The Topology of The Cosmic Microwave Background Anisotropy on The Scale $\sim 1^o$

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Abstract

In this paper we develop the theory of clusterization of peaks in a Gaussian random field. We have obtained new mathematical results from this theory and the theory of percolation and have proposed a topological method of analysis of sky maps based on these results. We have simulated $10^o \times 10^o$ sky maps of the cosmic microwave background anisotropy expected from different cosmological models with $0.5^o - 1^o$ resolution in order to demonstrate how this method can be used for detection of non-Gaussian noise in the maps and detection of the Doppler-peak in the spectrum of perturbation of $\Delta T/T$.

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1 Introduction

Observations of the cosmic microwave background (CMB) anisotropy are fundamental in understanding the formation of structures and the nature of the dark matter in the Universe. The distribution of the fluctuations of ΔT on angular scales larger than a few degrees characterizes fluctuations of the primordial density on scales larger than the acoustic horizon at the moment of the last scattering. In this paper we discuss the CMB fluctuations on the angular scale $\theta \sim 1^o$. This angular size corresponds to the scale at recombination. The distribution of the CMB anisotropy on this scale preserves the information about the ionization history of the Universe and about the nature of the dark matter. Various scenarios of recombination and different amounts of baryons in different cosmological models lead to different properties of the anomaly in the ΔT distribution. (Namely, the different height of the so-called Doppler-peak in the power spectrum C_l at the multipole number $l \approx 220$). (Efstathiou, Bond, White, 1992; Coulson et al., 1994; Gorski, 1993; Scott et al., 1995).

Several groups have reported on observational data on angular scales about 1^o (Devlin et al., 1994; Clapp et al., 1994; Dragovan et al., 1994; De Bernardis et al., 1994; Cheng et al., 1994; Schuster et al., 1993; Wollack et al., 1994). The interpretations of these experimental results and the comparison with the expected power spectrum of the $\Delta T/T$ fluctuations in different cosmological models are unclear.

Several authors propose different methods to detect the peak-like anomaly in the angular distribution. These methods are based on correlation analysis (Hinshaw et al., 1995; Naselsky and I.Novikov 1993, Jørgensen et al., 1995). Recently, the first attempt to understand the topological properties of $\Delta T/T$ maps caused by the presence of the Doppler-peak, was made by Naselsky and D. Novikov (1995). Many important questions related to this problem were listed in the paper by Hinshaw, Bennett and Kogut (1994). Some of them are: 1) how to distinguish the effect caused by the presence of the Doppler-peak from the isolated point sources, and 2) what is the role of the cosmic variance in a statistical analysis of small regions of the sky.

In this paper, we propose the topological method of analysis of the sky maps. This method is based on the assumption that the initial fluctuations are Gaussian. In this case, the background radiation will form a two-dimensional Gaussian random field. The statistical properties of a Gaussian random field were first investigated by Rice (1944,1945) for a one-dimensional field to analyze electrical noise in communication devices. A.Doroshkevich (1970) was the first who applied this theory extensively in the study of formation of cosmic structures. J.Bardeen J.Bond, N.Kaiser and A.Szalay (1986) (referred to as BBKS in what follows) in their classical paper developed the theory for a three-dimensional field and J.Bond and G.Efstathiou (1987) (referred to as BE in what follows) developed the theory for a two-dimensional field.

In the work presented here, we develop the theory of clusterisation of peaks in a random Gaussian field, investigate the influence of the spectral parameters on the $\Delta T/T$ fluctuations for different cosmological models, propose methods of filtering of the sky maps, investigate the properties of the two-point correlation function for a small region of the sky, investigate the topological properties of the $\Delta T/T$ angular distribution caused by the presence of the Doppler-peak in the spectrum of fluctuations of $\Delta T/T$, and propose the method of percolation and cluster analysis which allows us to detect the unresolved point sources (non-Gaussian noise which can closely imitate the presence of a Doppler-peak in the spectrum) and remove them from the observational data.

Note, that percolation has become a popular term among astronomers and cosmologists. It was first introduced in cosmology by Zel'dovich (1982) and has been succesfully applied during several years for the investigation of density distributions in the non-linear stage of gravitational instability. (See the review by Dominik and

Shandarin, 1992; Einasto et al., 1984; Klypin, 1985). The $\Delta T/T$ maps percolation as a first test of the presence of non-Gaussian noise was first discussed by Naselsky and D.Novikov (1995).

The outline of this paper is as follows. In Section 2, we review some general properties of $\Delta T/T$ fluctuations and investigate the properties of the correlation function for different cosmological models, namely with and without a Doppler-peak. In Section 3, we introduce different kinds of filtering of the $\Delta T/T$ maps and discuss the influence of filtering on spectral parameters and the correlation function. In Section 4, we develop the properties of one-dimensional slices of a two-dimensional Gaussian random field and of a two-dimensional Gaussian random field itself and obtain what we consider to be the most important result of this paper: that the clusterisation of peaks in a Gaussian random field depends only on one spectral parameter γ and is determined by the Gaussian nature of the field. In the same section we also propose a method for detecting "noise peaks". In Section 5, we propose a method of analysis of the sky maps. We summarize our main results in Section 6. Details of the derivations may be found in the appendices.

2 The power spectrum and correlation function on intermediate scales

In this paper we assume that the fluctuations in the cosmic microwave background (CMB) are the result of a random Gaussian process. This hypothesis has been adopted by many authors (see the review by B.E.Hinshaw et al., 1995) and may be argued for in the following way: since the fluctuations of the microwave background were imprinted in the linear regime, $\Delta T/T$ will be a linear combination of the initial perturbations amplitudes, and will, therefore, have the same probability distribution. Such a field with a zero mean is completely characterized by its two-point correlation function or by the equivalent power spectrum.

Let us consider the distribution of the temperature of the cosmic microwave background on the celestial sphere. As it was mentioned above, we assume that the field is a two-dimensional random Gaussian field on a sphere. This field is completely characterized by the power spectrum C_l. Using this description one can write the following well known expression for the temperature of the relic radiation

$$T(\overline{q}) = \langle T(\overline{q})\rangle + \sum_{l=1}^{\infty}\sum_{m=-l}^{l} a_l^m C_l^{\frac{1}{2}} Y_l^m(\overline{q}), \tag{2.1}$$

where $\overline{q}$ is the unit vector tangential to the direction of photon motion; a_l^m are independent random Gaussian numbers; $\langle T(\overline{q})\rangle$ is the average temperature of the relic radiation, so that $\langle T(\overline{q})\rangle = \frac{1}{4\pi}\int T(\overline{q})d\Omega$; Y_l^m are the spherical harmonics.

We introduce the following expression for the anisotropy of the CMB: $\Delta T(\overline{q}) = (T(\overline{q}) - \langle T(\overline{q})\rangle)/\langle T(\overline{q})\rangle$. The two-point correlation function $C(\theta)$ can be found by averaging of $\Delta T(\overline{q}) \cdot \Delta T(\overline{q}')$ over the whole sky under the condition that the angle between the directions of $\overline{q}$ and $\overline{q}'$ is a constant:

$$C_{obs}(\theta) = \langle \Delta T(\overline{q}) \cdot \Delta T(\overline{q}') \rangle, \quad \overline{q} \cdot \overline{q}' = \cos\theta \ . \tag{2.2}$$

Taking into account Eq.(2.1) and that $\langle a_l^m a_{l'}^{m'} \rangle = \delta_{ll'}\delta_{mm'}$ we get:

$$C_{obs}(\theta) = \frac{1}{4\pi}\sum_{l=2}^{\infty}\sum_{m=-l}^{l}(a_l^m)^2 C_l P_l(\cos\theta). \tag{2.3}$$

The average value of the observable correlation function is

$$C(\theta) = \overline{C_{obs}(\theta)} = \frac{1}{4\pi}\sum_{l=2}^{\infty}(2l+1)C_l P_l(\cos\theta) \ . \tag{2.4}$$

We begin the summation in Eq.(2.4) from $l = 2$. The term with $l = 1$ has to be removed before the calculation of the correlation function because the contribution to this term in the $\Delta T/T$ fluctuations cannot be separated from the motion of the observer relative to the background radiation.

