

SCATTERING MATRIX IN EXTERNAL FIELD PROBLEMS

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ABSTRACT: We discuss several aspects of second quantized scattering operators \hat{S} for fermions in external time dependent fields. We derive our results on a general, abstract level having in mind as a main application potentials of the Yang–Mills type and in various dimensions. We present a new and powerful method for proving existence of \hat{S} which is also applicable to other situations like external gravitational fields. We also give two complementary derivations of the change of phase of the scattering matrix under generalized gauge transformations which can be used whenever our method of proving existence of \hat{S} applies. The first is based on a causality argument i.e. \hat{S} (including phase) is determined from a time evolution, and the second exploits the geometry of certain infinite-dimensional group extensions associated with the second quantization of 1-particle operators. As a special case we obtain a Hamiltonian derivation of the the axial Fermion-Yang-Mills anomaly and the Schwinger terms related to it via the descent equations, which is on the same footing and traces them back to a common root.

1. INTRODUCTION

The main difficulty when quantizing fermions in higher than two space-time dimensions in background (gauge) fields is that the interaction term generically is too large to allow a naive application of the standard methods of canonical quantization. More precisely, if ϵ is the sign of the 'free' Hamiltonian, then only those one-particle operators A are well-defined in the free Fock space which satisfy the condition that $[\epsilon, A]$ is Hilbert-Schmidt. For example, the minimal gauge interaction operator does not satisfy this condition when the space-time dimension is higher than 2. The same holds for gauge transformation operators which makes the implementation of these operators somewhat tricky, [M2].

The one-particle time evolution operator can be constructed for example by the Dyson expansion provided that the potential is smooth and appropriate boundary conditions are satisfied. However, the time evolution cannot be quantized because of the remarks above. The asymptotic scattering operator S is better behaving.

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One can show that it satisfies the Hilbert-Schmidt condition. The existing proofs are rather involved, [P,R2]. In this paper we shall give a much simpler proof using the methods introduced earlier for the construction of the quantum gauge transformations and computation of commutator anomalies, [M2]. The method is based on the observation that the interaction Hamiltonians can be conjugated by unitary operators such that the resulting equivalent Hamiltonians satisfy the Hilbert-Schmidt condition with respect to fixed free Hamiltonian. Moreover, we give an effective method for an actual construction of such unitary conjugations, as a function of (time dependent) background fields. This method is very general and does not use the specific properties of gauge interactions. In general, it applies to any bounded interactions such that its commutator with the absolute value of the free Hamiltonian does not have worse fall-off properties in the momentum space than the original operator. Gravitational background fields can be also treated using a somewhat modified form of the conjugation (Appendix A).

In sections 3 and 4 we discuss the determination of the phase of the quantum scattering operator. It is shown that the phase is uniquely determined by causality (section 3), or, alternatively, by the geometric structure of the central extension of the group of one-particle (renormalized) time evolution operators (section 4). Our treatment relies heavily on the theory of infinite-dimensional linear groups. Some of the basic aspects of the theory of these groups in quantum field theory are recalled on the way; for further reading we recommend [CR] and [GV].

2. EXISTENCE OF QUANTUM SCATTERING OPERATORS

Consider a family of Hamiltonians of the form $H_A(t) = D_0 + A(t)$ acting in a one-particle Hilbert space H where $t \mapsto A(t)$ is a smooth and compactly supported ($t, t' \in \mathbb{R}$ here and in the following). We assume that D_0 is a self-adjoint operator and $D_0 + A(t)$ is essentially self-adjoint in the same domain for all t ; the $A(t)$ are bounded self-adjoint operators. We study the time evolution equation

$$(2.1) \quad i\partial_t U_A(t, t') = H_A(t)U_A(t, t'), \quad U_A(t, t) = 1.$$

Writing $V_A(t, t') = e^{itD_0}U_A(t, t')e^{-it'D_0}$ we obtain an equivalent equation

$$(2.2) \quad i\partial_t V_A(t, t') = h_A(t)V_A(t, t'), \quad V_A(t, t) = 1$$

where $h_A(t) = e^{itD_0}A(t)e^{-itD_0}$. Since $h_A(t)$ is bounded, this equation has a solution for all finite times given by the Dyson expansion

$$(2.3) \quad V_A(t, t') = \sum_{n=0}^{\infty} V_n(t, t'), \quad V_0(t, t') = 1, \quad V_{n+1}(t, t') = -i \int_{t'}^t ds h_A(s)V_n(s, t')$$

(it is easy to see that this series converges absolutely in the operator norm, see Appendix B).

Let $\epsilon = D_0/|D_0|$. (This is well-defined even if zero is in the spectrum of D_0 if we set $x/|x| = 1$ and -1 for $x \geq 0$ and $x < 0$, respectively, and use the spectral theorem of self-adjoint operators [RS].)

The spectral decomposition $H = H_+ \oplus H_-$ corresponding to the splitting of the spectrum of D_0 to positive and negative part fixes an irreducible representation of the canonical anticommutation relations (CAR), uniquely defined up to unitary equivalence, in a Fock space \mathcal{F} with a vacuum $|0\rangle$ which is annihilated by the elements $a^*(v_-)$ and $a(v_+)$, $v_\pm \in H_\pm$, of the CAR algebra

$$(2.4) \quad a^*(v)a(v') + a(v')a^*(v) = (v, v'),$$

and all the other anticommutators are equal to zero. Let $\{e_n\}_{n \in \mathbb{Z}}$ be an orthonormal basis in H such that $\{e_n\}_{n \geq 0}$ span H_+ and $\{e_n\}_{n < 0}$ span H_- . Set $a_n = a(e_n)$ and $a_n^* = a^*(e_n)$. Fix the usual normal ordering for the products of creation and annihilation operators by: $a_n^* a_m := -a_m a_n^*$ if $n = m < 0$ and all the other products remain unchanged.

It is known that a bounded one-particle operator $X = (X_{nm})$ can be canonically quantized as

$$(2.5) \quad d\Gamma(X) = \sum X_{nm} : a_n^* a_m :$$

iff $[\epsilon, X]$ is Hilbert-Schmidt, [SS,A,CR]. This quantization is such that $[d\Gamma(X), a^*(v)] = a^*(Xv)$ for all $v \in H$, and preserves the commutation relations of the Lie algebra of linear operators on H except for a complex valued cocycle ('Schwinger term'), see section 3. Similarly, a unitary operator U on H can be second quantized to an operator $\Gamma(U)$ obeying $\Gamma(U)a^*(v)\Gamma(U)^{-1} = a^*(Uv)$ if and only if $[\epsilon, U]$ is Hilbert-Schmidt [SS,R1].

If we have a time evolution with $[\epsilon, A(t)]$ Hilbert Schmidt for all t , then it is easy to see that $[\epsilon, V_A(t, t')]$ is always Hilbert Schmidt (see Appendix B), and this trivially implies that the scattering operator

$$(2.6) \quad S_A = \lim_{t_f \rightarrow \infty} \lim_{t_i \rightarrow -\infty} V_A(t_f, t_i).$$

can be second quantized (note that due to our compactness assumption, $S_A = V_A(T, -T)$ for some $T < \infty$).

In many interesting situations $[\epsilon, A(t)]$ is not Hilbert-Schmidt, and $[\epsilon, V_A(t, t')]$ is not Hilbert-Schmidt either, and the canonical quantum operator $\Gamma(V_A(t, t'))$ does therefore not exist. Nevertheless the scattering operator can be still second quantized in many such cases. We will proof below a general, abstract result for this. As a motivation for our abstract setting, we first discuss a special case. We shall use some basic facts about pseudodifferential operators (PSDO) [H]; see Appendix C for notation.

We assume that spacetime is $M^n \times \mathbb{R}$ where M^n is a n -dimensional compact manifold with spin structure, and $H = L^2(M^n) \otimes V$ where V is a vector space carrying the spin and color indices of the fermions. (Actually, the following discussion applies also to noncompact situations like $M = \mathbb{R}^n$ but then one has to assume suitable fall-off properties of the interaction as $|x| \rightarrow \infty$. For example, in the case of a gauge interaction the requirement that the vector potential and all its derivatives fall off faster than $|x|^{-n/2}$ as $|x| \rightarrow \infty$ would be sufficient.) Moreover, the free hamiltonian D_0 is a self-adjoint PSDO of order ≥ 1 .

