 and conclude $H^{\mu\,+-}=0$. Including them we would have local symmetries which are not included in the BRST operator. If we would include these symmetries in the BRST operator with new ghosts c_{+-}, the vectors $H^{\mu\,+-}$ would not be cohomological solutions, but rather determine the extra term in the extended action at antifield number 1: $S_{extra} = X^*_\mu H^{\mu\,+-} c_{+-}$.

It is interesting to note how the different types of symmetries are all organised in terms of the new solutions $\mathcal{H}$: all the rigid symmetries make use of the $H(X)$ component, those rigid symmetries related to the chiral symmetries occur in $H^\pm(X)$, and the extra gauge symmetries would show up in $H^{+-}(X)$.

We have now analysed all conditions imposed by $\mathcal{S}\omega = 0$ and have used part of the freedom to add trivial solutions for fixing the form of ω. We give a summary of all solutions in the form of a theorem:

Theorem 9.1 *The cohomology of $\mathcal{S}$ on local functions is given by*

$$\begin{aligned}\mathcal{S}\omega = 0 \;\Leftrightarrow\; \omega \;&=\; \mathcal{S}\eta + \omega^0 + a_{++}T^+\Theta^+ + a_{--}T^-\Theta^- \\ &\quad -\tfrac{1}{2}T^AT^B\mathcal{F}_{BA}(z)(-)^A + \tfrac{1}{2}R^\mu T^\nu \mathcal{N}_{\mu\nu}(z) \\ &\quad +T^\mu\mathcal{B}_\mu(z) - R^\mu\mathcal{H}_\mu(z) + T^*_\mu\mathcal{H}^\mu(z)\end{aligned}\qquad(9.15)$$

where ω^0, a_{++} and a_{--} are constants and the superfields $\mathcal{B}_\mu(z)$, $\mathcal{H}^\mu(z)$ and $\mathcal{H}_\mu(z)$ are given in terms of the solutions of (9.9) and (9.13) according to

$$\begin{aligned}\mathcal{B}_\mu(z) &= \lambda^a b_{\mu a}(X) - \lambda^{a^+}\zeta_{\mu a^+}(X)\Theta^+ + \lambda^{a^-}\zeta_{\mu a^-}(X)\Theta^- \;; \\ \mathcal{H}^\mu(z) &= \lambda^a\zeta^\mu_a(X) + \lambda^{a^+}\zeta^\mu_{a^+}(X)\Theta^+ + \lambda^{a^-}\zeta^\mu_{a^-}(X)\Theta^- \;,\end{aligned}\qquad(9.16)$$

where the λ's are arbitrary constants. There are still trivial solutions which can be added to (9.15) without changing its form for fixed choices (9.10) and (9.14). (9.5) shows that they are given by

$$\mathcal{S}[T^A\mathcal{B}'_A(z) + T^*_\mu\mathcal{H}'^\mu(z) - R^\mu\mathcal{H}'_\mu(z)]\qquad(9.17)$$

and give rise to the following redefinitions of the superfields in (9.15):

$$\begin{aligned}\mathcal{N}_{\mu\nu}(z)+\mathcal{F}_{\mu\nu}(z) &\to \mathcal{N}_{\mu\nu}(z)+\mathcal{F}_{\mu\nu}(z)+2\partial_{[\mu}\mathcal{B}'_{\nu]}(z)+D^-_\mu\mathcal{H}'_\nu(z)+D^+_\nu\mathcal{H}'_\mu(z)\;; \\ \mathcal{F}_{\mu\pm}(z) &\to \mathcal{F}_{\mu\pm}(z)+\partial_\mu\mathcal{B}'_\pm(z)-\partial_\pm\mathcal{B}'_\mu(z)\mp\partial_\pm\mathcal{H}'_\mu(z)\;; \\ \mathcal{F}_{+-}(z) &\to \mathcal{F}_{+-}(z)+\partial_+\mathcal{B}'_-(z)+\partial_-\mathcal{B}'_+(z)\;,\end{aligned}\qquad(9.18)$$

where $\mathcal{B}'_\mu(z)$ and $\mathcal{H}'^\mu(z)$ denote arbitrary superfields and $\mathcal{H}'_\nu(z) = G_{\mu\nu}(X)\mathcal{H}'^\mu(z)$.

Hence, the different inequivalent types of solution are:

1. *The constants $\omega = \omega^0$.*

2. *The two solutions $T^+\Theta^+$ and $T^-\Theta^-$ stemming from lemma 7.1.*

3. *The terms involving the superfields $\mathcal{F}_{AB}$ and $\mathcal{N}_{\mu\nu}$, in so far as they are not of the form (9.18).*

4. The solutions involving the N Killing vectors ζ^μ_a and the respective $b_{\mu a}$.

5. The terms involving the N^c covariantly constant Killing vectors ζ^μ_{a+}, ζ^μ_{a-}.

The numbers $N^c \leq N$ can be zero.

We will now order the solutions according to ghost number and reduce the remaining arbitrariness by removing trivial solutions. Recall that a generic superfield $\mathcal{F}(z)$ contains parts with ghost number ranging from 0 to 2. This is due to the nilpotency of the $\Theta^\alpha = c^0_\alpha$. Since T^μ, R^μ and T^*_μ have ghost number 1 respectively and $T^\pm$ has ghost number 2, the various superfields and constants occurring in (9.15) contribute only to solutions ω with specific ghost number G:

$$\begin{array}{c|c|c|c|c|c|c} & \omega^0 & a_{\pm\pm} & \mathcal{F}_{\mu\nu}, \mathcal{N}_{\mu\nu} & \mathcal{F}_{\pm\mu} & \mathcal{F}_{+-} & \mathcal{B}_\mu, \mathcal{H}^\mu \\ \hline G & 0 & 3 & 2,3,4 & 3,4,5 & 4,5,6 & 1,2 \end{array} . \tag{9.19}$$

Note that the cohomology groups $H^G(\mathcal{S},\mathcal{C})$ are infinite dimensional for $G = 2,\ldots,6$ due to the presence of arbitrary functions of the X's in the results for these ghost numbers. It is therefore more instructive to compare the number of arbitrary functions occurring for the various values of G rather than the dimensions of the $H^G(\mathcal{S},\mathcal{C})$ themselves. Of course one should subtract from this number the number of arbitrary functions contained in the remaining trivial solutions, and add again zero modes of the trivial solutions. In addition there are extra solutions or zero modes. An overview is given in table 1.

Table 1: Overview of the cohomology at fixed ghost number. The upper indices $\pm$ and $+-$ refer to the component of the superfield as in (6.12). The numbers indicate the number of arbitrary functions that characterise the solution. The numbers in square brackets refer to the number of extra constants. In the counting we assumed an invertible target space metric (otherwise e.g. H'_μ does not subtract 2D solutions).

G	soln.	number	zero modes	number	zero for zero	number	result
0	ω^0	[1]					[1]
1	$(\zeta^\mu_a, b_{\mu a})$	$[N]$					$[N]$
2	$\zeta^\mu_{a\pm}$	$[N^c]$					$[N^c]$
	$F_{\mu\nu} + N_{\mu\nu}$	D^2	B'_μ, H'_μ	$2D$	A'', ζ^μ_a	$1+[N]$	$(D-1)^2+[N]$
3	$a_{\pm\pm}$	[2]					[2]
	$F^\pm_{\mu\nu} + N^\pm_{\mu\nu}$	$2D^2$	$B'^\pm_\mu, H'^\pm_\mu$	$4D$	$A''^\pm$	2	$2D(D-1)$
	$F_{\mu\pm}$	$2D$	$B'_\pm$	2	$\zeta^\mu_{a\pm}$	$[N^c]$	$[N^c]$
4	$F^{+-}_{\mu\nu} + N^{+-}_{\mu\nu}$	D^2	H'^{+-}_μ	D			
	$F^\pm_{\mu\pm}$	$4D$	B'^{+-}_μ	D			
	F_{+-}	1	$B'^\pm_\pm$	4	A''^{+-}	1	$D^2+2(D-1)$
5	$F^{+-}_{\mu\pm}$	$2D$					$2D$
	$F^\pm_{+-}$	2	$B'^{+-}_\pm$	2			
6	F^{+-}_{+-}	1					1

We now present the explicit solution for each ghost number.

$\underline{G=0.}$ In this case the only solution is $\omega^0 = constant$.

$\underline{G=1.}$ The possible solutions are those of type 4 in theorem 9.1. We can write the result for ω^1 in terms of the Killing vectors as

$$\omega^1 = \lambda^a \omega^1_a \ , \tag{9.20}$$

where λ^a are arbitrary constants and

$$\omega^1_a = T^\mu b_{\mu a} - R^\mu G_{\mu\nu} \zeta^\nu_a + T^*_\mu \zeta^\mu_a \ . \tag{9.21}$$

$\underline{G=2.}$ There are two types of solutions with $G = 2$: those of type 3 involving the component fields $N_{\mu\nu}$ and $F_{\mu\nu}$ of $\mathcal{N}_{\mu\nu}$ and $\mathcal{F}_{\mu\nu}$, and secondly there are the possible solutions of type 5.

Up to trivial solutions we therefore obtain in the case $G = 2$

$$\omega^2 = \omega^2_{(0)} + \lambda^{a^+} \omega^2_{a+} + \lambda^{a^-} \omega^2_{a-} \tag{9.22}$$

where

$$\begin{aligned} \omega^2_{(0)} &= \tfrac{1}{2} T^\mu T^\nu F_{\mu\nu}(X) + \tfrac{1}{2} R^\mu T^\nu N_{\mu\nu}(X) \\ &= c^+ c^- X^\mu_{1,0} X^\nu_{0,1} [F_{\mu\nu}(X) + N_{\mu\nu}(X)] \ ; \end{aligned} \tag{9.23}$$

$$\begin{aligned} \omega^2_{a\pm} &= [T^*_\mu - (R^\nu \pm T^\nu) G_{\nu\mu}] \zeta^\mu_{a\pm} \Theta^\pm \\ \Leftrightarrow \quad & \begin{cases} \omega^2_{a+} = (T^*_\mu - 2c^+ X^\nu_{1,0} G_{\nu\mu}) \zeta^\mu_{a+} \Theta^+ \\ \omega^2_{a-} = (T^*_\mu + 2c^- X^\nu_{0,1} G_{\nu\mu}) \zeta^\mu_{a-} \Theta^- \ . \end{cases} \end{aligned} \tag{9.24}$$

Specialising to $N_{\mu\nu} = G_{\mu\nu}$, $F_{\mu\nu} = B_{\mu\nu}$ and $\lambda^{a^\pm} = 0$, this reproduces the result derived in section 8 for the classical action. The remaining arbitrariness resulting from (9.18) reads:

$$\begin{aligned} N_{\mu\nu}(X) + F_{\mu\nu}(X) \quad &\rightarrow \\ & N_{\mu\nu}(X) + F_{\mu\nu}(X) + 2\partial_{[\mu} B'_{\nu]}(X) + D^-_\mu H'_\nu(X) + D^+_\nu H'_\mu(X) \end{aligned} \tag{9.25}$$

where $H'^\mu(X)$ and $B'_\mu(X)$ are arbitrary functions (they are the lowest component fields of the superfields $\mathcal{H}'^\mu(z)$ and $\mathcal{B}'_\mu(z)$ occurring in (9.18)). Note that (9.25) represents a larger arbitrariness than its analog (8.2) found in section 8 since there we did not take the antifields into account (see the remark at the end of section 8 for the interpretation of this additional freedom in the case that $F_{\mu\nu} = B_{\mu\nu}$ and $N_{\mu\nu} = G_{\mu\nu}$).

