Classical and Quantum Nonultralocal Systems on the Lattice

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Abstract

We classify nonultralocal Poisson brackets for 1-dimensional lattice systems and describe the corresponding regularizations of the Poisson bracket relations for the monodromy matrix . A nonultralocal quantum algebras on the lattices for these systems are constructed.For some class of such algebras an ultralocalization procedure is proposed.The technique of the modified Bethe-Anzatz for these algebras is developed.This technique is applied to the nonlinear sigma model problem.

Introduction

This article is devoted to an old problem, which arose in the beginning of the development of the Classical Inverse Scattering Method (CISM) [**?**]. An important point of CISM is the calculation of the Poisson brackets relations for the monodromy matrix of an auxiliary linear problem. This calculation is usually performed under the technical assumption of 'ultralocality' of the Poisson brackets for local variables (this condition means simply that the Poisson operator defining the bracket is a multiplication operator and does not contain any derivations). In many interesting models this condition is violated, and in this case getting consistent Poisson brackets relations for the monodromy becomes nontrivial. Technically, the trouble is that the Frechet derivative of the monodromy has a discontinuity, and so one has to extend a differential operator to functions with a jump. It is easy to observe that Poisson operators are nonultralocal precisely for the models with non-skew-symmetric r-matrices. A naive calculation of the Poisson brackets for the monodromy in this case gives:

$$\begin{aligned} \{M_1, M_2\} &= aM_1M_2 - M_1M_2a, \\ a &= \tfrac{1}{2}(r - r^*). \end{aligned} \tag{1}$$

This bracket does not satisfy the Jacobi identity, since the skew part of r usually does not satisfy the Yang-Baxter identity (in fact, the bracket(1) is inconsistent even if it does). A natural way to regularize the monodromy brackets in this case has been proposed in [?]. This method allows to regularize some (though not all) of the Poisson brackets of the type (1). The idea is that to extend the Poisson operator to functions with a jump one has to add to it a boundary form sensitive to the jump, which is well in the spirit of the operator extensions theory. In this article we classify all regularized r-matrices and all regularizations of this kind using the Belavin-Drinfeld classification theorem for the modified Yang-Baxter equation [?]. Unfortunately, our classification is given in an implicit form because the Belavin-Drinfeld classification theorem describes solutions of the modified Yang-Baxter equation only up to automorphisms of the corresponding affine Lie algebra. This fact doesn't enable to write all regularizations in the explicit form. But we give a natural way to find all regularizations. We define the corresponding quantum algebras by means of the Faddeev-Reshetikhin-Takhtajan approach [?]. The same class of Poisson structures and of the corresponding quantum algebras has been recently studied in a slightly different way by J.M.Maillet and L.Freidel [?] and by S.Parmentier [?]; to describe them we use the unified approach based on the notion of the twisted double (cf.[?], [?])

The second goal of the present work is to construct quantum nonultralocal systems on the lattice ,which possess infinite series of conservation laws and to calculate the spectrum of the corresponding commuting operators. For this calculation we develop a generalization of the Bethe-Ansatz construction.

Some words about the contents of this paper.

In section 1 we review the construction of Poisson algebras on the lattice arising in the study of Lax equations on the lattice with non-ultralocal Poisson brackets.

In section 2 we remind the main construction of [?]. We reformulate the Belavin-Drinfeld classification theorem [?] in terms of the affine root systems. This reformulation is convenient for our purposes. We reduce the classification of regularizations to the search of some class of solutions of the Yang-Baxter equation on the square of a finite-dimensional Lie algebra.

In section 3 we discuss the main examples of regularizations and the corresponding nonultralocal algebras and investigate their algebraic properties. In particular, we determine their centers; under some additional conditions it is possible to find a new system of generators of these algebras which already satisfy *local* commutation relations This ultralocalization procedure has been discussed earlier in [?]. We present new examples of ultralocalization; the new system of generators is related to the original one by an appropriate quantum lattice gauge transformation. At the end of section 3 we describe a generalization of the algebraic Bethe Ansatz for nonultralocal algebras.

In section 4 we apply the technique developed in the previous sections to the nonlinear sigma model problem. It is well known that integrable models usually admit several different Poisson structures; the simplest one for the nonlinear sigma model is associated with its standard Lagrangian formulation. We were unable to find a regularization of this Poisson structure; however, the general scheme introduced in section 2 may be applied to another, and a fairly natural Poisson structure which we introduce in this section for a nonlinear sigma model with values in an arbitrary Riemannian symmetric space. We explicitly describe the corresponding quantum lattice systems . For the n-field (i.e., the sigma model with values in the unit sphere S^2) we get a representation of the local quantum lattice Lax operator via the canonical Weyl pairs. It turns out that the n-field with this Poisson structure is gauge equivalent to the lattice Sine-Gordon model.

In the conclusion we discuss some open problems.

