The classical r-matrix method for nonlinear sigma-model

Alexey Sevostyanov
Steklov Mathematical Institute, St. Petersburg

September 8, 1995

Abstract

The canonical Poisson structure of nonlinear sigma-model is presented as a Lie-Poisson r-matrix bracket on coadjoint orbits. It is shown that the Poisson structure of this model is determined by some 'hidden singularities' of the Lax matrix.

Introduction

Two-dimensional nonlinear sigma models were studied for almost 20 years. Their full treatment in quantum case proved to be much more difficult than for other related relativistic models, e.g. the Sine Gordon equation. Surprisingly, a consistent r-matrix formulation of these models (which is a neccessary prerequisite of the study of the quantum case) was lacking, although a Lax pair for chiral model was found by Zakharov and Mikhailov many years ago. The purpose of this note is to explain the origin of the Zakharov-Mikhailov Lax pair and its generalizations and of the associated Poisson structures in the r-matrix language. As it appears, the relevant r-matrices are non-unitary; they belong to the hierarchy associated with the standard rational r-matrix. Our method also applies to nonlinear sigma models with values in a riemannian symmetric space; it represents a first step towards a solution of the corresponding quantum problem. (Although the particle spectrum and the corresponding factorized scattering matrices have been guessed many years ago by Zamolodchikov and later Faddeev and Reshetikhin [5] were able to reproduce this result using an infinite spin limit in an appropriate lattice model, a systematic treatment of chiral models on the lines of the Quantum Inverse Scattering Method still seems to be lacking.) It should be noted that the non-unitarity of the r-matrix poses additional problems in our approach. The crucial point is to find consistent Poisson bracket relations for the monodromy matrix of the Lax operator. The anomalies in these relations are directly related to the non-unitarity of the r-matrix, and although their regualrization in some cases is possible, for chiral models this technique fails.

In this paper we shall discuss sigma models with values in a semisimple Lie group (following a suggestion of L.D.Faddeev we shall call them principal chiral models), as well as sigma models with values in Riemannian symmetric spaces.

I am grateful to L.D.Faddeev, F.A.Smirnov, and A.Yu.Alekseev for helpful discussions.

I would like to thank M.A.Semenov-Tian-Shansky who pointed out to me the elegant adèlic construction of the rational r-matrices and of the related linear hierarchies of Poisson brackets [12], [9].

1 Hamiltonian formulation of chiral fields.

We remind some standard facts on the canonical Poisson structures on cotangent bundles of Lie groups [6]. Let G be a Lie group, $\mathbf{g}$ its Lie algebra, and $\mathbf{g}^*$ the dual space of $\mathbf{g}$. The cotangent bundle T^*G admits two canonical trivializations by left and right translations, respectively. If $L_g : h \longmapsto gh$ is a left translation on $\widehat{G}$, then

$$dL_g^* : T_g^*G \to T_e^*G \simeq \mathbf{g}^*.$$

Similarly, if $R_g : h \longmapsto hg$ is a right translation, then

$$dR_g^* : T_g^*G \to T_e^*G \simeq \mathbf{g}^*.$$

Thus if $(g, \xi_g) \in T^*G, \xi_g \in T_g^*G$, then

$$
\begin{gathered}
(g,\xi_g) \overset{L}{\mapsto} (g,l_t)\,, l_t = dL_g^*\xi_g,\\
(g,\xi_g) \overset{R}{\mapsto} (g,r_t)\,, r_t = -dR_g^*\xi_g,\\
T^*G \overset{L,R}{\to} G\times \mathbf{g}^*
\end{gathered}
\tag{1}
$$

are the left and right trivializations. Clearly, we have

$$
r_t = -Ad^*g\,(l_t)\,. \tag{2}
$$

The minus sign in (1), (2) reflects the fact that the Lie algebras of left- and right-invariant vector fields on G are anti-isomorphic. Let $\widehat{\mathbf{g}} = C^\infty(S^1,\mathbf{g})$ be the current algebra and $\widehat{G}$ the corresponding current group. Using the trivializations of T^*G described above the cotangent bundle $T^*\,\widehat{G}$ may be identified with $\widehat{G}\times\widehat{\mathbf{g}}^*$. The canonical Poisson bracket on $T^*\,\widehat{G}$ [1], [7] is described as follows:

Every functional on $T^*\,\widehat{G}$ may be represented as a function of two variables $g\in\widehat{G}$ and $l_t\in\widehat{\mathbf{g}}^*$ (or $r_t\in\widehat{\mathbf{g}}^*$, depending on the trivialization chosen) . The derivatives with respect to these variables are defined by:

$$
\begin{gathered}
\langle \mathrm{D}'\varphi\,(g,l_t)\,,X\rangle = \tfrac{d}{dt}\,|_{t=0}\,\varphi\left(ge^{tX},l_t\right),\\
\langle X_\varphi\,(g,l_t)\,,Y\rangle = \tfrac{d}{dt}\,|_{t=0}\,\varphi\,(g,l_t+tY)\,,\\
\varphi\in Fun\left(T^*\widehat{G}\right),\mathrm{D}'\varphi,Y\in\widehat{\mathbf{g}}^*;X_\varphi,X\in\widehat{\mathbf{g}},
\end{gathered}
\tag{3}
$$

where $\langle\cdot,\cdot\rangle$ is the natural pairing between $\widehat{\mathbf{g}}$ and $\widehat{\mathbf{g}}^*$. In our case we may identify $\widehat{\mathbf{g}}$ and $\widehat{\mathbf{g}}^*$ using an invariant scalar product on $\widehat{\mathbf{g}}$:

$$
\langle X\,(x)\,,Y\,(x)\rangle = \int_0^{2\pi} tr\,(X\,(x)\,Y\,(x))\,dx, \tag{4}
$$

where tr is an invariant bilinear form on $\mathbf{g}$.