$\underline{G=3.}$ There are two types of solutions with $G = 3$ arising from (9.15): first there are the solutions of type 2 and secondly there are solutions of type 3 containing the component fields $F^\pm_{\mu\nu}$, $N^\pm_{\mu\nu}$ and $F_{\mu\pm}$ of the superfields $\mathcal{F}_{\mu\nu}$, $\mathcal{N}_{\mu\nu}$ and $\mathcal{F}_{\mu\pm}$. In fact the $F_{\mu\pm}$ can be assumed to be zero with no loss of generality since they can be removed by choosing

$$B'^\pm_\mu = -F_{\mu\pm} + \partial_\mu B'_\pm \mp H'^\pm_\mu \tag{9.26}$$

in (9.18). Here $B'_\pm$ is irrelevant since contributions $\partial_\mu B'_\pm$ to $B'^\pm_\mu$ drop out of (9.18), and thus this term can actually be omitted in (9.26) (it corresponds to the zero for zero entry $A''^{\pm}$ in table 1). On the other hand $H'^{\pm}_{\mu}$ will appear again below. Up to trivial contributions the solution for $G=3$ thus reads

$$\omega^3 = a_{++}\omega^3_+ + a_{--}\omega^3_- + \omega^3_{X+} + \omega^3_{X-}\ , \tag{9.27}$$

where $a_{\pm\pm}$ are the constants occurring in (9.15) and

$$\begin{aligned}
\omega^3_{\pm} &= T^{\pm}\Theta^{\pm} = -c^{\pm}\,\partial_{\pm}c^{\pm}\cdot\partial^2_{\pm}c^{\pm}\ ; && (9.28)\\
\omega^3_{X\pm} &= \tfrac{1}{2}R^{\mu}T^{\nu}N^{\pm}_{\mu\nu}(X)\Theta^{\pm} + \tfrac{1}{2}T^{\mu}T^{\nu}F^{\pm}_{\mu\nu}(X)\Theta^{\pm} \\
&= c^+c^-X^{\mu}_{1,0}X^{\nu}_{0,1}[F^{\pm}_{\mu\nu}(X) + N^{\pm}_{\mu\nu}(X)]\Theta^{\pm}\ . && (9.29)
\end{aligned}$$

The solutions ω^3_+ and ω^3_- are nontrivial, but $\omega^3_{X\pm}$ can still have trivial contributions. The remaining arbitrariness is given by those transformations (9.18) which preserve the form (9.27), i.e. which do not reintroduce $F_{\mu\pm}$. These transformations involve only $H'^{\pm}_{\mu}$ since $B'^{\pm}_{\mu}$ is completely determined in terms of $H'^{\pm}_{\mu}$ according to (9.26) which yields, setting $F_{\mu\pm}=0$ and dropping $\partial_{\mu}B'_{\pm}$,

$$B'^{\pm}_{\mu} = \mp H'^{\pm}_{\mu}\ . \tag{9.30}$$

One easily verifies that therefore $\omega^3_{X\pm}$ are determined only up to

$$\begin{aligned}
N^+_{\mu\nu}(X) + F^+_{\mu\nu}(X) &\rightarrow N^+_{\mu\nu}(X) + F^+_{\mu\nu}(X) + 2D^+_{\nu}H'^+_{\mu}(X)\ ;\\
N^-_{\mu\nu}(X) + F^-_{\mu\nu}(X) &\rightarrow N^-_{\mu\nu}(X) + F^-_{\mu\nu}(X) + 2D^-_{\mu}H'^-_{\nu}(X) && (9.31)
\end{aligned}$$

where $H'^{\mu\pm}(X)$ are arbitrary functions. They trivialize parts of the solutions $N^{\pm}_{\mu\nu}+F^{\pm}_{\mu\nu}$, unless they are themselves covariantly constant Killing vectors, in which case they do not contribute to (9.31).

$\underline{G=4.}$ All solutions are of type 3 and involve the component fields $F^{+-}_{\mu\nu}$, $N^{+-}_{\mu\nu}$, $F^{\pm}_{\mu\pm}$, $F^{\mp}_{\mu\pm}$ and F_{+-} of the corresponding superfields respectively. Using (9.18) one verifies that one can always achieve

$$F_{+-}=0\ ;\quad F^-_{\mu+} = F^+_{\mu-} \equiv F_{\mu} \tag{9.32}$$

by choosing B'^-_+, B'^+_- and B'^{+-}_{μ} appropriately. Note that, again, only $D+1$ out of the $D+2$ component fields B'^-_+, B'^+_- and B'^{+-}_{μ} are needed for the choice (9.32), which is related to the zero for zero mode A''^{+-} in table 1. Hence, one finds up to trivial solutions

$$\begin{aligned}
\omega^4 = {}& \tfrac{1}{2}\Theta^+\Theta^-[R^{\mu}T^{\nu}N^{+-}_{\mu\nu}(X) + T^{\mu}T^{\nu}F^{+-}_{\mu\nu}(X)] + T^{\mu}(T^-\Theta^+ + T^+\Theta^-)F_{\mu}(X)\\
&+T^{\mu}[T^-\Theta^-F^-_{\mu-}(X) + T^+\Theta^+F^+_{\mu+}(X)]\ . && (9.33)
\end{aligned}$$

The remaining arbitrariness is given by

$$\begin{aligned}
N^{+-}_{\mu\nu}(X) + F^{+-}_{\mu\nu}(X) &\rightarrow N^{+-}_{\mu\nu}(X) + F^{+-}_{\mu\nu}(X) + D^-_{\mu}H'^{+-}_{\nu}(X) + D^+_{\nu}H'^{+-}_{\mu}(X)\ ;\\
F_{\mu}(X) &\rightarrow F_{\mu}(X) - H'^{+-}_{\mu}(X)\ ;\\
F^{\pm}_{\mu\pm}(X) &\rightarrow F^{\pm}_{\mu\pm}(X) + \partial_{\mu}B'^{\pm}_{\pm}(X) && (9.34)
\end{aligned}$$

where $H'^{\mu+-}(X)$ and $B'^{\pm}_{\pm}(X)$ are arbitrary functions. If $G_{\mu\nu}$ is invertible, we can simplify the result and simultaneously reduce the remaining freedom. Namely in that case we can remove F_μ by choosing $H'^{\mu+-} = G^{\mu\nu}F_\nu$. Since this fixes $H'^{\mu+-}$ completely, we are then left with

$$\begin{aligned}\det(G_{\mu\nu}) \neq 0: \quad \omega^4 &= \tfrac{1}{2}\Theta^+\Theta^-[R^\mu T^\nu N^{+-}_{\mu\nu}(X) + T^\mu T^\nu F^{+-}_{\mu\nu}(X)] \\ &\quad + T^\mu[T^-\Theta^- F^-_{\mu-}(X) + T^+\Theta^+ F^+_{\mu+}(X)]\end{aligned} \tag{9.35}$$

with the only remaining arbitrariness

$$F^\pm_{\mu\pm}(X) \quad \rightarrow \quad F^\pm_{\mu\pm}(X) + \partial_\mu B'^\pm_\pm(X)\ . \tag{9.36}$$

<u>$G = 5.$</u> Analogously one verifies that the result is, up to trivial solutions,

$$\omega^5 = \Theta^+\Theta^- T^\mu \left[T^- F^{+-}_{\mu-}(X) + T^+ F^{+-}_{\mu+}(X)\right]\ . \tag{9.37}$$

No arbitrariness is left, i.e. (9.37) is nontrivial for any non–vanishing choice of $F^{+-}_{\mu\pm}(X)$.

<u>$G = 6.$</u> Any non–vanishing function in $\mathcal{C}$ with ghost number 6 is $\mathcal{S}$–invariant, nontrivial and of the form

$$\omega^6 = T^+T^-\Theta^+\Theta^- F^{+-}_{+-}(X) \tag{9.38}$$

where $F^{+-}_{+-}(X)$ is an arbitrary (non–vanishing) function.

10 Results and their interpretation

In this section we spell out the results for the antibracket cohomology on local functionals with ghost numbers $g = -1, 0, 1$ implied by the computation of the previous sections. We give their physical interpretation too. Of course the results of the previous sections provide also a complete list of solutions of the cohomology problem for functionals of all other ghost numbers but no physical interpretation of them is known yet. We just recall here that our results imply the absence of such functionals for all ghost numbers $g < -1$ and $g > 4$, and that the results for $g = 2, 3, 4$ can be easily obtained from (9.33) (or (9.35)), (9.37) and (9.38) by means of the 'ascent prescription' described in section 4. That prescription is given by equations (4.4) resp. (4.6) which 'integrate' the descent equations by converting $\mathcal{S}$–invariant functions with ghost number G to $\mathcal{S}$–invariant functionals with ghost number $g = G - 2$. As the analysis in section 9 shows, the following results are valid for any given action of the form (8.4).

