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Abstract. A quantum theory is developed for a difference-difference system which can serve as a toy-model of the quantum Korteveg-de-Vries equation.

Introduction

This Letter presents an example of a completely integrable 'discrete-space-time quantum model' whose Heisenberg equations of motion have the form

$$\begin{aligned}&\phi(\tau,n)\,\phi(\tau,n-1)+\lambda\,\phi(\tau,n-1)\,\phi(\tau-1,n-1)\\&=\lambda\,\phi(\tau,n)\,\phi(\tau-1,n)+\phi(\tau-1,n)\,\phi(\tau-1,n-1).\end{aligned}\tag{1}$$

By 'discrete...model' we mean

(i) an algebra 'of observables' Φ,

whose generators ϕ_n are labeled by integer numbers n which are regarded as a (discrete) spatial variable; together with

(ii) an automorphism $\mathcal{Q}$,

whose sequential action

$$\mathsf{x}(0)\equiv\mathsf{x}\in\Phi$$
$$\mathcal{Q}:\quad\ldots\mapsto\mathsf{x}(\tau-1)\mapsto\mathsf{x}(\tau)\mapsto\mathsf{x}(\tau+1)\mapsto\ldots$$

is viewed as the (discrete) time evolution. Thus, we intend to produce a pair $\Phi\&\mathcal{Q}$ such that the evolution $\phi_n(\tau)$ of generators, in the natural notation

$$\phi(\tau,n)\equiv\phi_n(\tau),$$

obeys the system (1).

Complete integrability is understood as the existence of a commutative subalgebra 'of conservation laws' preserved under time evolution and spanning, in a sense, half of the algebra of observables: it is commonly believed that a Hamiltonian system may be either 'completely' nonintegrable possessing only a few conservation laws due to its manifest symmetries, or completely integrable enjoying a whole lot of conservation laws, one per degree of freedom. The commutative subalgebra which we encounter in this Letter definitely contains a lot of conservation laws but the question of how many is left to be answered elsewhere.

Actually, we deal here not with a single model but rather with a family of them $\Phi\&\mathcal{Q}(\lambda)$, each model being related to a certain value of a complex parameter λ in (1). Moreover, all their evolution automorphisms $\mathcal{Q}(\lambda)$ turn out to be mutually commuting and sharing the common subalgebra of conservation laws.

The order of presentation is as follows. In Section 1 we introduce an algebra of observables which is basically the same lattice $U(1)$ exchange algebra which appeared already in [FV93, 94]. Naturally, the behaviour of that algebra depends on the value of a constant q involved in the commutation relations. For simplicity we shall assume that q is a root of unity.

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After some preliminaries (Sections 2 and 3) we pick in Section 4 from the algebra of observables a set of 'Fateev-Zamolodchikov R-matrices' $\mathsf{R}_n(\lambda)$ which satisfy a chain of Yang-Baxter equations

$$\mathsf{R}_{n-1}(\lambda)\,\mathsf{R}_n(\lambda\mu)\,\mathsf{R}_{n-1}(\mu) = \mathsf{R}_n(\mu)\,\mathsf{R}_{n-1}(\lambda\mu)\,\mathsf{R}_n(\lambda)$$

amounting to mutual commutativity

$$\mathsf{Q}\,(\lambda)\,\mathsf{Q}\,(\mu) = \mathsf{Q}\,(\mu)\,\mathsf{Q}\,(\lambda)$$

of their properly defined 'ordered products'. The family $\mathsf{Q}\,(\lambda)$ provides the demanded commuting (inner) evolution automorphisms

$$\boldsymbol{\phi}\,(\tau, n) \equiv \mathsf{Q}^{-\tau}\,\boldsymbol{\phi}_n\,\mathsf{Q}^{\tau}$$

and doubles as their common conservation laws.

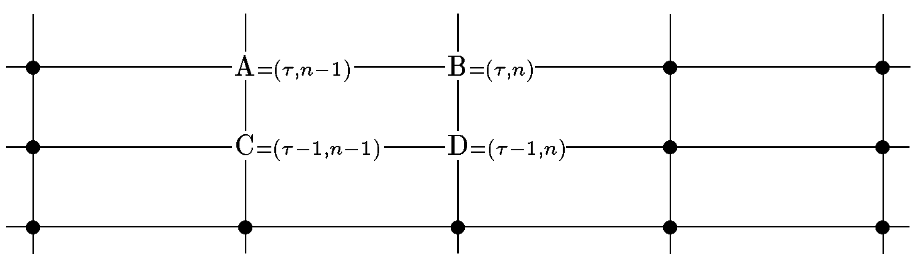
In Section 6 we eventually establish that these evolutions do solve the equations (1). Prior to that, in Section 5, we discover one more face of the family $\mathsf{Q}\,(\lambda)$. As a function of λ it proves to satisfy the Baxter equation [Bax]

$$\mathsf{q}^{N\ell^2}\,\mathsf{Q}\,(\lambda)\,\mathsf{t}\,(\lambda) = \alpha^N(\lambda)\,\mathsf{Q}\,(\mathsf{q}^{-1}\lambda) + \delta^N(\lambda)\,\mathsf{Q}\,(\mathsf{q}\,\lambda)$$

which in turn makes Bethe ansatz equations to emerge in a purely algebraic context.

While the quantum system (1) seems to be a recent invention (it has been looked at, albeit from a somewhat different angle, in [FV92, 94]), its classical counterpart has been around for quite a while. It was introduced (in a somewhat different form) by Hirota back in 1977 [H] as an integrable difference-difference approximation of the sine-Gordon equation but eventually proved far more universal making perfect sense as a lattice counterpart of numerous integrable equations including that of Korteveg and de Vries. To conclude the Introduction we shall list various continuous limits and alternative forms of the (classical) system (1). This should give some idea of where our model fits into the scheme of things accepted in Soliton Theory. For more of that and a comprehensive list of relevant references see [NC].