1 General construction of lattice algebras

It is natural to assume that the phase space of a mechanical system associated with a 1-dimensional lattice $\Gamma = \mathbf{Z}/N\mathbf{Z}$ is the direct product $\mathcal{M}^N$ of "1-particle spaces". In applications to integrable systems these "elementary" phase spaces are parametrized by Lax matrices and hence are modeled on submanifolds of an appropriate Lie group (usually, a loop group associated with a finite-dimensional semisimple Lie group). In simple cases the Poisson structure on $\mathcal{M}^N$ is the product structure. (The corresponding Poisson bracket is called *ultralocal.*) The auxiliary linear problem associated with Lax equations on the lattice is

$$\psi_{n+1} = L^n \psi_n. \tag{2}$$

The associated monodromy map is the product map

$$M : G^N \to G : (L^1, ..., L^N) \mapsto \prod_{n=1}^{N} L^n. \tag{3}$$

It is natural to demand that M is a Poisson map. In ultralocal case this condition means that G should be a Poisson Lie group. It is interesting (and also important for applications) to study the most general Poisson structures on G^N which are compatible with this property of the monodromy. The corresponding Poisson algebras are referred to as *lattice algebras.* First examples of nonultralocal lattice algebras appeared in [?]; further examples and a classification (for finite dimensional semisimple Lie algebras) appeared in [?], [?], [?]. In this section we briefly recall the construction of lattice algebras using the approach proposed in [?], [?].

Fix an affine Lie algebra $\mathbf{g}$ with a normalized invariant bilinear form $\langle \cdot, \cdot \rangle$. It is well known that $\mathbf{g}$ admits the structure of a quasitriangular Lie bialgebra (the corresponding classical r-matrices are listed in [?]). Put $\mathbf{d} = \mathbf{g} \oplus \mathbf{g}$. We define the bilinear invariant form on the square of $\mathbf{g}$ in the following way:

$$\langle\langle (X_1, Y_1), (X_2, Y_2) \rangle\rangle = \langle X_1, X_2 \rangle - \langle Y_1, Y_2 \rangle, \tag{4}$$

so that the diagonal subalgebra is isotropic. As a Lie algebra, $\mathbf{d}$ is isomorphic to the double of $\mathbf{g}$. (This isomorphism does not depend on a particular choice of the r-matrix.) Hence $\mathbf{d}$ carries a natural r-matrix, the r-matrix of the double; for our present goals, however, we shall need *arbitrary* classical r-matrices on $\mathbf{d}$ which define on it the structure of a quasitriangular Lie bialgebra. In other words, we are interested in r-matrices which are skew with respect to(4) and satisfy the modified classical Yang-Baxter equation on $\mathbf{g} \oplus \mathbf{g}$.

Let $\mathcal{R}$ be such a solution; it may be written in the block form:

$$\mathcal{R} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \ A^* = -A, \ D^* = -D. \tag{5}$$

For $\varphi \in Fun(D), D = G \times G$ let $\mathrm{D}\varphi$, $\mathrm{D}'\varphi \in (\mathbf{g} \oplus \mathbf{g})^*$ be the left and right derivatives of φ:

$$\begin{gathered}\langle\langle \mathrm{D}\varphi(g), X\rangle\rangle = \tfrac{d}{dt}\mid_{t=0} \varphi\left(e^{tX}g\right), \\ \langle\langle \mathrm{D}'\varphi(g), X\rangle\rangle = \tfrac{d}{dt}\mid_{t=0} \varphi\left(ge^{tX}\right), \\ g \in D,\, X \in \mathbf{g} \oplus \mathbf{g}.\end{gathered} \tag{6}$$

It is well known that for any two solutions $\mathcal{R}, \mathcal{R}'$ of the modified classical Yang-Baxter equation the bracket

$$\{\varphi, \psi\}_{\mathcal{R},\mathcal{R}'} = \langle\langle \mathcal{R}_1 \mathrm{D}\varphi, \mathrm{D}\psi\rangle\rangle + \langle\langle \mathcal{R}_2 \mathrm{D}'\varphi, \mathrm{D}'\psi\rangle\rangle \tag{7}$$

satisfies the Jacobi identity. Let us take, in particular, $\mathcal{R}_1 = \mathcal{R}, \mathcal{R}_2 = \pm\mathcal{R}$ we get the following important brackets

$$\{\varphi, \psi\}_{D_\pm} = \langle\langle \mathcal{R}\mathrm{D}\varphi, \mathrm{D}\psi\rangle\rangle \pm \langle\langle \mathcal{R}\mathrm{D}'\varphi, \mathrm{D}'\psi\rangle\rangle. \tag{8}$$

We denote by $D_\pm$ the group D with the bracket $\{\cdot,\cdot\}_{D_\pm}$. The bracket $\{,\}_{D_-}$ equips D with the structure of a Poisson-Lie group, while the "+" sign corresponds to an almost nondegenerate Poisson structure on D_+. (It is symplectic on an open cell in D containing the unit element, see [?] for the description of the symplectic leaves of D_+.)