Note, that due to the presence of the Doppler-peak in the spectrum of perturbations of $\Delta T/T$, the two-point correlation function for the CMBR anisotropy has a very characteristic feature on the angular scale of the last scattering, $\theta \sim \theta_{ls}$. This characteristic feature looks like a local antipeak on this scale.

In practise the Doppler-peak in the spectrum of perturbations leads to an effective decrease of the correlation radius. On Fig.1 we have plotted the two-point correlation function for three different models ($\Omega_b = 0.03$, $h = 0.5$; $\Omega_b = 0.1$, $h = 0.5$; and the model without a Doppler-peak) in order to illustrate their properties on a scale $\theta \sim 1^o$.

Let us consider the influence of a finite size of the considered region on the property of the observed correlation function. It is obvious, that the observed correlation function differs from its average ensemble approximately by the value of the dispersion

$$D_o(\theta) = \overline{C_{obs}^2}(\theta) - \overline{C_{obs}}^2 = \left(\frac{1}{4\pi}\right)^2 \sum_l (2l+1)C_l^2 P_l^2(\cos\theta) \ . \tag{2.5}$$

The subscript 0 on the left-hand side means that this value is obtained by averaging over the whole sky. The value $D_0(\theta)$ is very small for $\theta \sim 1^o$, but if we consider only a small part of the sky, then this value increases to:

$$D_\Omega(\theta) \sim \sqrt{\frac{4\pi}{\Omega}} D_0(\theta), \tag{2.6}$$

where Ω is the angular size of the considered region. On Fig.2 we have plotted the correlation function for $\Omega_b = 0.03$, its dispersion and realizations of the random process for a part of the sky of $10^o \times 10^o$ for the unfiltered and filtered cases presented on Maps 1a and 1b. This result demonstrates the difficulty to detect the Doppler-peak using only small regions of the sky.

Note, that this difficulty is caused only by "cosmic variance". If the presence of the noise is taken into account, then this difficulty becomes even worse (see Hinshaw et al., 1995). Further we are interested in detection of the peak by using not only correlation analysis of the observational data, but also by topological analysis of the sky maps. For the further investigations we introduce the filtering of sky maps.

3 Filtering of the sky maps

a. Filtered correlation functions

If we only consider a small part of the sky, then the geometry is approximately flat, and we can introduce the Dekart coordinates on a small area of a unit radius sphere ($\theta \times \theta$, $\theta \ll \pi$) and describe $\Delta T(x, y)$ as a sum of the Fourier series (B.E.):

$$\Delta T(x,y) = \sum_{ij} a_{ij} C^{\frac{1}{2}}(k) \cos\left(2\pi \frac{ix + jy}{L} + \varphi_{ij}\right), \tag{3.7}$$

where $C(k)$ is the power spectrum; $k = \frac{2\pi}{L}\sqrt{i^2 + j^2}$; a_{ij} are independent random Gaussian values; φ_{ij} are random phases homogeneously distributed in the interval $(0, 2\pi)$; $L \approx \theta_0$ is the size of the investigated area.

Note, that Eq.(3.7) takes into account only modes smaller than L and cannot be applied for simulation of the map $L \times L$. Nevertheless, after simulation of the map $L \times L$ by (3.7), it is possible to use only the part of this map $l \times l$, where $l \ll L$ since the simulated distribution of ΔT in this part contains all modes and can be used for imitation of the ΔT distribution on the celestial sphere.

The correlation function $C_{obs}(r) = \langle \Delta T(x, y) \cdot \Delta T(x', y') \rangle$ can be found by averaging over the square $L \times L$ similar to Eq.(2.2).

$$C_{obs}(r) = \frac{1}{2} \sum_{ij} a_{ij}^2 C(k) J_0(kr), \tag{3.8}$$

where $r = \sqrt{(x - x')^2 + (y - y')^2}$. After averaging over the ensemble, we have

$$C(r) = \overline{C_{obs}(r)} = \frac{1}{2} \sum_{ij} C(k) J_0(kr). \tag{3.9}$$

where k was given above as function of i and j. Eq.(3.9) is in good agreement with Eq.(2.4) because if $\theta \ll \pi$ and $l \gg 1$ then $P_l(\cos\theta) \approx J_0(l\theta)$, and $l\theta \approx kr$.

Thus, the fluctuations of ΔT can be described by Eq.(3.7), where $C(k) \approx C_l$, $k \sim l/\xi_n$ (ξ_n is the present horizon), and C_l are the well known coefficients of the correlation function decomposition to series of Legendre polinomials, for different models of the Universe. It is obvious that the observational correlation function differs from its average over the ensemble by the value of the dispersion:

$$D(r) = \overline{C_{obs}^2(r)} - (\overline{C_{obs}(r)})^2 = \frac{1}{2}\sum_{ij} C^2(k) J_0^2(kr) \ . \tag{3.10}$$

For ($r \ll L$) the correlation function and its dispersion can be written as follows:

$$\begin{array}{l} C(r) = \pi \int kC(k)J_0(kr)dk \\ D(r) = \pi \int kC^2(k)J_0^2(kr)dk \\ C_{obs}(r) \sim C(r) \pm \sqrt{D(r)}. \end{array} \tag{3.11}$$

Note that for investigation of scales as the size of the recombination horizon, $R = R_{LS}$, the size of area has to be $\theta_{LS} \ll l \approx \theta_0 \ll \pi$, $\theta_{LS} \simeq \frac{R_{LS}}{\xi_n}$

For the further discussion we introduce smoothing of the map by a Gaussian filter:

$$H_0(\delta, r) = \frac{1}{4\pi\delta^2} e^{-\frac{r^2}{4\delta^2}}; \qquad r^2 = (x' - x)^2 + (y' - y)^2 \ . \tag{3.12}$$

Then the filtered field $\Delta\widetilde{T}(x, y)$ is calculated from the initial, i.e., observable field $\Delta T(x, y)$ by:

$$\Delta\widetilde{T}(x, y) = \frac{1}{4\pi\delta^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Delta T(x', y') e^{-\frac{r^2}{4\delta^2}} dx' dy' \ . \tag{3.13}$$

Formally, the limits of integration in Eq.(3.13) are $(-l/2, l/2)$, but, if $\delta \ll L$, we can use $\pm\infty$ instead of $(-l/2, l/2)$ as the limits of integration. Thus, the correlation function for the filtered map is:

$$C(r, \delta) = \langle \Delta\widetilde{T}(x, y) \cdot \Delta\widetilde{T}(x', y') \rangle = \pi \int_0^\infty kC(k) e^{-\frac{k^2\delta^2}{2}} J_0(kr) dk \ . \tag{3.14}$$

Another kind of filtering is the difference between two Gaussian filters:

$$H_*(\delta_{min}, \delta_{max}, r) = H_0(\delta_{min}, r) - H_0(\delta_{max}, r) \ . \tag{3.15}$$

This filter removes the influence of the long modes. The spectrum of multipoles C_l for many popular models has a specific shape:

$$C_l \sim \frac{1}{l(l+1)} \ . \tag{3.16}$$

Therefore, it is convenient to introduce a filter $H_2(\delta, r)$ which has the following characteristic feature: the spectrum of the filtered map, $\widetilde{C}(k)$, can be obtained from the spectrum of unfiltered map $C(k)$ by:

$$\widetilde{C}(k) = k^2 C(k) e^{-\frac{k^2\delta^2}{2}} \ . \tag{3.17}$$

If $C(k) = \frac{1}{k^2}$ (spectrum without Doppler-peak), then from Eq.(3.14) and Eq.(3.17) we can easily see, that the correlation function for the filtered map has an especially simple (Gaussian) form:

$$\frac{C(r,\delta)}{C(0,\delta)} = e^{-\frac{r^2}{2\delta^2}} \ . \tag{3.18}$$

The deviation from the Gaussian in the observed correlation function for a filtered map can be caused either by presence of the Doppler-peak or by "cosmic variance" and noise. On Fig.1(a,b,c) we have plotted filtered correlation functions with different kinds of filters. This filter also improves the statistical properties of the map, which is convenient for the further cluster analysis. On the Map.1a,b we have shown the ΔT distribution after filtering and unfiltered, respectively.

