SPECIAL GEOMETRY AND TWISTED MODULI IN ORBIFOLD THEORIES WITH CONTINUOUS WILSON LINES.

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ABSTRACT

Target space duality symmetries, which acts on Kähler and continuous Wilson line moduli, of a $\mathbf{Z}_N$ $(N \neq 2)$ 2-dimensional subspace of the moduli space of orbifold compactification are modified to include twisted moduli. These spaces described by the cosets $\frac{SU(n,1)}{SU(n)\times U(1)}$ are *special* Kähler, a fact which is exploited in deriving the extension of tree level duality transformation to include higher orders of the twisted moduli. Also, restrictions on these higher order terms are derived.

Heterotic string theories compactified on orbifolds are of phenomenological importance as they give rise to an $N = 1$ space-time supersymmetric semi-realistic four dimensional quantum field theories [1, 2]. A phenomenologically appealing feature of string compactifications lies in the fact that the physical couplings in the low-energy effective action are dependent on the moduli fields which parametrize the shape and size of the orbifold and possible continuous Wilson lines. Also, such an action has a novel symmetry known as target space duality. This is a discrete symmetry of the moduli space which leaves the underlying conformal field theory invariant. This symmetry restricts the low energy effective action and connects it to the theory of modular forms [3]. The low-energy action is an $N = 1$ supergravity coupled to Yang-Mills and matter fields. If terms with up to two derivatives in the bosonic fields are included, the theory is then defined in terms of the Kähler potential K which encodes the kinetic terms for the massless fields, the superpotential W containing the Yukawa couplings and the $f-$function whose real part, at the tree level, determines the gauge couplings [13]. In fact the lagrangian of the theory depends on K and W via the target space duality invariant combination

$$\mathcal{G} = K + log|W|^2. \tag{1}$$

The method of calculating the superpotential of the low-energy effective action directly from the underlying conformal field theory was given in [4]. Moreover, the untwisted moduli dependence of the tree level Kähler potential has been addressed in the literature by several methods [5-9].

It is our purpose in this letter to derive the explicit dependence of the Kähler potential on the full moduli space, untwisted and twisted, of a 2-dimensional subspace of an orbifold with continuous Wilson lines [12]. We will consider the cases where the moduli space is given by the *special* Kähler manifold $\frac{SU(n,1)}{SU(n)\otimes U(1)}$ [10]. This coset is parametrized by one Kähler modulus and $n-1$ complex continuous Wilson lines of a plane where the twist has a complex eigenvalue* [8]. The real part

* this means that the twist is not $\mathbf{Z}_2$

of the Kähler or toroidal modulus describes the size of the 2-dimensional space and the imaginary part describes a possible internal axion field (antisymmetric tensor). The Wilson lines are homotopically non-trivial flat gauge connections.

In the process of deriving the Kähler potential, the higher order corrections, in terms of the twisted moduli, to the target space duality symmetry of the underlying conformal field theory are derived. Moreover, conditions on these higher order terms are also determined. For simplicity we will concentrate on the case with one complex Wilson line, *i.e.*, the moduli space $\frac{SU(2,1)}{SU(2)\otimes U(1)}$, and determine the full Kähler potential of the moduli space in the presence of a generic twisted modulus. Generalization to more than one Wilson line or twisted modulus is straightforward. The fact that the moduli spaces $\frac{SU(n,1)}{SU(n)\otimes U(1)}$ are special Kähler facilitates the calculation of the higher order duality transformations for the moduli and their associated Kähler potential as has been demonstrated in [11] for the special Kähler coset $\left[\frac{SU(1,1)}{U(1)}\right]^3$. This can be explained as follows. For special Kähler manifolds [10], the Kähler potential can be expressed in terms of a holomorphic function of the moduli, if we denote the moduli by ϕ^i then

$$K = -logY; \qquad Y = \sum_i (\phi^i + \bar{\phi}^i)(F_{\phi^i} + \bar{F}_{\bar{\phi}^i}) - 2(F + \bar{F}). \tag{2}$$