$g = -1$: *Rigid symmetries and conserved Noether currents.*

For $\mathcal{S}$–invariant functionals with ghost number -1 we have to start from (9.20), (9.21). The only term for which the ascent prescription (4.6) can lead to $dx^2 \equiv dx^+dx^- = -dx^-dx^+$ is the antifield dependent one, as one needs a term quadratic in ghosts. The only solutions in the cohomology are then linear combinations of

$$W_a^{-1} = \int d^2x\, X^*_\mu\, \zeta^\mu_a(X) \tag{10.1}$$

where the ζ_a are the Killing vectors of the target space, satisfying (9.9). The interpretation of these solutions is well–known: according to [2] the nontrivial $\mathcal{S}$–invariant functionals with ghost number (-1) correspond one-to-one to the nontrivial rigid symmetries of the classical action generated by field transformations which are *local*, i.e. polynomial in the derivatives of all fields. We conclude that the linearly independent solutions of (D.6) provide *all* nontrivial rigid symmetries of that type which leave the corresponding action functional (8.4) invariant.[19] In particular this implies that any rigid symmetry generated by local field transformations is independent of the two dimensional metric, and of derivatives of the matter fields and does not contain explicit dependence on the coordinates x^α of the two dimensional base manifold. For instance, Kač–Moody symmetries do not occur here since they are non–local in the space–time metric or zweibein field, see remarks in appendix D.3. That the Killing vectors indeed generate rigid symmetries can be easily verified, see e.g. appendix D.2. We also note that the corresponding conserved Noether currents j^α whose divergence vanishes on-shell can be obtained from the 1–forms ω_1^0 arising from (9.21) by the ascent procedure (4.6) through the identification [2]

$$\omega_1^0|_{X^*=0} \equiv dx^\alpha \varepsilon_{\alpha\beta} j^\beta \ ; \quad \varepsilon_{-+} = -\varepsilon_{+-} = 1 \ . \tag{10.2}$$

We obtain

$$\begin{aligned} j_a^\pm &= \tfrac{1}{1-y}\left[(\zeta_{\mu a} \pm b_{\mu a})\nabla_\mp X^\mu - h_{\mp\mp}(\zeta_{\mu a} \mp b_{\mu a})\nabla_\pm X^\mu\right] \\ \Leftrightarrow \quad j_a^\alpha &= \left(\sqrt{g} g^{\alpha\beta}\zeta_{\mu a} + \varepsilon^{\alpha\beta} b_{\mu a}\right)\partial_\beta X^\mu \ ; \quad \varepsilon^{+-} = -\varepsilon^{-+} = 1 \ . \end{aligned} \tag{10.3}$$

One can check that this agrees with (D.11).

$g = 0$: *Action and background charges.*

The antifield–independent solutions with ghost number 0 arise from (9.23) and have the same form as the action itself,

$$W^0_{(0)} = \int d^2x \ \left(\tfrac{1}{2}\sqrt{g}\, g^{\alpha\beta} N_{\mu\nu}(X)\partial_\alpha X^\mu \cdot \partial_\beta X^\nu + F_{\mu\nu}(X)\partial_+ X^\mu \cdot \partial_- X^\nu\right) \tag{10.4}$$

where $N_{\mu\nu}$ and $F_{\mu\nu}$ are arbitrary symmetric resp. antisymmetric functions. The equation (9.25) implies now that (10.4) is cohomologically trivial iff

$$N_{\mu\nu}(X) + F_{\mu\nu}(X) = \partial_{[\mu} B'_{\nu]}(X) + D^-_\mu H'_\nu(X) + D^+_\nu H'_\mu(X) \ , \tag{10.5}$$

for some B'_μ and H'^μ. In particular, two actions (8.4) differing only by a shift in $G_{\mu\nu} + B_{\mu\nu}$ given by (10.5) are thus cohomologically equivalent. Indeed they should be regarded also as physically equivalent since contributions $\partial_{[\mu} B'_{\nu]}$ to $B_{\mu\nu}$ give rise to a total derivative in the Lagrangian while the other contributions in (10.5) are generated by infinitesimal target space reparametrizations $X^\mu \to X^\mu + H'^\mu(X)$.

The antifield–dependent solutions with ghost number 0 arise via the ascent prescription (4.6) from (9.24). One gets

$$W^0_{a\pm} = \int d^2x \left[X^*_\mu\left(\partial_\pm \xi^\pm + h_{\mp\mp}\partial_\pm \xi^\mp\right) - \tfrac{2}{1-y}\partial_\pm h_{\mp\mp} \cdot \nabla_\pm X^\nu \cdot G_{\nu\mu}(X)\right] \cdot \zeta^\mu_{a\pm}(X) \tag{10.6}$$

where the $\zeta_{a\pm}$ are special ('covariantly constant') Killing vectors of the target space, satisfying (9.13). Hence, the solutions (10.6) correspond one–to–one to these Killing vectors whose existence and particular form depends on the choice of $G_{\mu\nu}$ and $B_{\mu\nu}$.

[19] A rigid symmetry is called trivial in this context if the field transformations reduce on–shell to gauge transformations, possibly with field dependent parameters.

The interpretation of (10.6) is familiar in the chiral gauge. Taking $h_{++}=0$, dropping the corresponding ξ^- ghost, and specialising to $G_{\mu\nu}=\delta_{\mu\nu}$, we get

$$\int d^2x (X^*_\mu \partial_+ \xi^+ - 2\partial_+ X_\mu \cdot \partial_+ h_{--}) \cdot \zeta^\mu_+ \ ; \quad \zeta^\mu_+ = \lambda^{a^+} \zeta^\mu_{a+}$$

in which one recognises the so–called 'background charge' terms (see [29, 30] for their inclusion in the BV formalism). To reproduce the well–known form of these background charge terms in the conformal gauge, one has to include both chiralities, and add an appropriate BRST–trivial term. Therefore, $W^0_{a\pm}$ constitute the generalisation of this chiral gauge treatment, and will be called background charge terms henceforth.

As we will show in detail in [11] these background charge terms have in general two different interesting applications: a) appropriate linear combinations of them can be used to construct generalizations of the corresponding action (8.4) (consistent deformations in the terminology of [12]) such that the generalized action is invariant under corresponding extensions of the BRST (resp. gauge) transformations (2.7), and b) other linear combinations represent indeed background charges in the usual sense, i.e. they can cancel (matter field independent) anomalies if regarded formally of order $\hbar^{1/2}$.

In fact we will show in [11] that the actions obtained from a) generalize the well-known Liouville actions.

$g=1$*: Anomalies.*

(9.27) provides two types of solutions: matter field independent ones arising from the $\omega^3_\pm$, and matter field dependent ones arising from the $\omega^3_{X\pm}$. The former read, after performing a partial integration,

$$W^1_\pm = \mp 2 \int d^2x\, c^\pm \partial^3_\pm h_{\mp\mp} = \mp 2 \int d^2x\, (\xi^\pm + h_{\mp\mp}\xi^\mp)\partial^3_\pm h_{\mp\mp} \ , \tag{10.7}$$

whereas the latter are given by

$$W^1_{X\pm} = \int d^2x\, \tfrac{1}{1-y} \left(\partial_\pm \xi^\pm + h_{\mp\mp}\partial_\pm \xi^\mp\right) \cdot \nabla_+ X^\mu \cdot \nabla_- X^\nu \cdot \left(N^\pm_{\mu\nu}(X) + F^\pm_{\mu\nu}(X)\right) \ . \tag{10.8}$$

Some of these are cohomologically trivial. This is the case if

$$N^+_{\mu\nu}(X) + F^+_{\mu\nu}(X) = -2D^+_\nu H'^+_\mu(X) \ ; \quad N^-_{\mu\nu}(X) + F^-_{\mu\nu}(X) = -2D^-_\mu H'^-_\nu(X) \tag{10.9}$$

respectively, with $D^\pm_\mu$ as in (A.18) ($H'^{\mu\pm}(X)$ are arbitrary functions).

The physical interpretation of the solutions (10.7) and (10.8) is well-known: they are the candidate anomalies. Those which are of the form (10.9) can still be cancelled by local counterterms. In section 11 we will show that particular linear combinations of these anomaly candidates indeed reproduce the well-known Weyl anomalies.

Finally we conclude that (10.1),(10.4),(10.6),(10.7) and (10.8) provide, up to the (locally) trivial solutions (10.5) and (10.9), a complete list of $\mathcal{S}$–invariant functionals with ghost numbers $-1,0,1$. More precisely, they represent all the inequivalent nontrivial cohomology classes of these ghost numbers (neglecting "topological" solutions which are locally but not globally trivial).

11 Weyl anomalies and the dilaton

The expressions (10.7) and (10.8) provide the candidate anomalies, up to the $\mathcal{S}$ variations of local counterterms. All these solutions of the consistency condition can be grouped in two chirality classes ('right' and 'left' ones), given by $\{W^1_+, W^1_{X+}\}$ and $\{W^1_-, W^1_{X-}\}$ respectively. Since the theories under consideration are governed by left–right symmetric actions (8.4), at most left–right symmetric combinations of the solutions (10.7) and (10.8) are expected to occur as true anomalies of the theories. We will therefore now compute those linear combinations of solutions (10.7) and (10.8) which are left–right symmetric. It will turn out that, by subtracting appropriate cohomologically trivial pieces, all of them can be cast in the form $\int d^2x(c\,\Omega)$ where c denotes the Weyl ghost and Ω is a density which does not depend on antifields at all. This form suggests to interpret them as candidate Weyl anomalies. The latter are of course the only anomalies that can be present if one uses a regularization scheme which preserves the diffeomorphism invariance.

The left–right symmetric combination of the solutions (10.7) reproduces precisely (1.1), up to a trivial solution:

$$W^1_+ - W^1_- + \mathcal{S}M^0 = \mathcal{A}_0 \tag{11.1}$$

where the counterterm M^0 is given by

$$M^0 = \int dx^2 \tfrac{1}{1-y}\left[-\nabla_+ L\cdot\nabla_- L + \partial_- h_{++}\cdot(2\nabla_- L - r_-) + \partial_+ h_{--}\cdot(2\nabla_+ L - r_+)\right] . \tag{11.2}$$

(We have used (B.13) and (A.12).)

To get the left–right symmetric combinations of the chiral solutions (10.8) we have to impose

$$N^+_{\mu\nu} + F^+_{\mu\nu} = N^-_{\mu\nu} + F^-_{\mu\nu} \equiv f_{\mu\nu} . \tag{11.3}$$

Then the left–right symmetric matter field dependent candidate anomalies are given by the sum $W^1_X = W^1_{X+} + W^1_{X-}$ which indeed can be transformed to a Weyl anomaly,

$$\begin{aligned} &W^1_X - \mathcal{S}\int d^2x\, \tfrac{1}{1-y}\, L\ \nabla_+ X^\mu\cdot\nabla_- X^\nu\cdot f_{\mu\nu}(X) \\ &\quad = -\int d^2x\, c\left(\tfrac{1}{2}\sqrt{g}\, g^{\alpha\beta}\partial_\alpha X^\mu\cdot\partial_\beta X^\nu\cdot f_{(\mu\nu)}(X) + \partial_+ X^\mu\cdot\partial_- X^\nu f_{[\mu\nu]}(X)\right) . \end{aligned} \tag{11.4}$$

In fact, the anomalies of the general action (8.4) have been investigated in [10], for invertible $G_{\mu\nu}$, including a non–Weyl invariant dilaton term, and all above types of anomalies do appear there. The dilaton term will be discussed below. Dropping it for the moment, they get in [10], up to one loop, anomalies of the form (11.4) with $f_{(\mu\nu)}$ and $f_{[\mu\nu]}$ given by

$$f_{(\mu\nu)}(X) = R_{\mu\nu}(X) - \tfrac{1}{4}H_\mu{}^{\lambda\sigma}(X)H_{\nu\lambda\sigma}(X)\ ; \qquad f_{[\mu\nu]}(X) = D_\lambda H^\lambda{}_{\mu\nu}(X) \tag{11.5}$$

where D_μ denotes the target space covariant derivative defined with torsionless connection $\Gamma_{\mu\nu}{}^\rho(X)$ and $R_{\mu\nu}(X)$ is the corresponding Ricci tensor of the target space. They also get an anomaly of the form (1.1) with coefficient (the second term is now a two–loop contribution)

$$\frac{D-26}{48\pi^2} + \frac{\alpha'}{16\pi^2}\left\{-R(X) + \tfrac{1}{12}H^2(X)\right\} \tag{11.6}$$

(where $(4\pi\alpha')^{-1}$ was put in front of the action, and the expansion in α' is thus the loop expansion). It was noted in [10] that the vanishing of the functions in (11.5) already implies that (11.6) is a constant. According to our analysis it is anyway only this constant which is cohomologically nontrivial, and thus is the relevant part of the result.