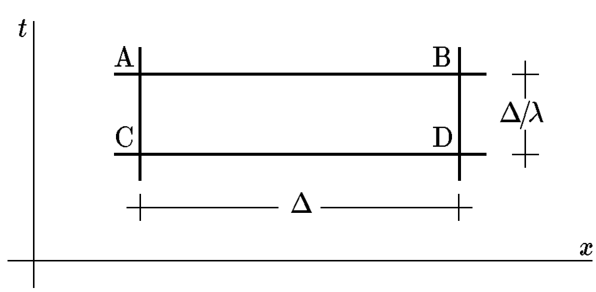
Pairs of integers (τ, n) may naturally be viewed as vertices of a plane lattice. In the shorthand notation



for a quartet of vertices enclosing some elementary cell of that lattice, with subscripts instead of parenthesized arguments and without bold letters reserved for the quantum case, the classical equations (1) read

$$\phi_{\mathrm{B}}\,\phi_{\mathrm{A}} - \phi_{\mathrm{D}}\,\phi_{\mathrm{C}} - \lambda\,(\phi_{\mathrm{B}}\,\phi_{\mathrm{D}} - \phi_{\mathrm{A}}\,\phi_{\mathrm{C}}) = 0.$$

• At least one continuous limit is already quite apparent. Let us put the lattice on the coordinate plane (t, x) in such a way that vertices (τ, n) go to points $(\lambda^{-1}\Delta\tau, \Delta n)$:

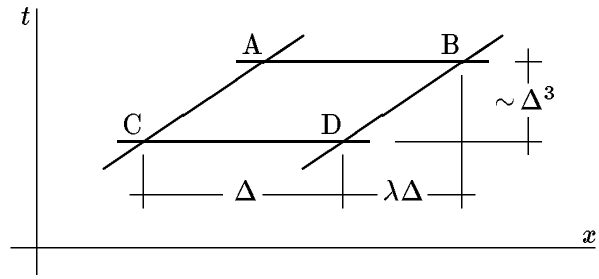


If now one manages to find a family of solutions to (1) depending on the lattice spacing Δ and tending to a smooth function $v(t, x)$ as Δ goes to zero then that function can be easily seen to satisfy the equation

$$\phi_t - \phi_x = 0.$$

This equation hardly needs any comment although at this point the use of a nonlinear difference-difference equation to model a linear differential one seems difficult to justify. Once, however, one takes it as a Minkowsky version of the Cauchy-Riemann equations things start to look like a unified approach to conformal invariance and integrability.

• Let us now perform a more sophisticated superimposition $(\tau, n) \longrightarrow (\frac{\lambda-\lambda^3}{24}\,\Delta^3\tau, \lambda\Delta\tau + \Delta n)$:



This time we come to the close relative of the KdV equation

$$\phi_t + \phi_{xxx} - 3\,\frac{\phi_{xx}\phi_x}{\phi} = 0$$

for any solution of which the potential

$$u = \frac{\phi_{xx}}{\phi}$$

solves the KdV equation itself

$$u_t + u_{xxx} - 6\,uu_x = 0.$$

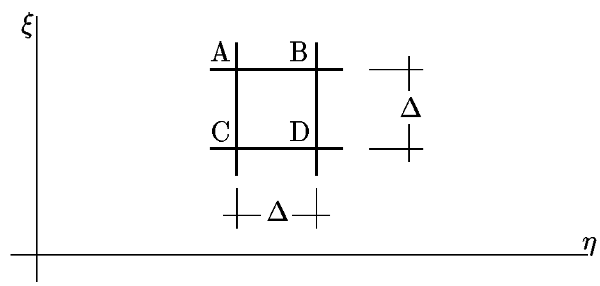
• To demonstrate a somewhat different scenario let us recall how to turn (1) into the sine-Gordon equation. Before performing a continuous limit we switch in (1) to the function $\varphi(\tau, n)$

$$\phi(\tau, n) = e^{(-1)^n i\varphi(\tau,n)}$$

that brings (1) back to its original form

$$\sin\tfrac{1}{2}(\varphi_{\mathrm{A}} - \varphi_{\mathrm{B}} - \varphi_{\mathrm{C}} + \varphi_{\mathrm{D}}) + \lambda\sin\tfrac{1}{2}(\varphi_{\mathrm{A}} + \varphi_{\mathrm{B}} + \varphi_{\mathrm{C}} + \varphi_{\mathrm{D}}) = 0.$$

Let the lattice cells now look like



and rescale the constant in the equation making it dependent on the lattice spacing

$$\lambda = -\left(\tfrac{\mathtt{m}\Delta}{2}\right)^2.$$

If now for a fixed value of the constant $\mathtt{m}$ one finds a family of solutions φ tending, as $\Delta \to 0$, to a smooth function $\varphi(\xi, \eta)$ then this function satisfies the sine-Gordon equation

$$\varphi_{\xi\eta} + \tfrac{\mathtt{m}^2}{2}\sin 2\varphi = 0.$$

Of course the original function ϕ can not survive under this continuous limit becoming badly oscillating.

• Although the mere ability of (1) to unify KdV and SG equations makes it a reasonable prospect, one is still left to wonder why the would-be universal lattice equation does not look special enough for that. And indeed there exists a far more spectacular version of (1). One can try to guess its form recollecting that the KdV equation looks best in its Krichever-Novikov reincarnation[†]

$$f_t + S[f] f_x = 0$$

where $S[f]$ stands for the Schwarz derivative

$$S[f] = \frac{f_{xxx}}{f_x} - \frac{3}{2}\left(\frac{f_{xx}}{f_x}\right)^2 .$$

Any decent lattice approximation for the f-equation should employ the cross-ratio as a difference counterpart of the Schwarz derivative and it is not difficult to spot the right one:

$$\frac{(f_{\rm A} - f_{\rm B})(f_{\rm C} - f_{\rm D})}{(f_{\rm A} - f_{\rm C})(f_{\rm B} - f_{\rm D})} = \text{const.}$$

Indeed, this equation does produce, provided const$= \lambda^{-2}$, in the above 'parallelogram' continuous limit exactly the f-equation being as well a completely integrable difference-difference model in its own right. And just as the continuous f- and ϕ-equations are tied up by the Fuchs formula

$$f_x = \frac{1}{\phi^2}$$

their lattice counterparts are connected by the map defined by

$$\frac{1}{\Delta}(f_n - f_{n-1}) = \frac{1}{\phi_{n-1}\phi_n}$$

where we omitted the argument τ common for all entries and moved the remaining one to the subscript position.

