BLACK HOLES WITH WEYL CHARGE

AND

NON-RIEMANNIAN WAVES

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ABSTRACT

A simple modification to Einstein's theory of gravity in terms of a non-Riemannian connection is examined. A new tensor-variational approach yields field equations that possess a covariance similar to the gauge covariance of electromagnetism. These equations are shown to possess solutions analogous to those found in the Einstein-Maxwell system. In particular one finds gravi-electric and gravi-magnetic charges contributing to a spherically symmetric static Reissner-Nordström metric. Such Weyl "charges" provide a source for the non-Riemannian torsion and metric gradient fields instead of the electromagnetic field. The theory suggests that matter may be endowed with gravitational charges that couple to gravity in a manner analogous to electromagnetic couplings in an electromagnetic field. The nature of gravitational coupling to spinor matter in this theory is also investigated and a solution exhibiting a plane-symmetric gravitational metric wave coupled via non-Riemannian waves to a propagating spinor field is presented.

1. Introduction

In the absence of matter Einstein's theory of gravity allows an elegant formulation in terms of (pseudo-)Riemannian geometry. The field equations follow as the local extremum of an action integral under metric variations. The integrand of this action is simply the curvature scalar associated with the curvature of the Levi-Civita connection times the (pseudo-)Riemannian volume form of spacetime. Such a connection ∇ is torsion-free and metric compatible. Thus for all vector fields X, Y on the spacetime manifold, the tensors given by:

$$\mathbf{T}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{1}$$

$$\mathbf{S} = \nabla \mathbf{g} \tag{2}$$

are zero where $\mathbf{g}$ denotes the metric tensor, $\mathbf{T}$ the 2-1 torsion tensor and $\mathbf{S}$ the gradient tensor of $\mathbf{g}$ with respect to ∇. Such a Levi-Civita connection provides a useful reference connection since it depends entirely on the metric structure of the manifold. Einstein's description provides a well tested theoretical edifice for describing the large scale structure of gravitational phenomena and although numerous supergravity and superstring variants of the theory have hinted at a more general geometry there has been little evidence to suggest that the additional torsion and metric-gradient fields can be given an immediate interpretation. Although Weyl [1]saw the potential inherent in certain non-Riemannian geometries his efforts to relate the metric-gradient to the electromagnetic field were regarded as unsuccessful and in the light of subsequent developments the unification of the fundamental interactions has been sought elsewhere. In particular the promise of string unification raises questions concerning the viability of a classical geometrical description on all scales. At the level of effective theories however (where one has a chance of testing their predictions) there are hints that a non-Riemannian geometry may offer a more economical and more elegant description of gravitational interactions [2], [3], [4], [5].

It appears that low energy dilaton and axi-dilaton interactions can be accommodated in terms of a connection that gives rise to a particular torsion and metric-gradient field. Whether one should treat seriously such models in astrophysical contexts is open to debate but it is of interest to enquire whether non-Riemannian fields could give rise to observational effects [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17]. In this paper we seek a *simple* modification to the traditional Einstein-Hilbert action that provides a dynamical prescription for a non-Riemannian geometry that might be considered as a viable alternative to Einstein's metric theory. A minimal requirement in this direction is the existence of a static spherically symmetric solution that can compete with the Schwarzschild metric in such a theory.

There exists a large body of literature that approaches non-Riemannian theories of gravitation in the language of affine structure groups of principal fibre bundles over spacetime. In this article we offer an alternative language in terms of the metric $\mathbf{g}$ and the connection ∇. Such a connection is in 1-1 correspondence with the notion of a $gl(4,R)-$ algebra valued connection on the bundle of linear frames over spacetime. However, unlike the metric-affine gauge approach [6] ,[18] we do not require a reduction of the $R^4 \times Gl(4,R)$ gauge group to formulate the theory, relying rather on the traditional definitions above of torsion and non-metricity in terms of ∇ and the metric $\mathbf{g}$. A notable feature of the underlying gauge symmetry of the actions discussed in this paper is its similarity to the gauge group of electromagnetism. Although of gravitational origin, the metric tensor remains invariant just as in Maxwell's $U(1)$ gauge covariant theory while ∇ experiences a transformation. This distinguishes our gauge symmetry from Weyl's original theory [1] in which the connection remained invariant under a dilation of the gravitational metric and a transformation of the non-metricity tensor. The papers by Gregorash and Papini [19], [20] are concerned with extensions of Weyl's original theory as interpreted by Dirac [21] and construct conformally invariant actions with the aid of additional scalar fields that compensate the transformation of the curvature scalar induced by a local scaling of the metric. Many models in the metric-affine gauge approach are similarly based on "conformally invariant" actions in which the metric also transforms under the gauge group. In the approach below the emphasis is on the analogy with the electromagnetic gauge group rather than conformal invariance. Since the metric remains inert under our gauge group we designate the associated conserved charges "Weyl charges" rather than "dilation charges".

In section 2 we summarise the essential prerequisites of non-Riemannian geometry from a modern viewpoint. In section 3 we describe a tensorial variational principle. Working tensorially offers a number of advantages over traditional variational techniques. No reliance is made on particular fields of frames or coframes and the resulting formulae, once derived, are unambiguous and easy to apply. Transition to more traditional component oriented manipulations is of course possible but great care must then be exercised in raising and lowering indices with the metric since the latter is no longer regarded as covariantly constant. In section 4 we explicitly derive our field equations in the absence of matter. Spherically symmetric static solutions are derived in section 5 and their properties under a Weyl "gauge" group made manifest. We interpret these solutions as Reissner-Nordström black holes with gravitational "Weyl charge". We justify this nomenclature in section 6 which deals with the conservation of this charge. In section 7 the nature of gravitational coupling to spinor matter in this theory is investigated and a solution exhibiting a plane-symmetric gravitational metric wave coupled via a non-Riemannian geometry to a propagating spinor field is presented.

2. Non-Riemannian Geometry

A linear connection on a manifold provides a covariant way to differentiate tensor fields. It provides a type preserving derivation on the algebra of tensor fields that commutes with contractions. Such a connection will be denoted ∇. Given an arbitrary local basis of vector fields $\{X_a\}$ the most general linear connection is specified locally by a set of n^2 1-forms $\Lambda^a{}_b$ where n is the dimension of the manifold:

$$\nabla_{X_a} X_b = \Lambda^c{}_b(X_a)\, X_c. \tag{3}$$

Such a connection can be fixed by specifying a (2, 0) symmetric metric tensor $\mathbf{g}$, a (2-antisymmetric, 1) tensor $\mathbf{T}$ and a (3, 0) tensor $\mathbf{S}$, symmetric in its last two arguments. If we require that $\mathbf{T}$ be the torsion of ∇ and $\mathbf{S}$ be the gradient of $\mathbf{g}$ then it is straightforward to determine the connection in terms of these tensors. Indeed since ∇ is defined to commute with contractions and reduce to differentiation on scalars it follows from the relation

$$X(\mathbf{g}(Y,Z)) = \mathbf{S}(X,Y,Z) + \mathbf{g}(\nabla_X Y, Z) + \mathbf{g}(Y, \nabla_X Z) \tag{4}$$