Now let us consider, how the filtering influences the spectral parameters.

b. Spectral parameters

Using, Eq.(3.11), we can introduce the spectral parameters similarly to (BBKS):

$$\begin{aligned} &\sigma_0^2 = \pi \int kC(k)dk \\ &\sigma_1^2 = \pi \int k^3C(k)dk \\ &\sigma_2^2 = \pi \int k^5C(k)dk \\ &R_* = \frac{\sigma_1}{\sigma_2}, \qquad \gamma = \frac{\sigma_1^2}{\sigma_0\sigma_2} \ . \end{aligned} \tag{3.19}$$

These parameters are completely defined by the value of correlation function and its second and fourth derivatives at the zero.

In the next section we will demonstrate, that the value of γ determines the geometry of the ΔT distribution seen on the map. In Fig.3a,b we have plotted the dependence of γ on the value of resolution for different kinds of filter. From this figure we can easily see that the parameter γ, for the model with a Doppler-peak, is different from γ for the model without a peak for an appropriate value of δ (for a resolution angle which corresponds to $l \sim 150$). From Fig.1 we also see that the differences between the correlation functions for the different models is particularly evident for this specific value of δ. We name this value as the resonance filter value, δ_{res}.

4 Clusterisation of peaks in a random Gaussian field

A Gaussian random field is the field for which a joint Gaussian probability distribution for random variables x_i is:

$$P(x_1, ..., x_n)dx_1....dx_n = \frac{e^{-Q}}{((2\pi)^n \det M)^{1/2}} dx_1...dx_n,$$

$$2Q = \sum_{ij} \Delta x_i (M^{-1})_{ij} \Delta x_j \ . \tag{4.20}$$

Only the means of the random variables $\langle x_i \rangle$ and their variances are required to specify completely the covariance matrix M and the distribution

$$M_{ij} = \langle \Delta x_i \Delta x_j \rangle, \qquad \Delta x_i = x_i - \langle x_i \rangle. \tag{4.21}$$

4.1 One-dimensional cross-section of the two-dimensional field

The problem of the clusterisation of peaks is especially easy in one-dimension. We demonstrate it in order to illustrate the properties of clusters of maxima in one dimension. Note, that results of this section can be applied to the analysis of one-dimensional observation. We now introduce the definition of a cluster of maxima of $\Delta T(x)$. Let us consider two points, x_1 and x_2, $(x_1 < x_2)$ and a threshold level ν_t (where $\nu = \frac{\Delta T}{\sigma_0}$), such that $\Delta T(x) > \nu_t$ for all x in the interval $[x_1; x_2]$. We call these two points connected to each other with respect to the threshold ν_t. A collection of k maxima (of $\Delta T(x)$) located at the points $x_1, x_2, ..., x_k$ is called a cluster of k maxima if all the points $x_1, x_2, ..., x_k$ are connected to each other with respect to the threshold level ν_t:

$$\Delta T(x) > \nu_t, \qquad x_1 \leq x \leq x_k \ . \tag{4.22}$$

In what follows, we will estimate the probability of existence of such a configuration.

a. One-dimensional distribution

If we consider a one-dimensional slice of the two-dimensional field, then the distribution of $\Delta T(x)$ can be written using Eq.(3.7)

$$\Delta T(x) = \sum_{ij} a_{ij} \sqrt{C(k)} \cos\left(\frac{2\pi}{L} ix + \varphi_{ij}\right) \qquad k = \frac{2\pi}{L}\sqrt{i^2 + j^2} \ .$$

The two-point correlation function in this case is:

$$C(r) = \langle \Delta T(x) \Delta T(x') \rangle = \pi \int kC(k) J_0(kr) dk, \qquad r = |x - x'|,$$

similar to the two-dimensional case. Next, we introduce new variables:

$$\nu(x) = \frac{\Delta T(x)}{\sigma_0}; \qquad \eta(x) = \frac{\Delta T'(x)}{\sigma_1}; \qquad \xi(x) = \frac{\Delta T''(x)}{\sigma_2} \ . \tag{4.23}$$

b. Conditional probability

We shall estimate the probability that two given maxima at the points x_1 and x_2 are connected together. A way to solve this problem is to find the total num of minima above some level ν_t, $n^+_{min}(r)$, and the total number of all minima, $n_{min}(r)$, between these two points. The ratio of these two values is an estimate of the desired probability.

We introduce the following notation. The event that the minimum with value above ν_t is located at the point x is called the event A. The event that the maximum with value above ν_t is located at the point x_1 is called the event B. The event that such a maximum is located at the point x_2 is called the event C. Then the conditional probability that minimum at the point x between two given maxima is above some threshold ν_t can be calculated via the Bayes formula:

$$P(A|B;C) = \frac{P(A;B;C)}{P(B;C)} \ . \tag{4.24}$$

The obvious joint probabilities

$$P(A;B;C) = (\Delta x)^3 \int |\xi_1 \xi_2 \xi| P(\nu_1, \eta_1 = 0, \xi_1, \nu_2, \eta_2 = 0, \xi_2, \nu, \eta = 0, \xi) \times$$

$$\times d\nu_1 d\nu_2 d\nu d\xi_1 d\xi_2 d\xi, \tag{4.25}$$

$$P(B;C) = (\Delta x)^2 \int |\xi_1 \xi_2| P(\nu_1, \eta_1 = 0, \xi_1, \nu_2, \eta_2 = 0, \xi_2) d\nu_1 d\nu_2 d\xi_1 d\xi_2.$$

The limits of the integration in Eq.(4.25) are
1) for ν_1, ν_2, ν – from ν_t to $+\infty$,
2) for ξ_1, ξ_2 – from $-\infty$ to 0,
3) for ξ – from 0 to $+\infty$.
The conditional probability is the ratio of these two integrals:

$$\frac{P(A;B;C)}{P(B;C)} = N^+_{min}(x)\Delta x, \tag{4.26}$$

where $N^+_{min}(x)\Delta x$ is the differential density of minima above ν_t (here and below x and r are in the units of R_*). The constraint that $\nu(x)$ is a minimum of any arbitrary value leads to identical equations except that the integration $d\nu$ is over all ν from $-\infty$ to $+\infty$.

The number of minima $n^+_{min}(r)$ and the total number of minima $n_{min}(r)$ of arbitrary value between x_1 and x_2 can be found by substituting Δx in (4.21) by dx and integrating from x_1 to x_2:

$$n^+_{min}(r) = \int_{x_1}^{x_2} N^+_{min}(x) dx,$$

$$n_{min}(r) = \int_{x_1}^{x_2} N_{min}(x) dx, \tag{4.27}$$

where $N_{min}(x)$ is the differential density of minima of an arbitrary value at the point x. The probability that any given minimum (from this collection of minima) has a value above ν_t is, therefore, $N^+_{min}(r)/N_{min}$. The probability that two given maxima (at points x_1 and x_2) are connected together is equal to the probability that all $n_{min}(r)$ minima between them are above ν_t. All points of considered minima can be labeled from x_1^{min} and $x_{n(r)}^{min}$ and the desired probability is

$$P(x_1 \longleftrightarrow x_2) = \frac{N^+_{min}(x_1^{min})}{N_{min}(x_1^{min})} \; ... \; ... \; \frac{N^+_{min}(x_{n(r)}^{min})}{N_{min}(x_{n(r)}^{min})} \; . \tag{4.28}$$

Here, $P(x_1 \longleftrightarrow x_2)$ is the probability that $\nu(x) > \nu_t$, for $x_1 \leq x \leq x_2$ (i.e. the probability that x_1 and x_2 are connected). In order to estimate this value, we substitute $n^+_{min}(r)/r$ and $n_{min}(r)/r$ by $N^+_{min}(x_i)$ and $N_{min}(x_i)$ respectively and Eq.(4.28) simplifies to:

$$P(x_1 \longleftrightarrow x_2) \approx \left(\frac{n^+_{min}(r)}{n_{min}(r)}\right)^{n_{min}(r)} \; . \tag{4.29}$$