The cross-ratio version will find extensive use in the forthcoming paper addressing higher-order equations (KdV hierarchy), equations with two fields (NLS hierarchy) and 2+1-dimensional equations (KP hierarchy). Unfortunately, despite of its virtues this version has not yet been really useful in the quantum theory where at present we are only able to handle a free-field sort of algebra of observables associated with the ϕ-equation.

• By the way, at this point one finds himself well prepared to design a lattice version of the u-equation. Introducing

$$u_n = \frac{2\phi_n}{\phi_{n-1} + \phi_{n+1}}$$

so that

$$u_{n-1}u_n = 4\,\frac{(f_{n+1} - f_n)(f_{n-1} - f_{n-2})}{(f_{n+1} - f_{n-1})(f_n - f_{n-2})}$$

and

$$u_{lat} \sim 1 - \tfrac{1}{2}\Delta^2 u_{cont}$$

one eventually transforms (1) into

$$(u_{\rm A} - u_{\rm D})(u_{\rm B} - u_{\rm C}) + \tfrac{1}{4}(\lambda^{-2} - 1)(u_{\rm A}u_{\rm B} - u_{\rm C}u_{\rm D})^2 = 0$$

approximating the KdV equation in its original form.

[†]As this is already the third form of the KdV equation we have met, so far, and the arrival of a fourth one, the so-called modified KdV equation, is imminent, let us recall how all these forms interact

$$\begin{array}{llll}
f & & \qquad\qquad & f_t + S[f]f_x = 0 \\
\downarrow & & & \\
\phi & = \frac{1}{\sqrt{f_x}} & & \phi_t + \phi_{xxx} - 3\frac{\phi_{xx}\phi_x}{\phi} = 0 \\
\downarrow & & & \\
p & = \frac{\phi_x}{\phi} = -\frac{1}{2}\frac{f_{xx}}{f_x} & & p_t + p_{xxx} - 6p^2 p_x = 0 \\
\downarrow & & & \\
u & = p^2 + p_x = \frac{\phi_{xx}}{\phi} = -\frac{1}{2}S[f] & & u_t + u_{xxx} - 6uu_x = 0
\end{array}$$

and for some while rename them from their traditional names to $f/\psi/p/u$-equations correspondingly.

1 Algebra of observables

All the way through this Letter a constant q will be an odd root of unity

$$\mathsf{q}^{2\ell+1} = 1$$

and an integer number $N(\geq 3)$ called the spatial period will be **odd**. The only algebra of observables in use will be the 'exchange' algebra Φ with generators $\boldsymbol{\phi}_n$ subject to

(i) commutation relations

$$\boldsymbol{\phi}_m \boldsymbol{\phi}_n = \mathsf{q}^{-\epsilon(m-n)} \boldsymbol{\phi}_n \boldsymbol{\phi}_m,$$

with

$$\epsilon(n) = 1 \quad n = 1, 3, 5, \ldots, N-2$$
$$\epsilon(n) = 0 \quad n = 0, 2, 4, \ldots, N-1$$
$$\epsilon(n+N) = \epsilon(n) + 1,$$

complemented by conditions

(ii)

$$\boldsymbol{\phi}_n^{2\ell+1} = 1,$$

(iii)

$$\boldsymbol{\phi}_n^{-1} \boldsymbol{\phi}_{n+N} \text{ does not depend on } n.$$

The simplifying condition (ii) is natural if not really essential. In (iii) one easily recognizes a quasiperiodic boundary condition leaving in the algebra of observables only $N+1$ independent generators[‡], for instance, $\boldsymbol{\phi}_0, \boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_N$.

That algebra contains a useful 'current' subalgebra W with generators

$$\mathsf{w}_n = \frac{\boldsymbol{\phi}_{n+1}}{\boldsymbol{\phi}_{n-1}}$$

that are easily seen to obey

(i) commutation relations

$$\mathsf{w}_{n-1} \mathsf{w}_n = \mathsf{q}^2 \mathsf{w}_n \mathsf{w}_{n-1}$$
$$\mathsf{w}_m \mathsf{w}_n = \mathsf{w}_n \mathsf{w}_m \quad |m-n| \neq 1 \,(\mathrm{mod}\, N),$$

and conditions

(ii)

$$\mathsf{w}_n^{2\ell+1} = 1,$$

(iii)

$$\mathsf{w}_{n+N} = \mathsf{w}_n.$$

Of course, (iii) is just the periodic boundary condition. The whole algebra Φ is, in a sense, one degree of freedom larger than that current subalgebra which has N independent generators and a central element

$$\mathsf{c} = \mathsf{q}\,(\boldsymbol{\phi}_n^{-1} \boldsymbol{\phi}_{n+N})^2 = \mathsf{q}\; \mathsf{w}_1 \mathsf{w}_3 \ldots \mathsf{w}_N \mathsf{w}_2 \mathsf{w}_4 \ldots \mathsf{w}_{N-1}.$$

A remark is in order here. The algebra of observables was designed with the 'second' periodic KdV bracket (aka the Virasoro algebra)

$$\tfrac{1}{\gamma}\{u(x), u(y)\} = 2(u(x)+u(y))\delta'(x-y) - \delta'''(x-y)$$

[‡] $N+1$ is even, all right.

in mind. Indeed, that bracket corresponds to the sign-function-bracket (the exchange algebra)[§]

$$\tfrac{1}{\gamma}\{\phi(x),\phi(y)\} = -\tfrac{1}{2}\,\mathrm{sign}(x-y)\phi(x)\phi(y)$$

in the ϕ-language and the delta-prime-function-bracket (the current algebra)

$$\tfrac{1}{\gamma}\{p(x),p(y)\} = \delta'(x-y)$$

in the p-language (see Introduction). Both are easily seen to be classical ($\mathrm{q} = e^{i\hbar\gamma}$, $\hbar \to 0$) continuous ($\phi_{lat} \sim \phi_{cont}$, $w_{lat} \sim 1 + 2\Delta p_{cont}$) limits of the above commutation relations. A natural question arises whether the Virasoro algebra itself has a reasonable lattice counterpart. The answer seems to be affirmative [FT, V, B, Fe] but this is another story.