Now we are ready to define the canonical Poisson bracket on $T^*\widehat{G}$:

$$
\{\varphi,\psi\}\,(g,l_t) = \langle\mathrm{D}'\psi,X_\varphi\rangle - \langle\mathrm{D}'\varphi,X_\psi\rangle + \langle l_t,[X_\varphi,X_\psi]\rangle\,. \tag{5}
$$

In tensor notations we have the following formulas for the brackets of matrix elements of g and l_t :

$$
\begin{gathered}
\{l_t\,(x)_1\,,l_t\,(y)_2\} = \tfrac{1}{2}\,[t,l_t\,(x)_1 - l_t\,(y)_2]\,\delta\,(x-y)\,,\\
\{g_1\,(x)\,,l_t\,(y)_2\} = -g_1\,(x)\,t\delta\,(x-y)\,,\\
\{g_1\,(x)\,,g_2\,(y)\} = 0,
\end{gathered}
\tag{6}
$$

where $l_t(x)_1 = l_t(x) \otimes I, l_t(x)_2 = I \otimes l_t(x)$, and t is the tensor Casimir in $\mathbf{g} \otimes \mathbf{g}$.

Our sign convention in formula (1) makes the mapping (2) which converts left trivialization into right trivialization a Poisson mapping.

The following series of results is connected with the Hamiltonian reduction on $T^*\widehat{G}$ [1], [2]. The natural actions of $\widehat{G}$ on itself by left and right translations may be canonically lifted to $T^*\widehat{G}$; if we use the trivialization of $T^*\widehat{G}$ by left translations the lifted actions are given by

$$\begin{gathered} \widehat{G} \times T^*\widehat{G} \xrightarrow{L,R} T^*\widehat{G}, \\ L_h(g, l_t) = (hg, l_t), \\ R_h(g, l_t) = (gh, Ad^*h^{-1}(l_t)) \end{gathered} \tag{7}$$

These actions are Hamiltonian; the Hamiltonians which correspond to $X \in \widehat{\mathbf{g}}$ are given by

$$\begin{gathered} H_X^L(g, l_t) = -\langle X, Ad^*g(l_t)\rangle, \\ H_H^R = \langle X, l_t\rangle, \\ X \in \widehat{\mathbf{g}}. \end{gathered} \tag{8}$$

The corresponding moment maps are

$$\begin{gathered} \mu_L = -Ad^*g(l_t) = r_t, \\ \mu_R(g, l_t) = l_t. \end{gathered} \tag{9}$$

Let θ_L, θ_R be the left-invariant (respectively, the right-invariant) Maurer-Cartan form on G. The left-invariant (right-invariant) current associated with $g \in \hat{G}$ is defined by

$$l_x dx = g^*\theta_L, r_x dx = -g^*\theta_R$$

If G is a matrix group we have simply

$$l_x(x) = g^{-1}\partial_x g(x), r_x(x) = -\partial_x g(x) g^{-1}.$$

Remark. The notation we use suggests that (l_t, l_x) and (r_t, r_x) are two components of a single two-dimensional Noether current (left-invariant or right-invariant, respectively). This notation will be motivated later.

The currents l_x, r_x parametrize the quotient space obtained by reduction of $T^*\widehat{G}$ over the action of the subgroup G of constant loops by left (respectively, right) translations. The moment map which coresponds to this action is given by

$$\mu_L^G(g, l_t) = \int_0^{2\pi} r_t(x)\, dx. \tag{10}$$

The quotient space $G \backslash T^*\widehat{G}$ may be identified with $\widehat{\mathbf{g}} \times \widehat{\mathbf{g}}^*$ via the mapping:

$$(g, l_t) \mapsto (l_x, l_t)$$

The coordinates l_x, l_t have the following Poisson brackets in the tensor notations:

$$\begin{gathered}
\{l_t(x)_1, l_x(y)_2\} = \tfrac{1}{2}[t, l_x(x)_1 - l_x(y)_2]\,\delta(x-y) - \\
-t\delta'(x-y), \\
\{l_t(x)_1, l_t(y)_2\} = \tfrac{1}{2}[t, l_t(x)_1 - l_t(y)_2]\,\delta(x-y), \\
\{l_x(x)_1, l_x(y)_2\} = 0.
\end{gathered} \tag{11}$$

This Poisson structure is slightly degenerate and the space $G \setminus T^*\widehat{G}$ decomposes into symplectic leaves; among these latter there is one which corresponds to the zero moment. This leaf is the reduced phase space over the point $\mu_L^G(g, l_t) = 0$ [1] ,[2] .We denote this symplectic manifold by $\mathcal{M}$.