We denote as B_2 the ideal of Hilbert Schmidt operators in the algebra of bounded operators on H . In case zero is in the spectrum of D_0 we interpret D_0^{-1} as $D_0(D_0^2 + \lambda)^{-2}$ for some $\lambda > 0$, and similarly for $|D_0|^{-1}$. We use this simplified notation since the precise value of λ is irrelevant (the essential regularizations concern the ultraviolet and λ is a harmless infrared regulator). If not evident to the reader already at this point, this will become clear in the following.

Denote as O_{-k} the PSDOs of order $\leq -k$. We assume that the free hamiltonian D_0 satisfies $[|D_0|, a] \in O_{-k}$ for $a \in O_{-k}$ and for each k ; this is the case for example when D_0 is a Dirac operator. The algebra of bounded PSDO's is equal to O_0 . We state the basic properties of PSDO's, on a compact manifold M or in \mathbb{R}^n with the asymptotic conditions discussed above, which we shall need in the proof of the main theorem:

$$\begin{aligned}
(2.7) \quad & O_{-\ell} \subset O_{-k} \quad \forall \ell > k & (i) \\
& \forall a \in O_{-k} \text{ and } b \in O_0 : \quad ab, ba \in O_{-k} & (ii) \\
& \forall a \in O_{-k} : \quad [|D_0|, a] \text{ and } |D_0|^{\pm 1} a |D_0|^{\mp 1} \in O_{-k} & (iii) \\
& \forall a \in O_{-k} : \quad |D_0|^{-1} a \text{ and } a |D_0|^{-1} \in O_{-k-1} & (iv) \\
& \exists p < \infty : O_{-p} \subset B_2. & (v)
\end{aligned}$$

In fact, we shall prove a result implying that S_A can be quantized whenever there are operator families $\{O_{-k}\}_{k=0,1,\dots}$ with the properties (2.7) and $A \in O_0$. The idea is to construct a time dependent family of operators $T(A) = T_t(A)$ for a regularization at the one particle level [M2] i.e. consider the modified time evolution $T_t(A)U_A(t, t')T_{t'}(A)^{-1}$ which can be second quantized even if $U_A(t, t')$ cannot. It is easy to see that the latter is generated by the Hamiltonian $H'_A(t) = T_t(A)H_A(t)T_t(A)^{-1} + i(\partial_t T_t(A))T_t(A)^{-1} = D_0 + A'(t)$ where

$$(2.8) \quad A'(t) = i(\partial_t T_t(A))T_t(A)^{-1} - [D_0, T_t(A)]T_t(A)^{-1} + T_t(A)A(t)T_t(A)^{-1}.$$

Our strategy thus is to choose $T(A)$ in such a way that A' is better behaved than the original interaction A , i.e. that $[\epsilon, A'] \in B_2$. Note that a conjugation $T_t(A)$ which becomes the identity as $|t| \rightarrow \infty$ does not alter the scattering matrix, $S_{A'} = S_A$.

All differentiations of operators in the following are meant with respect to the operator norm.

Definition 2.9. *Let D_0 be a self-adjoint operator such that there are operator families $\{O_{-k}\}_{k=0,1,\dots}$ with the properties (2.7). We call an interaction A **regular** (w.r.t. to D_0 and $\{O_{-k}\}_{k=0,1,\dots}$) if $A(t)$ and the derivatives $(\partial_t)^k[\epsilon, A(t)]$, $k = 1 \dots p$, are in O_0 for all $t \in \mathbb{R}$. We denote the set of all such interactions as \mathcal{A} .*

Theorem 2.10. *For all interactions $A(t) \in \mathcal{A}$, there is a family of unitary operators $T_t(A)$ differentiable in t and such that the transformed time evolution $V_{A'}(t, t')$, A' eq. (2.8), can be second quantized, $[\epsilon, V_{A'}(t, t')] \in B_2$ for all $t, t' \in \mathbb{R}$. Moreover, $T(A)$ can be chosen local in time, i.e. $T_t(A) = 1$ if $A(t) = 0$ and $(\partial_t)^k A(t) = 0$ for $k = 1 \dots p$.*

Corollary 2.11. *For all interactions $A(t) \in \mathcal{A}$ compactly supported in t , the scattering operator S_A exists and can be second quantized, $[\epsilon, S_A] \in B_2$.*

Before proving this theorem, we give one other example of such operator families $\{O_{-k}\}_{k=0,1,\dots}$ with the properties (2.7). It is easy to see that it generalizes the PSDO setting for the Dirac operator D_0 on a compact spin manifold M^n discussed above: Let D_0 be such that for some $\lambda \in \mathbb{R}$, $(D_0 + \lambda)^{-1}$ exists and is in the Schatten class $B_p = \{a \in B \mid (a^*a)^{p/2} \text{ trace class}\}$ for some $p < \infty$. Let O_0 be all bounded operators a such that $(D_0 + \lambda)^{-\ell} a (D_0 + \lambda)^\ell$ is bounded for all integers ℓ . One can then check that the operator families $O_{-k} = \{a \in O_0 \mid |a| |D_0|^k \text{ is bounded}\}$, $k = 0, 1, \dots$, satisfy (2.7).

Proof of Theorem. We say a map $a : \mathbb{R} \rightarrow O_{-k}$ is C^r if it is r times differentiable with all derivatives $\partial_t^\ell a$, $\ell = 1, 2, \dots, r$, continuous maps $\mathbb{R} \rightarrow O_{-k}$. We first prove the following key lemma providing the recipe for constructing $T(A)$:

Lemma 2.12. *Let $A : \mathbb{R} \rightarrow O_0$ such that $[\epsilon, A] : \mathbb{R} \rightarrow O_{-k}$ be C^r with $r \geq 1$. Then $A'(t)$, defined by (2.8), with the unitary operator*

$$(2.13) \quad T_t(A) = e^{\alpha(t)}, \quad \alpha(t) = -\frac{1}{8} (|D_0|^{-1}[\epsilon, A(t)] + [\epsilon, A(t)]|D_0|^{-1})$$

defines a map $A' : \mathbb{R} \rightarrow O_0$ such that $[\epsilon, A']$ maps \mathbb{R} into O_{-k-1} and is C^{r-1} .

Proof of Lemma. We write $A'(t) = A'_1(t) + A'_2(t)$ where

$$A'_1 = A + [D_0, \alpha]$$

is the leading terms in an expansion in powers of $|D_0|$, and

$$A'_2 = -iT(A)^{-1} \partial_t(T(A) - 1) + T(A)^{-1} [D_0, T(A) - \alpha - 1] + T(A)^{-1} [A, T(A) - 1]$$

is the rest. In the following we refer to maps $a : \mathbb{R} \rightarrow O_{-k}$ also as $a \in O_{-k}$ etc.

Since obviously α and $\partial_t(\alpha)$ are in O_{-k-1} , $A'_2 \in O_{-k-1}$ trivially follows from $\alpha D_0, D_0 \alpha \in O_{-k}$ and

$$T(A) - 1 = \alpha T_1 = T_1 \alpha, \quad T(A) - 1 - \alpha = \alpha^2 T_2 = T_2 \alpha^2$$

where $T_{1,2}$ and $\partial_t(T_1)$ all are in O_0 .

The nontrivial part thus is to show that $[\epsilon, A'_1] \in O_{-k-1}$. This can be seen by the following calculation,

$$\begin{aligned} [\epsilon, A'_1] &= [\epsilon, A] - \frac{1}{8} [\epsilon, [D_0, |D_0|^{-1}[\epsilon, A(t)] + [\epsilon, A]|D_0|^{-1}]] \\ &= \frac{1}{8} (8[\epsilon, A] - [\epsilon, \epsilon[\epsilon, A] - |D_0|^{-1}[\epsilon, A(t)]|D_0| \epsilon + \epsilon |D_0| [\epsilon, A(t)]|D_0|^{-1} - [\epsilon, A(t)]\epsilon]) \\ &= \frac{1}{8} (8[\epsilon, A] - 4[\epsilon, A] - 2|D_0|[\epsilon, A]|D_0|^{-1} - 2|D_0|^{-1}[\epsilon, A]|D_0|) \\ &= \frac{1}{4} (|D_0|^{-1} [|D_0|, [\epsilon, A]] - [|D_0|, [\epsilon, A]] |D_0|^{-1}) \end{aligned}$$

where we used $[\epsilon, A] = -\epsilon[\epsilon, A]\epsilon$ and $\epsilon D_0^{\pm 1} = D_0^{\pm 1} \epsilon = |D_0^{\pm 1}|$. Thus

$$[\epsilon, A'_1] = \frac{1}{4} [|D_0|^{-1}, [|D_0|, [\epsilon, A]]]$$

which is in O_{-k-1} by definition. If we replace D_0^{-1} by $D_0(D_0^2 + \lambda)^{-1}$ a similar calculation leads to the same conclusion,

$$[\epsilon, A'_1] = \frac{1}{4} [|D_0|(D_0^2 + \lambda)^{-1}, [|D_0|, [\epsilon, A]]] + \frac{1}{2} \lambda ((D_0^2 + \lambda)^{-1}[\epsilon, A] + [\epsilon, A](D_0^2 + \lambda)^{-1}).$$

This proves our Lemma.