In terms of the homogeneous coordinates $x^I, I = 0, \cdots, j$, where the physical moduli are expressed by the *special* coordinates, $\phi^i = \frac{x^i}{ix^0}$ (with $i = 1, \cdots j$), (2) is expressed in the form

$$K = -log\Big(\frac{\bar{\mathcal{F}}_I x^I + \mathcal{F}_I \bar{x}^I}{x^0\bar{x}^0}\Big) = -log\Big[\frac{-2i}{x^0\bar{x}^0}\begin{pmatrix} x^I \\ \frac{1}{2}i\mathcal{F}_J \end{pmatrix}^\dagger \begin{pmatrix} 0 & \delta^L{}_I \\ -\delta^J{}_K & 0 \end{pmatrix} \begin{pmatrix} x^K \\ \frac{1}{2}i\mathcal{F}_L \end{pmatrix}\Big], \tag{3}$$

where $\mathcal{F}(x) = -(x^0)^2 F(\phi)$. From (3) it is clear that the Kähler potential is invariant up to a Kähler transformation under

$$\begin{aligned} x^I &\to U^I{}_J x^J + \frac{1}{2}iV^{IJ}\mathcal{F}_J, \\ \frac{1}{2}i\mathcal{F}_I &\to W_{IJ}x^J + \frac{1}{2}iZ_I{}^J\mathcal{F}_J, \end{aligned} \tag{4}$$

with

$$S = \begin{pmatrix} U & V \\ W & Z \end{pmatrix}, \qquad S^\dagger \eta S = \eta, \qquad \eta = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \tag{5}$$

where $\mathbf{1}$ is the j-dimensional identity. The transformations in (5) act as holomorphic field redefinitions provided that

$$\begin{aligned} W_{IJ}x^J + \frac{1}{2} i Z_I{}^J \frac{\partial \mathcal{F}(x)}{\partial x^I} &= \frac{1}{2} i \frac{\partial \tilde{\mathcal{F}}(y)}{\partial y^I}, \\ y^I = U^I{}_J x^J + \frac{1}{2} i V^{IJ} \frac{\partial \mathcal{F}(x)}{\partial x^J}. & \end{aligned} \tag{6}$$

The integrability is guaranteed if U, V, W and Z are real and satisfy [11]

$$U^t W - W^t U = 0 \quad V^t Z - Z^t V = 0 \quad U^t Z - W^t V = I. \tag{7}$$

These transformatons define duality transformations if $\mathcal{F} = \tilde{\mathcal{F}}$.

The importance of this formalism lies in the fact that the duality transformations act linearly and are field independent. Therefore, the lowest order duality transformations fix S completely. The second equation in (6) determine the modified duality transformations of the moduli, and corrections to the tree level transformations comes from the higher order terms in the holomorphic function $\mathcal{F}$. Moreover the first eq in (6) restricts the higher order terms of the function $\mathcal{F}$ itself. Finally the knowledge of the function $\mathcal{F}$ is sufficient to determine the Yukawa couplings of the theory [11].

We now get back to the study of $\frac{SU(2,1)}{SU(2)\otimes U(1)}$ moduli space. To lowest order in the twisted modulus $\mathbf{C}$, the tree level Kähler potential is given by

$$K = -log(\mathbf{T} + \bar{\mathbf{T}} - \frac{k}{2}\mathbf{A}\bar{\mathbf{A}} - \mathbf{C}\bar{\mathbf{C}}), \tag{8}$$

where $\mathbf{T}$ is the Kähler complex modulus and $\mathbf{A}$ is a complex Wilson line and c is a model-dependent constant (for $\mathbf{Z}_3$ twist $k = \sqrt{3}$). We would like now to generalize