Now we define the principal chiral field as a Hamiltonian system on this symplectic manifold with the Hamiltonian function:

$$H = \frac{1}{2}\int_0^{2\pi} tr\,(l_x l_x + l_t l_t)\, dx. \tag{12}$$

The equations of motion have the form:

$$\begin{gathered}
\partial_t l_\mu = \{H, l_\mu\}, \\
\partial_x l_x = \partial_t l_t, \\
\partial_x l_t - \partial_t l_x + [l_x, l_t] = 0.
\end{gathered} \tag{13}$$

The last equation is the zero curvature condition which serves to restore the group variable $g(x) \in \widehat{G}$ using the variables l_x, l_t modulo the left action of constant loops.

To justify the notation introduced above let us observe that the action functional which corresponds to our choice of the Hamiltonian is given by

$$S(g) = \frac{1}{2}\int tr\,(l_x l_x - l_t l_t)\, dx dt. \tag{14}$$

where $l_t = g^{-1}\partial_t g, l_x = g^{-1}\partial_x g$; clearly, the Legendre transform associated with (14) identifies l_t, l_x with our canonical variables and hence our notation is consistent. The same remark applies, of course, to right currents $r_t = -\partial_t g g^{-1}, r_x = -\partial_x g g^{-1}$, since both the Hamiltonian (12) and the action functional (14) are Ad-invariant.

We turn to the study of chiral fields with values in symmetric spaces. Our exposition follows [4]; we add some details on the canonical formalism.

Let $\mathbf{g} = \mathbf{k} \dot{+} \mathbf{p}$ be a Cartan decomposition [3], i.e.

$$[\mathbf{k},\mathbf{k}] \subset \mathbf{k}, [\mathbf{k},\mathbf{p}] \subset \mathbf{p}, [\mathbf{p},\mathbf{p}] \subset \mathbf{k}, \tag{15}$$

so that $\mathbf{k}$ is a Lie subalgebra in $\mathbf{g}$. Put $\widehat{\mathbf{k}} = C^\infty(S^1,\mathbf{k})$, and let $\widehat{K}$ be the corresponding subgroup in $\widehat{G}$. There is a Hamiltonian action of $\widehat{K}$ on $T^*\widehat{G}$ which arises from right translations by elements of $\widehat{G}$. Its moment map is given by

$$\mu_R^K(g,l_t) = P_{\mathbf{k}} l_t, \tag{16}$$

where $P_{\mathbf{k}}$ is the orthogonal projection operator onto the subspace $\mathbf{k} \subset \mathbf{g}$

.

Since the left and right actions of $\widehat{G}$ on $T^*\widehat{G}$ commute, the above defined action of $\widehat{K}$ on $T^*\widehat{G}$ generates the action of $\widehat{K}$ on $G\backslash T^*\widehat{G}$ and even on the symplectic submanifold $\mathcal{M}$ in this space with the same moment map (16).We introduce the quotient space over this action $G\backslash T^*\widehat{G}/\widehat{K}$. The quotient Poisson structure on this space is degenerate. The phase space of the chiral field is the symplectic leaf corresponding to the zero values of the left and right moments of the actions of G and $\widehat{K}$ $\mu_L^G(g,l_t) = \mu_R^K(g,l_t) = 0$. This is the reduced phase space over these actions. We denote it by $\mathcal{M}_1$.

Let us choose the following coordinates on the space $G\backslash T^*\widehat{G}$:

$$P_{\mathbf{k}} l_\mu = A_\mu, P_{\mathbf{p}} l_\mu = B_\mu, \mu = t, x. \tag{17}$$

Now we describe the Poisson structure of the phase space. First of all we calculate the Poisson brackets of the variables A_x, B_x, B_t .

Let $t_{A=}(P_{\mathbf{k}} \otimes P_{\mathbf{k}})t, t_{B=}(P_{\mathbf{p}} \otimes P_{\mathbf{p}})t$ be the $\mathbf{k}$ and $\mathbf{p}$ components of the Casimir element of $\mathbf{g}$,. $t = t_A + t_B$. In this realization we have the following Poisson brackets for A_x and B_μ :

$$
\begin{array}{c}
\left\{B_t(x)_1, B_t(y)_2\right\} = 0, \\
\left\{B_t(x)_1, B_x(y)_2\right\} = \frac{1}{2}\left[t_B, A_x(x)_1 - A_x(y)_2\right]\delta(x-y) - \\
-t_B\delta'(x-y), \\
\left\{B_t(x)_1, A_x(y)_2\right\} = \left[t_A, B_x(x)_1\right]\delta(x-y).
\end{array} \tag{18}
$$

These brackets give the Poisson structure on the subspace in $G\backslash T^*\widehat{G}$ (and hence on $\mathcal{M}$) on which $\mu_R^K(g,l_t) = 0$ .The space $\mathcal{M}_1$ carries the quotient Poisson structure of the Poisson structure (18) under the right action of the group $\widehat{K}$ on $\mathcal{M}$.