Let us now discuss how the dilaton terms of [10] arise in our results. As we pointed out in sections 9 and 10, not all solutions (10.8) are nontrivial but among them there are trivial ones given by (10.9). Furthermore recall that these are the only trivial solutions. Let us now investigate the trivial left–right symmetric solutions (11.4). (11.3) imposes

$$D^+_\nu H'^+_\mu = D^-_\mu H'^-_\nu \quad \Leftrightarrow \quad D^-_\mu \zeta_\nu + D^+_\nu \zeta_\mu + \partial_\mu b_\nu - \partial_\nu b_\mu = 0 \tag{11.7}$$

where

$$\zeta^\mu = H'^{\mu-} - H'^{\mu+} \; ; \quad b_\mu = H'^-_\mu + H'^+_\mu \; . \tag{11.8}$$

(11.7) states that ζ^μ solves the Killing vector equations (9.9). Hence, it is a linear combination of the Killing vectors ζ^μ_a, and b_μ is the corresponding linear combination of $b_{\mu a}$, up to a piece $2\partial_\mu\phi$ containing an arbitrary function $\phi(X)$ which drops out of (11.7) (a factor of 2 has been introduced to compare with the results of [10]),

$$\zeta^\mu = \lambda^a \zeta^\mu_a \; ; \quad b_\mu = \lambda^a b_{\mu a} + 2\partial_\mu \phi \; . \tag{11.9}$$

If we insert this result in the triviality condition $f_{\mu\nu} = -2D^+_\nu H'^+_\mu$ and assume $G_{\mu\nu}$ to be invertible in order to make contact with [10], we find that (11.4) is trivial if

$$\det(G_{\mu\nu}) \neq 0 : \quad \begin{aligned} f_{(\mu\nu)} &= -2D_\mu \partial_\nu \phi - \lambda^a D_{(\mu} b_{\nu)a} \; ; \\ f_{[\mu\nu]} &= -H_{\mu\nu}{}^\rho \partial_\rho \phi - \lambda^a (\partial_{[\mu}\zeta_{\nu]a} + \tfrac{1}{2} H_{\mu\nu}{}^\rho b_{\rho a}) \end{aligned} \tag{11.10}$$

where

$$D_\mu \partial_\nu \phi = \partial_\mu \partial_\nu \phi - \Gamma_{\mu\nu}{}^\rho \partial_\rho \phi \; .$$

In absence of Killing vectors (11.10) reduces precisely to the anomaly cancellation condition found in [10][20]. Notice however that in presence of Killing vectors we find in fact that the anomaly cancellation condition is more general than the one imposed in [10]. It should also be noted that the covariantly constant Killing vectors drop out of (11.10) due to (D.16), i.e. these Killing vectors do *not* contribute to that anomaly cancellation condition (rather, they provide the background charges!).

Finally we compute the counterterm whose $\mathcal{S}$–variation leads to the anomaly cancellation (11.10). To that end we recall that the latter arose from (9.17) where we have to use (9.30). Hence, the *function* whose $\mathcal{S}$–variation leads to (11.10) is given by

$$\eta = -T^*_\mu (H'^{\mu+}\Theta^+ + H'^{\mu-}\Theta^-) + (T^\mu + R^\mu) H'^+_\mu \Theta^+ - (T^\mu - R^\mu) H'^-_\mu \Theta^- \; . \tag{11.11}$$

The integrand of the *counterterm* we are looking for arises from (11.11) through the ascent prescription (4.4) which converts η to a 2–form with ghost number 0. The resulting counterterm is

$$\begin{aligned} W^0 = &- \int d^2x \, H'^+_\mu(X) \left[X^{*\mu}(\partial_+ \xi^+ + h_{--}\partial_+ \xi^-) - \tfrac{2}{1-y} \partial_+ h_{--} \cdot \nabla_+ X^\mu \right] \\ &- \int d^2x \, H'^-_\mu(X) \left[X^{*\mu}(\partial_- \xi^- + h_{++}\partial_- \xi^+) - \tfrac{2}{1-y} \partial_- h_{++} \cdot \nabla_- X^\mu \right] \end{aligned} \tag{11.12}$$

[20]We have a difference in the factor in front of the $H_{\mu\nu}{}^\rho \partial_\rho \phi$–term.

where one has to insert the expressions for $H'^{\mu\pm}$ which result from (11.8) and (11.9), i.e.

$$H'^{\pm}_{\mu} = \partial_\mu\phi + \tfrac{1}{2}\lambda^a(b_{\mu a} \mp \zeta_{\mu a}) \ . \tag{11.13}$$

Then W^0 is the general form of the counterterm which can cancel the left-right symmetric anomalies W^1_X in (11.4) at the one loop level when added to the action and multiplied with $\hbar$. The reader can check that the covariantly constant Killing vectors ζ_{a+} contribute only to H'^+ (but not to H'^-) whereas the ζ_{a-} contribute only to H'^-. Hence, these Killing vectors occur in (11.12) only through the functionals (10.6). Since the latter are $\mathcal{S}$–invariant, the covariantly constant Killing vectors do not contribute to $\mathcal{S}W^0$ at all, in accordance with the above observation that they drop out of the anomaly cancellation condition (11.10).

Let us finally discuss those terms in (11.12) which contain the "dilaton" $\phi(X)$. After performing a partial integration they read

$$\begin{aligned} W^0_\phi \ &= \ \int d^2x \Big[-2\phi(\nabla_+ \tfrac{1}{1-y}\partial_+ h_{--} + \nabla_- \tfrac{1}{1-y}\partial_- h_{++}) \\ &\qquad -X^{*\mu}\partial_\mu\phi(\partial_\alpha\xi^\alpha + h_{--}\partial_+\xi^- + h_{++}\partial_-\xi^+)\Big] \ . \end{aligned} \tag{11.14}$$

Using (A.12), (A.7) and (B.13) and partial integrations we can cast (11.14) in the form

$$\begin{aligned} W^0_\phi \ &= \ \int d^2x \Big[-2\phi(\nabla_- \tfrac{1}{1-y}\nabla_+ L - \tfrac{1}{2}\, eR) - X^{*\mu}\partial_\mu\phi(\mathcal{S}L - \xi^\alpha\partial_\alpha L - c)\Big] \\ &= \ \int d^2x \Big[\sqrt{g}(\phi R + g^{\alpha\beta}\partial_\alpha\phi\cdot\partial_\beta L) - X^{*\mu}\partial_\mu\phi(\mathcal{S}L - \xi^\alpha\partial_\alpha L - c)\Big] \ . \end{aligned} \tag{11.15}$$

Finally we split off an $\mathcal{S}$-exact piece in the last term in (11.15) and end up with

$$\begin{aligned} W^0_\phi \ &= \ \int d^2x \Big[\sqrt{g}\,\phi R - \tfrac{2}{1-y}\, L\nabla_+ X^\nu\cdot\nabla_- X^\mu\cdot D^+_\mu\partial_\nu\phi \\ &\qquad +\partial_\mu\phi\cdot X^{*\mu}c + \mathcal{S}(L\, X^{*\nu}\partial_\nu\phi)\Big] \end{aligned} \tag{11.16}$$

where the last term may be omitted since it does not contribute to $\mathcal{S}W^0_\phi$ at all. Combining this with eq.(11.4), and using eq.(11.10), we see that the dilaton dependence of the counterterm that can cancel the matter field dependent Weyl anomaly is just

$$W^0_{\phi,Weyl} = \int d^2x\,(\sqrt{g}\,\phi R - c\,X^{*\mu}\partial_\mu\phi) \ . \tag{11.17}$$

12 Conclusions and final remarks

We investigated the BRST–antibracket cohomology for two-dimensional theories with given field content (two-dimensional metric and scalar matter fields) and given gauge invariances (Weyl and diffeomorphism invariance). We have solved that cohomology completely both on local functions and on local functionals, where the latter arises from the former via the descent equations. Neglecting global aspects, we found that nontrivial cohomology exists only for ghost numbers ranging from 0 to 6 in the case of local functions resp. from (-1) to 4 in the case of local functionals. In particular we obtained the following results:

1. The most general classical action functional describing the models in question is given by (8.4).

2. The rigid symmetries of the models which are generated by local field transformations (i.e. by field transformations which are polynomial in the derivatives of the fields) correspond one-to-one to the target space isometries, i.e. they are given by the independent Killing vectors of the target space, solving (D.6). In particular, Kač–Moody symmetries are *not* present among these symmetries since they are non-local in the two-dimensional metric. They are only symmetries, strictly speaking, after gauge–fixing the metric.

3. The background charges correspond one-to-one to the covariantly constant Killing vectors of the target space. There are in general two types of such Killing vectors, distinguished by the connection ($\Gamma^{+}_{\mu\nu,\rho}$ resp. $\Gamma^{-}_{\mu\nu,\rho}$) which occurs in the respective equation (D.15) defining these Killing vectors. The general form of the corresponding background charge terms in the BV–formalism is given by (10.6).

4. There are two types of candidate anomalies. Both are independent of antifields (up to cohomologically trivial contributions), and both are subdivided in two chirality classes. Those of the first type do not depend on the matter fields at all and are represented by the two solutions (10.7) which are cohomologically nontrivial and inequivalent. The left-right symmetric combination of these two candidate anomalies provides the Weyl anomaly (1.1). The candidate anomalies of the second type involve the matter fields and are given by (10.8). They depend on arbitrary functions $N^{\pm}_{\mu\nu}$ and $F^{\pm}_{\mu\nu}$ of the matter fields, and are cohomologically trivial if and only if these functions are of the form (10.9).