(8) to include higher orders of the expectation values of the twisted moduli $\mathbf{C}$. To be concrete we take the twist to be $\mathbf{Z}_3$. A duality symmetry of the theory is given by the subgroup $SL(2,Z)$ which acts on on the moduli as follows

$$\mathbf{T} \to \frac{a\mathbf{T} - ib}{ic\mathbf{T} + d}, \quad \mathbf{A} \to \frac{\mathbf{A}}{ic\mathbf{T} + d}, \quad \mathbf{C} \to \frac{\mathbf{C}}{ic\mathbf{T} + d} \qquad ad - bc = 1. \tag{9}$$

This acts on (8) by a Kähler transformation $K \to K + \ln |(ic\mathbf{T} + d)|^2$. However, it is more convenient to work with a different basis for the moduli space. Perform a change of variable as follows:

$$\frac{\mathbf{T}}{\sqrt{3}} = \frac{1-t}{1+t}, \qquad \mathbf{A} = \frac{2\mathcal{A}}{(1+t)}, \qquad \mathbf{C} = \frac{\sqrt{2}\mathcal{C}}{(1+t)}, \tag{10}$$

then in terms of the new variables t, $\mathcal{A}$ and $\mathcal{C}$, the duality transformations (9) takes the form

$$t \to \tilde{t} = \frac{\alpha}{\gamma} = \frac{At+B}{Ct+D}, \quad \mathcal{A} \to \tilde{\mathcal{A}} = \frac{\mathcal{A}}{\gamma}, \quad \mathcal{C} \to \frac{\mathcal{C}}{\gamma}, \tag{11}$$

where

$$\begin{aligned} &A = D^* = \frac{1}{2}\Big((a+d) + i(b' - c')\Big), \quad B = C^* = \frac{1}{2}\Big((d-a) + i(b' + c')\Big) \\ &b' = \frac{b}{\sqrt{3}}, \; c' = \sqrt{3}c. \end{aligned} \tag{12}$$

The Kähler potential in terms of the new coordinates can be written in the form

$$K = -log(1 - t\bar{t} - \mathcal{A}\bar{\mathcal{A}} - \mathcal{C}\bar{\mathcal{C}}). \tag{13}$$

The holomorphic function which produces (13) via (2) is given by

$$F(t,\mathcal{A},\mathcal{C}) = -\frac{1}{4}(1 + t^2 + \mathcal{A}^2 + \mathcal{C}^2) = -\frac{1}{4(x^0)^2}\Big((x^0)^2 - \sum_i^3 (x^i)^2\Big), \tag{14}$$

where $(t,\mathcal{A},\mathcal{C}) = (\frac{x^1}{ix^0}, \frac{x^2}{ix^0}, \frac{x^3}{ix^0})$. The duality transformation (11) acts on the homogeneous coordinates as in (4) where the matrices U, V, W and Z are given by

$$Z = \frac{1}{2}\begin{pmatrix} (a+d) & (b'+c') & 0 & 0 \\ (b'+c') & (a+d) & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad U = \frac{1}{2}\begin{pmatrix} (a+d) & -(b'+c') & 0 & 0 \\ -(b'+c') & (a+d) & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$W = -\frac{1}{8}\begin{pmatrix} (c'-b') & (a-d) & 0 & 0 \\ (a-d) & (c'-b') & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad V = 2\begin{pmatrix} (c'-b') & (d-a) & 0 & 0 \\ (d-a) & (c'-b') & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

If one expands $\mathcal{F}$ in terms of higher orders of the twisted moduli,

$$\mathcal{F} = \frac{1}{4}\Big((x^0)^2 - \sum_i^3 (x^i)^2 - \sum_{n=0}^{\infty} f_n(t,\mathcal{A})\frac{(x^3)^{n+3}}{(ix^0)^{n+1}}\Big), \tag{16}$$

then using (3) the Kähler potential is now given by

$$\begin{aligned} K = &-ln\Big[1 - t\bar{t} - \mathcal{A}\bar{\mathcal{A}} - \mathcal{C}\bar{\mathcal{C}} - \\ &\frac{1}{4}\sum\Big([(t+\bar{t})\frac{\partial f_n}{\partial t} + (\mathcal{A}+\bar{\mathcal{A}})\frac{\partial f_n}{\partial \mathcal{A}} + (n+1)f_n]\mathcal{C}^{n+3} + (n+3)f_n\bar{\mathcal{C}}\mathcal{C}^{n+2} + c.c\Big)\Big]. \end{aligned} \tag{17}$$