Multiplication map $D \times D \to D$ defines a Poisson group action $D_- \times D_+ \to D_+$; its restriction to the diagonal subgroup $G \subset D$ is *admissible* [?], and hence it is possible to perform Poisson reduction over the action of G. The quotient space is canonically identified with G itself; in fact, it is clear that the map $\pi : D \to G : (g_1, g_2) \mapsto g_1 g_2^{-1}$ is constant on the right coset classes of G.

To calculate the explicit form of the quotient Poisson structure on G choose $\varphi \in Fun(G)$ and put $\hat{\varphi} = \pi^*\varphi$; let $\nabla\varphi, \nabla'\varphi$ be the left and right derivatives of φ.

$$\begin{gathered}\langle \nabla\varphi(g), X\rangle = \tfrac{d}{dt}\mid_{t=0} \varphi\left(e^{tX}g\right), \\ \langle \nabla'\varphi(g), X\rangle = \tfrac{d}{dt}\mid_{t=0} \varphi\left(ge^{tX}\right), \\ g \in G,\, X \in \mathbf{g}.\end{gathered} \tag{9}$$

Then

$$\mathrm{D}\hat{\varphi}(g_1, g_2) = \left(\nabla\varphi\left(g_1 g_2^{-1}\right), \nabla'\varphi\left(g_1 g_2^{-1}\right)\right). \tag{10}$$

After a short computation this yields:

$$\{\varphi,\psi\}_G = \langle A\nabla\varphi,\nabla\psi\rangle - \langle D\nabla'\varphi,\nabla'\psi\rangle + \langle B\nabla'\varphi,\nabla\psi\rangle - \langle B^*\nabla\varphi,\nabla'\psi\rangle\,. \quad (11)$$

In general, this Poisson structure is degenerate.

Suppose that τ is an automorphism of $\mathbf{g}$;then $\tau\oplus\tau$ is an automorphism of $\mathbf{g}\oplus\mathbf{g}$. Let us assume that $\tau\oplus\tau$ commutes with $\mathcal{R}$. To twist the r-matrix on $\mathbf{d}$ we shall use another extension of τ to $\mathbf{d}$; namely, we put $\hat{\tau}=\tau\oplus\tau^{-1}$. Put

$$\mathcal{R}^\tau = \hat{\tau}\mathcal{R}\hat{\tau}^{-1} = \begin{pmatrix} A & B\tau^{-1} \\ \tau B^* & D \end{pmatrix}. \quad (12)$$

$\mathcal{R}^{\ \tau}$ satisfies the Yang-Baxter equation. Put $\mathcal{R}_1 = \mathcal{R}^\tau, \mathcal{R}_2 = \mathcal{R}$ in (7) and denote by D_τ the group D with the corresponding Poisson structure:

$$\{\varphi,\psi\}_{D_\tau} = \langle\langle \mathcal{R}^\tau \mathrm{D}\varphi, \mathrm{D}\psi\rangle\rangle + \langle\langle \mathcal{R}\mathrm{D}'\varphi, \mathrm{D}'\psi\rangle\rangle\,. \quad (13)$$

(If $\mathcal{R}$ is the r-matrix of the double of $\mathbf{g}$, the group D_τ is usually referred to as the *twisted double*.) This Poisson structure on D_τ also admits reduction with respect to the action of the diagonal subgroup; the quotient structure on $G=\pi(D_\tau)$ is given by

$$\{\varphi,\psi\}_\tau = \langle A\nabla\varphi,\nabla\psi\rangle - \langle D\nabla'\varphi,\nabla'\psi\rangle + \left\langle B\tau^{-1}\nabla'\varphi,\nabla\psi\right\rangle - \langle \tau B^*\nabla\varphi,\nabla'\psi\rangle\,. \quad (14)$$

In particular, let us apply this construction to the group $\mathsf{G}=G^N$;in this case $\mathsf{D}=(G\times G)^N$, and τ is the cyclic shift in the direct sum $\bigoplus_1^N \mathbf{g}$. Let

$$r = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \quad (15)$$

be a solution of the modified classical Yang-Baxter equation on $\mathbf{g}\oplus\mathbf{g}$; put $\mathcal{R}=\bigoplus_1^N r$. Evidently, $\mathcal{R}$ commutes with $\tau\oplus\tau$. To describe the resulting lattice Poisson algebra it is convenient to introduce tensor notations. Fix an exact matrix representation ρ_V of G and denote

$$L^n = \rho_V\ (g_n)\,, L_1^n = L^n\otimes I, L_2^n = I\otimes L^n, \quad g=(g_{1,\dots,}g_n)\in G^N. \quad (16)$$

The reduced Poisson brackets of the matrix coefficients of L^n have the form:

$$\begin{gathered}\{L_1^n, L_2^n\} = -AL_1^n L_2^n + L_1^n L_2^n D, \\ \left\{L_1^n, L_2^{n+1}\right\} = L_1^n B^* L_2^{n+1}, \\ \{L_1^n, L_2^m\} = 0, |n-m| \geq 2.\end{gathered} \tag{17}$$

Here we denote $(\rho_V \otimes \rho_V) A$ as well as A ,etc. The main property of the Poisson bracket (17) is given by the following assertion:

Theorem 1.1 *[?]*

Equip $\mathsf{G} = \mathrm{G}^N$ *with the Poisson structure (11); then the monodromy map*

$$M : \mathsf{G}_\tau \to \mathrm{G},\ M(g_1, \ldots, g_N) = g_1 \cdot \ldots \cdot g_N$$

is a Poisson mapping if and only if the r-matrix (15) satisfies the additional constraint

$$A + B = B^* + D. \tag{18}$$

In that case the Poisson structure in the target space of the monodromy map is given by (11).