that

$$\begin{aligned} 2\mathbf{g}(Z,\nabla_X Y) = {} & X(\mathbf{g}(Y,Z)) + Y(\mathbf{g}(Z,X)) - Z(\mathbf{g}(X,Y)) - \\ & \mathbf{g}(X,[Y,Z]) - \mathbf{g}(Y,[X,Z]) - \mathbf{g}(Z,[Y,X]) - \\ & \mathbf{g}(X,\mathbf{T}(Y,Z)) - \mathbf{g}(Y,\mathbf{T}(X,Z)) - \mathbf{g}(Z,\mathbf{T}(Y,X)) - \end{aligned}$$

$$\mathbf{S}(X,Y,Z)-\mathbf{S}(Y,Z,X)+\mathbf{S}(Z,X,Y) \tag{5}$$

where X, Y, Z are any vector fields. The general curvature operator $\mathbf{R}_{X,Y}$ defined in terms of ∇ by

$$\mathbf{R}_{X,Y}Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z \tag{6}$$

is also a type-preserving tensor derivation on the algebra of tensor fields. The general (3, 1) curvature tensor $\mathbf{R}$ of ∇ is defined by

$$\mathbf{R}(X,Y,Z,\beta)=\beta(\mathbf{R}_{X,Y}Z) \tag{7}$$

where β is an arbitrary 1-form. This tensor gives rise to a set of local curvature 2-forms

$$R^a{}_b(X,Y)=\frac{1}{2}\,\mathbf{R}(X,Y,X_b,e^a) \tag{8}$$

where $\{e^c\}$ is any local basis of 1-forms dual to $\{X_c\}$, $\quad e^a(X_b)=\delta^a{}_b$. In terms of the connection forms

$$R^a{}_b=\mathrm{d}\Lambda^a{}_b+\Lambda^a{}_c\wedge\Lambda^c{}_b. \tag{9}$$

In a similar manner the torsion tensor gives rise to a set of local torsion 2-forms T^a:

$$T^a(X,Y)\equiv\frac{1}{2}\,e^a(\mathbf{T}(X,Y)) \tag{10}$$

which can be expressed in terms of the connection forms as

$$T^a=\mathrm{d}e^a+\Lambda^a{}_b\wedge e^b. \tag{11}$$

Since the metric is symmetric the tensor $\mathbf{S}$ can be used to define a set of local non-metricity 1-forms Q_{ab} symmetric in their indices:

$$Q_{ab}(Z)=\mathbf{S}(Z,X_a,X_b). \tag{12}$$

It will prove useful in the following to make use of the exterior covariant derivative D. Its definition may be found in [22]. With $g_{ab}\equiv\mathbf{g}(X_a,X_b)$, it follows from (2) that

$$Q_{ab}=\mathrm{D}g_{ab} \tag{13}$$

$$Q^{ab}=-\mathrm{D}g^{ab} \tag{14}$$

where, as usual, indices are raised and lowered with components of the metric in the ambient local basis or cobasis. We shall denote the metric trace of these forms

$$Q\equiv Q^a{}_a \tag{15}$$

and refer to this as the Weyl 1-form. Riemannian geometry chooses a metric-compatible torsion-free connection where $\mathbf{S}$ and $\mathbf{T}$ are zero. A geometry with a torsion-free connection that preserves a conformal metric is known as a Weyl space [23].

It is sometimes useful to decompose $\Lambda^a{}_b$ into its Riemannian and non-Riemannian parts, $\Omega^a{}_b$ and $\lambda^a{}_b$ respectively:

$$\Lambda^a{}_b=\Omega^a{}_b+\lambda^a{}_b. \tag{16}$$

It follows from (5) that

$$2\Omega_{ab}=(g_{ac}i_b-g_{bc}i_a+e_c i_a i_b)\,\mathrm{d}e^c+(i_b\mathrm{d}g_{ac}-i_a\mathrm{d}g_{bc})\,e^c+\mathrm{d}g_{ab} \tag{17}$$

and

$$2\lambda_{ab} = i_a\, T_b - i_b\, T_a - (i_a i_b T_c + i_b Q_{ac} - i_a Q_{bc})\, e^c - Q_{ab} \tag{18}$$

where, for any frame $\{X_a\}$, we abbreviate the contraction operator with respect to X_a by i_a.

3. The Variational Principle

We wish to find equations for the local extrema of action functionals $\mathcal{S}$ of a general connection and tensor fields. We consider 1-parameter deformations of these fields in determining the variational equations in the traditional manner and denote the variational derivative of a tensor W with respect to a field F by $\underbrace{\dot{W}}_{F}$. Since a specification of a metric $\mathbf{g}$ and a connection ∇ determines a non-Riemannian geometry we shall first consider functionals $\mathcal{S}[\mathbf{g}, \nabla]$. If we take $\mathcal{S}[\mathbf{g}, \nabla] = \int \Lambda(\mathbf{g}, \nabla)$ for some n-form Λ the field equations of the theory follow from

$$\underbrace{\dot{\Lambda}}_{\mathbf{g}} = 0 \tag{19}$$

$$\underbrace{\dot{\Lambda}}_{\nabla} = 0 \tag{20}$$

where the variations of Λ are induced respectively from the tensor variations

$$\mathbf{h} = \dot{\mathbf{g}} \tag{21}$$

in the metric and

$$\boldsymbol{\gamma} = \dot{\nabla} \tag{22}$$

in the connection. Since we shall consider functionals constructed from the curvature tensor and its various contractions we need a basic formula for the variation of the curvature operator. Regarding ∇ as a curve in the space of all connections its tangent induces a variation in the curvature operator given by:

$$\underbrace{\dot{\mathbf{R}}_{X,Y} Z}_{\nabla} = \dot{\nabla}_X \nabla_Y Z - \dot{\nabla}_Y \nabla_X Z + \nabla_X \dot{\nabla}_Y Z - \nabla_Y \dot{\nabla}_X Z - \dot{\nabla}_{[X,Y]} Z. \tag{23}$$

Since $\dot{\nabla}$ is a (2, 1) tensor, which we have denoted by $\boldsymbol{\gamma}$, then

$$\underbrace{\dot{\mathbf{R}}_{X,Y} Z}_{\nabla} = (\nabla_X \boldsymbol{\gamma})(Y, Z, -) - (\nabla_Y \boldsymbol{\gamma})(X, Z, -) + \boldsymbol{\gamma}(\mathbf{T}(X, Y), Z, -) \tag{24}$$

This formula simplifies considerably if we introduce a set of local 1-forms $\gamma^a{}_b$ by

$$\boldsymbol{\gamma}(X, X_b, e^a) = \gamma^a{}_b(X) \tag{25}$$

and write it in terms of the exterior covariant derivative

$$e^a(\dot{\mathbf{R}}_{X,Y} X_b) = 2(\mathrm{D}\gamma^a{}_b)(X, Y). \tag{26}$$

Finally we denote by $\star$ the Hodge map of the metric $\mathbf{g}$ and write the n-form measure $\star 1$ as $\mu_{\mathbf{g}}$ to remind us of its metric dependence. It is a standard result that

$$\underbrace{\dot{\mu}_{\mathbf{g}}}_{\mathbf{g}} = \frac{1}{2} \mathbf{h}(X_a, X^a) \mu_{\mathbf{g}}. \tag{27}$$