We can apply this method successively: Starting from some interaction $A_0 = A$ such that $[\epsilon, A] \in O_0$ we get a new interaction $A_1 = A'$ using the conjugation $T(A)$, with $[\epsilon, A_1] \in O_{-1}$. We can then insert A_1 as an argument to $T(\cdot)$ and obtain an unitary operator $T(A_1)$. This defines again a new interaction $A_2 = A'_1$ such that $[\epsilon, A_2] \in O_{-2}$. Continuing this way we obtain, after p steps, an unitary operator $T^{(p)}(A) = T(A_{p-1}) \dots T(A_0)$ such that the time evolution for the operator $T_t^{(p)}(A)U(t, t')T_{t'}^{(p)}(A)^{-1}$ is determined by an interaction A_p such that $[\epsilon, A_p(t)] \in O_{-p}$ for all t . For sufficiently big p the new interaction satisfies the Hilbert-Schmidt condition, and thus the corresponding scattering operator can be second quantized. Since $T_t^{(p)}(A)$ by construction is equal to the identity for times t where $A(t)$ and all its t -derivatives vanish, the latter scattering operator is equal to S_A . This implies theorem 2.10.

Remark 1. As a particular case, our result gives the existence of the scattering operators for Dirac (or Weyl) fermions in external Yang-Mills fields, on a compact space manifold M or on \mathbb{R}^n with sufficient fall-off properties for the vector potential as $|x| \rightarrow \infty$. Here our discussion above implies $p > n/2$, but for n odd one can show that actually $p = (n - 1)/2$ is already sufficient (e.g. for $n = 1$ no regularization is necessary).

Remark 2. We stress the Hilbert-Schmidt property of the scattering operator since only this is of primary interest for quantum field theory. However, our the argument above shows that usually $[\epsilon, S_A]$ is much better behaved: e.g. in the fermion-Yang-Mills case it is in *all* Schatten classes B_q for $q > 0$. (This follows from $O_{-k} \subset B_{\frac{n+1}{k}}$.)

3. PHASE OF QUANTUM SCATTERING OPERATOR: CAUSAL APPROACH

In the previous section we have shown that the one-particle scattering operator S satisfies the Hilbert-Schmidt condition for $[\epsilon, S]$ and therefore it can be promoted to an unitary operator $\hat{S} = \Gamma(S)$ in the Fock space \mathcal{F} . However, by this the operator \hat{S} is uniquely defined only up to a phase. In this section we show that the regularization for the time evolution operators in the previous section fixes the phase in a natural causal manner.

We denote the group of unitary operators U on H with $[\epsilon, U]$ Hilbert-Schmidt as U_1 . All $U \in U_1$ can be second quantized, and the second quantization $\Gamma(U) = \Gamma(U^{-1})^{-1}$ of $U \in U_1$ is unique up to a phase (= element in $U(1)$) which implies that for some local (near the unit element in U_1 , e.g.) choice of phases

$$(3.1) \quad \Gamma(U)\Gamma(V) = \chi(U, V)\Gamma(UV) \quad \forall U, V \in U_1,$$

where $\chi : U_1 \times U_1 \rightarrow U(1)$ is only defined locally (a derivation of an explicit, locally valid formula of χ is given in [PS]; in the second quantization setting see also [L1]). The latter is a nontrivial local 2-cocycle providing a central extension \hat{U}_1 of U_1 by $U(1)$. Similarly the (complexification of the) Lie algebra \mathfrak{u}_1 of U_1 contains all bounded operators X on H with $[\epsilon, X]$ Hilbert-Schmidt, and its second quantization $\mathfrak{u}_1 \ni X \rightarrow d\Gamma(X) = d\Gamma(X^*)^*$ gives a representation of a central extension $\hat{\mathfrak{u}}_1 = \mathfrak{u}_1 \oplus \mathbb{C}$ of \mathfrak{u}_1 ,

$$(3.2) \quad [d\Gamma(X), d\Gamma(Y)] = d\Gamma([X, Y]) + c_L(X, Y),$$

with a Lie algebra 2-cocycle [Lu]

$$(3.3) \quad c_L(X, Y) = \frac{1}{4} \text{tr} \epsilon[\epsilon, X][\epsilon, Y]$$

which is the infinitesimal version of the Lie group 2-cocycle χ above. It is possible to choose phases such that

$$(3.4) \quad \Gamma(e^{iX}) = e^{id\Gamma(X)} \quad \forall X = X^* \text{ close to } 0 \in \mathfrak{u}_1$$

(the existence of the $e^{id\Gamma(X)}$ as a unitary operator follows from Stone's theorem [RS] since $d\Gamma(X)$ is self-adjoint [CR]). This equation actually is true for all $X \in \mathfrak{u}_1$, but it fixes the phase of $\Gamma(U)$ for only for $U \in U_1$ sufficiently close to the identity where local bijectivity of the exponential mapping is guaranteed. We will assume this phase convention in the following. Then (3.2) implies

$$(3.5) \quad \Gamma(e^{-i\delta s X} e^{-i\delta t Y} e^{i\delta s X}) = e^{\delta s \delta t c_L(X, Y)} \Gamma(e^{-i\delta s X}) \Gamma(e^{-i\delta t Y}) \Gamma(e^{i\delta s X}) + \mathcal{O}(\delta s^2, \delta t^2)$$

for all $X, Y \in \mathfrak{u}_1$ and sufficiently small $\delta s, \delta t \in \mathbb{R}$ (to see this, use (3.4) and expand both sides of this equation in powers of δs and δt).

We now consider a time evolution $V_A(t, t') = V(t, t')$ defined in eq. (2.2) with $h_A = h : \mathbb{R} \rightarrow \mathfrak{u}_1$ smooth and compactly supported. We first consider the simple case where $h(t) \in \mathfrak{u}_1$ so that $V(t, t') \in U_1$ for all $t, t' \in \mathbb{R}$. As shown in the last section, many interesting cases can be brought to this simplest situation using the conjugation by a family of operators $T(A)$ (we will discuss this in more detail further below).

We first note the essential group property of the time evolution,

$$(3.6) \quad V(t, t')V(t', t'') = V(t, t'') \quad \forall t, t', t'' \in \mathbb{R},$$

which follows from (2.2); it is this what we mean by *causality*. Somewhat parallel to our discussion, the use of the causality condition in the renormalization of a quantum field theory has been stressed by Scharf and his coworkers, [S].

To construct the second quantization of the scattering operator $S = S_A$ (2.6) including the phase, we first second quantize the time evolution. The naive guess $\Gamma(V(t, t'))$ for this is not right since this is not a time evolution: it does not obey equations similar to (3.6) due to the Schwinger term χ in (3.1) which gives nontrivial contributions in general. One can, however, define $\widehat{V}(t, t') = \lim_{N \rightarrow \infty} \widehat{V}^{(N)}(t, t')$ with

$$(3.7) \quad \widehat{V}^{(N)}(t, t') = \prod_{N \geq \nu \geq 1} \Gamma(V(t_\nu, t_{\nu-1})), \quad t_\nu = t' + \frac{(t-t')\nu}{N}$$

where $\prod_{N \geq \nu \geq 1} F(t_\nu, t_{\nu-1})$ is the ordered product $F(t_N, t_{N-1})F(t_{N-1}, t_{N-2}) \cdots F(t_2, t_1)$ for any operator valued function F on $\mathbb{R} \times \mathbb{R}$. This is a time evolution by construction, and with (3.1)

$$(3.8a) \quad \widehat{V}(t, t') = \eta(t, t')\Gamma(V(t, t'))$$

where η a phase valued function on $\mathbb{R} \times \mathbb{R}$ which can be explicitly computed in terms of χ (for $V(t, t')$ in some neighborhood of the identity) [L1]. This allows to calculate the scattering operator $\hat{S} = \widehat{V}(T, -T)$ including phase as follows (here and in the following we assume that T is big enough so that $h(t)$ vanishes for $|t| > T/2$, say): choose some partition $t_0 = -T < t_1 < \cdots < t_n = T$ of the time interval $[-T, T]$ such that all $V(t_i, t_{i-1})$ are in the neighborhood of the identity for which $\eta(t, t')$ is defined. Then

$$(3.8b) \quad \hat{S} = \prod_{n \geq i \geq 1} \eta(t_i, t_{i-1})\Gamma(V(t_i, t_{i-1}))$$

can be shown to be independent of which particular partition is chosen.