Note, that we always have $n_{min}(r) \geq 1$ and $n^+_{min}(r)/n_{min}(r) \leq 1$. If we consider two maxima which are spaced at $r \gg 1$, then $P(A;B;C) = P(A) \cdot P(B) \cdot P(C)$, and $P(B;C) = P(B) \cdot P(C)$ because events A, B, C become independent. In this case we have:

$$\begin{aligned} &N^+_{min}(x) = \widetilde{N}^+_{min} = const, \\ &N_{min}(x) = \widetilde{N}_{min} = const, \\ &n^+_{min}(r) = N^+_{min} \cdot r, \\ &n_{min}(r) = N_{min} \cdot r, \end{aligned} \tag{4.30}$$

where $\widetilde{N}^+_{min}$ and $\widetilde{N}_{min}$ are the number of minima above ν_t and the number of minima of arbitrary height respectively, without the condition that there are maxima at x_1 and x_2. Combining Eqs.(4.29) and (4.30) we get

$$P(x_1 \longleftrightarrow x_2) = \left(\frac{N^+_{min}}{N_{min}}\right)^{N_{min} \cdot r} \; . \tag{4.31}$$

If $r \leq 1$, then most probably only one minimum is located between two maxima ($n_{min}(r) = 1$) and

$$P(x_1 \longleftrightarrow x_2) \approx N^+_{min}(r/2) \; . \tag{4.32}$$

c. Shape around two maxima

The expected value of the field around two maxima at the point x_1 and x_2 can be obtained from the covariance matrix of 7×7 variables: $\nu_{1,2} = \nu(x_{1,2})$; $\eta_{1,2} = \eta(x_{1,2})$; $\xi_{1,2} = \xi(x_{1,2})$ and $\nu(x)$. The derivation is given in detail in Appendix A2. The steps

are as follows:
1) The conditional probability that the field in point x falls in the range from ν to $\nu + d\nu$ on condition that $\nu(x_{1,2}) = \nu_{1,2}$, $\eta(x_{1,2}) = 0$, $\xi(x_{1,2}) = \xi_{1,2}$, is

$$P(\nu|B;C)d\nu = \frac{P(\nu(x_{1,2}) = \nu_{1,2},\ \ \eta(x_{1,2}) = 0,\ \ \xi(x_{1,2}) = \xi_{1,2},\ \ \nu)}{P(\nu(x_{1,2}) = \nu_{1,2},\ \ \eta(x_{1,2}) = 0,\ \ \xi(x_{1,2}) = \xi_{1,2})} d\nu, \tag{4.33}$$

where events B and C are the conditions that there are maxima at the points x_1 and x_2 respectively.
2) Multiplication of the right hand side of Eq.(4.33) by ν and integration over ν gives the expected value of the field at an arbitrary point x, $\langle \nu(x) \rangle$. Multiplication of the right hand side of Eq.(4.33) by ν^2 and integration over ν results in the variance $\langle \nu(x)^2 \rangle$.
The result (shown in Fig.4) can be applied to the local analysis of one-dimensional data, since we can estimate the probability for an appropriate pair of extrema to be Gaussian or not.

d. Two-point peak-peak correlation function

If a maximum with the height above ν_t is located at the point $r = 0$, then the differential density of maxima $N^+_{max}(r)$ above this threshold at the distance r from the given maximum can be calculated using the technique of subsection **b**:

$$N^+_{max}(r)dr = \frac{\int |\xi_1 \xi| P(\nu_1, \eta_1 = 0, \xi_1, \nu, \eta = 0, \xi) d\nu_1 d\nu d\xi_1 d\xi}{\int |\xi_1| P(\nu_1, \eta_1 = 0, \xi_1) d\nu_1 d\xi_1} \frac{\sigma_2}{\sigma_1} dr\ . \tag{4.34}$$

This density has to be compared with $\widetilde{N^+_{max}}$, i.e. with the density of maxima above ν_t without the condition that there is a maximum at $r = 0$:

$$\widetilde{N^+_{max}} = \int |\xi| P(\nu, \eta = 0, \xi) d\nu d\xi. \tag{4.35}$$

Then, the two-point peak-peak correlation function is:

$$\Psi_{p-p}(r) = \frac{N^+_{max}(r) - \widetilde{N^+}_{max}}{\widetilde{N^+}_{max}}. \tag{4.36}$$

The two-point peak-antipeak correlation function can be obtained similarly to Eq.(4.36):

$$\Psi_{p-ap}(r) = \frac{N^+_{min}(r) - \widetilde{N^+}_{min}}{\widetilde{N^+}_{min}}\ . \tag{4.37}$$

e. Clusters of peaks

The problem of clusterisation of peaks has a few difficult points. Calculation of the probability of appearance of a cluster of length k is quite difficult when peak-peak and peak-antipeak correlations are taking into account. However the calculation of the mean length of a cluster for an appropriate level ν_t is much easier and can be done analytically.

Let us consider a maximum above the threshold ν_t at the point $x = x_*$, and calculate the total numbers of maxima and minima above ν_t in the vicinity of this peak for x from $x_* - r$ to $x_* + r$:

$$n^+_{max}(r) = \widetilde{N}^+_{max} \int_{x_*-r}^{x_*+r} (\Psi_{p-p}(x) + 1)dx, \tag{4.38}$$

$$n^+_{min}(r) = \widetilde{N}^+_{min} \int_{x_*-r}^{x_*+r} (\Psi_{p-ap}(x) + 1)dx.$$

In the vicinity of this maximum clusters of the different length k appear with the appropriate probability, which is difficult to determine. However, the total number of clusters in this vicinity is given exactly by

$$\sum_k N_k(r) = n^+_{max}(r) - n^+_{min}(r) + 1 \ , \tag{4.39}$$

where N_k is the number of clusters of the length k. Obviously, the summation of the number of maxima in all clusters gives the total number of maxima above ν_t from $x_* - r$ to $x_* + r$:

$$\sum_k kN_k = n^+_{max}(r) + 1. \tag{4.40}$$

The unit in the right hand side of Eqs.(4.39) and (4.40) accounts for one additional maximum at $x = x_*$. Therefore, the mean length of a cluster $\overline{k}(r)$, is:

$$\overline{k}(r) = \frac{\sum_k k \cdot N_k(r)}{\sum_k N_k(r)} = \frac{n^+_{max}(r) + 1}{n^+_{max}(r) - n^+_{min}(r) + 1}. \tag{4.41}$$

Note, that if $n^+_{max}(r) = n^+_{min}(r)$ (with $n^+_{max}(r) \geq n^+_{min}(r)$ always being true), then only one cluster of the length $n^+_{max}(r) + 1$ appears. We are interested in the mean value $\overline{k}$ for the whole considered region $[0; L]$. This value can be found by using Eq.(4.38) and Eq.(4.41) and averaging over all maxima, and letting the integration in Eq.(4.38) go from 0 to L. If the statistical properties of the region are "good enough" (namely $r_c \ll L$), then

$$n^+_{max}(r) = \widetilde{N}^+_{max} \int_0^L (\Psi_{p-p}(x) + 1)dx \approx \widetilde{N}^+_{max} L \gg 1, \tag{4.42}$$

$$n^+_{min}(r) = \widetilde{N}^+_{min} \int_o^L (\Psi_{p-ap}(x) + 1)dx \approx \widetilde{N}^+_{min} L \gg 1,$$

and

$$\overline{k} = \frac{\widetilde{N}^+_{max}}{\widetilde{N}^+_{max} - \widetilde{N}^+_{min}} = \frac{1}{1-\alpha(\nu_t,\gamma)} \ , \tag{4.43}$$

where $\alpha(\nu_t,\gamma) = \widetilde{N}^+_{min}/\widetilde{N}^+_{max}$. The values $\widetilde{N}^+_{max}(\nu_t)$ and $\widetilde{N}^+_{min}(\nu_t)$ can be found analytically, see Rice (1944, 1945):

$$\widetilde{N}^+_{max} = \frac{1}{\sqrt{(2\pi)^3(1-\gamma^2)}}\frac{\sigma_2}{\sigma_1}\int_{\nu_t}^{\infty} d\nu \int_{-\infty}^{0} |\xi| e^{-\frac{\nu^2}{2}-\frac{(\xi+\gamma\nu)^2}{2(1-\gamma^2)}} d\xi, \tag{4.44}$$

$$\widetilde{N}^+_{min} = \frac{1}{\sqrt{(2\pi)^3(1-\gamma^2)}}\frac{\sigma_2}{\sigma_1}\int_{\nu_t}^{\infty} d\nu \int_{0}^{\infty} |\xi| e^{-\frac{\nu^2}{2}-\frac{(\xi+\gamma\nu)^2}{2(1-\gamma^2)}} d\xi. \tag{4.45}$$