4. The Field Equations

Einstein's field equations in the absence of matter follow as the variational equations deduced from the Einstein-Hilbert action. This is the integral of the curvature scalar of the Levi-Civita connection. This scalar is obtained by a metric contraction of the Ricci tensor. The latter is a trace of the curvature operator. When the metric has a gradient and the connection is not torsion-free there are basicly two distinct second rank tensors that can be obtained by taking different traces:

$$\mathbf{Ric}(X,Y) = e^a(\mathbf{R}_{X_a,X}Y) \tag{28}$$

$$\mathbf{ric}(X,Y) = e^a(\mathbf{R}_{X,Y}X_a). \tag{29}$$

The first has no symmetry in general while **ric** is a 2-form. It follows from (17), (18) and (9) that

$$\mathbf{ric} = 2R^a{}_a = -\mathrm{d}Q. \tag{30}$$

The symmetric part of **Ric** can be contracted with the symmetric metric tensor to define a generalised curvature

$$\mathcal{R} = \mathbf{Ric}(X_a, X_b)\mathbf{g}(X^a, X^b). \tag{31}$$

In view of the comments in the introduction concerning the gauge structure of Maxwell's action for electromagnetism, by analogy we adopt as the gravitational action functional the integral

$$\int \Lambda = \int \kappa_1\, \mathcal{R}\mu_{\mathbf{g}} + \kappa_2\, \mathbf{ric} \wedge \star\mathbf{ric} \tag{32}$$

where κ_1 and κ_2 are couplings. Regarding this as a functional of $\mathbf{g}$ and ∇ we may exploit the variational derivatives in the last section to deduce field equations for the metric and connection. The relation

$$\begin{aligned}\underbrace{(\mathbf{ric})^{\cdot} \wedge \star\mathbf{ric}}_{\nabla} &= 4\,\mathrm{D}\gamma^a{}_a \wedge \star\mathbf{ric} \\ &= 4\,\gamma^a{}_a \wedge \mathrm{d} \star \mathbf{ric} + 4\,\mathrm{d}(\gamma^a{}_a \wedge \star\mathbf{ric})\end{aligned} \tag{33}$$

isolates the connection variation from the second term in the action. The connection variation of the generalised Einstein-Hilbert term is

$$\begin{aligned}\underbrace{\dot{\mathcal{R}} \star 1}_{\nabla} &= g^{ab}\ \underbrace{e^c(\dot{\mathbf{R}}_{X_c,X_x} X_b)}_{\nabla} \star 1 = g^{ab}(\mathrm{D}\gamma^c{}_b)(X_c, X_a) \star 1 = g^{ab}(i_{X_a} i_{X_c} \mathrm{D}\gamma^c{}_b) \star 1 \\ &= \quad -g^{ab}\mathrm{D}\gamma^c{}_b \wedge \star(e_a \wedge e_c) = -\gamma^c{}_b \wedge \mathrm{D}(\star(e^b \wedge e_c)) - \mathrm{d}(\gamma^c{}_b \wedge \star(e^b \wedge e_c)).\end{aligned} \tag{34}$$

Hence for variations with compact support:

$$\int \underbrace{\dot{\Lambda}}_{\nabla} = \int -\kappa_1\, \gamma^b{}_a \wedge \mathrm{D} \star (e^a \wedge e_b) + 4\kappa_2\, \gamma^a{}_a \wedge \mathrm{d} \star \mathbf{ric}. \tag{35}$$

In a similar manner the metric variations of the generalised Einstein-Hilbert term give

$$\begin{aligned}\underbrace{(\mathcal{R} \star 1)^{\cdot}}_{\mathbf{g}} &= -(\mathbf{Ric})(X^b, X^c)\,\mathbf{h}(X_b, X_c)\mu_{\mathbf{g}} + \mathcal{R}\ \underbrace{\dot{\mu}_{\mathbf{g}}}_{\mathbf{g}} \\ &= -\mathbf{h}(X_b, X_c)\,\mathbf{Ein}(X^b, X^c)\mu_{\mathbf{g}}\end{aligned} \tag{36}$$

while the second term in the action contributes a term involving the **ric** stress tensor $\mathcal{T}_{[\mathbf{ric}]}$:

$$\int \underbrace{\dot{\Lambda}}_{\mathbf{g}} = -\int \mathbf{h}(X^b, X^c)\,(\kappa_1\,\mathbf{Ein}(X_b, X_c) + \kappa_2\,\mathcal{T}_{[\mathbf{ric}]}(X_b, X_c))\mu_{\mathbf{g}}. \tag{37}$$

In these relations $\mathrm{ric}_{ab} \equiv \mathbf{ric}(X_a, X_b)$ and

$$\mathbf{Ein} = \mathrm{Sym}(\mathbf{Ric}) - \frac{1}{2}\mathbf{g}\mathcal{R} \tag{38}$$

$$\tau_{[\mathbf{ric}]a} \equiv \frac{1}{2}(i_{X_a}\mathbf{ric} \wedge \star\mathbf{ric} - i_{X_a} \star \mathbf{ric} \wedge \mathbf{ric}) \tag{39}$$

$$\mathcal{T}_{[\mathbf{ric}]ab} \equiv \mathcal{T}_{[\mathbf{ric}]}(X_a, X_b) = \star^{-1}(e_a \wedge \tau_{[\mathbf{ric}]b}) \tag{40}$$

$$= -4\,\mathrm{ric}_{ac}\ \mathrm{ric}^c{}_b - g_{ab}\,\mathrm{ric}_{cd}\ \mathrm{ric}^{cd}. \tag{41}$$

Thus the field equations are

$$\kappa_1\,\mathbf{Ein} + \kappa_2\,\mathcal{T}_{[\mathbf{ric}]} = 0 \tag{42}$$

$$\mathcal{C}_b{}^a = 0 \tag{43}$$

where

$$\mathcal{C}_b{}^a = -\kappa_1\,\mathrm{D}\star(e^a \wedge e_b) + 4\,\delta^a{}_b\,\kappa_2\,\mathrm{d}\star\mathbf{ric}. \tag{44}$$

Equation (43) gives immediately

$$\mathrm{d}\star\mathbf{ric} = 0 \tag{45}$$

$$\mathrm{D}\star(e^a \wedge e_b) = 0. \tag{46}$$

In general, the variational derivative of the integral

$$\int \mathcal{R}\mu_{\mathbf{g}} + \mathcal{F}(\mathbf{g}, \nabla, \ldots) \tag{47}$$

with respect to the connection gives rise to the equation

$$\mathrm{D}\star(e^a \wedge e_b) = \mathcal{F}^a{}_b \tag{48}$$

for some $(n-1)$-forms $\mathcal{F}^a{}_b$ defined by

$$\underbrace{\dot{\mathcal{F}}}_{\nabla} = \gamma^b{}_a \wedge \mathcal{F}^a{}_b. \tag{49}$$