The chiral field is a Hamiltonian system on the phase space $\mathcal{M}_1$described above with the Hamiltonian function:

$$
H = \frac{1}{2}\int_0^{2\pi} tr\left(B_x B_x + B_t B_t\right)dx, \tag{19}
$$

This function is invariant under the right action $\widehat{K}$ on $\mathcal{M}$ so that it is a well defined function on the phase space. The equations of motion have the form:

$$
\begin{array}{c}
\partial_t B_t = \partial_x B_x + \left[A_x, B_x\right], \\
-\partial_t A_x + \left[B_x, B_t\right] = 0, \\
\partial_x B_t - \partial_t B_x + \left[A_x, B_t\right] = 0.
\end{array} \tag{20}
$$

As above, the two last equations are zero curvature conditions which serve to restore the field variable $g(x) \in \widehat{G}/\widehat{K}$ given the currents A_x and B_μ (this correspondence is unique modulo constant loops).

Note that the functions A, B are coordinate functions on the space $G\backslash T^*\widehat{G}$ which is larger than the phase space $\mathcal{M}_1$; the genuine observables for the reduced system are gauge invariant functionals of A, B. However it is convenient to look at the evolution on this larger space as well; as we shall see it admits a Lax representation.

As well as for the principal chiral field one can calculate the Lagrangian function given the Hamiltonian function and the Poisson structure of the phase space. This calculations gives the well known result:

$$
S(g) = \frac{1}{2}\int tr\left(B_x B_x - B_t B_t\right)dxdt. \tag{21}
$$

This shows that A_x, B_x, B_t are the components of the current and S is a well defined function on $\widehat{G}/\widehat{K}$.

If $\mathbf{g} = su(2)$ and $\mathbf{k} = \mathbf{h}$ is the Cartan subalgebra of $su(2)$ our description of the nonlinear sigma-model is equivalent to the formulation given in [6]. This formulation uses the space $T^*\widehat{G}/\widehat{K}$ as the phase space of our model. In this realization the coordinates on $T^*\widehat{G}/\widehat{K}$ are:

$$\begin{gathered} n = gXg^{-1}, \pi = -\left[r_t, n\right], \\ \text{here } X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \end{gathered} \tag{22}$$

and it is supposed that $\mu_R^K(g,l) = P_{\mathbf{k}} l_t = 0$. We don't need the explicit form of the equations of motion in this realization nor the formulas expressing the Poisson structure in terms of n and π . The reader may easily restore these formulas using [6]. For instance, we rewrite the action (21) in the terms of n and π :

$$S = \frac{1}{2}\int \left((\partial_x n)^2 - (\pi)^2 \right) dxdt. \tag{23}$$

2 The coadjoint orbits formulation for principal chiral fields.

In this section we shall develope the Lie-algebraic point of view on the Poisson structure of the principal chiral field. Our goal is to propose an r-matrix formulation of the Zakharov-Mikhailov Lax pair [6], [10]. The Lax matrix of Zakharov and Mikhailov is a rational function on CP_1 with two poles at $\lambda = \pm 1$; there is also a 'hidden' singularity at $\lambda = 0$. As we shall see, it is this latter singularity that determines the Poisson structure of the chiral model. By contrast, apparent singularities at $\lambda = \pm 1$ do not influence the Poisson structure; instead each of them produces a series of local conservation laws. A convenient formalism allowing to work with Lax matrices with arbitrary poles uses the algebra of adèles of rational functions; the rational r-matrix is associated with the canonical decomposition of this algebra into complementary subalgebras.

We remind some definitions [9]. Let $\mathbf{g}$ is a semisimple Lie algebra, $\mathbf{g}_-$ the Lie algebra of rational functions on $\overline{\mathbf{C}}$ with values in $\mathbf{g}$. Let λ_ν be the local parameter at $\nu \in \overline{\mathbf{C}}, \lambda_\nu = \lambda - \nu, \nu \in \mathbf{C}, \lambda_\infty = -\frac{1}{\lambda}$. Let $\mathbf{g}_\nu = \mathbf{g}\left(\left(\lambda_\nu\right)\right)$ be the algebra of formal Laurent series with values in $\mathbf{g}$. Put $\mathbf{g}_{\mathcal{A}} = \coprod_{\nu \in \overline{\mathbf{C}}} \mathbf{g}_\nu$. In this formula it is supposed that the sum is taken over the extended complex plane $\overline{\mathbf{C}}$ and for every element of $\mathbf{g}_{\mathcal{A}}$ only a finite number of series in the direct sum are Laurent series and all the others are Taylor series. The algebra $\mathbf{g}_-$ of rational functions with values in $\mathbf{g}$ is embedded in $\mathbf{g}_{\mathcal{A}}$ in the following way:

$$X(\lambda) \rightarrow \bigoplus_{\nu \in \overline{\mathbf{C}}} X(\lambda)_\nu ; \tag{24}$$

here $X(\lambda) \in \mathbf{g}_-$ and $X(\lambda)_\nu$ is the Laurent series of $X(\lambda)$ at the point ν . The subalgebra $\mathbf{g}_-$ is isotropic in $\mathbf{g}_{\mathcal{A}}$ with respect to the invariant scalar product:

$$\begin{gathered}
(X(\lambda), Y(\lambda)) = \sum_{\nu \in \overline{\mathbf{C}}} Res_\nu \; tr X_\nu(\lambda_\nu) Y_\nu(\lambda_\nu) d\lambda, \\
X(\lambda) = \bigoplus_{\nu \in \overline{\mathbf{C}}} X_\nu(\lambda_\nu), Y(\lambda) = \bigoplus_{\nu \in \overline{\mathbf{C}}} Y_\nu(\lambda_\nu).
\end{gathered} \tag{25}$$