Remark 1. We note our formulas (3.8a,b) still do not fix the phase of \hat{S} completely since the function $\eta(t, t')$ is unique only up to

$$(3.9) \quad \eta(t, t') \mapsto \exp\left(-i \int_{t'}^t d\bar{t} E(\bar{t})\right) \eta(t, t')$$

with E a smooth real-valued functions on \mathbb{R} . This is due to the ambiguity of the second quantization map $\mathbf{u}_1 \ni X \rightarrow d\Gamma(X)$ which can be changed by smooth, linear functions $b : \mathbf{u}_1 \rightarrow \mathbb{C}$ with $b(X^*) = b(X)^*$ and $b(0) = 0$. A shift $d\Gamma(X) \rightarrow d\Gamma(X) + b(X)$ changes (3.3) by a trivial 2-cocycle, $c_L(X, Y) \rightarrow c_L(X, Y) - b([X, Y])$, and this implies (3.9) with $E(t) = b(h(t))$.

Remark 2. Since $h(t) \in \mathbf{u}_1$ for all t , the second quantized Hamiltonian $\hat{h}(t) = d\Gamma(h(t))$ (in the interaction picture) always exists, and it should be the generator of the second quantized time evolution $\widehat{V}(t, t')$. Moreover, the ambiguity (3.9) of the phase of $\widehat{V}(t, t')$ corresponds to a shift $\hat{h}(t) \rightarrow \hat{h}(t) + E(t)$ which physically amounts to a change of the zero-point energy. It would be difficult to construct $\widehat{V}(t, t')$ directly from $\hat{h}(t)$ since the latter is unbounded which makes the existence

of a Dyson series nontrivial. This technical problem is avoided in our approach above.

In the following we are interested in the change of the second quantized time evolution operator under transformations

$$(3.10) \quad V(t, t') \mapsto (g.V)(t, t') \equiv g(t)V(t, t')g(t')^{-1}$$

where $g : \mathbb{R} \mapsto U_1$ where $g(t)$ is assumed to be sufficiently smooth and such that $g(t) = 1$ for $|t| > T/2$. We will derive an explicit formula for the gauge anomaly of the time evolution,

$$(3.11) \quad \lambda(t, g) \equiv \Gamma(g(t))^{-1} \widehat{(g.V)}(t, -T) \widehat{V}(-T, t),$$

which is a phase factor according to our discussion above (since the r.h.s. is the second quantization of $g(t)^{-1}(g.V)(t, -T)V(-T, t)$ equal to the identity). Especially, $\lambda(g) \equiv \lambda(T, g)$ is the change of the quantum scattering operator \hat{S} under the transformation g .

We first consider only infinitesimal gauge transformations $g(t) = e^{-i\delta s X(t)}$ for $\delta s \rightarrow 0$. We calculate $\lambda(t, g)$ as $\lim_{N \rightarrow \infty} \lambda^{(N)}$ where

$$\lambda^{(N)} = \Gamma(g(t))^{-1} \left\{ \prod_{N \geq \nu \geq 1} \Gamma(g(t_\nu)V(t_\nu, t_{\nu-1})g(t_{\nu-1})^{-1}) \right\} \left\{ \prod_{1 \leq \nu \leq N} \Gamma(V(t_{\nu-1}, t_\nu)) \right\}$$

with $t_\nu = -T + (t + T)\nu/N$. Now (3.1) implies

$$\Gamma(g(t + \delta t)V(t + \delta t, t)g(t)^{-1}) = \lambda^{(t+\delta t, t)}(g)\Gamma(g(t + \delta t))\Gamma(V(t + \delta t, t))\Gamma(g(t))^{-1}$$

for some phase factors $\lambda^{(t+\delta t, t)}$, and we explicitly see that the various factors $\Gamma(g(t_\nu))$ and $\Gamma(g(t_{\nu-1}))^{-1}$ cancel each other leaving only phase factors. Using $V(t + \delta t, t) \simeq e^{-i\delta t h(t)}$, $g(t + \delta t) \simeq e^{-i\delta s X(t)}$ and (3.5), we get $\lambda^{(t+\delta t, t)}(g) \simeq e^{\delta s \delta t c_L(X(t), h(t))}$ (' \simeq ' means 'equal up to irrelevant higher order terms in δs and δt '). Thus $\lambda^{(N)}$ is just the exponent of a Riemann sum, and in the limit $N \rightarrow \infty$

$$\lambda(t, e^{-i\delta s X}) = \exp\left(\delta s \int_{-T}^t d\bar{t} c_L(X(\bar{t}), h(\bar{t}))\right) + \mathcal{O}(\delta s^2).$$

We now consider the case of finite gauge transformations $g(t)$ and introduce a homotopy $g_s(t)$, $0 \leq s \leq 1$, smoothly deforming it to the identity,

$$(3.12) \quad g_1(t) = g(t) \text{ and } g_0(t) = 1 \quad \forall t, \quad g_s(t) = 1 \quad \text{for } |t| > T/2.$$

To be specific, we first restrict ourselves to gauge transformations $g(t) = e^{-iX(t)}$ with $X(t) \in \mathbf{u}_1$ for all t , and $g_s(t) = g(t) = e^{-isX(t)}$. We define $V_s(t, t') \equiv (g_s.V)(t, t')$ and

$$\lambda_{s, s'} \equiv \Gamma(g_s(t))^{-1} \widehat{V}_s(t, -T) \widehat{V}_{s'}(-T, t) \Gamma(g_{s'}(t))$$

so that $\lambda(g) = \lambda_{1,0}$. We observe that these phases have the group property, $\lambda_{s,s'}\lambda_{s',s''} = \lambda_{s,s''}$ for all $0 \leq s, s', s'' \leq 1$, thus we can evaluate $\lambda(g)$ as $\lim_{M \rightarrow \infty} \lambda_M$ where

$$\lambda_M = \prod_{M \geq \mu \geq 1} \lambda_{s_\mu, s_{\mu-1}}, \quad s_\mu = \frac{\mu}{M}.$$

Now $\lambda_{s+\delta s, s}$ is the change of phase of $\widehat{V}_s(T, -T)$ under an infinitesimal gauge transformation $g_{s+\delta s}(t)g_s(t)^{-1} \simeq e^{-i\delta s X_s(t)}$ and thus equal to $\exp(\delta s \int_{-T}^t d\bar{t} c_L(X_s(\bar{t}), h_s(\bar{t})))$ with

(3.13)

$$h_s(t) = i \{ \partial_t V_s(t) \} V_s(t)^{-1}, \quad X_s(t) = i \{ \partial_s V_s(t) \} V_s(t)^{-1}, \quad V_s(t) = g_s(t) V(t, -T)$$

(we used $g_s(-T) = 1$). Again λ_M becomes the exponential of a Riemann sum, and in the limit $M \rightarrow \infty$ we obtain

Theorem 3.14.

$$\lambda(t, g) = \exp \left(\int_0^1 ds \int_{-T}^t d\bar{t} c_L(X_s(\bar{t}), h_s(\bar{t})) \right).$$

Note that this result was derived for the special homotopy $g_s(t) = e^{-isX(t)}$, but our derivation can be immediately generalized to arbitrary gauge transformations $g(t)$ and homotopies $g_s(t)$ (sufficiently smooth in s and t) obeying (3.12). For $t \geq T$, $\lambda(t, g) = \lambda(g)$ (3.14) is then actually independent of the homotopy chosen (this follows from its definition (3.11) which does not depend on the homotopy). For intermediate times $-T < t < T$ this is not true. The reason is that then the phase of the implementors $\Gamma(g(t))$ in (3.11) depends on the homotopy: our derivation above implies that this phase has to be chosen such that

$$\Gamma(g(t)) = \lim_{M \rightarrow \infty} \prod_{M \geq \mu \geq 1} \Gamma(g_{s_\mu}(t)g_{s_{\mu-1}}(t)^{-1}), \quad s_\mu = \frac{\mu}{M},$$

and this coincides with our phase convention (3.4) only for homotopies $s \mapsto g_s(t) = e^{-isX(t)}$.