In tensor notations we have the following brackets for M :

$$\{M_1, M_2\} = -AM_1M_2 + M_1M_2D + M_1B^*M_2 - M_2BM_1 \tag{19}$$

Later we shall describe symplectic leaves of the bracket (17) in the main examples. In the particular case when $B = 0, A = D$ the bracket is ultralocal. The reader may keep in mind this possibility as a degenerate case.

The remainder of this section is devoted to the quantization of the Poisson brackets (17), (19). It may be easily performed on the lines of [?] provided that we know the quantum R-matrix which corresponds to the chosen classical r-matrix on $\mathbf{d}$. More precisely, let $U_q(\mathbf{d};\mathcal{R})$ be the quantized universal enveloping algebra of $\mathbf{d}$ which corresponds to $\mathcal{R}$ [?]; note that its description is not quite obvious since in the existing literature only the standard algebras $U_q(\mathbf{d};\mathcal{R})$ which correspond to simplest solutions of the classical Yang-Baxter equation are usually considered. It is widely believed that all solutions from the Belavin-Drinfeld list [?] give rise to quasitriangular Hopf algebras. (Examples in section 3 below give evidence to support this belief.) Assuming that the algebra $U_q(\mathbf{d};\mathcal{R})$ exists, let

$$\mathcal{R}_q = \begin{pmatrix} A_q & B_q \\ B_q^* & D_q \end{pmatrix} \in U_q(\mathbf{d};\mathcal{R}) \otimes U_q(\mathbf{d};\mathcal{R}) \tag{20}$$

be its universal quantum R-matrix. We omit the explicit form of the relations in the algebra $U_q(\mathbf{d};R)$. Let ρ_V be some representation of the algebra $U_q(\mathbf{d};\mathcal{R})$ in the space V and let

$$\mathcal{R}_q^{VV} = (\rho_V \otimes \rho_V)\,\mathcal{R}_q = \begin{pmatrix} A_q & B_q \\ B_q^* & D_q \end{pmatrix} \tag{21}$$

The following theorem is parallel to the description of the twisted quantum double and of the lattice current algebra [?], [?].

Theorem 1.2 *The free algebra $Fun_q^{\mathcal{R}}(\mathsf{G}_\tau)$ generated by the matrix elements of the matrices $L^n \in Fun_q^{\mathcal{R}}(\mathsf{G}_\tau) \otimes End(V)$, satisfying the following relations:*

$$\begin{array}{c} A_q L_1^n L_2^n = L_2^n L_1^n D_q \\ L_1^n B_q^{*^{-1}} L_2^{n+1} = L_2^{n+1} L_1^n \end{array}, \tag{22}$$

is the quantization of the Poisson algebra (17).

Theorem 1.3 *The free algebra $Fun_q^{\mathcal{R}}(\mathrm{G})$ generated by the matrix elements of the matrix $M \in Fun_q^{\mathcal{R}}(\mathrm{G}) \otimes End(V)$, satisfying the relations:*

$$A_q M_1 B_q^{*^{-1}} M_2 = M_2 B_q^{-1} M_1 D_q \tag{23}$$

is the quantization of the Poisson algebra (19).

Finally, we formulate the quantum version of theorem 1.1

Theorem 1.4 *The map*

$$\begin{array}{c} M : Fun_q^{\mathcal{R}}(\mathsf{G}_\tau) \to Fun_q^{\mathcal{R}}(\mathrm{G}), \\ \left(L^1, \ldots, L^N\right) \mapsto L^1 \cdot \ldots \cdot L^N \end{array}$$

is an homomorphism of algebras.

The algebras (22), (23) are the principal objects of our investigation.

2 Regularization of nonultralocal Poisson brackets

The goal of this section is to link the construction of lattice algebras with Hamiltonian systems on coadjoint orbits of current algebras. This approach is outlined in [?] where a regularization procedure for the Poisson brackets of the monodromy matrix is proposed which matches naturally with lattice Poisson algebras described in section 1. This will also allow us to construct consistent lattice approximations of nonultralocal systems on the circle.

We remind some details of the construction of dynamical systems on coadjoint orbits [?], [?].