Clearly this equation implies

$$\mathcal{F}^a{}_a = 0. \tag{50}$$

It is convenient to decompose Q_{ab} into its trace and trace-free parts: [24]

$$Q_{ab} = \widehat{Q}_{ab} + \frac{1}{n}\,g_{ab}\,Q \tag{51}$$

so that $\widehat{Q}^a{}_a = 0$. Equation (48) can be then decomposed as

$$i_b\,\widehat{Q}_a{}^c - \delta^c{}_b\,i_d\widehat{Q}_a{}^d + (\delta^c{}_b\,\delta^d{}_a - \delta^c{}_a\,\delta^d{}_b)\,(\frac{n-2}{2n}\,i_dQ - i_d\,i_h\,T^h) - i_b\,i_a\,T^c + f^c{}_{ab} = 0 \tag{52}$$

that is,

$$i_a\,\widehat{Q}_{bc} - i_a\,i_b\,T_c = -\frac{1}{2n}\,g_{bc}\,i_a\,Q + \frac{1}{2n}\,g_{ac}\,i_b\,Q - f_{cba}$$

$$-\frac{1}{n(n-2)}\,g_{ac}\,f^{d}{}_{db}+\frac{n-1}{n(n-2)}\,g_{bc}\,f^{d}{}_{da}+\frac{n-1}{n(n-2)}\,g_{ac}\,f^{d}{}_{bd}-\frac{1}{n(n-2)}\,g_{bc}\,f^{d}{}_{ad} \qquad (53)$$

where $\mathcal{F}^{a}{}_{b}=f^{ca}{}_{b}\star e_{c}$. Using the symmetry of $\widehat{Q}_{bc}$ and the antisymmetry of $i_a\,i_b\,T_c$ in $a\,b$, and splitting off the trace-free part from the torsion form:

$$T^{a}=\widehat{T}^{a}+\frac{1}{n-1}\,e^{a}\wedge T \qquad (54)$$

where

$$T\equiv i_a\,T^{a} \qquad (55)$$

such that $i_a\,\widehat{T}^{a}=0$, it follows that

$$i_a\,\widehat{Q}_{bc}=\frac{1}{n}\,g_{bc}\,(f^{d}{}_{da}+f^{d}{}_{ad})-\frac{1}{2}\,(f_{bac}+f_{bca}+f_{cab}+f_{cba}-f_{abc}-f_{acb}) \qquad (56)$$

$$i_a\,i_b\,\widehat{T}_{c}=\frac{1}{n-1}\,(g_{bc}\,f^{d}{}_{ad}-g_{ac}\,f^{d}{}_{bd})-\frac{1}{2}\,(f_{bac}+f_{bca}+f_{cab}-f_{cba}-f_{abc}-f_{acb}) \qquad (57)$$

$$T-\frac{n-1}{2n}\,Q=\left(\frac{1}{n(n-2)}\,f^{c}{}_{ac}-\frac{n-1}{n(n-2)}\,f^{c}{}_{ca}\right)\,e^{a}. \qquad (58)$$

The equations (56) , (57) and (58) provide the general solution to (48). It will be observed that the trace-free parts are determined in terms of contributions from $\mathcal{F}$ while the trace of the torsion and Weyl forms remain correlated. If the contributions from $\mathcal{F}$ are algebraic in the components of the torsion and metric-gradient these equations enable one to immediately express the connection in terms of the metric, the components of $\star\mathcal{F}^{a}{}_{b}$ and the Weyl form Q.

Before analysing the field equations (45) and (46) corresponding to $\mathcal{F}=\kappa_2/\kappa_1\,\mathbf{ric}\wedge\star\mathbf{ric}$, it is of interest to verify that (48) is equivalent to the variational equations of any action regarded as the functional of $\mathbf{g}$, $\mathbf{T}$ and $\mathbf{S}$ obtained by replacing the connection explicitly by (5). We note that the $(n(n+1)+2n^3)/2$ component variations of the metric and connection may be considered as induced by the $n(n+1)/2$ component metric variations, the $n^2(n+1)/2$ component variations of the metric-gradient tensor and the $n^2(n-1)/2$ component variations of the torsion tensor in n dimensions. Thus the $\gamma^{a}{}_{b}$ variations in

$$\int \underbrace{\dot{\Lambda}}_{\mathbf{g},\mathbf{T},\mathbf{S}}=-\int \mathbf{h}(X^{b},X^{c})\,(\kappa_1\,\mathbf{Ein}(X_b,X_c)+\kappa_2\,\mathcal{T}_{[\mathbf{ric}]}(X_b,X_c))\mu_{\mathbf{g}}+\gamma^{a}{}_{b}\wedge\mathcal{C}_{a}{}^{b}. \qquad (59)$$

are to be induced by variations $\mathbf{h}$, $\dot{\mathbf{T}}$ and $\dot{\mathbf{S}}$ of $\mathbf{g}$, $\mathbf{T}$ and $\mathbf{S}$ respectively. By varying (5) we readily find

$$\gamma(X,Y,\beta)=\underbrace{\gamma(X,Y,\beta)}_{\mathbf{T}}+\underbrace{\gamma(X,Y,\beta)}_{\mathbf{S}}+\underbrace{\gamma(X,Y,\beta)}_{\mathbf{g}} \qquad (60)$$

where

$$2\,\underbrace{\gamma(X,Y,\beta)}_{\mathbf{T}}=g(X,\dot{\mathbf{T}}(\tilde{\beta},Y)+g(Y,\dot{\mathbf{T}}(\tilde{\beta},X)+g(\tilde{\beta},\dot{\mathbf{T}}(X,Y)) \qquad (61)$$

$$2\,\underbrace{\gamma(X,Y,\beta)}_{\mathbf{S}}=\dot{\mathbf{S}}(\tilde{\beta},X,Y)-\dot{\mathbf{S}}(X,Y,\tilde{\beta})-\dot{\mathbf{S}}(Y,\tilde{\beta},X) \qquad (62)$$

$$2\underbrace{\gamma(X,Y,\beta)}_{\mathbf{g}} = (\nabla_X \mathbf{h})(Y,\tilde{\beta}) + (\nabla_Y \mathbf{h})(X,\tilde{\beta}) - (\nabla_{\tilde{\beta}} \mathbf{h})(X,Y) \tag{63}$$

$$-\mathbf{h}(X,\mathbf{T}(Y,\tilde{\beta})) - \mathbf{h}(Y,\mathbf{T}(X,\tilde{\beta})) - \mathbf{h}(\tilde{\beta},\mathbf{T}(Y,X)) \tag{64}$$

for any vector fields X, Y and 1-form β with metric dual $\tilde{\beta}$ defined by $g(\tilde{\beta}, Y) = \beta(Y)$. Writing

$$\gamma^a{}_b \wedge \mathcal{C}_a{}^b = \gamma(X_c, X_b, e^a)\, e^c \wedge \mathcal{C}_a{}^b$$

we may express these relations in terms of $h_{ab} = \mathbf{h}(X_a, X_b)$, $T_{ab}{}^c = e^c(\mathbf{T}(X_a, X_b))$, $S_{abc} = \mathbf{S}(X_a, X_b, X_c)$ and $\gamma^a{}_b(X_c) = \gamma_{cb}{}^a$ where

$$\gamma_{cba} = \frac{1}{2}\{\dot{T}_{abc} + \dot{T}_{acb} + \dot{T}_{cba}\} + \frac{1}{2}\{-\dot{S}_{cba} - \dot{S}_{bac} + \dot{S}_{acb}\} +$$

$$\frac{1}{2}\{(\mathrm{D}h_{ab})(X_c) + (\mathrm{D}h_{ac})(X_b) - (\mathrm{D}h_{bc})(X_a) - h_{ck}\, T_{ba}{}^k - h_{bk}\, T_{ca}{}^k - h_{ak}\, T_{bc}{}^k\}. \tag{65}$$