5. The general conditions for the absence of matter field dependent Weyl anomalies are given by (11.10), expressing which of the corresponding BRST–cocycles are cohomologically trivial. On the one hand these conditions reproduce the dilaton terms well-known in the literature [10]. On the other hand they involve further terms which, to our knowledge, have not been discussed in the literature yet. These additional terms occur in presence of isometries of the target space and involve the corresponding Killing vectors. The general form of the counterterm which can cancel the matter field dependent Weyl anomalies is given by (11.12), with $H'^{\pm}_{\mu}$ as in (11.13). The part of this counterterm involving the dilaton can be cast in the form (11.17). Hence, the dilaton need not be introduced by hand but shows up naturally within the cohomological analysis (and in the counterterm), and there may exist novel anomaly free target space manifolds with suitable isometries.

Our presentation has been completely target space covariant. We started by a covariant transformation rule on the X coordinate, i.e. it was independent of the choice of coordinates. Then we took the most general solution for our action. This was then covariant too. Therefore the cohomology problem was also treated covariantly.

As far as we know, our computation is the first *complete* computation of the cohomology considered. Previous work [4, 5, 7] contains partial results, and is to some extent inaccurate. In particular concerning the anomalies, we disagree with [4] where it is claimed that *all* matter field dependent candidate anomalies become cohomologically trivial when the antifields are taken into account. We have given explicitly, eq. (10.8), the form of the remaining *nontrivial* candidates, see also the discussion under result 4 above. In [5] the splitting of all types of candidate anomalies in pairs of two cohomologically inequivalent solutions with different chirality does not stand out. Furthermore, the form of the matter field dependent candidate anomalies given in [5] is *not* the most general one, in that only candidate anomalies are presented there which are Lorentz invariant in target space. In [7] the classical action is not the most general one in that the torsion term is not present. Also, the chiral splitting of the matter dependent anomalies is not found either.

After the preliminary report of part of our work in [8], some of our methods have been used also by [6]. A first criticism on this work is that it ignores the indices of the matter fields and therefore overlooks the subtleties stemming from (anti-) symmetrization of these indices in $D > 1$ target space dimensions. But even in the case $D = 1$ the results in table 2 of [6] are not the same as ours in table 1. To compare these tables, one must omit in our table the zero modes indicated by 'H', as they arise from antifield dependent terms which have not been taken into account in [6] (see discussion below). Then table 1 would give us for $D = 1$ as number of solutions involving arbitrary functions for G =2, 3, 4, 5, 6 respectively 1, 2, 2, 2, 1. This still differs from table 2 in [6] for $G = 4$: in fact their first two types of solutions can be shown to be identical cohomologically in the case $D = 1$, using the "counterterm" $\mathcal{S}\left[T^\mu\Theta^+\Theta^- B'^{+-}_\mu(X)\right] = \mathcal{S}\left[(c^+X^\mu_{1,0} + c^-X^\mu_{0,1})c_0^+c_0^- B'^{+-}_\mu(X)\right]$.

A more serious criticism, which also applies to [4, 5], is that the antifields are not taken into account fully. This implies that they investigate strong BRST cohomology and that their results are in fact gauge–dependent.

To clarify this difference we recall some points about gauge fixing and BRST in the BV framework (for short reviews, see [31, 32]). All field quantities occur in field-antifield pairs. The terminology used throughout this paper is that we indicated as 'fields' all those which have non–negative ghost numbers, while the 'antifields' are those with the negative ghost numbers. This is referred to as the 'classical basis'. Using this basis, the 'BRST'–operator is given by $\Omega = s$, introduced in (2.4). Another possible choice is to choose as fields a set that has no zero modes in the propagators. This is referred to as the 'gauge–fixed basis'. For such a basis to exist, it is necessary that the extended action is proper, although this does not guarantee that the change of basis can be done in a local and covariant way. The latter sometimes requires the introduction of extra trivial sectors, although this is not necessary in our case, where in the gauge fixed basis the fields can be chosen to be

$$\{\Phi^A\} = \{X^\mu, c^+, c^-, b^{++} = h^{++\,*}, b^{--} = h^{--\,*}\}\ . \tag{12.1}$$

Of course, the antibracket cohomology does not depend on the basis in which it is computed, i.e. our results remain valid also in the gauge–fixed basis. What changes, however, is the BRST–operator Ω. On the fields it is defined through

$$\Omega F(\Phi) \equiv \mathcal{S}F|_{\Phi^*=0}\ . \tag{12.2}$$

In general $\Omega^2 \approx 0$, where $\approx$ means equality up to field equations, namely the field equations of the extended action with the appropriate antifields set equal to zero. (In our case $\Omega^2 = 0$.) It can be proven in general [27, 30] that the 'weak cohomology' of Ω (in the definition of that cohomology all equalities are replaced by $\approx$) for local functions is equal to the cohomology of $\mathcal{S}$. For integrals of local functions, this statement holds in the classical basis also for non–negative ghost numbers, but there is no such statement for the gauge–fixed basis. However, we can circumvent this problem using the descent equations, which relate in each case the cohomologies of local integrals to those of local functions.

The work cited above was concerned with the BRST cohomology. The antibracket cohomology, which we have calculated, is related by the considerations above to the *weak* BRST cohomology (in the classical basis). It is the relevant one for anomalies, physical states, ... This remains true if, as in our case, $\Omega^2 = 0$. Thus our results are more complete than those of [4, 5, 6] where antifields have not been taken into account seriously. This confirms once more that the inclusion of the antifields in the cohomological analysis gives more insight into the properties of a theory than the antifield independent (strong) BRST cohomology alone and is thus superior to latter, even if the gauge algebra is closed. It constitutes another good reason for computing the antibracket cohomology directly, keeping all the antifields, as we have done.

The advantage of our treatment stands out if one considers that the Killing vectors enter in the cohomological analysis only if one includes the antifields (resp. investigates the weak cohomology). The same holds for the dilaton terms. Our results show that the isometries (Killing vectors) of the target space play an important role in the theory. They provide all the rigid symmetries of the models, all background charges and occur in the most general anomaly cancellation condition. In fact we will show in the companion paper [11] that they also give rise to interesting deformations of the models, possibly providing new non-critical string theories.

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A Useful formulae

Let us first remark that we use symmetrization and antisymmetrization of indices with 'total weight 1', i.e.

$$A_{[\mu}B_{\nu]} = \tfrac{1}{2}\left(A_\mu B_\nu - A_\nu B_\mu\right)\ ; \qquad A_{(\mu}B_{\nu)} = \tfrac{1}{2}\left(A_\mu B_\nu + A_\nu B_\mu\right)\ . \tag{A.1}$$

We use covariant derivatives in the world–sheet

$$\begin{aligned}\nabla_+ &\equiv \partial_+ - h_{++}\partial_- + \lambda(\partial_- h_{++}\cdot)\ ;\\ \nabla_- &\equiv \partial_- - h_{--}\partial_+ - \lambda(\partial_+ h_{--}\cdot)\end{aligned} \tag{A.2}$$

and λ is the number of lower $+$ indices of the expression on which the operator acts ($-$ the number of lower $-$ indices $+$ the number of upper $-$ indices $-$ the number of upper $+$ indices). To express ordinary derivatives in terms of covariant ones, we have

$$\partial_\pm Z^{(\lambda)} = \frac{1}{1-y}\left(\nabla_\pm + h_{\pm\pm}\nabla_\mp \mp \lambda r_\pm\right) Z^{(\lambda)}\ , \tag{A.3}$$

where $Z^{(\lambda)}$ is an arbitrary tensor of weight λ, and

$$r_\pm = \partial_\mp h_{\pm\pm} - h_{\pm\pm}\partial_\pm h_{\mp\mp}\ . \tag{A.4}$$

Eq.(A.3) is often used for the ghosts:

$$\nabla_\pm c^\pm + h_{\pm\pm}\nabla_\mp c^\pm = (1-y)\partial_\pm c^\pm - c^\pm r_\pm\ . \tag{A.5}$$

Other useful formulae concerning the covariant derivatives are the commutators

$$[\nabla_\pm, \partial_\mp] Z^{(\lambda)} = \mp\lambda\partial^2_\mp h_{\pm\pm}\cdot Z^{(\lambda)} \tag{A.6}$$

and the following identities:

$$\nabla_+ \tfrac{1}{1-y}\nabla_- Z^{(0)} = \nabla_- \tfrac{1}{1-y}\nabla_+ Z^{(0)} = \tfrac{1}{2}\partial_\alpha\left(\sqrt{g}\, g^{\alpha\beta}\partial_\beta Z^{(0)}\right)\ ; \tag{A.7}$$

$$\nabla_\alpha\left(c^\alpha \tfrac{1}{1-y} e Z^{(0)}\right) = \partial_\alpha\left(\xi^\alpha\sqrt{g} Z^{(0)}\right)\ . \tag{A.8}$$

For the inverse of the metric and some other conversions of functions of the metric to the chiral basis we have

$$\begin{aligned}\sqrt{g}\, g^{+-} &= \frac{g_{+-}}{\sqrt{g}} = \frac{1+y}{1-y}\ ; \qquad \sqrt{g}\, g^{\pm\pm} = -\frac{g_{\mp\mp}}{\sqrt{g}} = \frac{-2h_{\mp\mp}}{1-y}\ ;\\ 1+y &= \frac{2g_{+-}}{g_{+-}+\sqrt{g}}\ ; \qquad 1-y = \frac{2\sqrt{g}}{g_{+-}+\sqrt{g}}\ .\end{aligned} \tag{A.9}$$

For the ghosts, important translation formulae are

$$\begin{aligned}&\xi^\pm = \tfrac{1}{1-y}\left(c^\pm - h_{\mp\mp}c^\mp\right)\ ;\\ &\nabla_\pm c^\mp = \xi^\alpha\partial_\alpha h_{\pm\pm} + \partial_\pm\xi^\mp - (h_{\pm\pm})^2\partial_\mp\xi^\pm + h_{\pm\pm}(\partial_\pm\xi^\pm - \partial_\mp\xi^\mp)\ ;\\ &\left(\xi^\alpha\partial_\alpha + \frac{\lambda}{1-y}(\xi^+ r_+ - \xi^- r_-)\right) Z^{(\lambda)} = \tfrac{1}{1-y} c^\alpha\nabla_\alpha Z^{(\lambda)}\ .\end{aligned} \tag{A.10}$$