Moreover, making use of (16) and the second equation in (6), the modified duality transformations of the moduli can be determined and are given by

$$\begin{aligned} t \to t' &= \frac{\alpha - \frac{1}{4}\sum\Big((d-a)[t\frac{\partial f_n}{\partial t} + \mathcal{A}\frac{\partial f_n}{\partial \mathcal{A}} + (n+1)f_n] + i(c'-b')\frac{\partial f_n}{\partial t}\Big)\mathcal{C}^{n+3}}{\Gamma}, \\ \mathcal{A} \to \mathcal{A}' &= \frac{\mathcal{A}}{\Gamma}, \qquad \mathcal{C} \to \mathcal{C}' = \frac{\mathcal{C}}{\Gamma}, \\ \Gamma &= \gamma - \frac{1}{4}\sum\Big(i(c'-b')[t\frac{\partial f_n}{\partial t} + \mathcal{A}\frac{\partial f_n}{\partial \mathcal{A}} + (n+1)f_n] - (d-a)\frac{\partial f_n}{\partial t}\Big)\mathcal{C}^{n+3} \\ &= \gamma + \sum_n k_n \mathcal{C}^{n+3}. \end{aligned} \tag{18}$$

Clearly under these transformations the Kähler potential transforms as $K \to K +$

in $\Gamma + $ in Γ. Using (16) and the first equation (6) we obtain

$$\gamma - \frac{1}{4}\sum\Big((a+d)[t\frac{\partial f_n}{\partial t} + \mathcal{A}\frac{\partial f_n}{\partial \mathcal{A}} + (n+1)f_n] + i(b' + c')\frac{\partial f_n}{\partial t}\Big)\mathcal{C}^{n+3}$$
$$= \Gamma\Big\{1 - \frac{1}{2}\sum\Big(t'\frac{\partial f'_n}{\partial t'} + \mathcal{A}'\frac{\partial f'_n}{\partial \mathcal{A}'} + (n+1)f'_n\Big)\mathcal{C}'^{n+3}\Big\},$$

$$\alpha - \frac{1}{4}\sum\Big(i(b' + c')[t\frac{\partial f_n}{\partial t} + \mathcal{A}\frac{\partial f_n}{\partial \mathcal{A}} + (n+1)f_n] - (a+d)\frac{\partial f_n}{\partial t}\Big)\mathcal{C}^{n+3}$$
$$= \Gamma\Big(t' + \frac{1}{2}\sum\frac{\partial f'_n}{\partial t'}\mathcal{C}'^{n+3}\Big),$$

$$\mathcal{A} + \frac{1}{2}\sum\frac{\partial f_n}{\partial \mathcal{A}}\mathcal{C}^{n+3} = \Gamma\Big(\mathcal{A}' + \frac{1}{2}\sum\frac{\partial f'_n}{\partial \mathcal{A}'}\mathcal{C}'^{n+3}\Big),$$

$$\mathcal{C} + \frac{1}{2}\sum(n+3)f_n\mathcal{C}^{n+2} = \Gamma(\mathcal{C}' + \frac{1}{2}\sum(n+3)f'_n\mathcal{C}'^{n+2}\Big). \tag{19}$$

Expanding t' and $\mathcal{A}'$ around $\tilde{t}$ and $\tilde{\mathcal{A}}$ given in equation (11), we get