For every element $X \in \mathbf{g}_{\mathcal{A}}$ there exists a rational function $P_- X(\lambda)$ such that the principal parts of its Laurent series of at all points coincide with the principal parts of the Laurent series of $X_\nu(\lambda_\nu)$ at the same points. This function is given by the formula:

$$P_- X(\lambda) = \sum_{\nu \in \overline{\mathbf{C}}} \underset{\nu}{Res} \;\; tr \frac{t}{\lambda - \mu} X_\nu(\mu_\nu) d\mu. \tag{26}$$

So there exists a direct decomposition of the linear space $\mathbf{g}_{\mathcal{A}}$:$\mathbf{g}_{\mathcal{A}} = \mathbf{g}_- \dot{+} \mathbf{g}_+$, where $\mathbf{g}_-$ is the algebra of rational functions and $\mathbf{g}_+$ is the subalgebra of $\mathbf{g}_{\mathcal{A}}$ which consists of Taylor series. Evidently, the subalgebra $\mathbf{g}_+$ is isotropic with respect to the scalar product (25) , so that there are natural pairings: $\mathbf{g}_+^* \simeq \mathbf{g}_-, \mathbf{g}_-^* \simeq \mathbf{g}_+$. And if we define the projection operator P_- onto $g(\lambda)$ by the formula (26) and the complementary projection operator $P_+ = I - P_-$, then $P_+^* = P_-$ with respect to the scalar product (25). The rational r-matrix is defined by the standard formula [9], [6]:

$$\begin{gathered}
r = P_+ - P_-, \\
r^* = -r;
\end{gathered} \tag{27}$$

It satisfies the modified classical Yang-Baxter equation. In standard applications the space $\mathbf{g}_-$ is used as a model of the dual space $\mathbf{g}_+^*$; a generic point $L \in \mathbf{g}_-$ is regarded as a Lax matrix and the Lie-Poisson bracket of $\mathbf{g}_+$ provides a 'universal' hamiltonian structure for Lax equations in question. For our purposes we modify this construction in the following way. To identify the space $\mathbf{g}_{\mathcal{A}}$ with $\mathbf{g}_{\mathcal{A}}^*$ we use the modified scalar product on $\mathbf{g}_{\mathcal{A}}$:

$$(X(\lambda), Y(\lambda))_\phi = \sum_{\nu \in \overline{\mathbf{C}}} \underset{\nu}{Res} \;\; trX_\nu(\lambda_\nu) Y_\nu(\lambda_\nu) \phi(\lambda) d\lambda, \tag{28}$$

where $\phi(\lambda)$ is some rational function with numerical values. This scalar product allows to consider a nontrivial model for the space $\mathbf{g}_+^*$. Namely the space $\mathbf{g}_+$ is not isotropic with respect to this scalar product and

$$\mathbf{g}_+^* \simeq \left\{ \phi^{-1}(\lambda) X(\lambda), X(\lambda) \in \mathbf{g}_- \right\}. \tag{29}$$

The rational r-matrix is not skew-symmetric with respect to this scalar product:

$$r^* = -\phi^{-1} r \phi, \tag{30}$$

where ϕ denotes the operator of multiplication by ϕ .We use the scalar product (28) and the r-matrix (27) to define the r-matrix Lie-Poisson bracket for the central extension of the current algebra $\widehat{\mathbf{g}}_{\mathcal{A}} = C^\infty(S^1, \mathbf{g}_{\mathcal{A}})$. It has the form [6],[11],[8]:

$$\begin{gathered} \{\varphi, \psi\}(L) = \int_0^{2\pi} dy \left(L, [rX_\varphi, X_\psi] + [X_\varphi, rX_\psi]\right)_\phi - \\ - \int_0^{2\pi} dy \left((r + r^*)\partial_y X_\varphi, X_\psi\right)_\phi, \\ L \in \widehat{\mathbf{g}}_{\mathcal{A}}^*, \text{and } X_\varphi \text{ is a derivative of } \varphi : \\ \int_0^{2\pi} dy \left(X_\varphi, Y\right)_\phi = \tfrac{d}{dt} \mid_{t=0} \varphi(L + tY), Y \in \widehat{\mathbf{g}}_{\mathcal{A}}^*. \end{gathered} \tag{31}$$

The Jacobi identity for this bracket follows from the Yang-Baxter equation for r . It is well known that this bracket may be restricted to the space $\widehat{\mathbf{g}}_+^*$ and symplectic leaves of this bracket are coadjoint orbits of the algebra $\widehat{\mathbf{g}}_+$ in the space $\widehat{\mathbf{g}}_+^*$. The Lax operators of integrable models lie in this space. In our realization they have the form (29) with respect to the spectral parameter λ . It is evident from the definition (31) that the Poisson structure of such models will be connected only with the poles of $X(\lambda)$ in (29). while local conservation laws are connected with the asymptotic expansion of the monodromy matrix in the neighbourhood of the poles of the Lax operator and hence they depend on the poles of the function $\phi^{-1}(\lambda)X(\lambda)$ in (29).