The result of this integration is presented in Appendix A1. The integration over $d\xi$ in Eq.(4.44) gives the differential density of maxima and minima which are plotted in Fig.5a for different values of γ. From Eq.(4.43) we can see that the mean length of the clusters depends only on the density of minima and maxima above the threshold ν_t. For a large value of ν_t, $\overline{k} \to 1$ because, as we see from Fig.5a, almost all the high extrema are maxima and $\alpha \ll 1$. For a large negative value of ν_t $\overline{k} \to \infty$, because, in this case, $\widetilde{N}^+_{max}(\nu_t) \to \widetilde{N}^+_{min}(\nu_t)$ and $\alpha \to 1$.

The dependence of $\overline{k}$ on the level of slice ν_t for different γ are plotted in Fig.5b. A low value of γ corresponds to the high rate of clusterisation. Note, that γ lies in the interval $0 < \gamma \leq 1$. The value $\gamma = 1$ corresponds to the power spectrum $P(k) = const \cdot \delta(k-k_0)$. In this case we have $\overline{k} = 1$ for $-|\nu_0| < \nu_t < |\nu_0|$; $\overline{k} = \infty$ for $\nu_t < -|\nu_0|$; $\overline{k} = 0$ for $\nu_t \geq |\nu_0|$. Here, ν_0 is the random Gaussian value and depends on the appropriate realization.

4.2 Two-dimensional field

In this section, we develop the theory of the statistical properties of the two-dimensional field. These properties can be applied to further analysis of the sky maps. In what follows we obtain an important characteristic statistical feature of the two-dimensional Gaussian field: clusterisation of peaks depends only on one spectral parameter γ, just as in the one-dimensional case. This spectral parameter of a Gaussian field of CMB radiation depends effectively only on the model of recombination. Note, that this natural feature of the Gaussian field can be used not only for determination of the ionization history, but also as a test of the presence of noise in a map (which, most probably, is non-Gaussian). We derive equations for the field in the vicinity of two neighbouring maxima. These equations define the details of the structure of the cluster in the vicinity of the maxima and can be used for the local analysis of the appropriate part of the map.

a. Surface $\Delta T(x, y)$

A two-dimensional field can be described as a two-dimensional surface in a three-dimensional space, $\Delta T(x,y)$, Fig.6. We will cut this surface at the different levels ν_t. The surface intersects the plane $\Delta T(x,y) = \nu_t$ along the lines of level ν_t (see Map.2a,b). If, for example, two local maxima with the values above this threshold ν_t are confined by a closed line of level ν_t, then these maxima are considered connected together in one cluster. It is obvious that two neighbouring maxima on this surface can be connected together only through the saddle point between them, see Map.2. Therefore, if the saddle point between two maxima is above the threshold ν_t, then these maxima are connected together. Let us define a cluster of the length k similar to the one-dimensional case.

Definition: Cluster of the length k is a collection of k maxima confined by a closed level-line.

In what follows we describe the clusterisation of maxima, namely, we find the mean length of the clusters as a function of the level of the slice, ν_t, and spectral parameter, γ.

b. Extremal points

Let us introduce the following expressions:

$$\begin{aligned} &\nu = \frac{\Delta T}{\sigma_0}; \quad \eta_1 = \frac{\Delta T'_x}{\sigma_1}; \quad \eta_2 = \frac{\Delta T'_y}{\sigma_1}; \\ &\xi_{11} = \frac{\Delta T''_{xx}}{\sigma_2}; \quad \xi_{22} = \frac{\Delta T''_{yy}}{\sigma_2}; \quad \xi_{12} = \frac{\Delta T''_{xy}}{\sigma_2}. \end{aligned} \tag{4.46}$$

If $\eta_1 = \eta_2 = 0$ at the point (x_0, y_0), then this point is an extremum, and the field ν in the vicinity of (x_0, y_0) is

$$\nu(x,y) = \nu(x_0,y_0) + \xi_{11}x^2 + 2\xi_{12}xy + \xi_{22}y^2. \tag{4.47}$$

We rotate the x, y axes by the angle $\varphi = \frac{1}{2} arctg \frac{2\xi_{12}}{\xi_{11}-\xi_{22}}$ around the point (x_0, y_0) and find the eigen values of the second derivatives matrix ξ, λ_1 and λ_2:

$$\begin{aligned} &\xi_{11} = \lambda_1 \cos^2\varphi + \lambda_2 \sin^2\varphi, \\ &\xi_{22} = \lambda_1 \sin^2\varphi + \lambda_2 \cos^2\varphi, \\ &\xi_{12} = (\lambda_1 - \lambda_2)\sin\varphi\cos\varphi. \end{aligned} \tag{4.48}$$

All extrema can be divided in three types: maxima $\lambda_{1,2} < 0$, saddle points $(\lambda_1 > 0, \lambda_2 < 0)$ or $(\lambda_1 < 0, \lambda_2 > 0)$, and minima $\lambda_{1,2} > 0$.

c. Density of maxima, minima and saddle points.

Derivation of the density of saddle points is given in Appendix B1. (The density of maxima and minima for the two-dimensional Gaussian field was obtained by B.E., 1987). Here we present the results of our analytical calculations for the saddle points and mention the results of calculations by B.E. for the maxima and minima. The differential density of the saddle points is:

$$N_{sad}(\nu,\gamma)d\nu = \frac{1}{\sqrt{32\pi^3}}\frac{\sigma_2^2}{\sigma_1^2}\frac{1}{\sqrt{3-2\gamma^2}}e^{-\frac{3\nu^2}{2(3-2\gamma^2)}}d\nu \ . \tag{4.49}$$

The integration of Eq.(4.49) from ν_t to infinity gives the number of saddle points above the same threshold ν_t

$$n_{sad}(\nu_t) = \int_{\nu_t}^{\infty} N_{sad}(\nu)d\nu = \frac{1}{8\pi\sqrt{3}}\frac{\sigma_2^2}{\sigma_1^2}\left[1-\Phi\left(\frac{\nu_t\sqrt{3}}{\sqrt{2(3-2\gamma^2)}}\right)\right] \ . \tag{4.50}$$

where $\Phi(x)$ is the probability integral by Gradshteyn and Ryzhik, 1980. The full density of saddle points is:

$$n_{sad}(-\infty,\gamma) = \frac{1}{4\pi\sqrt{3}}\frac{\sigma_2^2}{\sigma_1^2}. \tag{4.51}$$

The differential density of maxima is:

$$N_{max}(\nu)d\nu = \frac{\sqrt{2}}{4\pi\sqrt{\pi}}\frac{\sigma_2^2}{\sigma_1^2}\cdot\left\{\frac{1}{4\sqrt{3-2\gamma^2}}e^{-\frac{3\nu^2}{2(3-2\gamma^2)}}\left[1+\Phi\left(\frac{\gamma\nu}{\sqrt{(2-2\gamma^2)(3-2\gamma^2)}}\right)\right]\right.$$

$$\left.+\frac{\gamma(\nu^2-1)}{4}e^{-\frac{\nu^2}{2}}\left[1+\Phi\left(\frac{\gamma\nu}{\sqrt{2(1-\gamma^2)}}\right)\right]+\frac{\gamma\nu\sqrt{1-\gamma^2}}{\sqrt{2\pi}}e^{-\frac{\nu^2}{2(1-\gamma^2)}}\right\} \ . \tag{4.52}$$