$$\begin{aligned} t' &= \tilde{t} + \frac{1}{4\gamma^2}\Big\{\sum\Big(i(c' - b')\alpha - (d-a)\gamma\Big)\Big(t\frac{\partial f_n}{\partial t} + \mathcal{A}\frac{\partial f_n}{\partial \mathcal{A}} + (n+1)f_n\Big) \\ &+ \Big((a-d)\alpha - i(c' - b')\gamma\Big)\frac{\partial f_n}{\partial t}\Big\}\mathcal{C}^{n+3}\sum_{m=0}^{m=\infty}\Big(\frac{l}{\gamma}\Big)^m \\ &= \tilde{t} + \sum_n(\Delta t)_n\mathcal{C}^{n+3}, \\ \mathcal{A}' &= \tilde{\mathcal{A}} + \frac{\mathcal{A}l}{\gamma^2}\sum_{m=0}^{m=\infty}\Big(\frac{l}{\gamma}\Big)^m = \tilde{\mathcal{A}} + \sum_n(\Delta\mathcal{A})_n\mathcal{C}^{n+3}. \end{aligned} \tag{20}$$

where

$$l = \frac{1}{4}\sum\Big(i(c' - b')[t\frac{\partial f_n}{\partial t} + \mathcal{A}\frac{\partial f_n}{\partial \mathcal{A}} + (n+1)f_n] + (a-d)\frac{\partial f_n}{\partial t}\Big)\mathcal{C}^{n+3}. \tag{21}$$

Using (19)-(21), one could determine the conditions that the functions f_n must satisfy and for the first five terms one gets,

$$
\begin{aligned}
f_0(\tilde{t},\tilde{\mathcal{A}}) =&\gamma f_0(t,\mathcal{A}),\\
f_1(\tilde{t},\tilde{\mathcal{A}}) =&\gamma^2 f_1(t,\mathcal{A}),\\
f_2(\tilde{t},\tilde{\mathcal{A}}) =&\gamma^3 f_2(t,\mathcal{A}),\\
f_3(\tilde{t},\tilde{\mathcal{A}}) =&\gamma^4 f_3 - \frac{\gamma^5}{2}\frac{\partial f_0}{\partial\mathcal{A}}(\Delta\mathcal{A}_0 + C\mathcal{A}\Delta t_0) - \frac{\gamma^5}{2}\Delta t_0 C f_0 - \frac{\gamma^6}{2}\Delta t_0\frac{\partial f_0}{\partial t} + \frac{\gamma^3}{2}k_0 f_0,\\
f_4(\tilde{t},\tilde{\mathcal{A}}) =&\gamma^5 f_4 + \frac{3}{7}\gamma^4 k_1 f_0 + \frac{8}{7}k_0\gamma^4 f_1 - \frac{4}{7}\Delta t_0\Big(2\gamma^6 C f_1 + \gamma^7\frac{\partial f_1}{\partial t} + \gamma^6 C\mathcal{A}\frac{\partial f_1}{\partial\mathcal{A}}\Big)\\
&- \frac{4}{7}\Delta\mathcal{A}_0\gamma^6\frac{\partial f_1}{\partial\mathcal{A}} - \frac{3}{7}\Delta t_1\Big(\gamma^6 C f_0 + \gamma^7\frac{\partial f_0}{\partial t} + \gamma^6 C\mathcal{A}\frac{\partial f_0}{\partial\mathcal{A}}\Big) - \frac{3}{7}\Delta\mathcal{A}_1\gamma^6\frac{\partial f_0}{\partial\mathcal{A}}.
\end{aligned}
\tag{22}
$$

The $SL(2,Z)$ duality symmetry is only a subgroup of the full duality symmetry of the theory. Next we employ another duality symmetry subgroup of the theory in order to derive further constraints on the functions f_n. A symmetry of the theory is given by*

$$
\begin{aligned}
&t\to\hat{t} = \frac{\hat{A}}{\hat{B}} = \frac{t - \frac{p}{2}\mathcal{A} - \frac{p^2}{8}(1+t)}{1 + \frac{p}{2}\mathcal{A} + \frac{p^2}{8}(1+t)}, \quad \mathcal{A}\to\hat{\mathcal{A}} = \frac{\hat{C}}{\hat{B}} = \frac{\mathcal{A} + \frac{p}{2}(1+t)}{1 + \frac{p}{2}\mathcal{A} + \frac{p^2}{8}(1+t)},\\
&\mathcal{C}\to\frac{\mathcal{C}}{\hat{B}}.
\end{aligned}
\tag{23}
$$