Thus in terms of the n-forms $\mathcal{C}_a{}^{bc} \equiv e^c \wedge \mathcal{C}_a{}^b$ in n-dimensions:

$$\gamma^a{}_b \wedge \mathcal{C}_a{}^b = \gamma_{cb}{}^a \mathcal{C}_a{}^{bc}$$

$$= \frac{1}{2}\dot{T}_{abc}\{\mathcal{C}^{abc} + \mathcal{C}^{acb} + \mathcal{C}^{cba}\} + \frac{1}{2}\dot{S}_{abc}\{-\mathcal{C}^{cba} - \mathcal{C}^{bac} + \mathcal{C}^{abc}\} +$$

$$\frac{1}{2} h_{ab}\{T_{ck}{}^b\, \mathcal{C}^{cka} - T_{ck}{}^b\, \mathcal{C}^{kac} + T_{ck}{}^b\, \mathcal{C}^{akc}\} +$$

$$+\frac{1}{2}(\mathrm{D}h_{ab})(X_c)\{\mathcal{C}^{abc} + \mathcal{C}^{acb} - \mathcal{C}^{cab}\}. \tag{66}$$

The independent torsion variations and metric-gradient variations give respectively:

$$\mathcal{C}^{abc} + \mathcal{C}^{acb} + \mathcal{C}^{cba} - \mathcal{C}^{bac} - \mathcal{C}^{bca} - \mathcal{C}^{cab} = 0 \tag{67}$$

$$\mathcal{C}^{abc} + \mathcal{C}^{acb} - \mathcal{C}^{cba} - \mathcal{C}^{bac} - \mathcal{C}^{bca} - \mathcal{C}^{cab} = 0. \tag{68}$$

Hence we immediately deduce that $\mathcal{C}^{abc} = 0$ corresponding to

$$\mathcal{C}_b{}^a = 0 \tag{69}$$

derived above. Furthermore the metric variations now reproduce the same Einstein equation so one recovers the same field equations for the metric, metric-gradient and torsion.

5. General Solutions

We first find the general solution to (46) . This is an algebraic equation relating the components of the metric-gradient and the torsion tensor. It has less than maximal rank and from (50) and (58) with the action (32) it follows that the general solution is expressible in terms of an arbitrary Weyl form $Q = \alpha$. In n-dimensions:

$$Q_{ab} = \frac{1}{n} g_{ab}\, \alpha \tag{70}$$

$$T^a = \frac{1}{2n} e^a \wedge \alpha \tag{71}$$

corresponding to the tensors

$$\mathbf{S} = \frac{1}{n}\alpha \otimes \mathbf{g} \tag{72}$$

$$\mathbf{T} = -\frac{1}{n}(\alpha \wedge e^a) \otimes X_a. \tag{73}$$

The choice $\alpha = 0$ corresponds to (pseudo-)Riemannian geometry with $\mathbf{ric} = 0$ and reduces the theory to Einsteinian gravity for n=4. The remaining system (42) and (45) is reminiscent of the Maxwell-Einstein system. Indeed it is not difficult to show that for any torsion and metric-gradient our action (32) is invariant under the "gauge" transformations:

$$\mathbf{g} \mapsto \mathbf{g}$$
$$\nabla \mapsto \nabla + \mathrm{d}f \otimes \tag{74}$$

when ∇ acts on vector fields and f is any 0-form. This implies that the curvature operator and hence $\mathcal{R}$ and $\mathbf{ric}$ are gauge invariant. In a fixed local coframe the connection forms of ∇:

$$\Lambda^c{}_b \mapsto \Lambda^c{}_b + \delta^c{}_b \, \mathrm{d}f. \tag{75}$$

Furthermore the Bianchi identity $\mathrm{d}\,\mathbf{ric} = 0$ implies that locally $\mathbf{ric} = -\mathrm{d}\alpha$ for some 1-form α. From (30) we may identify the potential α with the Weyl form Q. It is important to stress that in contradistinction to electromagnetism where the gauge potential is identified with a 1-form in the Lie algebra of the compact $U(1)$ group, the 1-form α is determined by the dynamical non-Riemannian geometry associated with a connection in the Lie-algebra of the general linear group, the structure group of the bundle of linear frames. With the solutions (70) and (71) it follows that

$$\lambda_{ab} = -\frac{1}{8} g_{ab} Q \tag{76}$$

and the tensor **Ein** reduces to the Levi-Civita Einstein tensor of Einstein's theory. Consequently (42) is isomorphic to the standard Einstein-Maxwell system with the gravitational field dQ corresponding to the Maxwell field strength. It follows that (42) admits *all* solutions to that system with an appropriate correspondence between couplings. For example, with $Q = \alpha$ we find the exact spherically symmetric static spacetime ($n = 4$) solution to the remaining field equations (42) and (45) :

$$\alpha = \frac{q\,\mathrm{d}t}{r} \tag{77}$$

$$\mathbf{g} = -\left(1 - \frac{2M}{r} - \frac{\kappa_2\, q^2}{2\kappa_1\, r^2}\right) \mathrm{d}t \otimes \mathrm{d}t + \left(1 - \frac{2M}{r} - \frac{\kappa_2\, q^2}{2\kappa_1\, r^2}\right)^{-1} \mathrm{d}r \otimes \mathrm{d}r +$$
$$r^2\, \mathrm{d}\theta \otimes \mathrm{d}\theta + r^2\, \sin^2\theta \mathrm{d}\phi \otimes \mathrm{d}\phi \tag{78}$$

in a chart with coordinates $\{t, r, \theta, \phi\}$. M and q are arbitrary constants and the metric is asymptotically flat. For $q \neq 0$ our metric takes the form of the Reissner-Nordström solution where the associated stress tensor is provided by contributions from the Weyl form (15). This particular solution has also featured in [25] where a considerably more complex non-Riemannian action has been analysed. We stress that the vacuum field equations from the action (32) above admit solutions that may be constructed from *all* Einstein-Maxwell solutions.

If we define a geodesic test particle to be one that follows a time-like geodesic associated with the metric $\mathbf{g}$ then both electrically neutral and electrically charged geodesic test particles would behave in the same way in the geometry of this solution. While it is by no means obvious that real test particles behave like geodesic test particles this phenomena suggests that massive particles may be endowed with a "gravitational charge" that couples to gravity in a manner similar to the way electrical charge couples to the electromagnetic field.