In the tensor notations introduced above this bracket has the form:

$$\begin{array}{c}\{L_1(x,\lambda), L_2(y,\mu)\} = [a, L_1(x,\lambda) + L_2(y,\mu)]\,\delta(x-y) + \\ +[s, L_1(x,\lambda) - L_2(y,\mu)]\,\delta(x-y) + 2s\delta'(x-y), \\ \text{here } a \text{ and } r \text{ are kernels of the operators:} \\ a = \frac{1}{2}(r - r^*), s = \frac{1}{2}(r + r^*) \\ \text{in the scalar product (28).}\end{array} \tag{32}$$

For instance, P_- has the following kernel

$$P_-(\lambda,\mu) = \frac{t}{\lambda-\mu}\phi(\mu)^{-1}. \tag{33}$$

Moreover,

$$s = -P_- + \phi^{-1}P_-\phi, \tag{34}$$

so that

$$s\mid_{\mathfrak{g}_-} = 0, s\mid_{\mathfrak{g}_+} = \phi^{-1}P_-\phi\mid_{\mathfrak{g}_+}, \tag{35}$$

$s\mid_{\mathfrak{g}_\nu} \neq 0$ if and only if ϕ has a pole at the point ν .

$$a = I - P_- - \phi^{-1}P_-\phi. \tag{36}$$

Thus we have

$$\begin{array}{c} s(\lambda,\mu) = \frac{t}{\lambda-\mu}\left(\phi(\lambda)^{-1} - \phi(\mu)^{-1}\right), \\ a(\lambda,\mu) = \frac{t}{\lambda-\mu}\left(\phi(\lambda)^{-1} - \phi(\mu)^{-1}\right). \end{array} \tag{37}$$

For another point of view on the bracket (31) see [8].

Now we are going back to the principal chiral field. For this model there exists the Zakharov-Mikhailov Lax pair [6]:

$$\begin{array}{c} L = -\frac{1}{1-\lambda^2}(l_x + \lambda l_t), \\ T = -\frac{1}{1-\lambda^2}(l_t + \lambda l_x). \end{array} \tag{38}$$

The equations of motion (13) are expressed as the zero curvature condition:

$$[\partial_x - L, \partial_t - T] = 0. \tag{39}$$

We propose the following Lie-algebraic interpretation of this pair. Let us consider the algebra $\mathbf{g}_A$ with the scalar product (28), where

$$\phi(\lambda) = 2\frac{\lambda^2 - 1}{\lambda^2}. \tag{40}$$

. Our main result is the theorem:

Theorem 2.1 *The r-matrix Lie-Poisson bracket (31) for the Lax operator (38) coincides with the brackets (11) for the canonical Poisson structure of the principal chiral field if ϕ in the definition (31) is given by the formula (40). In this case we must put in (32):*

$$\begin{aligned} s(\lambda, \mu) &= -\tfrac{1}{2}\tfrac{t(\lambda+\mu)}{(1-\lambda^2)(1-\mu^2)}, \\ a(\lambda, \mu) &= \tfrac{1}{2}\tfrac{t}{\lambda-\mu}\left(\tfrac{\lambda^2}{1-\lambda^2} + \tfrac{\mu^2}{1-\mu^2}\right). \end{aligned} \tag{41}$$

The theorem is verified by a direct computation. Thus the principal chiral field is described by the general Lie-algebraic scheme used in the Classical Inverse Scattering Method [6],[10]. We conclude this section with the formula allowing to restore the square of moment (10) using only the monodromy matrix of our model. It should be mentioned that the moment (10) is not well defined quantity on the space $G\backslash T^*\widehat{G}$ because we may restore the variable r_t on the space $G\backslash T^*\widehat{G}$ only up to constant loops. The well defined quantity is the square of the moment μ_L^G with respect to the scalar product tr on $\mathbf{g}$.

Let us consider the equation for the monodromy matrix:

$$\partial_x \Psi(x, \lambda) = -\frac{1}{1-\lambda^2}(l_x + \lambda l_t)\Psi(x, \lambda), \Psi(0, \lambda) = I, \tag{42}$$

so that $M(\lambda) = \Psi(2\pi, \lambda)$ .We have:

$$\begin{gathered} \partial_x \Psi(x, 0) = -l_x \Psi(x, 0) = -g^{-1}\partial_x g \Psi(x, 0), \\ \text{so that modulo left actions of constant loops } \Psi(x, 0) = g^{-1}(x). \end{gathered} \tag{43}$$

It means that there exist an element $g_0 \in G$ such that $\Psi(x,0)\, g_0 = g^{-1}(x)$

.