Let $\mathbf{G} = C^{\infty}\left(S^1, \mathbf{g}\right)$ be a current algebra with the values in some affine Lie algebra $\mathbf{g}$. Let us define the invariant scalar product on $\mathbf{G}$:

$$(X, Y) = \int_0^{2\pi} \langle X, Y \rangle \, dz, \tag{24}$$

where $X, Y \in \mathbf{G}, \langle \cdot, \cdot \rangle$ is a invariant bilinear form on $\mathbf{g}$. Let $\widehat{\mathbf{G}}$ be the central extension of the algebra $\mathbf{G}$ which corresponds to the 2-cocycle

$$\omega\left(X, Y\right) = \left(X, \partial_z Y\right). \tag{25}$$

By definition, $\widehat{\mathbf{G}}$ is the set of pairs $\left(X, a\right), X \in \mathbf{G}, a \in \mathbf{C}$ with the commutator

$$\left[\left(X.a\right), \left(Y, b\right)\right] = \left(\left[X, Y\right], \omega\left(X, Y\right)\right). \tag{26}$$

If r is a solution of the modified classical Yang-Baxter equation on $\mathbf{g}$, we put as usual

$$\left[X, Y\right]_r = \left[rX, Y\right] + \left[X, rY\right].$$

Let $\mathbf{g}_r$ be the algebra $\mathbf{g}$ equipped with this bracket. Put $\mathbf{G}_r = C^{\infty}\left(S^1, \mathbf{g}_r\right)$; it is easy to see that

$$\omega_r\left(X, Y\right) = \omega\left(rX, Y\right) + \omega\left(X, rY\right)$$

is a 2-cocycle on $\mathbf{G}_r$; thus we may define the second structure of a Lie algebra on $\widehat{\mathbf{G}}$

$$[(X,a),(Y,b)]_r = ([X,Y]_r, \omega_r(X,Y)). \tag{27}$$

In this formula it is not assumed that r is skew-symmetric with respect to the scalar product $\langle\cdot,\cdot\rangle$. (In fact, if it is, the cocycle ω_r vanishes identically.)

Let $\widehat{\mathbf{G}}^*$ be the dual space of $\widehat{\mathbf{G}}$; using the inner product (24) we may identify it with $\mathbf{G}\oplus\mathbf{C}$. The Poisson bracket used in the CISM is the Lie-Poisson bracket which corresponds to the commutator (27). The variable $e\in\mathbf{C}$ is central with respect to this bracket. If $X_\varphi\in\widehat{\mathbf{G}}$ is a derivative of a function $\varphi\in Fun\left(\widehat{\mathbf{G}}^*\right)$:

$$\begin{gathered}((X_\varphi, X)) = \tfrac{d}{dt}\,|_{t=0}\,\varphi(L+tX), \\ X, L\in\widehat{\mathbf{G}}^*, \end{gathered} \tag{28}$$

here $((\cdot,\cdot))$ is a natural pairing between $\widehat{\mathbf{G}}$ and $\widehat{\mathbf{G}}^*$.

then

$$\{\varphi,\psi\}(L,e) = \left(\left((L,e),[X_\varphi,X_\psi]_r\right)\right). \tag{29}$$

Without loss of generality we may assume that $e=1$ and suppress it in the notations. The bracket (29) may be represented as the bilinear form of the Poisson operator:

$$\mathcal{H} = adL\circ r + r^*\circ adL - (r+r^*)\,\partial_z, \tag{30}$$

$$\{\varphi,\psi\}(L) = (\mathcal{H}X_\varphi, X_\psi). \tag{31}$$

The operator $\mathcal{H}$ is unbounded, so the formula (31) requires some caution when the gradients are not smooth on the circle. This is precisely the case for the Poisson brackets of the monodromy matrix. Let ψ be the fundamental solution of the equation:

$$\partial_z\psi = L\psi \tag{32}$$

normalized by $\psi(0)=I$; then the monodromy matrix is equal to

$$M = \psi(2\pi)\in G. \tag{33}$$

Fix $\Phi\in Fun(G)$. According to [**?**], the Frechet derivative of the functional $L\mapsto\Phi(M[L])$ is given by

$$X_{\Phi}(z) = \psi(z)\,\nabla'\Phi(M)\,\psi(z)^{-1} \tag{34}$$

and in general is discontinuous at $z = 0$:

$$\begin{aligned} X_{\Phi}(0) &= \nabla'\Phi(M), \\ X_{\Phi}(2\pi) &= \nabla\Phi(M). \end{aligned} \tag{35}$$

To regularize Poisson brackets of the monodromy we shall use an idea borrowed from the theory of self-adjoint extensions [?]. Let $\triangle : C^{\infty}([0,2\pi];\mathbf{g}) \to \mathbf{g} \oplus \mathbf{g}$ be the map which associates to a function on $[0,2\pi]$ its boundary values,

$$\triangle X_{\varphi} = \begin{pmatrix} X_{\varphi}(0) \\ X_{\varphi}(2\pi) \end{pmatrix}. \tag{36}$$