Note, that $N_{min}(\nu) = N_{max}(-\nu)$ and

$$n_{max}(\nu_t,\gamma) = \int_{\nu_t}^{\infty} N_{max}(\nu,\gamma)d\nu \qquad n_{max}(-\infty,\gamma) = \frac{1}{8\pi\sqrt{3}}\frac{\sigma_2^2}{\sigma_1^2} \ . \tag{4.53}$$

The distribution of maxima, minima and saddle points are shown in Fig.7a for different values of ν. As it can be seen from this figure, this distribution is very sensitive to the value of γ. In what follows, we demonstrate the importance of sensitivity of $N_{sad}(\nu,\gamma)$, $N_{max}(\nu,\gamma)$ and $N_{min}(\nu,\gamma)$ to the value of γ. Namely, the rate of clusterisation of peaks in a random two-dimensional Gaussian field depends only on the value of γ. Note that the corresponding behaviour of $N_{max}(\nu,\gamma)$ and $N_{min}(\nu,\gamma)$ was discussed earlier in Section 3.1 for the one-dimensional cross-section. In the one-dimensional case, the rate of clusterisation (in a random Gaussian field) also depends only on the parameter γ as we demonstrated.

d. Clusters of peaks

Let us consider an appropriate realization of the two-dimensional random Gaussian process in a map $l \times l$, see Map.2. In this figure the maxima and the saddle points with values larger than the level of the slice, ν_t, together with the isolines $\nu = \nu_t$ are shown. If the value of ν_t is high, all maxima are separated and only clusters of the length $k = 1$ are observed. Reduction of the level ν_t leads to the appearance of big clusters (maxima begin to connect together and generate clusters). Let us consider a slice of the map and calculate the number of clusters N_k of the length k for $k = 1, 2, \ldots$. Summation of the number of maxima in all clusters of the map gives the total number of maxima:

$$\sum_{k=1}^{\infty} N_k \cdot k = n_{max}(\nu_t). \tag{4.54}$$

Now, let us calculate the total number of clusters (having arbitrary number of maxima). Each cluster can contain maxima, minima and saddle points. The numbers of maxima, minima and saddle points in one cluster are not independent values. If one particular cluster contains $k = k_{max}$ maxima and k_{min} minima, then it has to contain $k_{sad} = k_{max} + k_{min} - 1$ saddle points. Therefore, the total number of clusters of arbitrary length k for a given slice - level ν_t is:

$$\sum_k N_k = n_{max}(\nu_t) + n_{min}(\nu_t) - n_{sad}(\nu_t) \ . \tag{4.55}$$

Using Eqs.(4.54) and (4.55), one can write:

$$\overline{k} = \frac{\sum k N_k}{\sum N_k} = \frac{n_{max}}{n_{max} + n_{min} - n_{sad}} \ . \tag{4.56}$$

If only high levels of ν_t are considered, then the number of minima, $n_{min}(\nu_t)$, is small for γ sufficiently different from zero, (see Fig.7a) and can be disregarded.

In this case, two neighbouring maxima are counted in one cluster for the level ν_t, only if the saddle point between them is above this level. Therefore, each cluster containing k maxima, also contains $k - 1$ saddle points (clusters of the length 1 does not contain any saddle point). In this case the mean length of the clusters is

$$\overline{k} \approx 1 + \alpha(\nu_t, \gamma), \tag{4.57}$$

where

$$\alpha(\nu_t, \gamma) = \frac{n_{sad}(\nu_t, \gamma)}{n_{max}(\nu_t, \gamma)}, \tag{4.58}$$

similar to the one-dimensional case. The ratio n_{sad}/n_{max} depends not only on the level of ν_t, but also on spectral parameter γ. Thus, $\alpha = \alpha(\nu_t, \gamma)$ also depends on both ν_t and γ. The dependence of $\overline{k}$ on the level ν_t in the general case Eq.(4.57) is shown in Fig.7b, for different values of γ. For low values of γ the clusterisation occurs faster than for high values of γ. This can be concluded from Fig.7a, where the distributions of extrema for different values of γ are shown. Therefore, clusterisation of peaks in a two-dimensional random Gaussian field (namely the mean length of clusters) can be described in terms of only one parameter, γ.

For a high level of the slice, almost all extrema are maxima, and $n_{min} \to 0$ and $n_{sad} \to 0$ ($\overline{k} \to 1$ and we have separated maxima). For $\nu_t \to 0$, $n_{max} + n_{min} - n_{sad}$ tends to zero (see Appendix B1), and $\overline{k} \to \infty$. Infinite $\overline{k}$ is equivalent to the existence of an infinite cluster of maxima. However, the finite size of the region considered prevents the realization of an infinite cluster (a finite region has only a finite number of maxima). In an infinite region, the existence of an infinite cluster and the percolation are essentially equivalent. Since percolation is also defined for a finite region, the concept of an infinite cluster in a finite region can be addressed in terms of percolation.

e. Percolation

Let us colour the map of the considered region in two colours: black and white. The black area is for $\nu > \nu_t$, and the white area is for $\nu < \nu_t$ (see Map.3). Percolation through black (white) zones is usually defined as the possibility to "walk" from one side of the map to another (from the lower to the upper border and from the left to the right side) only through the black (white) zones. Roughly speaking, percolation means the existence of a black (white) cluster with the linear size of the considered map, or the existence of an infinite cluster in the case of an infinite map. In terms of extremal points, percolation over black (white) zones means the existence of an infinite cluster of maxima (minima). The change of percolation over black zones into percolation over white zones has to take place at the $\nu_p = 0$ in an infinite map. Gaussian nature of the distribution of the CMB anisotropy assures this property. In a particular realization of the Gaussian process in a finite map a change of percolation regimes arises when ν is slightly different from zero, because of the statistical character of the process. The difference of ν from zero can be evaluates as follows:

1. Each hotspot (black zone) and coldspot (white zone) can be considered as an approximately independent realization (correlations between regions of the size $r_c \times r_c$ is negligible).
2. The total number of such approximately independent realizations is $N \sim L^2/r_c^2$.
3. The variance is $D \sim 1/\sqrt{N} \sim r_c/L$.

Therefore, for the appropriate region of the sky the probability of ΔT fluctuations to be Gaussian can be estimated. This can be accomplished using the percolation technique: if $\nu_p \gg r_c/L$ then, most probably, the additional non-Gaussian noise is present in this map.

Note, that this obvious test can be considered only as the first step of the detection of the noise, because the critical point of percolation ν_p can be equal to zero even with non-Gaussian noise (for example, if the noise is symmetrical relatively to $\nu = 0$).

f. Shape around two maxima

The expected value of the field around two given maxima in the two-dimensional case can be derived similarly to one-dimensional case (Section 4.1). The derivation is given in Appendix B2. This result describes the clusterisation of two neighbouring maxima and can be applied to the local analysis of small patches of the map, which contain only the area of two neighbouring hotspots.

This area can be checked for presence of noise by evaluating the probability that these peaks have a Gaussian nature, as it is explained in the following. As it was mentioned (e.g. by B.E.) the contour curves in the neighbourhood of Gaussian peaks are ellipses. The directions of major and minor axes are different for different maxima. Thus, two neighbouring maxima have some relative orientation. Suppose that, in an appropriate part of the map, these maxima are located at the distance r from each other, and that the value of the field and its second derivatives (relative orientations) are ν_1, ξ^1_{11}, ξ^1_{12}, ξ^1_{22} and ν_2, ξ^2_{11}, ξ^2_{12}, ξ^2_{22} for the first and the second maximum respectively. Then it is possible to predict the mean value of the field and its variance in the vicinity of given maxima and to compare this result to observed pairs of hotspots in the map.

The mean value of the field around two maxima is presented in Fig.8, for different kinds of spectra (i.e., for the different values of the spectral parameter γ). Note, that this value depends on the behaviour of the correlation function on the scale of this area $\geq r_c$ mainly being determined by γ. For low values of γ two neighbouring hotspots stick together faster than for high values of γ. This fact is in good agreement with the results of subsection **d**. The increase of the rate of clusterisation for decreasing values of γ is favorable to faster production of big clusters.