This symmetry acts on the homogeneous coordinates as given in (4) where the matrices U, V, W and Z are given by

$$
U = Z = \begin{pmatrix} 1+\frac{p^2}{8} & 0 & 0 & 0\\ 0 & 1-\frac{p^2}{8} & -\frac{p}{2} & 0\\ 0 & \frac{p}{2} & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V = 16W = \begin{pmatrix} 0 & \frac{p^2}{2} & 2p & 0\\ -\frac{p^2}{2} & 0 & 0 & 0\\ 2p & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.
\tag{24}
$$

Using (24) and the second eq in (6), the modification of the duality transformations

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* In terms of the old basis $\mathbf{T}$ and $\mathbf{A}$, this is given by $\mathbf{A}\to\mathbf{A}+p$, $\mathbf{T}\to\mathbf{T}+\frac{\sqrt{3}}{2}p\mathbf{A}+\frac{\sqrt{3}}{4}p^2$, where p is a multiple of $\sqrt{2}$.

(25) can be calculated. These modified transformations are given by

$$
\begin{aligned}
t \to t' &= \frac{t - \frac{p}{2}\mathcal{A} - \frac{p^2}{8}(1+t) + \frac{p^2}{16}\sum\left(t\frac{\partial f_n}{\partial t} + \mathcal{A}\frac{\partial f_n}{\partial \mathcal{A}} + (n+1)f_n\right)\mathcal{C}^{n+3}}{\Lambda},\\
\mathcal{A} \to \mathcal{A}' &= \frac{\mathcal{A} + \frac{p}{2}(1+t) - \frac{p}{4}\sum\left(t\frac{\partial f_n}{\partial t} + \mathcal{A}\frac{\partial f_n}{\partial \mathcal{A}} + (n+1)f_n\right)\mathcal{C}^{n+3}}{\Lambda},\\
\mathcal{C} \to \mathcal{C}' &= \frac{\mathcal{C}}{\Lambda},\\
\Lambda &= 1 + \frac{p}{2}\mathcal{A} + \frac{p^2}{8}(1+t) + \sum\left(\frac{p^2}{16}\frac{\partial f_n}{\partial t} + \frac{p}{4}\frac{\partial f_n}{\partial \mathcal{A}}\right)\mathcal{C}^{n+3}.
\end{aligned}
\tag{25}
$$

Expanding t' and $\mathcal{A}'$ as given in (25) around $\hat{t}$ and $\hat{\mathcal{A}}$ we obtain

$$
\begin{aligned}
t' &= \frac{\hat{A} + X_1}{\Lambda} = \frac{\hat{A} + X_1}{\hat{B} - X_2} = \hat{t} + \frac{1}{\hat{B}^2}(\hat{A}X_2 + \hat{B}X_1)\sum_{m=0}^{\infty}(\frac{X_2}{\hat{B}})^m = \hat{t} + \sum_n (\delta t)_n \mathcal{C}^{n+3},\\
\mathcal{A}' &= \frac{\hat{C} + X_3}{\hat{B} - X_2} = \hat{\mathcal{A}} + \frac{1}{\hat{B}^2}(\hat{C}X_2 + \hat{B}X_3)\sum_{m=0}^{\infty}(\frac{X_2}{\hat{B}})^m = \hat{\mathcal{A}} + \sum_n (\delta\mathcal{A})_n \mathcal{C}^{n+3},\\
\mathcal{C}' &= \frac{\mathcal{C}}{\hat{B}} - \frac{\mathcal{C}}{\hat{B}^2}\sum_n s_n \mathcal{C}^{n+3},\\
\Lambda &= \hat{B} + \sum_n s_n \mathcal{C}^{n+3}.
\end{aligned}
\tag{26}
$$