Let

$$\overset{\bullet}{\Psi}(x,0) = \frac{\partial}{\partial \lambda}\, |_{\lambda=0}\ \Psi(x,\lambda). \tag{44}$$

For this function we have the equation:

$$\partial_x\ \overset{\bullet}{\Psi}(x,0) = -l_t g^{-1}(x) - l_x\ \overset{\bullet}{\Psi}(x,0). \tag{45}$$

This equation has the following general solution:

$$\overset{\bullet}{\Psi}(x,0) = -g^{-1}(x) \int_0^x g(y)\, l_t(y)\, g(y)^{-1}\, dy, \tag{46}$$

and

$$\overset{\bullet}{M}(0) = \frac{d}{d\lambda}\, |_{\lambda=0}\ M(\lambda) = M(0) \int_0^{2\pi} r_t(y)\, dy, \tag{47}$$

where we use (10). Finally we modulo left action of constant loops :

$$\mu_L^G(g, l_t) = \int_0^{2\pi} r_t(x)\, dx = M^{-1}(0)\ \overset{\bullet}{M}(0). \tag{48}$$

We remind that on the phase space of the principal chiral field one must put $\mu_L^G = 0$. Our formula allows to impose this condition only by means of the additional constraint for the monodromy matrix.

3 The coadjoint orbits formulation for a non-linear sigma-model.

We shall try to generalize the construction of the previous section for a non-linear sigma-model. To make this it is natural to consider so-called twisted algebra of adèles.

Let σ be the involution defined by the Cartan decomposition (15):

$$\sigma : \mathbf{g} \to \mathbf{g}, \sigma \mid_{\mathbf{k}} = id, \sigma \mid_{\mathbf{p}} = -id. \tag{49}$$

This involution gives rise to an involution $\widehat{\sigma}$ on the algebra $\mathbf{g}_{\mathcal{A}}$:

$$\begin{gathered}\widehat{\sigma} : \mathbf{g}_{\mathcal{A}} \to \mathbf{g}_{\mathcal{A}}, \widehat{\sigma}\left(X\left(\lambda\right)\right) = \sigma X\left(-\lambda\right), \\ X\left(\lambda\right) = \bigoplus_{\nu \in \overline{\mathbf{C}}} X_{\nu}\left(\lambda_{\nu}\right) \in \mathbf{g}_{\mathcal{A}}, \sigma X\left(-\lambda\right) = \bigoplus_{\nu \in \mathbf{C}} \sigma X_{\nu}\left(-\lambda-\nu\right) \oplus \sigma X_{\infty}\left(\frac{1}{\lambda}\right).\end{gathered} \tag{50}$$

We define the twisted algebra of adèles $\mathbf{g}^{\sigma}_{\mathcal{A}}$ as the subalgebra of elements of $\mathbf{g}_{\mathcal{A}}$ invariant under $\widehat{\sigma}$:

$$\mathbf{g}^{\sigma}_{\mathcal{A}} = \left\{X\left(\lambda\right) \in \mathbf{g}_{\mathcal{A}} : \widehat{\sigma}\left(X\left(\lambda\right)\right) = X\left(\lambda\right)\right\}. \tag{51}$$

We introduce the following notations:

$$\mathbf{g}^{\sigma}_{-} = \mathbf{g}^{\sigma}_{\mathcal{A}} \cap \mathbf{g}_{-}, \mathbf{g}^{\sigma}_{+} = \mathbf{g}^{\sigma}_{\mathcal{A}} \cap \mathbf{g}_{+}. \tag{52}$$

As above, there exists a direct decomposition of the linear space $g^{\sigma}_{\mathcal{A}}$:

$$\mathbf{g}^{\sigma}_{\mathcal{A}} = \mathbf{g}^{\sigma}_{+} \dot{+} \mathbf{g}^{\sigma}_{-}. \tag{53}$$

We define on $\mathbf{g}^{\sigma}_{\mathcal{A}}$ the invariant scalar product:

$$\left(X\left(\lambda\right), Y\left(\lambda\right)\right) = \sum_{\nu \in \overline{\mathbf{C}}} \underset{\nu}{Res}\ \ tr X_{\nu}\left(\lambda_{\nu}\right) Y_{\nu}\left(\lambda_{\nu}\right) \frac{d\lambda}{\lambda}. \tag{54}$$

Let $P^{\sigma}_{\pm}$ be the projectors on $\mathbf{g}^{\sigma}_{\pm}$ .Then we construct the standard r-matrix on $\mathbf{g}^{\sigma}_{\mathcal{A}}$:

$$r^{\sigma} = P^{\sigma}_{+} - P^{\sigma}_{-}. \tag{55}$$

This r-matrix is not skew-symmetric with respect to the scalar product (54), because the subalgebras $\mathbf{g}^{\sigma}_{\pm}$ are not isotropic with respect to this scalar product. The projector P^{σ}_{-} has the following kernel:

$$P^{\sigma}_{-}\left(\lambda, \mu\right) = t_{A} \frac{\mu^{2}}{\lambda^{2} - \mu^{2}} + t_{B} \frac{\lambda\mu}{\lambda^{2} - \mu^{2}}, \tag{56}$$

and r^{σ} has the symmetric part with the kernel:

$$s(\lambda,\mu) = t_A, \tag{57}$$

and the skew-symmetric part:

$$a(\lambda,\mu) = -t_A\frac{\lambda^2+\mu^2}{\lambda^2-\mu^2} - t_B\frac{2\lambda\mu}{\lambda^2-\mu^2}. \tag{58}$$

With the same purposes as in the previous section, we shall use the deformed scalar product:

$$(X(\lambda), Y(\lambda))_\phi = \sum_{\nu\in\overline{\mathbf{C}}} \underset{\nu}{Res}\ \ trX_\nu(\lambda_\nu)\, Y_\nu(\lambda_\nu)\, \phi\left(\lambda^2\right)\frac{d\lambda}{\lambda}, \tag{59}$$

here we must use only a scalar rational function ϕ depending on the variable λ^2 to define the scalar product correctly.