The Riemann tensor is defined by

$$\tfrac{1}{2}\,R^{\alpha}{}_{\beta\gamma\delta} = \partial_{[\gamma}\Gamma_{\delta]\beta}{}^{\alpha} + \Gamma_{\varepsilon[\gamma}{}^{\alpha}\Gamma_{\delta]\beta}{}^{\varepsilon}\ . \tag{A.11}$$

The curvature scalar can be written as

$$\tfrac{1}{2}eR = \tfrac{1}{2}eR^{\alpha\beta}{}_{\beta\alpha} = \nabla_{-}\tfrac{1}{1-y}\nabla_{+}L - \nabla_{-}\tfrac{1}{1-y}\partial_{-}h_{++} - \nabla_{+}\tfrac{1}{1-y}\partial_{+}h_{--}\ . \tag{A.12}$$

where we introduced

$$L = \ln\frac{e}{1-y} = \ln\frac{g_{+-}+\sqrt{g}}{2}\ . \tag{A.13}$$

For the fields, the first few of the basis of chiral tensor fields, defined in (5.11), are

$$\begin{aligned}
X^{\mu}_{0,0} &= X^{\mu}\ ; \\
(1-y)X^{\mu}_{1,0} &= \nabla_{+}X^{\mu}\ ; \qquad (1-y)X^{\mu}_{0,1} = \nabla_{-}X^{\mu}\ ; \\
(1-y)X^{\mu}_{1,1} &= \nabla_{+}\tfrac{1}{1-y}\nabla_{-}X^{\mu} = \nabla_{-}\tfrac{1}{1-y}\nabla_{+}X^{\mu} = \tfrac{1}{2}\partial_{\alpha}\left(\sqrt{g}\,g^{\alpha\beta}\partial_{\beta}X^{\mu}\right)\ ; \\
(1-y)X^{\mu}_{2,0} &= \nabla_{+}\tfrac{1}{1-y}\nabla_{+}X^{\mu} - \tfrac{1}{1-y}r_{+}\cdot\nabla_{+}X^{\mu}\ ; \\
(1-y)X^{\mu}_{0,2} &= \nabla_{-}\tfrac{1}{1-y}\nabla_{-}X^{\mu} - \tfrac{1}{1-y}r_{-}\cdot\nabla_{-}X^{\mu}\ .
\end{aligned} \tag{A.14}$$

Another useful equation is

$$(1-y)X^{\mu}_{1,0}X^{\nu}_{0,1} = \tfrac{1}{1-y}\nabla_{+}X^{\mu}\cdot\nabla_{-}X^{\nu} = \frac{1}{2}\sqrt{g}\,g^{\alpha\beta}\partial_{\alpha}X^{\mu}\cdot\partial_{\beta}X^{\nu} + \partial_{+}X^{[\mu}\cdot\partial_{-}X^{\nu]}\ , \tag{A.15}$$

and the last term is covariantly written as $\frac{1}{2}\varepsilon^{\alpha\beta}\partial_{\alpha}X^{\mu}\cdot\partial_{\beta}X^{\nu}$.

In target space we introduce the connections with torsion

$$2\Gamma_{\rho\sigma,\mu} \equiv \partial_{\rho}G_{\sigma\mu} + \partial_{\sigma}G_{\rho\mu} - \partial_{\mu}G_{\rho\sigma}\ ; \tag{A.16}$$

$$\begin{aligned}
H_{\rho\sigma\mu} &\equiv \partial_{\rho}B_{\sigma\mu} + \partial_{\sigma}B_{\mu\rho} + \partial_{\mu}B_{\rho\sigma}\ ; \\
\Gamma^{\pm}_{\rho\sigma,\mu} &\equiv \Gamma_{\rho\sigma,\mu} \pm \tfrac{1}{2}H_{\rho\sigma\mu} = \Gamma^{\mp}_{\sigma\rho,\mu}\ .
\end{aligned} \tag{A.17}$$

There are two types of covariant derivatives:

$$D^{\pm}_{\mu}V_{\nu} \equiv \partial_{\mu}V_{\nu} - \Gamma^{\pm}_{\mu\nu,\rho}V^{\rho}\ , \tag{A.18}$$

and we have that

$$2\partial_{[\mu}V_{\nu]} = D^{-}_{\mu}V_{\nu} - D^{+}_{\nu}V_{\mu}\ . \tag{A.19}$$

It is understood here that the fundamental quantities are the V^{μ} rather than the V_{μ} since we do not assume $G_{\mu\nu}$ to be invertible and thus cannot use it to raise indices but only to lower them. When written completely in terms of the V^{μ}, (A.18) reads

$$D^{\pm}_{\mu}V_{\nu} = G_{\nu\rho}\partial_{\mu}V^{\rho} + \Gamma^{\pm}_{\mu\rho,\nu}V^{\rho}\ . \tag{A.20}$$

Note that (A.20) would be obvious if we could define covariant derivatives on vectors with upper indices. However that requires an invertible metric.

B $\mathcal{S}$–transformations in Beltrami basis

On the variables introduced in section 3 the part s of $\mathcal{S}$ acts according to

$$
\begin{aligned}
sc^{+} &= c^{+}\partial_{+}c^{+} \;; && \text{(B.1)}\\
sh_{++} &= \nabla_{+}c^{-} \;; && \text{(B.2)}\\
sX^{\mu} &= c^{+}X^{\mu}_{1,0} + c^{-}X^{\mu}_{0,1} \;; && \text{(B.3)}\\
s\hat{X}^{*}_{\mu} &= c^{+}\tfrac{1}{1-y}(\nabla_{+} - r_{+})\hat{X}^{*}_{\mu} + \partial_{+}c^{+}\cdot\hat{X}^{*}_{\mu} && \\
&\quad + c^{-}\tfrac{1}{1-y}(\nabla_{-} - r_{-})\hat{X}^{*}_{\mu} + \partial_{-}c^{-}\cdot\hat{X}^{*}_{\mu} \;; && \text{(B.4)}\\
sh^{*++} &= c^{-}(\partial_{-}h^{*++} - h_{--}\hat{X}^{*}_{\mu}X^{\mu}_{0,1}) + c^{+}\hat{X}^{*}_{\mu}X^{\mu}_{0,1} + 2\partial_{-}c^{-}\cdot h^{*++} \;; && \text{(B.5)}\\
sc^{*}_{-} &= c^{-}\partial_{-}c^{*}_{-} + 2\partial_{-}c^{-}\cdot c^{*}_{-} && \text{(B.6)}
\end{aligned}
$$

with $\hat{X}^{*}_{\mu}$ as in (5.6). The transformations of c^{-}, h_{--}, h^{*--} and c^{*}_{+} are obtained from those of c^{+}, h_{++}, h^{*++} and c^{*}_{-} by interchanging all $+$ and $-$ indices. The weights (5.10) are read off (B.3)–(B.6) since these equations take the form

$$
sZ = w_{+}\partial_{+}c^{+}\cdot Z + w_{-}\partial_{-}c^{-}\cdot Z + c^{+}L^{+}_{-1}Z + c^{-}L^{-}_{-1}Z \;. \tag{B.7}
$$

This allows e.g. to determine

$$
\begin{aligned}
\mathcal{S}X^{\mu}_{1,0} &= \partial_{+}c^{+}\cdot X^{\mu}_{1,0} + c^{+}X^{\mu}_{2,0} + c^{-}X^{\mu}_{1,1}\\
&= \tfrac{1}{1-y}\left(\nabla_{+}c^{+} + c^{-}\nabla_{-} + h_{++}\nabla_{-}c^{+}\cdot\right)X^{\mu}_{1,0} \;,
\end{aligned} \tag{B.8}
$$

and to show that (A.15) is a density:

$$
\mathcal{S}\left(\tfrac{1}{1-y}\nabla_{+}X^{\mu}\cdot\nabla_{-}X^{\nu}\right) = \nabla_{\alpha}\left(c^{\alpha}\tfrac{1}{(1-y)^{2}}\nabla_{+}X^{\mu}\cdot\nabla_{-}X^{\nu}\right) \;. \tag{B.9}
$$

The part δ_{KT} of $\mathcal{S}$ is non-vanishing only on the antifields and for the general classical action (8.4) given by

$$
\begin{aligned}
\delta_{KT}\hat{X}^{*}_{\mu} &= -2G_{\mu\nu}X^{\nu}_{1,1} - 2\Gamma^{-}_{\rho\nu,\mu}X^{\rho}_{1,0}X^{\nu}_{0,1} = -2G_{\mu\nu}X^{\nu}_{1,1} - 2\Gamma^{+}_{\rho\nu,\mu}X^{\nu}_{1,0}X^{\rho}_{0,1} \;; && \text{(B.10)}\\
\delta_{KT}h^{*++} &= -G_{\mu\nu}X^{\mu}_{0,1}X^{\nu}_{0,1} \;; && \text{(B.11)}\\
\delta_{KT}c^{*}_{-} &= -\nabla_{+}h^{*++} + \hat{X}^{*}_{\mu}\nabla_{-}X^{\mu} && \text{(B.12)}
\end{aligned}
$$

with $\Gamma^{\pm}_{\rho\nu,\mu}$ as in (A.16). $\delta_{KT}h^{*--}$ and $\delta_{KT}c^{*}_{+}$ are obtained from $\delta_{KT}h^{*++}$ and $\delta_{KT}c^{*}_{-}$ by interchanging all $+$ and $-$ indices.

The useful quantity (A.13) transforms according to

$$
\begin{aligned}
\mathcal{S}L &= \xi^{\alpha}\partial_{\alpha}L + \partial_{\alpha}c^{\alpha} - \tfrac{1}{1-y}c^{\alpha}r_{\alpha} + c\\
&= \xi^{\alpha}\partial_{\alpha}L + \partial_{\alpha}\xi^{\alpha} + h_{--}\partial_{+}\xi^{-} + h_{++}\partial_{-}\xi^{+} + c \;.
\end{aligned} \tag{B.13}
$$

The transformation of eR reads

$$
\begin{aligned}
\mathcal{S}(eR) &= 2\nabla_{-}\tfrac{1}{1-y}\nabla_{+}c + \nabla_{\alpha}(c^{\alpha}\tfrac{1}{1-y}eR)\\
&= \partial_{\alpha}(\sqrt{g}g^{\alpha\beta}\partial_{\beta}c + \xi^{\alpha}\sqrt{g}R) \;.
\end{aligned} \tag{B.14}
$$

C $\mathcal{S}$–exactness of target space reparametrizations

Two actions (8.4) which differ only by a (regular) target space reparametrization should be regarded as physically equivalent. This fits nicely with the fact that two actions related by an infinitesimal target space reparametrization

$$X^\mu \to X^\mu + f^\mu(X) \tag{C.1}$$

are cohomologically equivalent since their difference is $\mathcal{S}$–exact. This statement is implied by the following more general one:

Lemma C.1 *The difference of two (local) classical actions $S_0[\phi]$, $S_0[\phi+\delta\phi]$ related by a (local) infinitesimal field redefinition $\delta\phi^i = f^i(\phi, \partial\phi, \ldots)$ is $\mathcal{S}$–exact in the space of (local) functionals of the fields and antifields if it is invariant under $\mathcal{S}$:*[21]

$$S_0[\phi+\delta\phi] - S_0[\phi] = \mathcal{S}\Gamma[\Phi,\Phi^*] \quad \Leftrightarrow \quad \mathcal{S}\left(S_0[\phi+\delta\phi] - S_0[\phi]\right) = 0\ . \tag{C.2}$$

Here Φ denotes collectively all fields (including the ghosts or ghosts for ghosts, ..., corresponding to the gauge symmetries of S_0).