Remark 3. Our derivation of (3.14) above was given for 1-parameter groups in U_1 for simplicity, but the result immediately generalizes to GL_1 which is the group of all (not only unitary) invertible operators U on H with $[\epsilon, U]$ Hilbert Schmidt: eq. (3.14) remains true for $h(t) \in \mathbf{u}_1$ not self-adjoint and $g(t) \in GL_1$. The technical problem for proving this more general result by the method above is that $e^{-i\delta t d\Gamma(X)}$ is unbounded if $d\Gamma(X)$ is not self-adjoint, thus one has to be careful with the domains of operators (the latter could, however, be handled by methods described in [GL]). Our alternative derivation of (3.14) in the next section is for GL_1 and bypasses such domain questions.

We consider now time evolutions generated by Hamiltonians $H_A(t) = D_0 + A(t)$ with $A(t) \in \mathcal{A}$ (cf. definition 2.9) and generalized gauge transformations

$$(3.15) \quad A(t) \rightarrow g.A(t) = i(\partial_t g(t))g(t)^{-1} - [D_0, g(t)]g(t)^{-1} + g(t)A(t)g(t)^{-1}$$

so that $U_{g.A}(t, t') = g(t)U_A(t, t')g(t')^{-1}$. We denote the group of all $g(t)$ which leave \mathcal{A} invariant as \mathcal{G} . Note that \mathcal{G} contains all $g(t)$ sufficiently smooth in t (i.e. C^{p+1}), which are unitary operators in O_0 for all t . We also introduce the Lie algebra $Lie\mathcal{G}$ of \mathcal{G} . In the following, all A are in \mathcal{A} , all g, g', g'' in \mathcal{G} , and all $X, Y, Z \in Lie\mathcal{G}$, except when stated otherwise. As before, we assume all these functions are trivial for $|t| > T/2$.

By theorem 2.10, there exist appropriate regularization operators $T(A)$ and $T(g.A)$ such that A' and $(g.A)'$, defined in (2.8), all lead to time evolutions which can be second quantized i.e. they are always in \mathbf{u}_1 . This also implies that the operators $T_t(g.A)g(t)T_t(A)^{-1} = U_{g.A}(t, -T)U_A(-T, t)$ all are in U_1 , and thus

$$(3.16) \quad \Gamma_t(g; A) \equiv \Gamma(g'_A(t)) , \quad g'_A(t) \equiv e^{itD_0} T_t(g.A) g(t) T_t(A)^{-1} e^{-itD_0}$$

always exist. These unitary operators have the natural interpretation as implementors of the generalized gauge transformations g at fixed time t . They are local in time i.e. only depend on g, A and t -derivatives thereof, at time t . We observe that they obey the relations

$$(3.17a) \quad \Gamma(g'; g.A) \Gamma(g; A) = \chi(g', g; A) \Gamma(g'g; A).$$

where we have dropped the common time argument t , with

$$(3.17b) \quad \chi_t(g', g; A) = \chi((g')'_{g.A}(t), g'_A(t))$$

defined locally (this follows from (3.1)). Note that (3.17a) and associativity of the operator product imply the 2-cocycle relation

$$(3.18) \quad \chi(g'', g'g; A) \chi(g', g; A) = \chi(g'', g'; g.A) \chi(g''g', g; A).$$

Our construction above can now be used to calculate

$$(3.19) \quad \lambda(t, g; A) \equiv \Gamma_t(g; A)^{-1} \widehat{V}_{(g.A)'}(t, -T) \widehat{V}_{A'}(-T, t)$$

which we define as the change of the quantum time evolution $\widehat{V}_{A'}$ under the generalized gauge transformation g . We immediately get the formula

$$(3.20a) \quad \lambda(t, g; A) = \exp \left(\int_0^1 ds \int_{-T}^t d\bar{t} c_L (\{\partial_s V_s(\bar{t})\} V_s(\bar{t})^{-1}, \{\partial_t V_s(\bar{t})\} V_s(\bar{t})^{-1}) \right)$$

with

$$(3.20b) \quad V_s(t) = e^{itD_0} T_t(g_s.A) g_s(t) U_A(t, -T) e^{iT D_0}$$

and $g_s(t) \in \mathcal{G}$ a homotopy interpolating between 1 and $g(t)$. Similarly as discussed above after theorem 3.14, for $t \geq T$ (but in general not for intermediate times $-T < t < T$) this formula is independent of the homotopy $s \mapsto g_s(t)$ chosen.

We observe that these phases are connected with the Schwinger terms in (3.17b) via the relation

$$(3.21) \quad \lambda(t, g'; g.A) \lambda(t, g; A) \chi_t(g', g; A) = \lambda(t, g'g; A)$$

(this follows from a simple calculations using the definition (3.19) and $(g'g).A = g'.(g.A)$),

$$\begin{aligned} & \Gamma_t(g'; g.A) \lambda(t, g'; g.A) \Gamma_t(g; A) \lambda(t, g; A) \\ &= \widehat{V}_{(g'g.A)'}(t, -T) \widehat{V}_{(g.A)'}(-T, t) \widehat{V}_{(g.A)'}(t, -T) \widehat{V}_{A'}(-T, t) \\ &= \Gamma_t(g'g; A) \lambda(t, g'g; A), \end{aligned}$$

and inserting (3.17a)). According to our derivation, this equation is valid only locally (i.e. $g(t)$ and $g'(t)$ close to identity).

Especially for $t = T$, $\lambda(g; A) \equiv \lambda(T, g; A)$ is equal to the change of the quantum scattering matrix \hat{S}_A under the transformation g , and equation (3.21) reduces to the 1-cocycle relation, $\lambda(g'; g.A) \lambda(g; A) = \lambda(g'g; A)$ (since $\chi(1, 1; 0) = 1$). The physical meaning of $\lambda(g; A)$ is as follows. We recall that the log of the vacuum expectation value of \hat{S}_A is equal to the Minkowskian action of the fermions in the time dependent external field A , thus $\log \lambda(g; A)$ is the change of the latter by the generalized time dependent gauge transformation $g(t)$. Especially for infinitesimal transformations $g(t) = 1 - i\delta s X(t) + \dots$ it gives the generalized gauge anomaly $Anom(X; A) = \left. \frac{d}{ds} \log \lambda(e^{-isX}; A) \right|_{s=0}$. We obtain

$$(3.22) \quad Anom(X; A) = \int_{-T}^T dt \bar{\omega}^1(X(t), A(t)), \quad \bar{\omega}^1(X(t), A(t)) = c_L(X'_A(t), h_{A'}(t))$$

where $h_{A'}(t) = e^{itD_0} A'(t) e^{-itD_0}$ with A' given in eq. (2.8), and

$$(3.23) \quad X'_A(t) \equiv e^{itD_0} (\{\mathcal{L}_X T_t(A)\} T_t(A)^{-1} + T_t(A) X(t) T_t(A)^{-1}) e^{-itD_0}$$

is in \mathbf{u}_1 for all $A \in \mathcal{A}$ and $X \in Lie\mathcal{G}$; we introduced the Lie derivative acting on functionals f on \mathcal{A} as

$$\mathcal{L}_X f(A) = \left. i \frac{d}{ds} f(e^{-isX} A) \right|_{s=0}.$$

Similarly, the infinitesimal version of (3.17a,b) is [M2]

$$(3.24a) \quad [G(X; A), G(Y; A)] = G([X, Y]; A) + S(X, Y; A)$$

where $G(X; A) = \mathcal{L}_X + d\Gamma(X'_A)$ are implementors of infinitesimal gauge transformations and

$$(3.24b) \quad S_t(X, Y; A) = c_L(X'_A(t), Y'_A(t))$$

a Schwinger term satisfying the 2-cocycle relation $\mathcal{L}_X S(Y, Z; A) + S(X, [Y, Z]; A) + cycl. = 0$ (the latter is the infinitesimal version of (3.18) and also follows from the Jacobi identity).

Especially, if we consider the Yang-Mills case and infinitesimal chiral gauge transformations, $Anom(X; A)$ is just the axial gauge anomaly and $S(X, Y; A)$ the Schwinger term appearing in the commutators of the chiral Gauss' law generators $G(X; A)$. We thus have obtained a Hamiltonian derivation of these two different manifestations of the gauge anomaly in a Hamiltonian framework which traces them back to a common root, i.e. the 2-cocycle c_L in (3.2).