Then with respect to this scalar product the kernel of the operator P_-^σ has the form:

$$P_-^\sigma(\lambda,\mu)_\phi = \left(t_A\frac{\mu^2}{\lambda^2-\mu^2} + t_B\frac{\lambda\mu}{\lambda^2-\mu^2}\right)\phi\left(\mu^2\right)^{-1}. \tag{60}$$

For the description of the Poisson structure of a nonlinear sigma-model we remind that , as well as for the principal chiral field , there exists a Lax pair for this model:

$$\begin{array}{c} L = -\left(A_x + \frac{\lambda}{2}(B_x+B_t) + \frac{1}{2\lambda}(B_x-B_t)\right), \\ T = -\left(+\frac{\lambda}{2}(B_x+B_t) - \frac{1}{2\lambda}(B_x-B_t)\right), \end{array} \tag{61}$$

and the equations of a motion have the form (39).

We choose the following form of ϕ :

$$\phi\left(\lambda^2\right) = -\frac{4\lambda^2}{\left(\lambda^2-1\right)^2}. \tag{62}$$

For our purposes we need to deform the bracket (31). Let us define the operator r_0 acting on the algebra $\mathbf{g}_{\mathcal{A}}^\sigma$:

$$r_0(X(\lambda)) = (P_{\mathbf{k}} r^\sigma X)(1). \tag{63}$$

This operator has the kernel:

$$P_{-}^{\mathbf{k}}(\mu) = t_A \frac{\mu^2}{1-\mu^2} \phi\left(\mu^2\right)^{-1} \tag{64}$$

The operator

$$\widehat{r} = r^{\sigma} - r_0 \tag{65}$$

does not satisfy the classical Yang-Baxter equation, so that this operators may not be used as an r-matrix for the definition of the Poisson brackets (31). But we may use this operator for the definition of the Poisson brackets (31) of the functions φ for which the derivatives X_{φ} lie in the kernel of the operator r_0 , because the Lie-Poisson bracket (31) depends on r only via combinations $\widehat{r}X_{\varphi}$. For instance it is not difficult to verify that derivatives of functions depending on the coefficients of the Lax operator (61) satisfy this condition.

Hence we may define the Poisson brackets of the Lax operator (61) by the formula (31) using the operator $\widehat{r}$. Our main result is

Theorem 3.1 *Let ϕ be given by the formula (62); choose the r-matrix $\widehat{r}$ as in (65) .Then for the Lax operator L (61) the r- bracket (31) coincides with the canonical Poisson structure of the nonlinear sigma-model (18). In this case we must put in (32):*

$$\begin{array}{c} s(\lambda,\mu) = \frac{t_B}{4}\left(\frac{1}{\lambda\mu} - \lambda\mu\right), \\ a(\lambda,\mu) = P_{-}^{\sigma}(\mu,\lambda)_{\phi} + P_{-}^{\mathbf{k}}(\lambda) - P_{-}^{\sigma}(\lambda,\mu)_{\phi} - P_{-}^{\mathbf{k}}(\mu). \end{array} \tag{66}$$

We conclude this section with a formula for the square of moment map (10). Without comments we present the straightforward calculation which is similar to the one in the previous section (42)-(48)

$$\begin{array}{c} \partial_x \Psi(x,\lambda) = -\left(A_x + \frac{\lambda}{2}(B_x + B_t) + \frac{1}{2\lambda}(B_x - B_t)\right)\Psi(x,\lambda), \Psi(0,\lambda) = I, \\ M(\lambda) = \Psi(2\pi,\lambda), \\ \partial_x \Psi(x,1) = -(A_x + B_t)\Psi(x,1) = -l_x \Psi(x,1), \\ \\ \Psi(x,1) = g^{-1}(x), \text{modulo left action of constant loops}, \\ \dot{\Psi}(x,1) = \frac{\partial}{\partial\lambda}\,|_{\lambda=1}\,\Psi(x,\lambda), \\ \partial_x \dot{\Psi}(x,1) = -l_x \dot{\Psi}(x,1) - B_t \Psi(x,1) = \end{array}$$

$$
\begin{aligned}
&= -l_x \overset{\bullet}{\Psi}(x,1) - B_t g^{-1}(x)\,, \\
\overset{\bullet}{\Psi}(x,1) &= -g^{-1}(x)\int\limits_0^x g(y)\,B_t(y)\,g^{-1}(y)\,dy = \Psi(x,1)\int\limits_0^x r_t(y)\,dy, \\
M^{-1}(1)\overset{\bullet}{M}(1) &= \int\limits_0^{2\pi} r(x)\,dx = \mu_L^G(g,l) \text{ modulo constant loops.}
\end{aligned}
$$

On the phase space of the chiral field $\mu_L^G(g,l) = 0$. Our formula for the moment allows to impose this constraint in explicit form as an additional condition on the monodromy matrix.

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