2 Fateev-Zamolodchikov R-matrix

Let two operators u and v satisfy the condition

$$\mathrm{u}^{2\ell+1} = \mathrm{v}^{2\ell+1} = 1$$

and obey Weyl's commutation relation

$$\mathrm{u}\,\mathrm{v} = \mathrm{q}^2\,\mathrm{v}\,\mathrm{u}\,.$$

Then, as was found in [FZ], the pair of functions of complex variable taking values in the algebra generated by u and v

$$R_1(\lambda) = r(\lambda,\mathrm{u}) \qquad R_2(\lambda) = r(\lambda,\mathrm{v}),$$

where

$$r(\lambda,z) = \sum_{k=-\ell}^{\ell} \rho_k(\lambda) z^k$$

$$\rho_k(\lambda) = \rho_{-k}(\lambda) = \mathrm{q}^{k^2} \prod_{j=1}^{k}(1-\lambda\mathrm{q}^{-2(j-1)}) \prod_{j=k+1}^{\ell}(1-\lambda\mathrm{q}^{2j}) \qquad 0 \le k \le \ell$$

and $\prod_1^0 \ldots = \prod_{\ell+1}^{\ell} \ldots = 1$,

satisfies the 'braid' Yang-Baxter equation

$$R_1(\lambda)\,R_2(\lambda\mu)\,R_1(\mu) = R_2(\mu)\,R_1(\lambda\mu)\,R_2(\lambda)\,.$$

Before going on let us compile a list of some useful properties of the function $r(\lambda,z)$. We shall often use them in remaining sections, sometimes not mentioning it explicitly. In what follows the second argument of r is always assumed to satisfy the condition

$$z^{2\ell+1} = 1.$$

(i) $r(\lambda,z)$ is, up to a constant factor, the only polynomial in λ of degree ℓ satisfying the functional equation

$$(\lambda+z)\,r(\lambda,\mathrm{q}\,z) = (1+\lambda z)\,r(\lambda,\mathrm{q}^{-1}z).$$

(ii) $r(\lambda,z)$ satisfies the functional equation

$$(1+\lambda)(1+\lambda\mathrm{q})\,r(\mathrm{q}\,\lambda,z) = (1+\lambda z)(1+\lambda z^{-1})\,r(\mathrm{q}^{-1}\lambda,z).$$

(iii)

$$r(\lambda,z) = r(\lambda,z^{-1})$$

[§] Strictly speaking, $\delta(\cdot)$ means here the 2π-periodic δ-function while $\mathrm{sign}(\cdot)$ is a function coinciding with the usual sign function in the interval $[-\pi,\pi]$ and extending quasiperiodically elsewhere: $\mathrm{sign}(x+2\pi) = \mathrm{sign}(x)+2$.

(iv) the product of $r(\lambda, z)$ and $r(\lambda^{-1}, z)$ does not depend on z.

$$r(\lambda, z)\, r(\lambda^{-1}, z) = \varrho(\lambda) = \text{const} \cdot \lambda^{-\ell} \prod_{\substack{k=-\ell \\ k \neq 0}}^{\ell} (1 + \lambda \mathrm{q}^{k}).$$

(v) At the point $\lambda = 0$ the function $r(\lambda, z)$ turns into a 'truncated' theta-function

$$r(0, z) = \theta(z) = \sum_{k=-\ell}^{\ell} \mathrm{q}^{k^2} z^k$$

satisfying the functional equation

$$z\, \theta(\mathrm{q}\, z) = \theta(\mathrm{q}^{-1} z).$$

The operators

$$\Theta_1 \equiv \theta(\mathrm{u}) \qquad \Theta_2 \equiv \theta(\mathrm{v})$$

obey Artin's commutation relation

$$\Theta_1 \Theta_2 \Theta_1 = \Theta_2 \Theta_1 \Theta_2$$

thus providing a 'free-field' model of the braid group B_2.

(vi)

$$r(1, z) = \text{const}$$

Some of these statements are quite transparent, some are less so. We will present their proofs in a more detailed paper.

3 Cyclic product

Let $\hat{W}$ be a free algebra[¶] with generators $\hat{\mathrm{w}}_n$. Denote

(i) by ς the homomorphism projecting $\hat{W}$ to W

$$\varsigma(\hat{\mathrm{w}}_n) = \mathrm{w}_n$$
$$\varsigma(\hat{\mathrm{x}}\, \hat{\mathrm{y}}) = \varsigma(\hat{\mathrm{x}})\, \varsigma(\hat{\mathrm{y}}),$$

(ii) by $_{+N}$ the shift-by-period automorphism of the algebra $\hat{W}$

$$(\hat{\mathrm{w}}_n)_{+N} = \hat{\mathrm{w}}_{n+N}$$
$$(\hat{\mathrm{x}}\, \hat{\mathrm{y}})_{+N} = \hat{\mathrm{x}}_{+N}\, \hat{\mathrm{y}}_{+N}.$$

Assign to monomials of $\hat{W}$ 'overlap' numbers

$$\nu\left(\hat{\mathrm{w}}_{n_1}^{k_1}\, \hat{\mathrm{w}}_{n_2}^{k_2} \ldots \hat{\mathrm{w}}_{n_L}^{k_L}\right) = \sum_{p=-\infty}^{+\infty} \sum_{i,j=1}^{L} p\, k_i k_j\, \delta_{n_j - n_i\,,\, pN-1}$$

and utilize them in the coboundary which defines another multiplication $\star$ in $\hat{W}$: for basis elements it reads

$$\hat{\mathrm{f}} \star \hat{\mathrm{g}} = \mathrm{q}^{2\left(\nu(\hat{\mathrm{f}}\, \hat{\mathrm{g}}) - \nu(\hat{\mathrm{f}}) - \nu(\hat{\mathrm{g}})\right)}\, \hat{\mathrm{f}}\, \hat{\mathrm{g}}$$

and extends bilinearly elsewhere. Then for any two elements $\hat{\mathrm{x}}$, $\hat{\mathrm{y}}$ of $\hat{W}$

$$\varsigma(\hat{\mathrm{x}} \star \hat{\mathrm{y}}) = \varsigma(\hat{\mathrm{y}} \star \hat{\mathrm{x}}_{+N}).$$

We omit the proof for it is a straightforward computation.