Choose $B \in End\left(\overset{\circ}{\mathbf{g}} \oplus \overset{\circ}{\mathbf{g}}\right)$, here $\overset{\circ}{\mathbf{g}} \subset \mathbf{g}$ is the spreading finite-dimensional Lie algebra which corresponds to an affine Lie algebra $\mathbf{g}$; we extend operator B to the space $\mathbf{g} \oplus \mathbf{g}$ as a zero operator outside $\overset{\circ}{\mathbf{g}} \oplus \overset{\circ}{\mathbf{g}}$ and define the regularized Poisson bracket in the following way:

$$\{\varphi,\psi\}(L,1) = \frac{1}{2}\left((\mathcal{H}X_{\varphi}, X_{\psi}) - (\mathcal{H}X_{\psi}, X_{\varphi})\right) + \langle\langle B\triangle X_{\varphi}, \triangle X_{\psi}\rangle\rangle. \tag{37}$$

The bracket (37) must coincide with the bracket (31) on smooth functions, hence B must satisfy the condition:

$$\left\langle\left\langle B\begin{pmatrix} X \\ X \end{pmatrix}, \begin{pmatrix} Y \\ Y \end{pmatrix}\right\rangle\right\rangle = 0. \tag{38}$$

The additional restriction on B imposed in [?] follows from the study of the linearized bracket for the monodromy (37) for $M \to 1$; it is natural to demand that this linearized bracket should coincide with the one defined by r. This gives, after a short computation:

$$\{\Phi,\Psi\}(M) = \left\langle\left\langle \mathcal{R}\begin{pmatrix} \nabla\Phi \\ \nabla'\Phi \end{pmatrix}, \begin{pmatrix} \nabla\Psi \\ \nabla'\Psi \end{pmatrix}\right\rangle\right\rangle, \tag{39}$$

where

$$\mathcal{R} = \begin{pmatrix} -a+\alpha & -\alpha-s \\ \alpha-s & -a-\alpha \end{pmatrix}, \alpha^* = -\alpha, a = \tfrac{1}{2}(r-r^*), s = \tfrac{1}{2}(r+r^*), \tag{40}$$
$$\text{where } \alpha \in \overset{\circ}{\mathbf{g}} \wedge \overset{\circ}{\mathbf{g}} \text{ because } B \in End\left(\overset{\circ}{\mathbf{g}} \oplus \overset{\circ}{\mathbf{g}}\right),$$

and our choice of B supposes that $s \in \overset{\circ}{\mathbf{g}} \otimes \overset{\circ}{\mathbf{g}}$.The Jacobi identity for this bracket will be valid if $\mathcal{R}$ satisfies the modified Yang-Baxter equation. In tensor notations a Poisson brackets of monodromy matrix have the form:

$$\{M_1, M_2\} = (a-\alpha)\,M_1M_2 - M_1M_2\,(a+\alpha) + M_1\,(\alpha-s)\,M_2 + M_2\,(\alpha+s)\,M_1. \tag{41}$$

The corresponding lattice Poisson algebra for which the monodromy matrix has the brackets (41) is:

$$\begin{gathered} \{L_1^n, L_2^n\} = (a-\alpha)\,L_1^nL_2^n - L_1^nL_2^n\,(a+\alpha), \\ \left\{L_1^n, L_2^{n+1}\right\} = L_1^n\,(\alpha-s)\,L_2^{n+1}, \\ \{L_1^n, L_2^m\} = 0, |n-m| \geq 2. \end{gathered} \tag{42}$$

Our next step is a classification of the Poisson brackets of type (31) for which the Poisson brackets of the monodromy matrix may be regularized. It is difficult to classify all non-skew solutions of the Yang-Baxter equation for which there exists an $\alpha \in \overset{\circ}{\mathbf{g}} \wedge \overset{\circ}{\mathbf{g}}$ such that a matrix $\mathcal{R}$ in (40) is a solution of the Yang-Baxter equation. But we can easily construct all solutions of the Yang-Baxter equation for $\mathbf{g} \oplus \mathbf{g}$ according to the Belavin-Drinfeld classification theorem [**?**], and then choose solutions of the form (40). To realize this program we start with an easy theorem:

Theorem 2.1 *If $\mathcal{R}$ is a solution of the modified Yang-Baxter equation for $\mathbf{g} \oplus \mathbf{g}$ of the type (40), then $a+s$ is a solution of the modified Yang-Baxter equation for $\mathbf{g}$.*

Proof. Let $r_1 = -a+\alpha, r_2 = -\alpha-s$,then $r = -(r_1+r_2)$. From the Yang-Baxter equation for $\mathcal{R}$ it follows:

$$\begin{array}{c}[r_1X, r_2Y] - r_1([r_1X, Y] + [X, r_1Y]) = -[X, Y], \\ [r_2X, r_2Y] - r_2([(r_1 - 2\alpha)X, Y] + [X, (r_1 - 2\alpha)Y]) = 0, \\ [r_1X, r_2Y] - r_1[X, r_2Y] - r_2[(r_2 + 2\alpha)X, Y] = 0, \\ X, Y \in \mathbf{g}.\end{array} \tag{43}$$

It is easy to check that the (43) implies the Yang-Baxter equation for r:

$$[rX, rY] - r([rX, Y] + [X, rY]) = -[X, Y]. \tag{44}$$

Let us turn now to the detailed study of the case of affine Lie algebras. We use the terminology and notations of the book [?].