It is interesting to consider also the infinitesimal version of the equation (3.21) which can be written as

$$(3.25) \quad \delta \bar{\omega}^1 + \partial_t S = 0$$

where

$$(\delta \omega^1)(X, Y; A) = \mathcal{L}_X \omega^1(Y; A) - \mathcal{L}_Y \omega^1(X; A) - \omega^1([X, Y]; A)$$

is defined on functions ω^1 on $Lie\mathcal{G} \times \mathcal{A}$. To interpret this equation, we recall that the above mentioned fermion-Yang-Mills anomalies are connected by descent equations [Z]: the axial anomaly on a $n + 1$ (even) dimensional space-time manifold M^{n+1} is the integral of a $(n + 1)$ - (de Rham) form $\omega_{n+1}^1(X; A)$ over M^{n+1} ; it depends on one infinitesimal gauge transformations X and the Yang-Mills field A . The corresponding Schwinger term is on n dimensional space M^n and an integral of a n -form $\omega_n^2(X, Y; A)$ over M^n depending on two infinitesimal gauge transformations X, Y and A . Embedding M^n in M^{n+1} , the descent equations are $\delta \omega_{n+1}^1 + d\omega_n^2 = 0$ where δ is defined as above and n is the usual exterior differentiation of de Rham forms. Setting $M^{n+1} = M^n \times \mathbb{R}$ and $\bar{\omega}^1 = \int_{M^n} \omega_{n+1}^1$ and $S = \int_{M^n} \omega_n^2$, one exactly obtains our equation (3.25). We thus have obtained an explicit field theory derivation of this descent equation for all odd dimensions n in the Hamiltonian framework. We stress, however, that our eq. (3.25) is not restricted to the Yang-Mills case but in fact is much more general.

Remark 4. As just mentioned, fermion-Yang-Mills anomalies are local de Rham forms, whereas our formulas (3.22) for the axial anomaly and (3.24b) for the Schwinger term are not explicitly local in space. In the Yang-Mills case one can prove, however, that they cohomologous to local de Rham forms. General arguments and mathematical techniques for showing this by explicit calculations have been given recently, [M2,LM,M4]. Nevertheless it would be interesting to explicitly do this latter calculations for all dimensions. In this paper we will only sketch the simplest case $n = 1$ (end of next section).

In the next Section we will give a different, more geometric approach to the phase of the scattering operator where the path independence of the anomaly becomes evident. Another important benefit in the geometric approach is that we can easily compute the cohomology class of the anomaly without going to the details of the renormalization $T(A)$.

4. THE QUANTUM PHASE AND PARALLEL TRANSPORT

Let \hat{G} be a central extension of a Lie group G by \mathbb{C}^\times . The Lie algebra $\hat{\mathfrak{g}}$ of \hat{G} is a vector space direct sum $\mathfrak{g} \oplus \mathbb{C}$. Let π be the projection on the second summand and let $\theta = \mathfrak{g}^{-1}d\mathfrak{g}$ be the left Maurer-Cartan one-form. We can then define a complex valued one-form ϕ on \hat{G} by $\phi = \pi(\theta)$. This is a connection form in the principal \mathbb{C}^\times bundle $\hat{G} \rightarrow G$. Its curvature is a left invariant two-form on G given by $\omega(X, Y) = c(X, Y)$, where left invariant vector fields X, Y on G are identified as elements of the Lie algebra and c is the 2-cocycle on \mathfrak{g} defining the central extension,

$$(4.1) \quad [(X, \lambda), (Y, \mu)] = ([X, Y], c(X, Y)).$$

Recall that GL_1 is the group of invertible linear transformations $\mathfrak{g} : H \rightarrow H$ such that $[\epsilon, \mathfrak{g}]$ is Hilbert-Schmidt and U_1 its unitary subgroup. Let us apply the above remarks to $G = U_1$, and to the Lie algebra cocycle c_L (3.3) arising when promoting the one-particle operators to operators (2.5) in the fermionic Fock space, as discussed in the last section.

The central extension \widehat{GL}_1 is a nontrivial \mathbb{C}^\times bundle over the base GL_1 , [PS]. The elements of the group \widehat{GL}_1 (containing the unitary subgroup \hat{U}_1) can be thought of equivalence classes of pairs (\mathfrak{g}, q) , where $\mathfrak{g} \in GL_1$ and $q : H_+ \rightarrow H_+$ is an invertible operator such that $a - q$ is a trace-class operator,

$$(4.2) \quad \mathfrak{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We have assumed that $\text{ind } a = 0$. If this is not the case, the subspace H_+ must be either enlarged or made smaller by a suitable finite-dimensional subspace in order to achieve $\text{ind } a = 0$. The equivalence relation is determined by $(\mathfrak{g}, q) \sim (\mathfrak{g}', q')$ if $\mathfrak{g} = \mathfrak{g}'$ and $\det(q'q^{-1}) = 1$. Thus the fiber of the extension is \mathbb{C}^\times and it is parameterized by (the nonexisting) determinant of q .

The product is defined simply $(\mathfrak{g}, q)(\mathfrak{g}', q') = (\mathfrak{g}\mathfrak{g}', qq')$. Near the unit element in G we can define a local section $\mathfrak{g} \mapsto (\mathfrak{g}, a)$, [PS]. Denoting

$$\mathfrak{g}^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

we can write the connection form as

$$(4.3) \quad \phi_{\mathfrak{g}, q} = \text{tr}[(\mathfrak{g}^{-1}d\mathfrak{g})_a - q^{-1}dq] = \text{tr}[\alpha da + \beta dc - q^{-1}dq].$$

The curvature of this connection is

$$(4.4) \quad \omega = \text{tr}(d\beta d\gamma)$$

and is easily checked to agree with c_L in (3.3).

We compute the parallel transport determined by the connection in the range of the local section. Let $\mathfrak{g}(t)$ be a path in GL_1 , $-T \leq t \leq T$, with $\mathfrak{g}(-T) = 1$. The lift $(\mathfrak{g}(t), q(t))$ is parallel if

$$0 = \phi_{\mathfrak{g}(t), q(t)}(d\mathfrak{g}, dq) = \text{tr}[\alpha(t)a'(t) + \beta(t)c'(t) - q(t)^{-1}q'(t)].$$

Thus the parallel transport, relative to the trivialization $\mathfrak{g} \mapsto (\mathfrak{g}, a)$, along the path $\mathfrak{g}(t)$ in the base is accompanied with the multiplication by the complex number

$$(4.5) \quad \exp\left\{-\int_{-T}^T \text{tr}[(\alpha(t) - a(t)^{-1})a'(t) + \beta(t)c'(t)]dt\right\}$$

in the fiber \mathbb{C} .

Formally,

$$\text{tr } q^{-1}q' = \text{tr}[\alpha a' + \beta c']$$

and so

$$\det q(T) = \exp \int_{-T}^T \text{tr}[\alpha(t)a'(t) + \beta(t)c'(t)]dt$$

and also

$$\det a(T) = \exp \int_{-T}^T \text{tr } a(t)^{-1}a'(t)dt.$$

Individually, the traces in these two expressions do not converge, but putted together the trace converges and gives

$$(4.6) \quad \det(a(T)q(T)^{-1}) = \exp\left\{\int_{-T}^T \text{tr}[(\alpha - a^{-1})a' + \beta c']dt\right\}.$$

Note that the exponent diverges outside of the domain of the local section, reflecting the fact that $\det a(T) = 0$ outside of the domain.

We can now apply the above results to the 'renormalized' one-particle time evolution operators $\mathfrak{g}(t) = V_{A'}(t) = e^{itD_0}T_t(A)U_A(t, -T)e^{iT D_0}$. For all times t , these are elements of the group U_1 . On the other hand, in the Fock representation of \widehat{GL}_1 these correspond to elements $\widehat{V}_{A'}(t)$ in the central extension \widehat{U}_1 . The phase of the quantum time evolution operator is then uniquely given by the parallel transport described above.

The Minkowskian effective action is by definition the vacuum expectation value of the quantum scattering operator \hat{S}_A . The vacuum is invariant under the free time evolution $\exp(itD_0)$ and taking into account the assumption that the interaction has essentially compact support in time, we can write

$$(4.7) \quad Z(A) = \langle 0|(V_{A'}(T), q(T))|0 \rangle .$$

The vacuum expectation value is given by a simple formula, [PS], [M3],

$$(4.8) \quad \langle 0|(\mathfrak{g}, q)|0 \rangle = \det(aq^{-1})$$

and therefore the parallel transport (4.5) (with respect to the given local trivialization) is equal the effective action $Z(A)$.