The purpose of this $\varsigma(\star)$ construction must be clear for those familiar with the Quantum Inverse Scattering Method: once we decide to do without an 'auxiliary space' and go for a purely algebraic version of the R-matrix approach we need some direct method of computing what used to be auxiliary space traces.

[¶]It is essential that the algebra $\hat{W}$, as opposed to the algebra W of Section 1, is devoid of the periodic boundary condition. It is not really necessary to get rid of other relations defining W.

4 Conservation laws

We are going to prove that 'cyclic ordered products'

$$\mathsf{U}(\lambda) \equiv \varsigma\,\Big(\hat{\mathsf{R}}_1(\lambda) \star \hat{\mathsf{R}}_2(\lambda) \star \ldots \star \hat{\mathsf{R}}_N(\lambda)\Big)$$

$$= \sum_{j,k=-\ell}^{\ell} \mathsf{q}^{2jk}\,(\rho_j(\lambda)\,\mathsf{w}_1^j)\;\mathsf{R}_2(\lambda)\,\mathsf{R}_3(\lambda)\ldots\mathsf{R}_{N-1}(\lambda)\,(\rho_k(\lambda)\,\mathsf{w}_N^k)$$

of FZ R-matrices

$$\mathsf{R}_n(\lambda) \equiv r(\lambda,\,\mathsf{w}_n) \qquad \hat{\mathsf{R}}_n(\lambda) \equiv r(\lambda,\,\hat{\mathsf{w}}_n)$$

commute with each other:

$$\mathsf{U}(\lambda)\,\mathsf{U}(\mu) = \mathsf{U}(\mu)\,\mathsf{U}(\lambda).$$

Indeed, the commutation relations of w's translate into N copies of the Yang-Baxter equation

$$\mathsf{R}_{n-1}(\lambda)\,\mathsf{R}_n(\lambda\mu)\,\mathsf{R}_{n-1}(\mu) = \mathsf{R}_n(\mu)\,\mathsf{R}_{n-1}(\lambda\mu)\,\mathsf{R}_n(\lambda)$$

$$\mathsf{R}_{n+N}(\lambda) = \mathsf{R}_n(\lambda)$$

that is believed to ensure the commutativity of ordered products of those R-matrices. It is not however evident how to make this idea work in the periodic case where a naive ordered product, say, $\mathsf{R}_1\mathsf{R}_2\ldots\mathsf{R}_N$, makes little sense. The $\varsigma\,(\star)$ 'product' has better chance to deliver, since, as we know from the preceding section, it does not at least depend on the starting point:

$$\ldots = \varsigma\,\Big(\hat{\mathsf{R}}_0 \star \hat{\mathsf{R}}_1 \star \ldots \star \hat{\mathsf{R}}_{N-1}\Big) = \varsigma\,\Big(\hat{\mathsf{R}}_1 \star \hat{\mathsf{R}}_2 \star \ldots \star \hat{\mathsf{R}}_N\Big) = \ldots.$$

And once we get the ordered product right the commutativity check becomes a matter of familiar R-matrix machinery complemented by the rules of $\varsigma\,(\star)$ 'multiplication' (and also by the definition $\hat{\mathsf{R}}_n^{-1}(\lambda) \equiv \hat{\mathsf{R}}_n(\lambda^{-1})/\varrho(\lambda)$ ensuring that $\varsigma\,\Big(\hat{\mathsf{R}}_n^{-1}(\lambda)\,\hat{\mathsf{R}}_n(\lambda)\Big) = 1$):