6. Conservation of Weyl Charge

We have identified the constant q in the Reissner-Nordström solution above with a new kind of "Weyl charge" associated with the gauge transformation (74) above. This is by analogy with the identification of electric charge of a black hole in the Einstein-Maxwell theory. It is of interest to consider how the conservation of "Weyl charge" arises in the context of a model of Weyl charged scalar fields. To this end consider the covariant derivatives of the exterior product of n orthonormal 1-forms $\star 1$ and n orthonormal vector fields $\sharp 1$:

$$\nabla \star 1 = \frac{1}{2} Q \otimes \star 1 \tag{79}$$

$$\nabla \sharp 1 = -\frac{1}{2} Q \otimes \sharp 1 \tag{80}$$

in terms of the Weyl form Q. Any real n-form Φ can be expressed in terms of the pseudo-scalar φ with respect to $\star 1$ by:

$$\Phi = \varphi \star 1. \tag{81}$$

Similarly any real n-multi-vector Θ can be expressed in terms of the pseudo-scalar ϑ with respect to $\sharp 1$ by:

$$\Theta = \vartheta \sharp 1. \tag{82}$$

Using (79) and (80) it follows that

$$\nabla \Phi = \mathcal{D}\varphi \otimes \star 1 \tag{83}$$

$$\nabla \Theta = \bar{\mathcal{D}}\vartheta \otimes \sharp 1 \tag{84}$$

where

$$\mathcal{D} \equiv \mathrm{d} + \frac{1}{2} Q\wedge \tag{85}$$

$$\bar{\mathcal{D}} \equiv \mathrm{d} - \frac{1}{2} Q\wedge \tag{86}$$

denote the appropriate Weyl exterior covariant derivatives. The action n-form:

$$\mathcal{F}_{\vartheta,\varphi}(\mathbf{g}, \nabla, \vartheta, \varphi) \equiv \nabla_{X_a}\Phi \, \sharp^{-1} \nabla_{X^a}\Theta + m^2 \Phi \, \sharp^{-1} \Theta = \mathcal{D}\varphi \wedge \star \bar{\mathcal{D}}\vartheta + m^2 \varphi \, \vartheta \star 1 \tag{87}$$

is invariant under the transformations:

$$\mathbf{g} \mapsto \mathbf{g}, \quad \Phi \mapsto \exp(nf)\,\Phi, \quad \Theta \mapsto \exp(-nf)\,\Theta \tag{88}$$

together with

$$\nabla \mapsto \nabla + \mathrm{d}f\otimes$$

acting on vector fields. These induce the corresponding transformations

$$\varphi \mapsto \exp(nf)\,\varphi, \quad \vartheta \mapsto \exp(-nf)\,\vartheta, \quad Q \mapsto Q - 2n\,\mathrm{d}f.$$

The field equations for φ and ϑ follow from the variations

$$\underbrace{\dot{\mathcal{F}}_{\vartheta,\varphi}}_{\varphi} = -\dot{\varphi}\,(\,\bar{\mathcal{D}} \star \bar{\mathcal{D}}\vartheta - m^2\vartheta \star 1\,) \;+\; \mathrm{d}(\dot{\varphi} \star \bar{\mathcal{D}}\vartheta)$$

$$\underbrace{\dot{\mathcal{F}}_{\vartheta,\varphi}}_{\vartheta} = -\dot{\vartheta}\,(\,\mathcal{D} \star \mathcal{D}\varphi - m^2\varphi \star 1\,) \;+\; \mathrm{d}(\dot{\vartheta} \star \mathcal{D}\varphi)$$

as

$$\bar{\mathcal{D}} \star \bar{\mathcal{D}}\vartheta - m^2\vartheta \star 1 = 0 \tag{89}$$

$$\mathcal{D} \star \mathcal{D}\varphi - m^2\varphi \star 1 = 0\,. \tag{90}$$

In these equations we may identify the real constant m as the mass of the real "Weyl doublet" (φ, ϑ).

The metric and connection variations follow from the relations

$$\underbrace{\dot{Q}}_{\mathbf{g}} = \mathrm{d}h \tag{91}$$

$$\underbrace{\dot{Q}}_{\nabla} = -2\gamma \tag{92}$$

where

$$h \equiv h^a{}_a \tag{93}$$

$$\gamma \equiv \gamma^a{}_a. \tag{94}$$

If we now add the integral of the action density (87) to (32) and consider variations in ∇ we generate the field equations

$$4\kappa_2\,\mathrm{d} \star \mathbf{ric} + \mathbf{j} = 0 \tag{95}$$

$$\mathrm{D} \star (e^a \wedge e_b) = 0 \tag{96}$$

where

$$\mathbf{j} \equiv \vartheta \star \mathcal{D}\varphi \;-\; \varphi \star \bar{\mathcal{D}}\vartheta \tag{97}$$

follows from the variation

$$\underbrace{\dot{\mathcal{F}}_{\vartheta,\varphi}}_{\nabla} = \frac{1}{2}\underbrace{\dot{Q}}_{\nabla}\,\varphi \wedge \star\bar{\mathcal{D}}\vartheta \;-\; \frac{1}{2}\underbrace{\dot{Q}}_{\nabla}\,\vartheta \wedge \star\mathcal{D}\varphi = -\gamma \wedge (\,\varphi \star \bar{\mathcal{D}}\vartheta \;-\; \vartheta \star \mathcal{D}\varphi\,).$$

The equation (96) may be solved as before to fix the connection. The closure of $\mathbf{j}$ is compatible with (89) and (90) since

$$\begin{aligned} \mathrm{d}\,\mathbf{j} &= \mathrm{d}\vartheta \wedge \star\mathcal{D}\varphi \;+\; \vartheta\,\mathrm{d} \star \mathcal{D}\phi \;-\; \mathrm{d}\varphi \wedge \star\bar{\mathcal{D}}\vartheta \;-\; \varphi\,\mathrm{d} \star \bar{\mathcal{D}}\vartheta \\ &= \mathrm{d}\vartheta \wedge \star\mathcal{D}\phi \;+\; \vartheta\,(-\frac{1}{2}\,Q \wedge \star\mathcal{D}\varphi + m^2\varphi \star 1) \\ &\quad - \mathrm{d}\varphi \wedge \star\bar{\mathcal{D}}\vartheta \;-\; \varphi\,(\frac{1}{2}\,Q \wedge \star\bar{\mathcal{D}}\vartheta + m^2\vartheta \star 1) \\ &= \bar{\mathcal{D}}\vartheta \wedge \star\mathcal{D}\varphi \;-\; \mathcal{D}\varphi \wedge \star\bar{\mathcal{D}}\vartheta = 0. \end{aligned}$$