Let $\mathbf{g}$ be an affine Lie algebra, $\triangle_+$ the set of its positive roots, $\overset{\circ}{\mathbf{g}}$ the corresponding spreading simple Lie algebra, $\overset{\circ}{\triangle}_+$ the set of its positive roots. Let $\triangle_{++} = \triangle_+ \setminus \overset{\circ}{\triangle}_+$. Using this notations we formulate some version of the Belavin-Drinfeld classification theorem.

Theorem 2.2 *[?] Up to an automorphism any solution of the modified classical Yang-Baxter equation for an affine Lie algebra* $\mathbf{g}$ *has the form:*

$$R = \sum_{\alpha \in \triangle_{++}} e_\alpha \wedge e_{-\alpha} + r,$$

where r *is a solution of the modified Yang-Baxter equation for* $\overset{\circ}{\mathbf{g}}$*.*

For an explicit form of such solutions see [?].

In [?] such solutions are called trigonometric.

Thus from theorem 2.2 we have the following ansatz for a:

$$a = \sum_{\alpha \in \triangle_{++}} e_\alpha \wedge e_{-\alpha} + a_0, a_0 \in \overset{\circ}{\mathbf{g}} \wedge \overset{\circ}{\mathbf{g}}. \tag{45}$$

and we reduce our problem to the Yang-Baxter equation on the square of a finite dimensional Lie algebra $\overset{\circ}{\mathbf{g}}$. Namely, $\mathcal{R}$ is a solution of the Yang-Baxter equation iff

$$\begin{pmatrix} -a_0 + \alpha & -\alpha - s \\ \alpha - s & -a_0 - \alpha \end{pmatrix} \tag{46}$$

is a solution of the Yang-Baxter equation for $\overset{\circ}{\mathbf{g}} \oplus \overset{\circ}{\mathbf{g}}$.

Theorem 2.3 *Let*

$$\begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, A^* = -A, D^* = -D$$

be a solution of the Yang-Baxter equation for $\overset{\circ}{\mathbf{g}} \oplus \overset{\circ}{\mathbf{g}}$. *It has the form (46) iff*

$$A + B = B^* + D. \tag{47}$$

Under this condition

$$\alpha = \frac{B^* - B}{2},\ a_0 = -\frac{A + D}{2},\ s = -\frac{B + B^*}{2}. \tag{48}$$

Notice that we again come to the condition (18).

Remark. It may be showed that it is not necessary to impose the condition $B \in End\left(\overset{\circ}{\mathbf{g}} \oplus \overset{\circ}{\mathbf{g}}\right)$ a priori. Actually, this condition follows from the generalization of theorem 2.2 for a direct sum of two copies of an affine Lie algebra, because $\alpha \in \overset{\circ}{\mathbf{g}} \wedge \overset{\circ}{\mathbf{g}}$, $s \in \overset{\circ}{\mathbf{g}} \otimes \overset{\circ}{\mathbf{g}}$ for every solution of the modified Yang-Baxter equation for $\mathbf{g} \oplus \mathbf{g}$ of the form (40).

Unfortunately, condition (47) is not stable under the automorphisms of $\overset{\circ}{\mathbf{g}}$. So we cannot use the Belavin-Drinfeld theorem to classify all regularizations. But this theorem gives a possibility to construct sufficiently general examples of such regularizations. These examples will be presented in the next section. In the $\widehat{sl(2)}$ case we shall able to classify all regularizations.

3 Main examples of regularizations

Now we are ready to discuss examples of regularizations using the results of the previous sections. We shall consider affine Lie algebras of type $X_N^{(1)}$ and $X_N^{(2)}$ in the loop realization. We shall describe the corresponding lattice quantum algebras and their Casimir elements. In the $\widehat{sl(2)}$ case we shall explain the Algebraic Bethe Ansatz construction for such algebras.

Example 3.1 Nontwisted loop algebras.

The first example is connected with the r-matrix of the double of a finite dimensional Lie algebra $\overset{\circ}{\mathbf{g}}$ equipped with the structure of a quasitriangular Lie bialgebra. Let $\mathbf{g}$ be an affine Lie algebra of type $X_N^{(1)}$, $\overset{\circ}{\mathbf{g}}$ the corresponding finite-dimensional Lie algebra. To apply theorem 2.3 consider the r-matrix of its double $\overset{\circ}{\mathbf{d}}=\overset{\circ}{\mathbf{g}} \oplus \overset{\circ}{\mathbf{g}}$; we have

$$r = \begin{pmatrix} \alpha & -2\alpha_+ \\ 2\alpha_- & -\alpha \end{pmatrix}, \tag{49}$$