$$\begin{aligned}
\mathsf{U}(\lambda)\,\mathsf{U}(\mu) &= \varsigma\,\Big(\hat{\mathsf{R}}_1(\lambda) \star \hat{\mathsf{R}}_2(\lambda) \star \ldots \star \hat{\mathsf{R}}_N(\lambda)\Big)\;\varsigma\,\Big(\hat{\mathsf{R}}_0(\mu) \star \hat{\mathsf{R}}_1(\mu) \star \ldots \star \hat{\mathsf{R}}_{N-1}(\mu)\Big)\\
&= \varsigma\,\Big(\hat{\mathsf{R}}_1(\lambda) \star \hat{\mathsf{R}}_0(\mu) \star \hat{\mathsf{R}}_2(\lambda) \star \hat{\mathsf{R}}_1(\mu) \star \ldots \star \hat{\mathsf{R}}_N(\lambda) \star \hat{\mathsf{R}}_{N-1}(\mu)\Big)\\
&= \varsigma\,\Big(\hat{\mathsf{R}}_0^{-1}(\tfrac{\lambda}{\mu}) \star \hat{\mathsf{R}}_0(\tfrac{\lambda}{\mu}) \star \hat{\mathsf{R}}_1(\lambda) \star \hat{\mathsf{R}}_0(\mu) \star \hat{\mathsf{R}}_2(\lambda) \star \hat{\mathsf{R}}_1(\mu) \star \ldots \star \hat{\mathsf{R}}_N(\lambda) \star \hat{\mathsf{R}}_{N-1}(\mu)\Big)\\
&= \varsigma\,\Big(\underline{\hat{\mathsf{R}}_0(\tfrac{\lambda}{\mu}) \star \hat{\mathsf{R}}_1(\lambda) \star \hat{\mathsf{R}}_0(\mu)} \star \hat{\mathsf{R}}_2(\lambda) \star \hat{\mathsf{R}}_1(\mu) \star \ldots \star \hat{\mathsf{R}}_N(\lambda) \star \hat{\mathsf{R}}_{N-1}(\mu) \star \hat{\mathsf{R}}_N^{-1}(\tfrac{\lambda}{\mu})\Big)\\
&= \varsigma\,\Big(\hat{\mathsf{R}}_1(\mu) \star \hat{\mathsf{R}}_0(\lambda) \star \underline{\hat{\mathsf{R}}_1(\tfrac{\lambda}{\mu}) \star \hat{\mathsf{R}}_2(\lambda) \star \hat{\mathsf{R}}_1(\mu)} \star \ldots \star \hat{\mathsf{R}}_N(\lambda) \star \hat{\mathsf{R}}_{N-1}(\mu) \star \hat{\mathsf{R}}_N^{-1}(\tfrac{\lambda}{\mu})\Big)\\
&= \varsigma\,\Big(\hat{\mathsf{R}}_1(\mu) \star \hat{\mathsf{R}}_0(\lambda) \star \hat{\mathsf{R}}_2(\mu) \star \hat{\mathsf{R}}_1(\lambda) \star \underline{\hat{\mathsf{R}}_2(\tfrac{\lambda}{\mu})} \star \ldots \star \hat{\mathsf{R}}_N(\lambda) \star \hat{\mathsf{R}}_{N-1}(\mu) \star \hat{\mathsf{R}}_N^{-1}(\tfrac{\lambda}{\mu})\Big)\\
&\;\;\vdots\\
&= \varsigma\,\Big(\hat{\mathsf{R}}_1(\mu) \star \hat{\mathsf{R}}_0(\lambda) \star \hat{\mathsf{R}}_2(\mu) \star \hat{\mathsf{R}}_1(\lambda) \star \ldots \star \underline{\hat{\mathsf{R}}_{N-1}(\tfrac{\lambda}{\mu}) \star \hat{\mathsf{R}}_N(\lambda) \star \hat{\mathsf{R}}_{N-1}(\mu)} \star \hat{\mathsf{R}}_N^{-1}(\tfrac{\lambda}{\mu})\Big)\\
&= \varsigma\,\Big(\hat{\mathsf{R}}_1(\mu) \star \hat{\mathsf{R}}_0(\lambda) \star \hat{\mathsf{R}}_2(\mu) \star \hat{\mathsf{R}}_1(\lambda) \star \ldots \star \hat{\mathsf{R}}_N(\mu) \star \hat{\mathsf{R}}_{N-1}(\lambda) \star \hat{\mathsf{R}}_N(\tfrac{\lambda}{\mu}) \star \hat{\mathsf{R}}_N^{-1}(\tfrac{\lambda}{\mu})\Big)\\
&= \varsigma\,\Big(\hat{\mathsf{R}}_1(\mu) \star \hat{\mathsf{R}}_0(\lambda) \star \hat{\mathsf{R}}_2(\mu) \star \hat{\mathsf{R}}_1(\lambda) \star \ldots \star \hat{\mathsf{R}}_N(\mu) \star \hat{\mathsf{R}}_{N-1}(\lambda)\Big)\\
&= \varsigma\,\Big(\hat{\mathsf{R}}_1(\mu) \star \hat{\mathsf{R}}_2(\mu) \star \ldots \star \hat{\mathsf{R}}_N(\mu)\Big)\;\varsigma\,\Big(\hat{\mathsf{R}}_0(\lambda) \star \hat{\mathsf{R}}_1(\lambda) \star \ldots \star \hat{\mathsf{R}}_{N-1}(\lambda)\Big) = \mathsf{U}(\mu)\,\mathsf{U}(\lambda).
\end{aligned}$$

We conclude the Section with two remarks. First, to make formulas easier on the eyes we shall adopt the notation $\overset{\circlearrowright}{\prod}\ldots$ for 'cyclic ordered products' like the ones above, for instance,

$$\mathsf{U}(\lambda) = \overset{\circlearrowright}{\prod}\,\mathsf{R}_n(\lambda)$$

$$\mathsf{U}(\lambda)\,\mathsf{U}(\mu) = \overset{\circlearrowright}{\prod}\,\mathsf{R}_n(\lambda)\,\mathsf{R}_{n-1}(\mu).$$

Second, for a reason that will become apparent in Section 6 the family $U(\lambda)$ should be properly 'normalized'

$$
Q(\lambda) = \frac{U(\lambda)}{U(0)}
$$

that, of course, does not spoil its commutativity

$$
Q(\lambda)\, Q(\mu) = Q(\mu)\, Q(\lambda)
$$

and polynomiality in λ.

5 Baxter equation

In the Quantum Inverse Scattering Method language the family $Q(\lambda)$ would be called the fundamental transfer-matrix [TTF] as opposed to the usual nonfundamental one which on this occasion has the form [G, V]

$$
t(\lambda) = \mathrm{tr} \overset{\circlearrowright}{\prod} (W_n L(\lambda))
$$

where the matrices W_n and $L(\lambda)$ are

$$
W_n = \begin{pmatrix} w_n^{\frac{1}{2}} & \\ & w_n^{-\frac{1}{2}} \end{pmatrix} \quad \text{with} \quad w_n^{\frac{1}{2}} \equiv w_n^{-\ell}
$$

$$
L(\lambda) = \begin{pmatrix} \lambda & 1 \\ 1 & \lambda \end{pmatrix},
$$

tr denotes the matrix trace and $\star$ (hidden in the product symbol) combines what it used to be with the standard matrix multiplication. In the decyphered form it reads

$$
t(\lambda) = \lambda^{\frac{N}{2}} \sum_{k_1, k_2, \ldots, k_N = \pm\frac{1}{2}} q^{2k_1 k_N} \lambda^{2(k_1 k_2 + k_2 k_3 + \ldots + k_{N-1} k_N + k_N k_1)} w_1^{k_1} w_2^{k_2} \ldots w_N^{k_N}
$$

with

$$
q^{\frac{1}{2}} \equiv q^{-\ell}.
$$

In other words, we have another polynomial in λ, this time of degree N[||], which, as we know from past experience,

(i) commutes with itself

$$
t(\lambda)\, t(\mu) = t(\mu)\, t(\lambda),
$$

(ii) commutes with $Q(\lambda)$

$$
t(\lambda)\, Q(\mu) = Q(\mu)\, t(\lambda)
$$

and, as we are going to see,

(iii) satisfies together with $Q(\lambda)$ the Baxter equation [Bax]

$$
q^{N\ell^2}\, Q(\lambda)\, t(\lambda) = \alpha^N(\lambda)\, Q(q^{-1}\lambda) + \delta^N(\lambda)\, Q(q\lambda)
$$

with

$$
\alpha(\lambda) = \lambda - 1
$$

$$
\delta(\lambda) = q\lambda + 1.
$$

The first two items of this list are actually superseded by the much stronger third one the proof of which may go as follows[**]:

[||] To be precise, only odd degrees are present.