The above formalism can be applied for computing the gauge anomaly in the space-time formalism starting from the commutator anomaly (3.3). Let $g(t) \in \mathcal{G}$

be a time-dependent gauge transformation such that at $t = \pm T$ it is equal to the identity. The change in the phase of the effective action is now

$$\lambda(g; A) = \exp \left(\int_{\gamma} \phi \right)$$

where γ is the closed loop in U_1 obtained by first following backwards in time from T to $-T$ the time evolution $U_{A'}(t)$, following then the gauge transformed time evolution operators $g(t)U_{A'}(t)$ back from $-T$ to T . The parallel transport around a closed loop can be written as an integral of the curvature ω over a surface S enclosed by the loop γ . By construction, the gauge anomaly λ satisfies the 1-cocycle condition $\lambda(gg'; A) = \lambda(g; g'.A)\lambda(g'; A)$.

Joining $g(t)$ to the identity by a homotopy $g_s(t)$, $0 \leq s \leq 1$, and writing $V_s(t) = g_s(t)U_{A'}(t)$ we get

$$(4.9) \quad \log \lambda(g; A) = \int_S c_L(\partial_t V V^{-1}, \partial_s V V^{-1}) = \frac{1}{4} \int \text{tr} \epsilon[\epsilon, \partial_t V V^{-1}][\epsilon, \partial_s V V^{-1}].$$

This result agrees with (3.14). For infinitesimal gauge transformations $g_s(t) \simeq 1 - isX(t) + \dots$ we get axial anomaly (3.22) as discussed in the last Section.

Let us complete the calculation for 1+1 spacetime dimensions in the case of chiral fermions in external Yang-Mills field. Now the chiral Hamiltonian on the circle S^1 acting on one-component spinors is $H(t) = -i\frac{\partial}{\partial x} - A_+$, where $A_+ = A_0 + A_1$. We now use that for $n = 1$ one can choose $T_t(A) = 1$ independent of A (see our remark 1 at the end of section 2). Thus, applying (3.22) derived either from (3.20) or (4.9), we get

$$Anom(X; A) = \frac{1}{4} \int_{-T}^T dt \text{tr} \epsilon[\epsilon, A_+][\epsilon, X(t)] = \frac{1}{2\pi i} \int_{-T}^T dt \int_{S^1} dx \text{tr} A_+(t, x) \frac{\partial}{\partial x} X(t, x).$$

Here we have used the general formula

$$(4.10) \quad \frac{1}{4} \text{tr} \epsilon[\epsilon, X][\epsilon, Y] = \frac{1}{2\pi i} \int_{S^1} dx \text{tr} X \partial_x Y$$

valid for smooth multiplication operators X, Y on the unit circle. Up to a coboundary (= a gauge variation of the local functional $\propto \int \text{tr} A_+ A_1$) this form of the anomaly is equal to the standard form of the two-dimensional chiral anomaly

$$(4.11) \quad Anom_s(X; A) = \frac{1}{4\pi i} \int_{S^1 \times \mathbb{R}} \text{tr} AdX.$$

We finally note that this same equation also allows to calculate the Schwinger term (3.24b),

$$(4.12) \quad S(X, Y; A) = \frac{1}{2\pi i} \int_{S^1} \text{tr} X dY$$

which actually is independent of A . This is the Kac-Moody cocycle and also the Schwinger term related to the axial anomaly (4.11) via the descent equations, as discussed at the end of the last section.

The cohomology class of the anomaly in dimensions $n + 1 > 2$

The group GL_p consists of all bounded invertible operators (4.2) in $H = H_+ \oplus H_-$ such that the off-diagonal blocks b, c are in the Schatten ideal B_{2p} . For any $p \geq 1$ the group GL_p contracts to the subgroup GL_1 , [Pa]. On the other hand, in GL_1 one can produce cohomologically equivalent cocycles $c_p \sim c_L$ such that c_p extends from GL_1 to GL_p . These are relevant for understanding the gauge group action in space-time dimension $n + 1 > 2$. The static gauge transformations are elements of GL_p for $p > n/2$. For example, when $n = 3$ the gauge group $\mathcal{G}_n = \text{Map}(M^n, G) \subset GL_2$ and one has, [MR],

$$(4.13) \quad c_2(X, Y; f) = \frac{1}{8} \text{tr} [\epsilon, f] f^{-1} [[\epsilon, X], [\epsilon, Y]],$$

where $c_2(X, Y; f)$ is the value of a 2-form on GL_2 at a point f to the directions of the left invariant vector fields (= Lie algebra elements) X, Y . This formula has been generalized for arbitrary p , [FT, L4].

In order to fix the cohomology class of the one-cocycle $\lambda(\mathfrak{g}; A)$ it is sufficient to look how λ winds around the circle when a family $f(t, s)$ of time dependent gauge transformations wraps around a closed surface S (parameterized by s, t) in the group \mathcal{G}_n of static gauge transformations. This follows from the fact that the cohomology class of any two-form is determined by giving its integral over all closed two-cycles. The winding number is given by the integral of the curvature c_L around the surface S in GL_1 defined by the family of gauge transformed renormalized evolution operators.

For any fixed potential A and a homotopy $f(t, s)$ of time dependent gauge transformations we have a map $S \rightarrow GL_p$ given by $(t, s) \mapsto f(t, s)U(t)$, where $U(t)$ is the nonrenormalized time evolution operator determined by A . The renormalization $T(A)$ does not change the homology class of the surface S in $GL_1 \subset GL_p$ since T is defined over a contractible parameter space. It follows that the integral over a *closed* surface S of the curvature on GL_1 is given by the integral of c_p of the nonrenormalized operators $f(t, s)U(t)$. Furthermore, the surface $(t, s) \mapsto f(t, s)U(t)$ contracts to $(t, s) \mapsto f(t, s)$. This follows from the fact that each component of U_p is simply connected and so $(t, s) \mapsto U(t)$ is contractible. Therefore, the final result for the anomaly around a closed surface is

$$\int_{f(t,s)} c_p.$$

In the case $M = S^1$ ($p = 1$) this gives

$$\begin{aligned} \frac{1}{4} \int_S \text{tr} \epsilon [\epsilon, (\partial_t f) f^{-1}] [\epsilon, (\partial_s f) f^{-1}] &= \frac{1}{2\pi i} \int_{S \times S^1} \text{tr} (\partial_t f) f^{-1} \partial_x ((\partial_s f) f^{-1}) \\ &= \frac{i}{12\pi} \int_{S \times S^1} \text{tr} (f^{-1} df)^3, \end{aligned}$$

where we have used (4.10) in the first step and performed integration by parts in the second step. In dimension $n = 3$ we insert $X = i(\partial_t f)f^{-1}$ and $Y = i(\partial_s f)f^{-1}$ into (4.13). A similar calculation gives, using a three dimensional equivalent of (4.10), [L2],

$$\frac{1}{240\pi^2} \int_{S \times M^3} \text{tr}(f^{-1}df)^5$$

and so on in higher dimensions. This agrees with the integral of the two form over S in a gauge orbit obtained by the descent equations. For example, in three space dimensions the form is

$$(4.14) \quad \frac{-i}{24\pi^2} \int_{M^3} \text{tr}dA(XdY - YdX)$$

which gives the commutator anomaly in three space dimensions, [M1, FS].

Conclusions: In section 2 we gave a new proof for the existence of the second quantized fermionic scattering operator in external Yang-Mills fields. The proof is valid also in a more abstract setting of generalized gauge interactions in the spirit of Connes' noncommutative geometry. In section 3 we derived a formula for the phase of the scattering operator and its gauge variation from the concept of causality by using the local 2-cocycle on the group GL_1 . In section 4 we gave an alternative geometric derivation using a connection on the global group extension \widehat{GL}_1 . A constructive interpretation for the descent equations was given in the hamiltonian framework, linking the anomaly of the Minkowskian effective action to the Schwinger terms. This is complementary to the standard approach which starts from the euclidean functional determinant.

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APPENDIX: THE CASE OF EXTERNAL METRIC

Let $g = (g_{ij}(x, t))$ be a time-dependent metric tensor in \mathbb{R}^n . We assume that space and time has been foliated by a choice of the time coordinate such that the space coordinates x_1, \dots, x_n are orthogonal with respect to the time t , i.e. $g_{0i} = g_{i0} = 0$ for $1 \leq i \leq n$. The Dirac equation is written as

$$(A1) \quad ig_{00}\partial_t\psi = \gamma^k h_{kj}(i\partial_j + \Gamma_j)\psi \simeq g_{00}D_h\psi$$

where $h_{kj}(x)$ are the components of an oriented orthonormal basis in (\mathbb{R}^n, g) ,

$$(A2) \quad h_{kj}h_{mj} = g_{km}.$$

The matrices Γ_j are the components of the spin connection (defined by the Levi-Civita connection of g), taking values in the Lie algebra of the spin group $Spin(n)$.