where α is some solution of the modified Yang-Baxter equation for $\overset{\circ}{\mathbf{g}}$ and $\alpha_\pm = \frac{1}{2}(\alpha \pm I)$. ($I$ is the identity operator in $\overset{\circ}{\mathbf{g}}$; its kernel is the Casimir element t.) According to theorem 2.3, in this case one gets:

$$s = I, a_0 = 0. \tag{50}$$

In this case $r = a + I$ is the rational r-matrix for $\mathbf{g}$. We choose for $\mathbf{g}$ the non-twisted loop realization [?]. We remind that in this realization $\mathbf{g} = \overset{\circ}{\mathbf{g}} \otimes \mathbf{C}[\lambda, \lambda^{-1}]$, and the invariant bilinear form is given by

$$\langle X(\lambda), Y(\lambda)\rangle = Res\ tr(X(\lambda)Y(\lambda))\frac{d\lambda}{\lambda}, \tag{51}$$

where tr is an invariant bilinear form on $\overset{\circ}{\mathbf{g}}$.

The kernel of a in this realization is:

$$a(\lambda, \mu) = -t\frac{\lambda+\mu}{\lambda-\mu}, \tag{52}$$

where we identify $\mathbf{g} \otimes \mathbf{g}$ with $\overset{\circ}{\mathbf{g}} \otimes \overset{\circ}{\mathbf{g}} \otimes \mathbf{C}[\lambda, \lambda^{-1}] \otimes \mathbf{C}[\mu, \mu^{-1}]$. Thus we have the following formulas for $-a \pm \alpha$:

$$\begin{aligned} -a + \alpha &= \tfrac{\lambda}{\lambda-\mu}2\alpha_+ - \tfrac{\mu}{\lambda-\mu}2\alpha_-, \\ -a - \alpha &= \tfrac{\mu}{\lambda-\mu}2\alpha_+ - \tfrac{\lambda}{\lambda-\mu}2\alpha_-. \end{aligned} \tag{53}$$

Let $\mathcal{R}_\pm$ be the finite-dimensional quantum R-matrix in the fundamental representation, which corresponds to $2\alpha_\pm$ after quantization. In the classical limit

$$\mathcal{R}_\pm = I + 2\alpha_\pm h + o(h), \tag{54}$$

where h is the deformation parameter.

We have:

$$\mathcal{R}_- = P\left(\mathcal{R}_+^{-1}\right), \tag{55}$$

where P is the permutation operator in the tensor square.

Using these data we may construct the quantum R-matrices corresponding to $-a \pm \alpha$. If we denote the quantum R-matrix corresponding to $-a-\alpha$ by $\mathcal{R}(\lambda,\mu)$, then

$$\mathcal{R}(\lambda,\mu) = \frac{\lambda}{\lambda-\mu}\mathcal{R}_-^{-1} - \frac{\mu}{\lambda-\mu}\mathcal{R}_+^{-1}, \tag{56}$$

and the quantum R-matrix $\mathcal{R}(\lambda,\mu)^T$ corresponds to $-a+\alpha$:

$$\mathcal{R}(\lambda,\mu)^T = \frac{\lambda}{\lambda-\mu}\mathcal{R}_+ - \frac{\mu}{\lambda-\mu}\mathcal{R}_-, \tag{57}$$

here T is the conjugation with respect to the scalar product tr.

Finally, we have the quantum R-matrix on the square of $\mathbf{g}$:

$$\mathcal{R}_q = \begin{pmatrix} \mathcal{R}(\lambda,\mu)^T & \mathcal{R}_+^{-1} \\ \mathcal{R}_- & \mathcal{R}(\lambda,\mu) \end{pmatrix}. \tag{58}$$

According to theorem 1.2 one can get the relations in the quantum lattice algebra $Fun_q^{\mathcal{R}}(\mathsf{G}_\tau)$, which gives a lattice approximation of the continuous system:

$$\begin{gathered} \mathcal{R}(\lambda,\mu)^T L_1^n(\lambda) L_2^n(\mu) = L_2^n(\mu) L_1^n(\lambda) \mathcal{R}(\lambda,\mu), \\ L_1^n(\lambda)\mathcal{R}_-^{-1} L_2^{n+1}(\mu) = L_2^{n+1}(\mu) L_1^n(\lambda), \\ L^n(\lambda) \in Fun_q^{\mathcal{R}}(\mathsf{G}_\tau) \otimes End(V), \end{gathered} \tag{59}$$

here V is the fundamental representation space.

From theorem 1.3 we get the following commutation relations for the monodromy matrix:

$$\mathcal{R}(\lambda,\mu)^T M_1(\lambda)\mathcal{R}_-^{-1} M_2(\mu) = M_2(\mu)\mathcal{R}_+ M_1(\lambda)\mathcal{R}(\lambda,\mu). \tag{60}$$

The algebra (59) is connected with the Lattice Kac-Moody algebra $\mathcal{A}_{LC}$ [?]. Namely, the algebra (59) admits a family of representation for which there exists the limit