Proof: The implication $\Rightarrow$ follows from the nilpotency of $\mathcal{S}$. In order to prove the implication $\Leftarrow$ we remark $S_0[\phi+\delta\phi] - S_0[\phi] = \int d^dx\,(S_0[\phi]\overleftarrow{\delta}/\delta\phi^i)\delta\phi^i$ which implies

$$S_0[\phi+\delta\phi] - S_0[\phi] = \delta_{KT}\int d^dx\,\phi_i^*\delta\phi^i = \mathcal{S}\int d^dx\,\phi_i^*\delta\phi^i + W[\Phi,\Phi^*] \tag{C.3}$$

where δ_{KT} denotes the Koszul–Tate differential ($\delta_{KT}\phi_i^* = S_0[\phi]\overleftarrow{\delta}/\delta\phi^i$). It contains all the terms of $\mathcal{S}\phi^*$ which have no antifields. Therefore W, a (local) functional of Φ, Φ^* of ghost number zero, contains an antifield and a ghost in all its terms. Since $\mathcal{S}\left(S_0[\phi+\delta\phi] - S_0[\phi]\right) = 0$ holds by assumption, we conclude from (C.3), using $\mathcal{S}^2 = 0$, that W is $\mathcal{S}$–invariant. General theorems on the cohomology of $\mathcal{S}$ [27, 33, 30] then imply that

$$W = \mathcal{S}\omega[\Phi,\Phi^*] \tag{C.4}$$

where ω is a (local) functional. This completes the proof of (C.2) since we have

$$S_0[\phi+\delta\phi] - S_0[\phi] = \mathcal{S}\left(\omega + \int d^dx\,\phi_i^*\delta\phi^i\right).$$

■

One may now verify that lemma C.1 applies to the target space reparametrizations (C.1) since $S_{cl}[X^\mu + f^\mu(X), g_{\alpha\beta}]$ is $\mathcal{S}$–invariant. That $S_{cl}[X^\mu + f^\mu(X), g_{\alpha\beta}] - S_{cl}[X^\mu, g_{\alpha\beta}]$ is indeed $\mathcal{S}$–exact can be seen from (9.25) and (D.9).

A few comments on the content of the above lemma seem to be in order here. Notice that the requirement $\mathcal{S}\left(S_0[\phi+\delta\phi] - S_0[\phi]\right) = 0$ imposes a highly nontrivial condition on the variations $\delta\phi^i$. For instance, if the gauge transformations form a closed algebra, then it requires $S_0'[\phi] \equiv S_0[\phi+\delta\phi]$ to be invariant under exactly the *same* gauge transformations

[21] $\mathcal{S}$ itself is defined with the extended action of which $S_0[\phi]$ constitutes the part in the fields of zero ghost number.

of the ϕ^i that leave $S_0[\phi]$ itself invariant (recall that for closed algebras $\mathcal{S}S_0[\phi]=0$ holds due to the gauge invariance of $S_0[\phi]$). Furthermore it should be noted that the symmetries of $S_0[\phi]$ (both the rigid and the gauge symmetries) are a subset of the transformations $\delta\phi^i$ satisfying the above condition. Namely, if $\delta\phi^i$ is a symmetry, then one even has $\delta S_0[\phi] \equiv S_0[\phi+\delta\phi]-S_0[\phi]=0$ which is evidently a stronger condition than $\mathcal{S}(\delta S_0[\phi])=0$. This suggests to call transformations $\delta\phi^i$ which fulfill $\mathcal{S}(\delta S_0[\phi])=0$ *generalized symmetries* or *pseudo–symmetries*.

Finally we remark that the lemma C.1 applies also to theories which do not possess a (nontrivial) gauge symmetry at all. However, in that particular case there are no conditions on the pseudo–symmetries since $S_0[\phi+\delta\phi]-S_0[\phi]$ is $\mathcal{S}$–invariant for arbitrary field redefinitions $\delta\phi^i$ because then $\mathcal{S}$ reduces to δ_{KT} and thus vanishes on all ϕ^i.

D Lie derivatives and Killing vectors

In this appendix we collect properties of Lie derivatives, Killing vectors, and finally special Killing vectors which are covariantly constant. Most of these properties were found already in [34] (where the target–space metric has been assumed to be invertible), but we stress that we will not assume that $G_{\mu\nu}$ is invertible. Instead, especially for the properties of covariantly constant Killing vectors we will use (8.8), i.e. assumption (iii) of section 2.

D.1 Lie derivatives

The Lie derivative along a vector H^μ is defined, for example for a 2–tensor $Y_{\mu\nu}$, by

$$\mathcal{L}_H Y_{\mu\nu} \equiv H^\rho\partial_\rho Y_{\mu\nu} + \partial_\mu H^\rho \cdot Y_{\rho\nu} + \partial_\nu H^\rho \cdot Y_{\mu\rho}\ . \tag{D.1}$$

The Lie–derivative commutes with the ordinary differential. For the metric and the anti-symmetric tensor $B_{\mu\nu}$ there are the identities

$$\begin{aligned}
\mathcal{L}_H G_{\mu\nu} &= 2D_{(\mu}H_{\nu)} = 2\partial_{(\mu}H_{\nu)} - 2\Gamma_{\mu\nu,\rho}H^\rho\ ; && \text{(D.2)}\\
\mathcal{L}_H B_{\mu\nu} &= -2\partial_{[\mu}\left(B_{\nu]\rho}H^\rho\right) + H_{\mu\nu\rho}H^\rho\ ; && \text{(D.3)}\\
\mathcal{L}_H H_{\mu\nu\rho} &= 3\partial_{[\mu}\left(H_{\nu\rho]\sigma}H^\sigma\right)\ , && \text{(D.4)}
\end{aligned}$$

where in (D.4) we used the Bianchi identity for the curl of $B_{\mu\nu}$. The commutator of two Lie derivatives gives a new Lie derivative:

$$[\mathcal{L}_H,\mathcal{L}_K]=\mathcal{L}_L \qquad \text{with} \qquad L^\mu = H^\nu\partial_\nu K^\mu - K^\nu\partial_\nu H^\mu\ , \tag{D.5}$$

and L^μ is called the Lie bracket of H and K.

D.2 Killing vectors

Killing vectors ζ^μ are defined by the condition that there exists a vector b_μ such that

$$D^-_\mu\zeta_\nu + D^+_\nu\zeta_\mu + 2\partial_{[\mu}b_{\nu]} = 0 \tag{D.6}$$

with $D^\pm_\mu$ as in (A.18). Splitting this condition in its symmetric and antisymmetric part using (D.2) and (D.3), we have

$$\begin{aligned} 0 &= \mathcal{L}_\zeta G_{\mu\nu} && \text{(D.7)} \\ 0 &= H_{\mu\nu\rho}\zeta^\rho + 2\partial_{[\mu} b_{\nu]} = \mathcal{L}_\zeta B_{\mu\nu} + 2\partial_{[\mu}\hat{b}_{\nu]} \quad \text{with} \quad \hat{b}_\mu = b_\mu + B_{\mu\nu}\zeta^\nu \ . && \text{(D.8)} \end{aligned}$$

(D.8) can also be written without making reference to a function b_μ as $\mathcal{L}_H H_{\mu\nu\rho} = 0$.

Killing vectors of a given metric and torsion generate rigid symmetries of the corresponding action (8.4). In general, replacing X^μ by $X^\mu + f^\mu(X)\epsilon$ for arbitrary $f(X)$ and infinitesimal ϵ leads (up to a total derivative) to a classical action with the replacement

$$G_{\mu\nu} + B_{\mu\nu} \quad \rightarrow \quad G_{\mu\nu} + B_{\mu\nu} + \epsilon(D^-_\mu f_\nu + D^+_\nu f_\mu) \ , \tag{D.9}$$

and an extra term

$$\int d^2x \sqrt{g}\, g^{\alpha\beta} G_{\mu\nu}(X) f^\nu(X) \partial_\alpha X^\mu \cdot \partial_\beta \epsilon \ . \tag{D.10}$$

Therefore if f^μ is replaced by ζ^μ, satisfying (D.6), the action is invariant provided ϵ does not depend on the world–sheet coordinates. For future reference, if ϵ depends on the world–sheet coordinates, we can use (8.5) to obtain ($\epsilon^{+-} = -\epsilon^{-+} = 1$)

$$\delta S_{cl} = \int d^2x \left(\sqrt{g}\, g^{\alpha\beta}\zeta_\mu(X) - \varepsilon^{\alpha\beta} b_\mu(X)\right) \partial_\alpha X^\mu \cdot \partial_\beta \epsilon \ . \tag{D.11}$$

The commutator of two (infinitesimal) rigid symmetries is again a rigid symmetry[22]. Therefore the Lie bracket of two Killing vectors gives a new Killing vector. Introducing a basis of the Killing vectors ζ^μ_a, we thus have that

$$\zeta^\mu_{[ab]} = \zeta^\nu_a \partial_\nu \zeta^\mu_b - \zeta^\nu_b \partial_\nu \zeta^\mu_a \tag{D.12}$$

defines again a Killing vector (or vanishes), i.e. it satisfies again (D.7) and (D.8) with

$$\begin{aligned} \hat{b}_{\mu[ab]} &= \mathcal{L}_a \hat{b}_{\mu b} - \mathcal{L}_b \hat{b}_{\mu a} \qquad \text{or} \\ b_{\mu[ab]} &= \mathcal{L}_a b_{\mu b} - \mathcal{L}_b b_{\mu a} - H_{\mu\nu\rho}\zeta^\nu_a \zeta^\rho_b \ , && \text{(D.13)} \end{aligned}$$

the latter modulo an irrelevant total derivative which drops out of (D.8). In (D.13) we have used the abbreviation

$$\mathcal{L}_a = \mathcal{L}_{\zeta_a} \ . \tag{D.14}$$

D.3 Covariantly constant Killing vectors

We shall now derive some useful properties of the special Killing vectors (9.14). They are defined by

$$D^\pm_\mu \zeta_{\nu a^\pm} = 0 \ , \tag{D.15}$$

i.e. they are covariantly constant. This definition is equivalent to the Killing equations (D.7) and (D.8) with the extra condition

$$b_{\mu a^\pm} = \mp \zeta_{\mu a^\pm} \ , \tag{D.16}$$

[22] A trivial symmetry cannot occur in this case since the Killing vectors do not involve partial derivatives of the X's whereas both the equations of motion as well as the gauge transformations would necessarily introduce derivatives of the X's.