We assume that the deviation of the metric g from the euclidean metric has only compact support in space and time. Furthermore, $g(x, t)$ is assumed to be smooth. If the dimension $n = 2N + 1$ then the g 's are $2^N \times 2^N$ complex matrices with the property

$$(A3) \quad \gamma^i \gamma^j + \gamma^j \gamma^i = 2\delta_{ij}.$$

The Lie algebra of $Spin(n)$ is spanned by the commutators $[\gamma_i, \gamma_j]$. If $n = 3$ the γ -matrices are just the 2×2 Pauli matrices which are also the generators of the Spin group $Spin(3) = SU(2)$.

The principal symbol of the Dirac Hamiltonian is $\gamma^k h_{kj} p_j$. The complete symbol is the sum of the principal symbol and of a symbol of order zero in the momenta.

Because for any given pair q, p of nonzero vectors there is a rotation R such that $q = Rp$, there exists an element $B(p, x) \in Spin(n)$ such that

$$(A4) \quad B \gamma^k p_k B^* = \lambda(x, p) \gamma^k h_{kj} p_j.$$

Here the product is a matrix product, no momentum space differentiation is involved, and the scale factor λ is the ratio of the euclidean lengths of the vectors $p = \gamma^k p_k$ and $q = \gamma^k h_{kj} p_j$. Both λ and B are homogeneous functions of order zero in momenta.

At the first sight it appears that it is not possible to construct B as a continuous function of the h field, the apparent obstruction being the hairy ball theorem: For a given direction q one can always choose a rotation R_q such that $R_q \cdot p = q$, but R_q is not a continuous function of q when n is odd and at least equal to 3. However, here we can profit from the information encoded in the matrix h .

The set of all orthogonal transformations which takes p to $q = h \cdot p$ (up to a scale) form a fiber $P_{p,q}$ in a principal bundle P with base $X = GL_+(n, \mathbb{R}) \times S^{n-1}$, consisting of the pairs $(h, p/|p|)$, and the fiber is isomorphic with $SO(n-1)$; the '+' refers to matrices with positive determinant. The base contracts to $X' = SO(n) \times S^{n-1}$ (by the Cartan decomposition). On the other hand, over X' the bundle P is trivial, the trivialization being given by $(h, p) \mapsto h$. Thus P is trivial. We choose a trivialization $(h, p) \mapsto R(h, p)$. We choose $B(h, p) \in Spin(n)$ which projects down to $R \in SO(n)$. There is a \mathbb{Z}_2 ambiguity in the choice which does not bother us since the transformation law for the Dirac operator is quadratic in B .

If we compute the left-hand side in (A4) with the complete star product instead of the matrix product, we generate symbols of order less than or equal to zero. Thus we have proven the following lemma:

Lemma. *There is a function $B(h)$ of the basis h taking values in the group of invertible PSDO's of order zero such that $B^* \gamma^k h_{kj} p_j B$ differs from $\lambda(x, p)^{-1} \gamma^k p_k$ by an operator of order zero.*

The unitarily equivalent Hamiltonian $B' = B^* D_h B$ has then the property that $[\epsilon, B^* D_h B]$ is a PSDO of order zero. One can now apply the recursive method in section 2 to obtain the renormalization operator $T = T(A)$ where now $A = B' - D_0$

and $D_0 = \gamma^k p_k$. This method applies as well to the case of combined background gauge and gravitational interactions.

APPENDIX B: ESTIMATES ON DYSON SERIES

We consider here the Dyson series (2.3) solving the time evolution equation (2.2) with $h_A(t) = e^{itD_0} A(t) e^{-itD_0}$. If $A(t)$ is bounded for all t , one can easily prove by induction that

$$\|V_n(t, t')\| \leq \frac{1}{n!} \left(\int_{t'}^t dr \|A(r)\| \right)^n$$

which shows that (2.3) converges in the operator norm $\|\cdot\|$ for all $t, t' \in \mathbb{R}$.

Similarly, if $[\epsilon, A(t)]$ is Hilbert Schmidt for all t , then the Hilbert Schmidt norm of V_n can be estimated as

$$\|[\epsilon, V_n(t, t')]\|_2 \leq \int_{t'}^t dr \|[\epsilon, A(r)]\|_2 \frac{1}{(n-1)!} \left(\int_{t'}^t dr \|A(r)\| \right)^{n-1}$$

showing that $[\epsilon, V(t, t')]$ is also Hilbert Schmidt.

APPENDIX C: PSEUDODIFFERENTIAL OPERATORS (PSDO)

To fix our notation we summarize here the basic definitions and facts about PSDOs [H]. A PSDO A on the Hilbert space $L^2(M^n) \otimes V$, M^n a smooth manifold and V a finite dimensional vector space, is given locally by its symbol $a(x, p) = \sigma(A)(x, p)$ which is a smooth matrix- ($\text{gl}(V, V)$ -) valued function of the local coordinates $x \in U \subset \mathbb{R}^n$ and momenta $p \in \mathbb{R}^n$, [H]. The action of A on a section ψ with support in U is given as

$$(C1) \quad (A\psi)(x) = \frac{1}{(2\pi)^{n/2}} \int a(x, p) \hat{\psi}(p) e^{-ip \cdot x} dp$$

where $\hat{\psi}$ is the Fourier transform of the function $\psi : U \rightarrow V$,

$$\hat{\psi}(p) = \frac{1}{(2\pi)^{n/2}} \int e^{ix \cdot p} \psi(x) dx.$$

We shall consider *the restricted class* of PSDO's which admit an asymptotic expansion of the symbol as

$$a(x, p) \sim a_k(x, p) + a_{k-1}(x, p) + a_{k-2}(x, p) + \dots$$

where k is an integer and each a_j is a homogeneous matrix valued function of the momenta, of order j , with $|a_j| \sim |p|^j$ as $\sqrt{p_1^2 + \dots + p_n^2} = |p| \rightarrow \infty$. The order of such a PSDO is $\text{orda} = k$.

The asymptotic expansion for the product of two PSDO's is given by the formula

$$(C2) \quad a * b \sim \sum \frac{(-i)^{|m|}}{m!} [(\partial_p)^m a(x, p)] [(\partial_x)^m b(x, p)],$$

where the sum is over all sets of nonnegative integers $m = (m_1, \dots, m_n =)$, $|m| = m_1 + \dots + m_n$, $\partial_x^m = (\frac{\partial}{\partial x_1})^{m_1} \dots (\frac{\partial}{\partial x_n})^{m_n}$, etc., and $m! = m_1! \dots m_n!$.

The order of $a * b$ is equal to the sum of $\text{ord} a + \text{ord} b$ since the leading term in $a * b$ is just the matrix product ab of the symbols.

The symbol of a massless Dirac operator D_A in an external vector potential A is $\gamma^k(p_k + A_k)$ where $\gamma_i \gamma_j + \gamma_j \gamma_i = g_{ij}$ are the Dirac gamma matrices and $g = (g_{ij})$ is the metric tensor. The symbol for the square D_A^2 is $p^2 +$ lower order terms in p and therefore the symbol of $|D_A|$ is $|p| +$ lower order terms. From this follows, using (A2), that the symbol of $[|D_A|, B]$ is $\frac{p_k}{|p|} \frac{\partial}{\partial x_k} b(x, p) +$ terms of order $\text{ord} B$ for any PSDO B with symbol b . In particular, the order of $[|D_A|, B]$ is at most equal to the order of B .

On a compact manifold of dimension n a PSDO is trace class if its order is strictly less than $-n$ and it is Hilbert-Schmidt if the order is $< -n/2$. In \mathbb{R}^n one has to assume in addition that the symbol is either compactly supported in x or at least the asymptotic behavior of the symbol and its derivatives at $|x| \rightarrow \infty$ is as $|x|^{-k}$, where $k > n$ in case of trace class operators and $k > n/2$ for Hilbert-Schmidt operators. In \mathbb{R}^n the trace (when it exists) of a PSDO is simply given as

$$\text{tr} A = \frac{1}{(2\pi)^n} \int \text{tr} a(x, p) dp dx.$$

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