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An Infinite Integral of Bessel Functions

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ABSTRACT. An infinite integral of Bessel functions J_0 and Y_0 containing three parameters is evaluated in closed form in terms of Bessel functions.

It is the purpose of this note to show that

$$\int_0^\infty \frac{J_0(ax)\,J_0(bx) - Y_0(ax)\,Y_0(bx)}{x^2+z^2}\,dx = \frac{\pi}{2z}\,H_0^{(1)}(iaz)\,H_0^{(1)}(ibz) \qquad (1)$$

$$(a>0,\ b>0,\ \operatorname{Re} z>0).$$

In particular for real $z = t$

$$\int_0^\infty \frac{J_0(ax)\,J_0(bx) - Y_0(ax)\,Y_0(bx)}{x^2+t^2}\,dx = -\frac{2}{\pi t}\,K_0(at)\,K_0(bt) \qquad (2)$$

$$(a>0,\ b>0,\ t>0).$$

For imaginary $z = it$, this integral can be written as

$$\int_0^\infty \frac{J_0(ax)\,J_0(bx) - Y_0(ax)\,Y_0(bx)}{x^2-t^2}\,dx \qquad (3)$$

$$= -\frac{\pi}{2t}\,\{J_0(at)\,Y_0(bt) + Y_0(at)\,J_0(bt)\} \qquad (a>0,\ b>0,\ t>0).$$

J_0 and Y_0 are the Bessel functions of order zero, $H_0^{(1)} = J_0 + iY_0$ is the Hankel function of the first kind, and K_0 is the modified Bessel function of the third kind. The integral (3) is to be understood as a Cauchy principal value. These integrals do not seem to be well-known.

Proof. In order to show (1), we make use of a complex ζ-plane which is cut along the negative real axis, and consider the integral $\oint_{\Gamma_1} f(\zeta)\,d\zeta$, with

$$f(\zeta) = \frac{H_0^{(1)}(a\zeta)\,H_0^{(1)}(b\zeta)}{\zeta^2+z^2},$$

over a closed contour Γ_1 in the upper half-plane $\operatorname{Im}\zeta > 0$. Since the singularity of $H_0^{(1)}$ at $\zeta = 0$ is logarithmic, the contour Γ_1 can be chosen as a straight line along the real axis from $-R$ to R and a semicircle C_R with radius R and centre zero, where $R > |z|$. Taking into account that $H_0^{(1)}$ has no other singularities, and that $\zeta = iz$ is the only pole of $f(\zeta)$ inside Γ_1, we obtain by applying the residue theorem

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(\zeta)\,d\zeta = 2\pi i \operatorname{res} f(\zeta)|_{\zeta=iz}. \qquad (4)$$

By using the well-known relation [2, No. 8.476 6.]

$$H_0^{(1)}(-x) = H_0^{(1)}(x) - 2J_0(x) \qquad (x\in\mathbb{R}) \qquad (5)$$

and the definition of $H_0^{(1)}$ given above, the integral from $-R$ to R can be written as

$$\begin{aligned}
&\int_{-R}^{R} \frac{H_0^{(1)}(ax)\, H_0^{(1)}(bx)}{x^2+t^2}\, dx \\
&= \int_0^R \frac{H_0^{(1)}(ax)\, H_0^{(1)}(bx) + H_0^{(1)}(-ax)\, H_0^{(1)}(-bx)}{x^2+t^2}\, dx \qquad (6) \\
&= 2\int_0^R \frac{J_0(ax)\, J_0(bx) - Y_0(ax)\, Y_0(bx)}{x^2+t^2}\, dx.
\end{aligned}$$

For the residue at $\zeta = iz$ we obtain

$$\operatorname{res} f(\zeta)|_{\zeta=iz} = -\frac{i}{2z}\, H_0^{(1)}(iaz)\, H_0^{(1)}(ibz). \qquad (7)$$

It remains to show that the integral over C_R vanishes as $R \to \infty$. By using the asymptotic expansion [3, p. 197]

$$H_0^{(1)}(\zeta) = \sqrt{\frac{2}{\pi\zeta}}\, e^{i(\zeta - \frac{1}{4}\pi)}\, [1 + O(\zeta^{-1})]$$

which is valid in the upper half-plane $\operatorname{Im}\zeta \geq 0$, we find using $|\zeta^2 + z^2| \geq |\zeta|^2 - |z|^2$ that, up to a factor $[1 + O(R^{-1})]$,

$$\begin{aligned}
\left|\int_{C_R} f(\zeta)\, d\zeta\right| &\leq \frac{R}{R^2 - |z|^2} \int_0^\pi \left|H_0^{(1)}\left(aRe^{i\phi}\right)\, H_0^{(1)}\left(bRe^{i\phi}\right)\right| d\phi \\
&\leq \frac{2}{\pi\sqrt{ab}}\, \frac{1}{R^2 - |z|^2} \int_0^\pi e^{-(a+b)R\sin\phi}\, d\phi \leq \frac{2}{\sqrt{ab}}\, \frac{1}{R^2 - |z|^2}.
\end{aligned}$$

As $R \to \infty$ this expression tends to zero and we obtain, from (4), (6) and (7), the result as stated in (1). For real $z = t$, formula (2) follows by using [1, No. 9.6.4]

$$K_0(x) = \tfrac{1}{2}\pi i\, H_0^{(1)}(ix).$$

In order to prove (3), we consider the integral $\oint_{\Gamma_2} f(\zeta)\, d\zeta$, with

$$f(\zeta) = \frac{H_0^{(1)}(a\zeta)\, H_0^{(1)}(b\zeta)}{\zeta^2 - t^2},$$

over a closed contour Γ_2, which differs from Γ_1 in that the two poles of $f(\zeta)$ at $\zeta = \pm t$ have to be avoided by small semicircles on the integration path from $-R$ to R. Again, by applying the residue theorem, we obtain

$$\int_{-R}^{R} f(x)\, dx + \int_{C_R} f(\zeta)\, d\zeta = \pi i \{\operatorname{res} f(\zeta)|_{\zeta=-t} + \operatorname{res} f(\zeta)|_{\zeta=t}\}. \qquad (8)$$

The integral from $-R$ to R can be handled in the same way as in (6), and here also the integral over C_R tends to zero as $R \to \infty$. For the sum of the residues at $\zeta = -t$ and $\zeta = t$ we obtain by using (5),

$$\begin{aligned} & \operatorname{res} f(\zeta)|_{\zeta=-t} + \operatorname{res} f(\zeta)|_{\zeta=t} \\ = \; & \frac{1}{2t}\{H_0^{(1)}(at)\,H_0^{(1)}(bt) - H_0^{(1)}(-at)\,H_0^{(1)}(-bt)\} \\ = \; & \frac{i}{t}\{J_0(at)\,Y_0(bt) + Y_0(at)\,J_0(bt)\}. \end{aligned} \tag{9}$$

As $R \to \infty$, we obtain from (6), (7) and (9) the result as stated in (3). We may add here that (3) can also be obtained by setting $z = it$ in (1) and taking the real part of the right-hand side.

REFERENCES

[1] M. Abramowitz and I.A. Stegun, eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing with corrections (Dover, New York, 1972).

[2] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products,* 5th edition, A. Jeffrey, ed., (Academic Press, New York, 1994).

[3] G.N. Watson, *A Treatise on the Theory of Bessel Functions* (University Press, Cambridge, 1966).