[**] The first proof of (iii) for the model in question was obtained by R. Kashaev [K] while the very idea that the fundamental transfer-matrix $Q(\lambda)$ can serve as a Baxter's Q-operator probably belongs to E. Sklyanin [S]. See also [PG] addressing similar matters.

$$\mathsf{U}(\lambda)\,\mathsf{t}(\lambda) = \mathrm{tr}\overset{\circlearrowright}{\prod}\left(\mathsf{R}_n(\lambda)\,\mathsf{W}_{n-1}L(\lambda)\right) = \mathrm{tr}\overset{\circlearrowright}{\prod}\,\mathsf{C}_n(\lambda)$$

with

$$\mathsf{C}_n(\lambda) = B^{-1}\,\mathsf{W}_{n-1}^{-1}\,\mathsf{R}_n(\lambda)\,\mathsf{W}_{n-1}L(\lambda)\,\mathsf{W}_nB,$$

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Using the obvious relation

$$\mathsf{w}_{n-1}^{\frac{1}{2}}\,r(\lambda,\,\mathsf{w}_n)\,\mathsf{w}_{n-1}^{-\frac{1}{2}} = r(\lambda,\mathsf{q}\,\mathsf{w}_n)$$

one gets

$$\mathsf{C}_n(\lambda) = C(\lambda,\,\mathsf{w}_n)$$

with

$$C(\lambda,z) = \begin{pmatrix} z^{\frac{1}{2}}(\lambda\,r(\lambda,\mathsf{q}^{-1}z) - r(\lambda,\mathsf{q}\,z)) & (\lambda z^{\frac{1}{2}} + z^{-\frac{1}{2}})\,r(\lambda,\mathsf{q}^{-1}z) - (z^{\frac{1}{2}} + \lambda z^{-\frac{1}{2}})\,r(\lambda,\mathsf{q}\,z) \\ z^{\frac{1}{2}}\,r(\lambda,\mathsf{q}\,z) & (z^{\frac{1}{2}} + \lambda z^{-\frac{1}{2}})\,r(\lambda,\mathsf{q}\,z) \end{pmatrix}.$$

The first property of the function $r(\lambda,z)$ (see Section 2) says that the upper off-diagonal element of this matrix vanishes (provided $z^{2\ell+1} = 1$ and $z^{\frac{1}{2}} \equiv z^{-\ell}$). This yields immediately

$$\mathsf{U}(\lambda)\,\mathsf{t}(\lambda) = \overset{\circlearrowright}{\prod} a(\lambda,\,\mathsf{w}_n) + \overset{\circlearrowright}{\prod} d(\lambda,\,\mathsf{w}_n)$$

where a and d denote the diagonal elements of the matrix C:

$$C(\lambda,z) = \begin{pmatrix} a(\lambda,z) & 0 \\ * & d(\lambda,z) \end{pmatrix}.$$

It remains to verify that

$$a(\lambda,z) = \mathsf{q}^{-\ell^2}(\lambda-1)\,r(\mathsf{q}^{-1}\lambda,z)$$
$$d(\lambda,z) = \mathsf{q}^{-\ell^2}(\mathsf{q}\,\lambda+1)\,r(\mathsf{q}\,\lambda,z).$$

We omit this part of the proof for it is neither difficult nor instructive.

To conclude, let us recall what use might be made of the Baxter equation. The function $\mathsf{Q}(\lambda)$ is a polynomial in λ with coefficients coming from the subalgebra of conservation laws. This allows, probably at the expense of considering a proper completion of the algebra of observables, the introduction of commuting 'roots' of $\mathsf{Q}(\lambda)$

$$\mathsf{Q}(\lambda) = \prod_k\left(1 - \frac{\lambda}{\mathsf{r}_k}\right)$$
$$\mathsf{r}_j\,\mathsf{r}_k = \mathsf{r}_k\,\mathsf{r}_j.$$

The Baxter equation says, in particular, that

$$\alpha^N(\lambda)\,\mathsf{Q}(\mathsf{q}^{-1}\lambda) + \delta^N(\lambda)\,\mathsf{Q}(\mathsf{q}\,\lambda)\Big|_{\lambda=\mathsf{r}_k} = 0.$$

This is just the famous system of the Bethe ansatz equations which look more familiar in the form

$$\left(\frac{\alpha(\mathsf{r}_k)}{\delta(\mathsf{r}_k)}\right)^N = -\prod_j\frac{\mathsf{r}_j - \mathsf{q}\,\mathsf{r}_k}{\mathsf{r}_j - \mathsf{q}^{-1}\,\mathsf{r}_k}\ .$$

6 Equations of motion

We are going to see that the generators of the algebra of observables evolve

$$\phi(\lambda\,|\,\tau, n) \equiv \mathsf{Q}^{-\tau}(\lambda)\,\phi_n\,\mathsf{Q}^{\tau}(\lambda)$$

according to the equations (1) which in ultimately accurate form read

$$\phi(\lambda\,|\,\tau,n)\,\phi(\lambda\,|\,\tau,n-1) + \lambda\,\phi(\lambda\,|\,\tau,n-1)\,\phi(\lambda\,|\,\tau-1,n-1)$$
$$= \lambda\,\phi(\lambda\,|\,\tau,n)\,\phi(\lambda\,|\,\tau-1,n) + \phi(\lambda\,|\,\tau-1,n)\,\phi(\lambda\,|\,\tau-1,n-1).$$

We shall cover the distance in four short steps.