i.e. (D.15) is equivalent to

$$\mathcal{L}_{a^\pm} G_{\mu\nu} = 0 \ ; \quad H_{\mu\nu\rho} \zeta^\rho_{a^\pm} = \pm 2 \partial_{[\mu} \zeta_{\nu] a^\pm} \ . \tag{D.17}$$

These Killing vectors determine the Kač–Moody symmetries. Indeed, in these cases (D.11) shows that the action is invariant for transformations with parameters $\epsilon^{a^\pm}$ satisfying

$$\left(\sqrt{g}\, g^{\alpha\beta} \pm \varepsilon^{\alpha\beta} \right) \partial_\beta \epsilon^{a^\pm} = 0 \ . \tag{D.18}$$

One can always find such $\epsilon^{a^\pm}(x)$ for any given metric $g_{\alpha\beta}(x)$ (In the zweibein formalism the above equation reduces[23] to $e^\alpha_\mp \partial_\alpha \epsilon^{a^\pm} = 0$). Hence, given an action (8.4) with a *fixed* metric $g_{\alpha\beta}(x)$ (keeping X^μ still arbitrary), $\delta_\epsilon X^\mu = \epsilon^{a^\pm}(x) \zeta^\mu_{a^\pm}(X)$ generates chiral (Kač–Moody) symmetries of that action, where $\epsilon^{a^\pm}$ are solutions of (D.18). However, in our actions the metric $g_{\alpha\beta}$ is a field and thus has to be regarded in (D.18) as a variable rather than as a specific function of the world sheet coordinates. One can then still solve (D.18) for $\epsilon^{a^\pm}$ but the solutions involve infinitely many derivatives of the $g_{\alpha\beta}$ and are thus nonlocal. Hence, diffeomorphism invariant actions (8.4) do not possess Kač–Moody symmetries generated by local field transformations, contrary to sigma models with non–gauged world–sheet diffeomorphisms, or to the gauge–fixed theory.

Now we derive some useful properties for the scalar products of the covariantly constant Killing vectors. We define

$$P_{a^\pm b^\pm} = \zeta^\mu_{a^\pm} G_{\mu\nu} \zeta^\nu_{b^\pm} \ ; \quad P_{a^+ b^-} = \zeta^\mu_{a^+} G_{\mu\nu} \zeta^\nu_{b^-} \ . \tag{D.19}$$

Using (A.20), (A.18) and (D.15) one easily verifies that $P_{a^+b^+}$ and $P_{a^-b^-}$ are constant:

$$\partial_\mu P_{a^\pm b^\pm} = \partial_\mu \zeta_{\rho a^\pm} \cdot \zeta^\rho_{b^\pm} + \zeta^\nu_{a^\pm} G_{\nu\rho} \partial_\mu \zeta^\rho_{b^\pm} = \zeta^\nu_{a^\pm} \zeta^\rho_{b^\pm} (\Gamma^\pm_{\mu\rho,\nu} - \Gamma^\pm_{\mu\rho,\nu}) = 0 \ . \tag{D.20}$$

Similarly one verifies that

$$\partial_\mu P_{a^+ b^-} = \zeta^\nu_{a^+} \zeta^\rho_{b^-} (\Gamma^+_{\mu\rho,\nu} - \Gamma^-_{\mu\rho,\nu}) = -H_{\mu\nu\rho} \zeta^\nu_{a^+} \zeta^\rho_{b^-} \ . \tag{D.21}$$

Lemma D.1 *The $\mathcal{L}_{a^+}, \mathcal{L}_{a^-}$ span a Lie algebra which is the direct sum of two subalgebras $\{\mathcal{L}_{a^+}\}$ and $\{\mathcal{L}_{a^-}\}$.*

Proof: We have to prove that

$$[\mathcal{L}_{a^\pm}, \mathcal{L}_{b^\pm}] = \lambda^{c^\pm}{}_{a^\pm b^\pm} \mathcal{L}_{c^\pm} \ ; \quad [\mathcal{L}_{a^+}, \mathcal{L}_{a^-}] = 0 \ , \tag{D.22}$$

for some constants $\lambda^{c^+}{}_{a^+b^+}$ and $\lambda^{c^-}{}_{a^-b^-}$. This is equivalent to showing that the Lie bracket of any two ζ_{a^+}'s is again a linear combination of the ζ_{a^+}'s (analogously for the ζ_{a^-}'s) and the Lie bracket of ζ_{a^+} and ζ_{b^-} vanishes for all pairs (a^+, b^-), i.e.

$$\zeta^\mu_{[a^+b^+]} = \zeta^\nu_{a^+} \partial_\nu \zeta^\mu_{b^+} - \zeta^\nu_{b^+} \partial_\nu \zeta^\mu_{a^+} = \lambda^{c^+}{}_{a^+b^+} \zeta^\mu_{c^+} \ ; \tag{D.23}$$

$$\zeta^\mu_{[a^-b^-]} = \zeta^\nu_{a^-} \partial_\nu \zeta^\mu_{b^-} - \zeta^\nu_{b^-} \partial_\nu \zeta^\mu_{a^-} = \lambda^{c^-}{}_{a^-b^-} \zeta^\mu_{c^-} \ ; \tag{D.24}$$

$$\zeta^\mu_{[a^+b^-]} = \zeta^\nu_{a^+} \partial_\nu \zeta^\mu_{b^-} - \zeta^\nu_{b^-} \partial_\nu \zeta^\mu_{a^+} = 0 \ . \tag{D.25}$$

[23] using $\eta^{+-} = 1$.

We note that (D.25) is proved in one line by means of (A.20) if $G_{\mu\nu}$ is invertible:

$$\det(G_{\mu\nu}) \neq 0 \ \Rightarrow \ \partial_\nu \zeta^\mu_{b\pm} = -\Gamma^{\pm}{}_{\nu\rho}{}^{\mu} \zeta^\rho_{b\pm} \ \Rightarrow \ \zeta^\mu_{[a+b-]} = \zeta^\nu_{a+}\zeta^\rho_{b-}(-\Gamma^{-}_{\nu\rho}{}^{\mu} + \Gamma^{+}_{\rho\nu}{}^{\mu}) = 0 \ .$$

For general $G_{\mu\nu}$ the proof of (D.23) and (D.25) is more involved. We first compute, using (A.20) and (D.15),

$$G_{\mu\nu}\zeta^\nu_{[a+b\pm]} = \zeta^\nu_{a+}\zeta^\rho_{b\pm}(-\Gamma^{\pm}_{\nu\rho,\mu} + \Gamma^{+}_{\rho\nu,\mu}) \ .$$

This gives

$$G_{\mu\nu}\zeta^\nu_{[a+b+]} = -H_{\mu\nu\rho}\zeta^\nu_{a+}\zeta^\rho_{b+} \ ; \tag{D.26}$$

$$G_{\mu\nu}\zeta^\nu_{[a+b-]} = 0 \ . \tag{D.27}$$

From (D.27) and $\mathcal{L}_{[a+b-]}G_{\mu\nu} = 0$ we conclude that

$$\Gamma_{\mu\nu,\rho}\zeta^\rho_{[a+b-]} = 0 \ . \tag{D.28}$$

We finally compute $H_{\mu\nu\rho}\zeta^\rho_{[a+b\pm]}$ in order to verify the second equation (D.17) for $\zeta^\rho_{[a+b\pm]}$. Since the latter is a Killing vector, it satisfies (D.8), i.e.

$$H_{\mu\nu\rho}\zeta^\rho_{[a+b\pm]} = -2\partial_{[\mu}b_{\nu][a+b\pm]} \tag{D.29}$$

with

$$\begin{aligned} b_{\mu[a+b\pm]} &= \mp\mathcal{L}_{a+}\zeta_{\mu b\pm} + \mathcal{L}_{b\pm}\zeta_{\mu a+} - H_{\mu\nu\rho}\zeta^\nu_{a+}\zeta^\rho_{b\pm} \\ &= -(1\pm 1)\zeta_{\mu[a+b\pm]} - H_{\mu\nu\rho}\zeta^\nu_{a+}\zeta^\rho_{b\pm} \end{aligned} \tag{D.30}$$

where we used (D.13), (D.16) and (D.12). By means of (D.26) respectively (D.21) we conclude from (D.30)

$$b_{\mu[a+b+]} = -\zeta_{\mu[a+b+]} \ ; \quad b_{\mu[a+b-]} = \partial_\mu P_{a+b-} \ . \tag{D.31}$$

Inserting this result in (D.29) we get

$$H_{\mu\nu\rho}\zeta^\rho_{[a+b+]} = \partial_\mu\zeta_{\nu[a+b+]} - \partial_\nu\zeta_{\mu[a+b+]} \ ; \tag{D.32}$$

$$H_{\mu\nu\rho}\zeta^\rho_{[a+b-]} = 0 \ . \tag{D.33}$$

(D.32) and $\mathcal{L}_{[a+b+]}G_{\mu\nu} = 0$ show that $\zeta_{[a+b+]}$ solves (D.17) and hence must be a linear combination of the ζ_{a+}'s. This proves (D.23) (of course (D.24) can be proved analogously). (D.27), (D.28) and (D.33) imply $\zeta^\mu_{[a+b-]} = 0$ due to (8.8). This proves (D.25). ■

D.4 Non–chiral covariantly constant Killing vectors

Consider now a Killing vector k^μ which is covariantly constant for both covariant derivatives (A.18). For such constant vector, we find that there is no torsion in this direction

$$H_{\mu\nu\rho}k^\rho = 0 \ , \tag{D.34}$$

and furthermore because of (A.19)

$$k_\mu \equiv G_{\mu\nu}k^\nu = \partial_\mu\Lambda \ ; \qquad \partial_\mu\partial_\nu\Lambda - \Gamma_{\mu\nu,\rho}k^\rho = 0 \ . \tag{D.35}$$

Note that according to (D.34) one has $b_\mu(X) = \partial_\mu b(X)$ in (D.8). Therefore it is clear from (D.11) that if the metric would be degenerate such that $k_\mu = 0$ for non–zero k^μ, then S_{cl} would have an additional local symmetry in contradiction to assumption (iii) of section 2.

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