5 How can we analyse the sky maps?

In this section we propose a method of analyzing of the observational data on CMB anisotropy for one- and two-dimensional experiments. Interpretation of observations of a specific patch of sky is complicated for the following reasons:
1. Most of the cosmological models predict the CMB anisotropy to be a single realization of a random Gaussian process for which the properties can be predicted only for an ensemble average of such realizations. Any single realization differs from its ensemble average approximately by the value of the dispersion (the so-called "cosmic variance").
2. Presence of a non-Gaussian noise can be interpreted as fluctuations of CMB itself and can give a wrong contribution to the correlation function or, equivalently, to the power spectrum.

As for the first reason, the following comments can be made. A small part of the sky may not be representative. This may lead to misinterpretation of properties of the two-point correlation function of the CMB as obtained from one patch only. Thus, the finite nature of the considered region may lead to misinterpretation of the correlation function of e.g. the galaxy distribution on scales comparable to the size of the region. This effect causes difficulties in detection of the so-called “long distance correlation” in the galaxy distribution which can appear in cosmological models with non-standard inflation (Starobinsky, 1992).

As for the second reason, the correlation function obtained by measuring the sum of the relic radiation itself and non-Gaussian noise will of course be the correlation function of the sum of the noise and relic radiation.

Note, that both of these two reasons may lead to a correlation function which closely can imitate the presence of a Doppler-peak in the spectrum of perturbations of CMB. In this chapter, we propose the topological method of analysis of the sky maps which is useful for (1) detection of non-Gaussian noise,(2) detection of a Doppler-peak in the spectrum.

a. One dimensional experiment

Assume that we have data from a one-dimensional experiment being a one-dimensional slice of the two-dimensional realization (Fig.4, Section 4.1). If the primordial signal is the result of perturbations of the CMB radiation only, then the correlation analysis is equivalent to the topological analysis because all statistical properties of the Gaussian field are completely characterized by its power spectrum. In this case, we should only estimate the dispersion of the correlation function (section 2) on the most interesting scales (about $15'$ to 1^o), which is determined by the size of the considered region (Hinshaw et al., 1995). This is necessary for the estimation of the probability that the peak-like anomaly (if it is detected in the correlation function of the observed data) occurs due to the presence of the Doppler-peak in the spectrum, and not due to “bad” statistical properties of the selected region.

Next, the rate of the clusterisation of peaks (Section 4.1) for different values of the filter (Section 3) can be checked. It is obvious that the clusterisation of minima is equivalent to the clusterisation of maxima if the sign of fluctuations is changed. This equivalence exists due to the symmetrical statistical properties of the Gaussian field with respect to zero. Variation of the value of the filter leads to a change of the spectral parameter γ. For a given parameter γ, the rate of clusterisation of peaks is determined only by the Gaussian nature of the field; this rate is plotted on (Fig.5b), for different values of γ. Note, that for different cosmological models, the dependence on $\gamma(r_0)$ (where r_0 is the resolution) is different (Fig.3a,b). If, for each value of γ (which corresponds to an appropriate value of r_0), the rate of clusterisation corresponds to the Gaussian statistic (Fig.5b), then this indicates a Gaussian random field and the observed correlation function is the correlation function of CMB.

Now let us discuss how the situation will change in case of non-Gaussian noise being present. In this case, the correlation and the topological analysis are not equivalent to each other. The presence of the noise shows up as presence of ”noise peaks“ in the random field. These “noise peaks” give a contribution to the observed two-point correlation function. If the observed correlation function is not analysed properly, the presence of the “noise peaks” in the observed data may lead to wrong conclusions.

In order to avoid a wrong interpretation of the observed data, we propose to perform the cluster analysis:
1) The rate of clusterisation (as discussed above) should be checked. If this test shows us that the rate does not correspond to one spectral parameter γ, then additional random noise is present in the data.
2) A search for "noise peaks" should be performed.
If these tests indicate the presence of noise, then how can we remove the "noise peaks"? We propose to check the probability to be Gaussian for each pair of neighbouring extrema, by using the technique of Section 4.2, Fig.4. The peaks with small probability should not be considered in the further analysis.

b. Two-dimensional data

Difficulties in the interpretation of the two-dimensional data are similar to that in one-dimensional case, but the statistical properties are usually much better. The analysis can be analogous to that for the one-dimensional case, and the steps are as follows:
1) Filtering of the sky map, see Section 3. This step can be useful for the following reasons. The use of the specific (for example non-Gaussian) filtering removes the influence of long modes and, therefore, leads to a decrease of the correlation radius. Since we are interested only in the relatively high modes, such a filtering improves the statistical properties of the map (for the purpose of the cluster analysis). A filtered map contains a larger number of peaks and is more representative than the unfiltered one. Thus we perform the cluster analysis on the filtered map.
2) Cluster analysis. This step consists of three substeps:
a. Percolation as the first test of the presence of non-Gaussian noise (see Section 4.2, Map.3).
b. Rate of clusterisation. This substep can be performed similarly to the one-dimensional case, but using the results of Section 4.2, Fig.7b.
c. Detection of the "noise peaks" (if the previous test indicates the presence of noise), as for the one-dimensional case, but using the results of Section 4.2, Fig.8.

c. Topology for different models

As it was shown above, the most important topological properties (namely clusterisation of peaks in a random Gaussian field) can be explained in terms of only one spectral parameter γ. This spectral parameter is very sensitive to the presence of Doppler-peaks in the spectrum of fluctuations of ΔT for the resonance filter (Section 3). It means that the rate of clusterisation will change from one model to the next, for the same value of the filter. A big value of γ corresponds to a low rate of clusterisation (Fig.7b) and, for the same level of the slice, the expected number of big clusters is less than for a small value of γ. For the model with a pronounced Doppler-peak, the value of γ is larger than it is for the model without a Doppler-peak in the spectrum (Fig.3a,b). Therefore, for such models, the clusterisation happens more slowly than for the model without a Doppler-peak (Map.4a,b). Thus, the number of big clusters for the model without a Doppler-peak is larger than for the model with a Doppler-peak.

Another obvious difference consists in that a number of maxima (minima) per unit square is greater for the model with a high Doppler-peak, because the presence of this peak leads to a decrease of the correlation angle.

6 Discussion

The results of this paper can naturally be separated into two parts. One is the presentation of new theoretical results on the theory of clusterisation of peaks in a Gaussian random field and on the theory of percolation. All mathematical results are analytical. The other is to suggest how such analytical calculations can be applied to the analysis of the observational data of the CMB anisotropy.

6.1 Theory

The most important new results on the clusterizarion of peaks in a Gaussian random filed, in this paper, are the following:

a. One-dimensional cross-section of the two-dimensional random field

1. Analytical calculations of the mean length of the cluster of peaks in, one-dimension, Eqs.(4.43-4.45).
2. The shape of the random field in the vicinity of two neighbouring maxima, namely the mean value of the field $\langle\nu(x)\rangle$ and its dispersion $\langle\nu(x)\rangle \pm \sqrt{\langle\nu^2(x)\rangle}$ about this mean. These values are to be determined from the Gaussian probability distribution of the value of $\nu(x)$, given that there are two maxima at the points x_1 and x_2 of given heights and curvatures (second derivatives), Eqs.(4.33, A2.8, A2.9).

b. Two-dimensional field

1. Calculation of the densities of saddle points: differential in height $N_{sad}(\nu)d\nu$,Eq.(4.49), and integral (above the threshold ν_t), Eq.(4.50).
2. The shape around two neighbouring maxima similar to the one-dimensional case, that is the expected average value and its dispersion, (Eqs.(B2.5,B2.6)).
3. Calculation of the mean length of the cluster of peaks in the two-dimensional case, Eq.(4.56).
4. We find, that the level of percolation, ν_p, and the level of appearance of an infinite cluster of peaks are the same: $\overline{k}(\nu_p) = \infty$. It is well known, that in the two-dimensional case we have $\nu_p = 0$ (because of the symmetrical properties with respect

to the zero). Similar analytical results can be obtained for the 3-dimensional case (now it is known only numerically), but is most probably only of theoretical interest and is beyond the scope of this paper.