(i) Commutation relations between ϕ's and w's have the form

$$\phi_n\,\mathsf{w}_n = \mathsf{q}^2\,\mathsf{w}_n\,\phi_n$$

$$\phi_m\,\mathsf{w}_n = \mathsf{w}_n\,\phi_m \quad m \neq n\,(\mathrm{mod}\,N).$$

This prompts us to read the first property of the function r from Section 2 as

$$(\phi_{n+1}\,\phi_n + \lambda\,\phi_n\,\phi_{n-1})\,\mathsf{R}_n(\lambda) = \mathsf{R}_n(\lambda)\,(\lambda\,\phi_{n+1}\,\phi_n + \phi_n\,\phi_{n-1}).$$

(ii) Viewing this equation as a fragment of the 'big' one

$$\sum_{j,k=-\ell}^{\ell} \mathsf{q}^{2jk}\,(\rho_j\,\mathsf{w}^j_{m+1})\,\mathsf{R}_{m+2}\ldots\mathsf{R}_{n-1}\,(\phi_{n+1}\,\phi_n + \lambda\,\phi_n\,\phi_{n-1})\,\mathsf{R}_n\,\mathsf{R}_{n+1}\ldots\mathsf{R}_{m+N-1}\,(\rho_k\,\mathsf{w}^k_{m+N})$$

$$= \sum_{j,k=-\ell}^{\ell} \mathsf{q}^{2jk}\,(\rho_j\mathsf{w}^j_{m+1})\,\mathsf{R}_{m+2}\ldots\mathsf{R}_{n-1}\mathsf{R}_n\,(\lambda\,\phi_{n+1}\,\phi_n + \phi_n\,\phi_{n-1})\,\mathsf{R}_{n+1}\ldots\mathsf{R}_{m+N-1}\,(\rho_k\mathsf{w}^k_{m+N})$$

and pulling ϕ's to the outside(s) we get

$$\phi_{n+1}\,\phi_n\,\mathsf{U}(\lambda) + \lambda\,\phi_n\,\mathsf{U}(\lambda)\,\phi_{n-1} = \lambda\,\phi_{n+1}\,\mathsf{U}(\lambda)\,\phi_n + \mathsf{U}(\lambda)\,\phi_n\,\phi_{n-1}.$$

(iii) At the point $\lambda = 0$ it turns into

$$\phi_{n+1}\,\phi_n\,\mathsf{U}(0) = \mathsf{U}(0)\,\phi_n\,\phi_{n-1}$$

suggesting more subtle

$$\phi_n\,\mathsf{U}(0) = \mathsf{U}(0)\,\phi_{n-1}.$$

As a matter of fact, the latter does hold. We omit the proof for it is too case-specific.

(iv) (ii) and (iii) combined yield

$$\phi_n\,\phi_{n-1}\,\mathsf{Q}(\lambda) + \lambda\,\phi_{n-1}\,\mathsf{Q}(\lambda)\,\phi_{n-1} = \lambda\,\phi_n\,\mathsf{Q}(\lambda)\,\phi_n + \mathsf{Q}(\lambda)\,\phi_n\,\phi_{n-1}$$

which is nothing but (1) with cut away common factors $\mathsf{Q}^{-\tau}(\lambda)\ldots\mathsf{Q}^{\tau-1}(\lambda)$.

So, we have met the last objective of the Letter. We conclude it with two remarks.

• A consistent approach to the subject should probably distinguish between observables and their automorphisms rather than mix them up as we did in this Letter. It would be only natural to deal not with the R-matrices but directly with automorphisms they represent:

$$\mathcal{R}_n(\lambda):\quad \begin{array}{lll} \phi_n & \mapsto & \frac{1+\lambda\,\mathsf{q}\,\mathsf{w}_n}{\lambda+\mathsf{q}\,\mathsf{w}_n}\,\phi_n \\ \phi_m & \mapsto & \phi_m \quad m \neq n\,(\mathrm{mod}\,N) \end{array}\quad.$$

Indeed, it would follow straight from the above definition that

$$\phi_{n+1}\,\phi_n + \lambda\,\phi_n\,\phi_{n-1} \overset{\mathcal{R}_n(\lambda)}{\longmapsto} \lambda\,\phi_{n+1}\,\phi_n + \phi_n\,\phi_{n-1}$$

and

$$\mathcal{R}_{n-1}(\lambda) \circ \mathcal{R}_n(\lambda\mu) \circ \mathcal{R}_{n-1}(\mu) = \mathcal{R}_n(\mu) \circ \mathcal{R}_{n-1}(\lambda\mu) \circ \mathcal{R}_n(\lambda).$$

These two relations are 'weaker' (but just sufficient!) substitutes for the two cornerstones of the whole scheme, which are item (i) of this Section and the Yang-Baxter equations of Section 4. Why then care whether those 'R-automorphisms' are inner or not? They happen to be inner in our particular case but even there some other important automorphisms are outer anyway [FV93]. In other cases one pays a dear price for 'inclusion' of R-matrices in the algebra of observables [BR, F, FV95]. Unfortunately, some difficulties of the 'automorphism' approach made the author to choose for this Letter the more familiar 'inner' route.

• It would be useful to know whether the equations of motion (1) provide the exhaustive information about the model. In other words, is $\phi(\tau, n)$ the only solution to the Cauchy problem

$$\phi(\tau, n)|_{\tau=0} = \phi_n$$

for the system (1)? Not quite, and it is easy to see why. First, in our solution the 'quasimomentum' c (see Section 1) does not evolve but the equations (1) do not know about that. Also, they do not feel whether we multiply $\mathsf{R}_n(\lambda)$'s 'from left to right' or $\mathsf{R}_n(-\lambda)$'s in the opposite order. As a matter of fact, there are no other sources of nonuniqueness. So, the local structure of (1) is good and all one needs to achieve the ultimate uniqueness is a couple of extra global conditions. This point will be described in more detail elsewhere.

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