The set of field equations is completed by taking the metric variation:

$$\underbrace{\dot{\mathcal{F}}_{\vartheta,\varphi}}_{\mathbf{g}} = \mathcal{D}\varphi \wedge \underbrace{\dot{\star}}_{\mathbf{g}}\,\bar{\mathcal{D}}\vartheta \;+\; m^2\,\varphi\,\vartheta\,\underbrace{\dot{\star 1}}_{\mathbf{g}} \;+\; \underbrace{\dot{\mathcal{D}}}_{\mathbf{g}}\,\varphi \wedge \star\bar{\mathcal{D}}\vartheta \;+\; \mathcal{D}\varphi \wedge \star\,\underbrace{\dot{\bar{\mathcal{D}}}}_{\mathbf{g}}\,\vartheta.$$

$$= -h^{ab}\,(\,i_{X_a}\,\mathcal{D}\phi\wedge\star i_{X_b}\bar{\mathcal{D}}\theta\;-\;g_{ab}\,\mathcal{F}_{\vartheta,\varphi}\,)\;+\;\frac{1}{2}\underbrace{\dot{Q}}_{\mathbf{g}}\wedge(\,\phi\star\bar{\mathcal{D}}\theta\;-\;\theta\star\mathcal{D}\phi\,). \tag{98}$$

The term in $\dot{Q}$ does not contribute since $\mathbf{j}$ is closed:

$$\underbrace{\dot{Q}}_{\mathbf{g}}\wedge(\,\varphi\star\bar{\mathcal{D}}\vartheta\;-\;\vartheta\star\mathcal{D}\varphi\,) = -\mathrm{d}h\wedge\mathbf{j} = h\,\mathrm{d}\mathbf{j} - \mathrm{d}(h\,\mathbf{j}).$$

Thus the Einstein equation for $\mathbf{g}$ becomes

$$\kappa_1\mathbf{Ein} + \kappa_2\mathcal{T}_{[\mathbf{ric}]} = \mathcal{T}_{[\vartheta,\varphi]}$$

where

$$\mathcal{T}_{[\vartheta,\varphi]} \equiv -\frac{1}{2}\left(\mathcal{D}\varphi\otimes\bar{\mathcal{D}}\vartheta + \bar{\mathcal{D}}\vartheta\otimes\mathcal{D}\varphi - \mathbf{g}(\widetilde{\mathcal{D}\phi},\widetilde{\bar{\mathcal{D}}\theta})\,\mathbf{g} - m^2\,\mathbf{g}\right) \tag{99}$$

and **Ein** remains the Einstein tensor associated with the Levi-Civita connection. Finally since $\mathbf{ric} = -\mathrm{d}Q$:

$$\mathrm{d}\,\mathbf{ric} = 0. \tag{100}$$

Thus we have demonstrated how the conservation of the "Weyl current" $\mathbf{j}$ follows from a local Weyl gauge covariant coupling of "Weyl charged" matter fields to gravity. This approach may be compared with the one given in [26] which deals with conservation laws induced by vector fields that generate *Killing symmetries* in non-Riemannian spacetimes.

7. Gravitational Wave Coupling to Spinor Matter

It is also of interest to examine the coupling of matter with zero Weyl charge to non-Riemannian gravity. To this end we examine the modification to the field equations $\mathcal{C}_b{}^a = 0$ produced when we consider a total action n-form

$$\Lambda_{\mathrm{Total}} = \Lambda + \mathcal{F}_{\Psi}(\mathbf{g},\nabla,\Psi) \tag{101}$$

As an example, consider a complex spinor field Ψ in spacetime with an action n-form

$$\mathcal{F}_{\Psi} = (\Psi\,,\not{S}\,\Psi)\mu_{\mathbf{g}}. \tag{102}$$

We shall employ the language of Clifford bundles to describe spinor fields [22] [27], although the transcription to the representation in terms of component spinors in a γ-matrix language is straightforward [28]. The *symmetric* inner product on spinors is defined as

$$(\Phi\,,\Psi) \equiv 4\,\mathrm{Re}\,\mathcal{P}_0(\widetilde{\Phi}\vee\Psi) \tag{103}$$

where $\mathcal{P}_l$ denotes the projection of a Clifford form to its l-form component and $\vee$ denotes the Clifford product between sections of the Clifford bundle associated with the spacetime metric. The familiar discussion of spinors is effected in terms of a local $\mathbf{g}$ *-orthonormal* coframe $\{e^a\}$. We choose as spinor adjoint $\widetilde{\Psi} \equiv C^{-1}\vee\Psi^{J}$ with $C = i\,e^0$ and the adjoint involution $J = \xi\eta^*$ where η and ξ are the main involution and anti-involution of the Clifford algebra respectively and $*$ signifies the complex conjugation. The operator $\not{S} \equiv e^a\vee S_{X_a}$ is the Dirac operator which is expressed in terms of a spinor covariant derivative S_X. Such a derivative may be expressed in terms of a connection on the bundle of spinor frames, whose structure group is a double cover of the spacetime Lorentz group. In terms of the local $\mathbf{g}$ *-orthonormal* coframe it is expressible in terms of the antisymmetric parts of the connection forms $\{\Lambda_{ab}\}$. For the action (102) one finds

$$\underbrace{\dot{\mathcal{F}}_{\Psi}}_{\nabla} = \gamma^a{}_b \wedge \frac{1}{4} (\Psi\,, [e^c \wedge e_a \wedge e^b] \vee \Psi) \star e_c. \tag{104}$$

It follows from (56), (57) and (58) that the traces of the torsion and non-metricity forms are correlated according

$$T - \frac{3}{8} Q = 0 \tag{105}$$

while the trace-free parts are specified to be

$$\widehat{Q}_{ab} = 0 \tag{106}$$

$$i_a\, i_b\, \widehat{T}_c = -\frac{1}{4\kappa_1} (\Psi\,, [e_c \wedge e_b \wedge e_a] \vee \Psi) \tag{107}$$

corresponding to the torsion forms:

$$\widehat{T}^c = \frac{1}{\kappa_1} \mathrm{Re}\, \mathcal{P}_2(e^c \vee \Psi \vee \widetilde{\Psi}). \tag{108}$$

With $Q = \alpha$ for some 1-form α we must solve $\mathcal{F}^a{}_a = 0$ or

$$\mathrm{d} \star \mathbf{ric} = -\mathrm{d} \star \mathrm{d}\alpha = 0 \tag{109}$$

together with the field equation arising from the vanishing of the variation of (102) induced by $\dot{\Psi}$:

$$\underbrace{\dot{\mathcal{F}}_{\Psi}}_{\Psi} = 2\,(\dot{\Psi}, \not{S}\Psi) - \left(\frac{1}{2}\, i^c \widehat{Q}_{ca} + (T - \frac{n-1}{2n}\, Q)(X_a)\right) (\dot{\Psi}, e^a \vee \Psi)\mu_{\mathbf{g}} - \mathrm{d}\left((\dot{\Psi}, e^a \vee \Psi) \star e_a\right). \tag{110}$$

Thus from (105) and (106) the spinor field equation in spacetime becomes simply:

$$\not{S}\Psi = 0. \tag{111}$$

From the metric variations one finds the Einstein equation:

$$\kappa_1\, \mathbf{Ein} + \kappa_2\, \mathcal{T}_{[\mathbf{ric}]} = \mathcal{T}_{[\Psi]} \tag{112}$$

where

$$\underbrace{\dot{\mathcal{F}}_{\Psi}}_{\mathbf{g}} = -\mathbf{h}(X^a, X^b) \mathcal{T}_{[\Psi]}(X_a, X_b) \mu_{\mathbf{g}} \tag{113}$$

with

$$\mathcal{T}_{[\Psi]} \equiv -\frac{1}{4} (\Psi, [e_a \vee S_{X_b} + e_b \vee S_{X_a}]\, \Psi)\, e^a \otimes e^b - \frac{1}{2} (\Psi, \not{S}\Psi)\, \mathbf{g}. \tag{114}$$