Using the first eq in (6) and (26) the following conditions on the first five of the functions f_n are obtained

$$
\begin{aligned}
&f_0(\hat{t},\hat{\mathcal{A}}) = B f_0(t,\mathcal{A}),\\
&f_1(\hat{t},\hat{\mathcal{A}}) = B^2 f_1(t,\mathcal{A}),\\
&f_2(\hat{t},\hat{\mathcal{A}}) = B^3 f_2(t,\mathcal{A}),\\
&f_3(\hat{t},\hat{\mathcal{A}}) = B^4 f_3 - \frac{B^6}{2}\delta\mathcal{A}_0\Big[(\frac{p}{2}f_0 + B\frac{\partial f_0}{\partial\mathcal{A}})(1+\frac{p}{2}\mathcal{A}) + (\frac{p^2}{8}f_0 + B\frac{\partial f_0}{\partial t})\frac{p}{2}(1+t)\Big]\\
&-\frac{B^6}{2}\delta t_0\Big[(\frac{p}{2}f_0 + B\frac{\partial f_0}{\partial\mathcal{A}})(-\frac{p}{2}-\frac{p^2}{8}\mathcal{A}) + (\frac{p^2}{8}f_0 + B\frac{\partial f_0}{\partial t})(1-\frac{p^2}{8}(1+t))\Big]\\
&+\frac{B^3}{2}s_0 f_0,\\
&f_4(\hat{t},\hat{\mathcal{A}}) = B^5 f_4 + \frac{3}{7}B^4 s_1 f_0 + \frac{8}{7}s_0 B^4 f_1\\
&-\frac{3}{7}B^7\delta\mathcal{A}_1\Big[(\frac{p}{2}f_0 + B\frac{\partial f_0}{\partial\mathcal{A}})(1+\frac{p}{2}\mathcal{A}) + (\frac{p^2}{8}f_0 + B\frac{\partial f_0}{\partial t})\frac{p}{2}(1+t)\Big]\\
&-\frac{3}{7}B^7\delta t_1\Big[(\frac{p}{2}f_0 + B\frac{\partial f_0}{\partial\mathcal{A}})(-\frac{p}{2}-\frac{p^2}{8}\mathcal{A}) + (\frac{p^2}{8}f_0 + B\frac{\partial f_0}{\partial t})(1-\frac{p^2}{8}(1+t))\Big]\\
&-\frac{4}{7}B^7\delta t_0\Big[(pf_1 + B\frac{\partial f_1}{\partial\mathcal{A}})(-\frac{p}{2}-\frac{p^2}{8}\mathcal{A}) + (\frac{p^2}{4}f_1 + B\frac{\partial f_1}{\partial t})(1-\frac{p^2}{8}(1+t))\Big]\\
&-\frac{4}{7}B^7\delta\mathcal{A}_0\Big[(pf_1 + B\frac{\partial f_1}{\partial\mathcal{A}})(1+\frac{p}{2}\mathcal{A}) + (\frac{p^2}{4}f_1 + B\frac{\partial f_1}{\partial t})\frac{p}{2}(1+t)\Big].
\end{aligned}
\tag{27}
$$

In conclusion, by employing the methods of special geometry, duality symmetries of the coset space $\frac{SU(2,1)}{SU(2)\otimes U(1)}$, parametrized by the complex Kähler moduli $\mathbf{T}$ and one complex Wilson moduli $\mathbf{A}$ of a $\mathbf{Z}_3$ two-dimensional plane of an orbifold compactifaction, are realized as symplectic transformations on the vector whose components are the homogeneous coordinates x^I and $\frac{1}{2}i\frac{\partial\mathcal{F}}{\partial x^I}$, where $\mathcal{F}$ is the holomorphic function describing the special Kähler geometry. Using the fact that such tansformations are exact to all orders in the expansion of $\mathcal{F}$ in the twisted moduli, constraints are derived on the moduli dependent coefficient of the expansion. The choice of these functions depends on the model under consideration. It should be mentioned that only a subspace of the duality symmetry of the theory have been implemented in deriving such constraints and it would be interesting to derive a set of constraints which are obtainable from the full duality symmetry group. We hope to report on this in a future publication.

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