We seek a solution in a local coordinate system $\{u, v, x, y\}$ in which the the metric takes the form

$$g = H\, \mathrm{d}u \otimes \mathrm{d}u + \mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y. \tag{115}$$

In terms of a local orthonormal coframe

$$g = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 \tag{116}$$

where

$$e^0 = \frac{H-1}{2}\, \mathrm{d}u + \mathrm{d}v$$

$$\begin{aligned} e^1 &= \frac{H+1}{2}\,\mathrm{d}u + \mathrm{d}v \\ e^2 &= \mathrm{d}x \\ e^3 &= \mathrm{d}y \end{aligned} \tag{117}$$

for some real function $H = H(u,x,y)$. Such a metric describes a propagating plane-symmetric gravitational wave. We seek a corresponding propagating tensor **ric** with Weyl form

$$\alpha = \rho\,\mathrm{d}x + \zeta\,\mathrm{d}y \tag{118}$$

where $\rho = \rho(u)$ and $\zeta = \zeta(u)$ are real. We find that, with the metric (116), the field equation (109) is satisfied for arbitrary ρ and ζ. We choose a complex spinor basis in a minimal left ideal of the complex Clifford algebra of spacetime in which the element

$$P = \frac{1}{4}(1 + e^0 \vee e^1) \vee (1 + i\,e^2 \vee e^3) \tag{119}$$

is a primitive idempotent. A spinor basis in this ideal can be chosen in which a general spinor takes the form

$$\Psi = (\Psi_1 + \Psi_2\,e^2 - \Psi_3\,e^0 + \Psi_4\,e^0 \vee e^2) \vee P \tag{120}$$

where Ψ_1, Ψ_2, Ψ_3, Ψ_4 are complex functions and in which the elements of the orthonormal coframe (117) are represented by the γ-matrices:

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}. \tag{121}$$

In this basis, Ψ becomes

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix}. \tag{122}$$

From (105) and (106) with $Q = \alpha$ we may construct the spinor covariant derivative that appears in (111). This is a non-linear equation for the spinor components Ψ_j since Ψ appears quadratically in the spinor connection. However the propagating ansatz

$$\Psi_1 = 0,\ \Psi_2 = 0,\ \Psi_3 = \Psi_3(u),\ \Psi_4 = \Psi_4(u) \tag{123}$$

is found to be a particular solution for arbitrary $\Psi_3(u)$ and $\Psi_4(u)$. Finally the Einstein equation (112) is satisfied if

$$\frac{\kappa_1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)H=\kappa_2\left(\left(\frac{\partial\,\rho}{\partial\,u}\right)^2+\left(\frac{\partial\,\zeta}{\partial\,u}\right)^2\right)+\frac{1}{2}\,\mathrm{Im}\left(\frac{\partial\Psi_3}{\partial\,u}\,\Psi_3^*+\frac{\partial\Psi_4}{\partial\,u}\,\Psi_4^*\right) \tag{124}$$

The metric function H is determined by this equation in terms of propagating spinor, torsion and metric-gradient waves with arbitrary profiles.

8. Conclusions

The consequences of a generalisation of Einstein's metric theory of gravitation has been examined in terms of an action functional dependent on a general linear connection in additional to the spacetime metric. A simple generalisation of the variational principle with the Einstein-Hilbert action permits one to determine such a connection. We have discussed spherically symmetric static solutions in which the Weyl form may be interpreted in terms of a gravitational analogue of the Maxwell potential and the Reissner-Nordström metric arises in terms of a gravi-electric source. Moreover we have shown that the vacuum theory admits solutions for the metric corresponding to *all* those in the standard Einstein-Maxwell theory. For example a more general solution in the presence of the torsion (71) and metric-gradient (70) can be generated from the Weyl form

$$\alpha=\frac{q\,\mathrm{d}t}{r}+\mu\cos\theta\,\mathrm{d}\phi \tag{125}$$

where μ is a constant. The solution for the metric is then (78) with $q^2\mapsto q^2+\mu^2$. The theory admits both gravi-electric and gravi-magnetic poles with a duality symmetry analogous to Einstein-Maxwell theory.

An interpretation of these solutions can be based on the singularity structure of frame and gauge invariant tensors. The tensor $T_a\wedge\star T^a$ is not gauge invariant. However, since $\mathrm{D}T^a=R^a{}_b\wedge e^b$ and the curvature forms are gauge invariant under (74) ,then $\mathrm{D}T^a$ is also gauge invariant. A gauge and frame independent invariant is

$$\mathrm{D}T_a\wedge\star\mathrm{D}T^a=\frac{q^2-\mu^2}{32\,r^4}. \tag{126}$$

Similarly a gauge and frame independent curvature invariant is

$$\star(R_{ab}\wedge\star R^{ab})=-\frac{24\,M^2}{r^6}+\frac{q^2-\mu^2}{16\,r^4}-\frac{24\,\kappa_2 M\,(q^2+\mu^2)}{\kappa_1\,r^7}-\frac{7\,\kappa_2^2\,(q^2+\mu^2)^2}{\kappa_1^2\,r^8}. \tag{127}$$

The generic black hole singularity at $r=0$ is clearly visible in these expressions.

The coupling of both Weyl charged and Weyl neutral matter to this generalised theory of gravitation has also been briefly examined. The inclusion of Weyl neutral spinor matter is of interest since the spin invariant matter action is sensitive to the connection variation used to determine dynamically the non-Riemannian geometry. A family of solutions has been presented describing propagating Weyl spinor fields coupled to propagating metric, torsion and metric-gradient plane symmetric waves. The inclusion of the Maxwell action is also straightforward. The associated Maxwell stress contributes to the Einstein equation without perturbing the non-Riemannian fields. Indeed an Einstein-Weyl-Maxwell solution exists with Weyl form (125), Maxwell 1-form

$$A = \frac{q_0\, \mathrm{d}t}{r} + \mu_0 \cos\theta\, \mathrm{d}\phi \tag{128}$$

with constants q_0 and μ_0 and metric given by (78) where $\kappa_2\, \mu^2 \mapsto \kappa_2\, \mu^2 + \mu_0^2$ and $\kappa_2\, q^2 \mapsto \kappa_2\, q^2 + q_0^2$.

The significance of non-Riemannian gravitational fields has long been recognised by a number of coworkers. Their relevance in recent low energy effective actions has stimulated a renewed interest in the physical significance of these fields. The solutions discussed above may offer an opportunity to confront theoretical predictions with some of the classical tests of general relativity and place bounds on the couplings that enter in the modifications to Einstein's theory. They also raise intriguing questions about the nature of the coupling of test particles with Weyl charge to the non-Riemannian components of the gravitational field. The relevance of such non-Riemannian fields in the astrophysical sector will be discussed elsewhere.

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