6.2 Applications to the analysis of the observational data.

In Section 5 we have presented our suggestions how analytical calculations can be used in the analysis of observational data. The proposed topological method can provide more information about the observational data than the correlation analysis.

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8 Appendix A1

Density of maxima and minima in one-dimension

The problem of the density of extrema is especially easy in one-dimension, and was solved by Rice (1944, 1945). For a review see also J.Bardeen, J.Bond, N.Kaiser and A.Szalay (1986, BBKS).

The differential density of maxima and minima is

$$N_{max}(\nu)d\nu =$$

$$\frac{1}{(2\pi)^{3/2}}\frac{\sigma_2}{\sigma_1}\left\{\sqrt{1-\gamma^2}e^{-\frac{\nu^2}{2(1-\gamma^2)}}+\gamma\nu\sqrt{\frac{\pi}{2}}e^{-\frac{\nu^2}{2}}\left[1+\Phi\left(\frac{\gamma\nu}{\sqrt{2(1-\gamma^2)}}\right)\right]\right\}d\nu, \quad (A1.1)$$

where $\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ is the probability integral (Gradshteyn and Ryzhik, 1980).

$$N_{min}(\nu) = N_{max}(-\nu).$$

The integrated density is:

$$\widetilde{N}^{+}_{max}(\nu_t) = \int_{\nu_t}^{+\infty} N_{max}(\nu(d\nu) =$$

$$= \frac{1}{4\pi}\frac{\sigma_2}{\sigma_1}\left\{1 - \Phi\left(\frac{\nu_t}{\sqrt{2(1-\gamma^2)}}\right) + \gamma e^{-\frac{\nu_t^2}{2}}\left[1 + \Phi\left(\frac{\nu_t}{\sqrt{2(1-\gamma^2)}}\right)\right]\right\}. \qquad (A1.2)$$

The density of maxima of arbitrary height is:

$$\widetilde{N}^{+}_{max}(-\infty) = \frac{1}{2\pi}\frac{\sigma_2}{\sigma_1}. \qquad (A1.3)$$

9 Appendix A2

Shape around two maxima in one-dimension.

Let us define the event A as the presence of two maxima at the points x_1 and x_2 respectively with parameters: $\nu(x_{1,2}) = \nu_{1,2}$; $\eta(x_{1,2}) = 0$; $\xi(x_{1,2}) = \xi_{1,2}$, and the event ν with the field at the point x located in the interval from ν to $\nu + d\nu$. The joint probabilities are:

$$P(\nu, B; C)d\nu(x)d\nu(x_1)...d\xi(x_2) = P(\nu(x), \nu(x_1), ..., \xi(x_2)d\nu(x)d\nu(x_1)...d\xi(x_2)\ ,$$

$$P(B; C)d\nu(x_1)...d\xi(x_2) = P(\nu(x_1), ..., \xi(x_2)d\nu(x)d\nu(x_1)...d\xi(x_2)\ ,$$

$$P(\nu, B; C) = \frac{1}{\sqrt{(2\pi)^7 \det M_1}} e^{-Q_1}, \qquad (A2.1)$$

$$P(\nu, B; C) = \frac{1}{\sqrt{(2\pi)^6 \det M_2}} e^{-Q_2},$$

where M_1 and M_2 are the 7×7 and 6×6 covariance matrices, respectively, Q_1 and Q_2 are the quadratic forms. The conditional probability for $\nu(x)$ to be in the range from $\nu(x)$ to $\nu(x) + d\nu(x)$ is:

$$P(\nu|B; C)d\nu(x) = \frac{\int P(\nu, B; C)\delta(\nu(x_1) - \nu_1)...\delta(\xi(x_2) - \nu_2)d\nu(x_1)...d\xi(x_2)}{\int P(B; C)\delta(\nu(x_1) - \nu_1)...\delta(\xi(x_2) - \xi_2)d\nu(x_1)...d\xi(x_2)} d\nu(x). (A2.2)$$

Using (A2.1) and (A2.2) we obtain:

$$P(\nu|B; C)d\nu(x) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\det M_2}{\det M_1}} e^{-(Q_1 - Q_2)} d\nu(x). \qquad (A2.3)$$

Now we define the functions:

$$\Psi(r) = \frac{C(r)}{\sigma_0^2}; \qquad \varphi(r) = \frac{C''(r)}{\sigma_1^2}; \qquad f(r) = \frac{C^{IV}(r)}{\sigma_2^2};$$

$$\delta(r) = \frac{C'(r)}{\sigma_0\sigma_1}; \qquad \lambda(r) = \frac{C'''(r)}{\sigma_1\sigma_2} \ . \tag{A2.4}$$

In terms of these functions the covariance matrix M_1 is

$$M_1 = \begin{pmatrix} 1 & \Psi & 0 & \delta & -\gamma/2 & \gamma\varphi & \Psi_1 \\ \Psi & 1 & -\delta & 0 & \gamma\varphi & -\gamma/2 & \Psi_2 \\ 0 & -\delta & 1/2 & -\varphi & 0 & -\lambda & -\delta_1\frac{x-x_1}{r_1} \\ \delta & 0 & 0 & 1/2 & \lambda & 0 & -\delta_2\frac{x-x_2}{r_2} \\ -\gamma/2 & \gamma\varphi & -\varphi & \lambda & 3/8 & f & \gamma\varphi_1 \\ \gamma\varphi & -\gamma/2 & -\lambda & 0 & f & 3/8 & \gamma\varphi_2 \\ \Psi_1 & \Psi_2 & -\delta_1\frac{x-x_1}{r} & -\delta_2\frac{x-x_2}{r} & \gamma\varphi_1 & \gamma\varphi_2 & 1 \end{pmatrix} , \tag{A2.5}$$

where functions with subscripts "1", "2" and without subscripts are evaluated at the points r_1, r_2 and r respectively. Here $r_1 = |x - x_1|$, $r_2 = |x - x_2|$, $r = |x_1 - x_2|$. M_2 can be obtained from M_1 by excluding the last row and the last column. If we introduce the new variables y_i, $i = 1, 6$, and $\widetilde{\nu}$, so that

$$y_1 = \nu_1 - \nu_2; \qquad y_2 = \nu_1 + \nu_2;$$

$$y_3 = \eta_1 - \eta_2 + \frac{\delta}{1+\Psi} y_2; \quad y_4 = \eta_1 + \eta_2 - \frac{\delta}{1-\Psi} y_1;$$

$$y_5 = \xi_1 - \xi_2 + \frac{\gamma(1+2\varphi)}{2(1-\Psi)} y_1 - \frac{2\lambda + \frac{\delta}{1-\Psi}\gamma(1+2\varphi)}{1 - 2\varphi - 2\frac{\delta^2}{1-\Psi}} y_4; \tag{A2.6}$$

$$y_6 = \xi_1 + \xi_2 + \frac{\gamma(1-2\varphi)}{2(1+\Psi)} y_2 + \frac{2\lambda + \gamma\delta\frac{1-2\varphi}{1+\Psi}}{1 + 2\varphi - 2\frac{\delta^2}{1+\Psi}} y_3;$$

$$\widetilde{\nu} = \nu - \sum_i \frac{\langle \nu y_i \rangle}{\langle y_1^2 \rangle} y_i \ ,$$

then $M_{1,2}$ can be written in a diagonal form. In M_1, $\langle y_i^2 \rangle$, $i = 1, ..., 6$ occupy the first six positions, and $\langle \widetilde{\nu}^2 \rangle$ stands on the seventh position. In matrix M_2, all six diagonal positions are occupied by $\langle y_i^2 \rangle$, $i = 1, ..., 6$. The difference between the two quadratic forms is then, $Q_1 - Q_2 = \frac{1}{2}\widetilde{\nu}/\langle \widetilde{\nu}^2 \rangle$. Also $\det M_1 = \det M_2 \cdot \langle \widetilde{\nu}^2 \rangle$. Eq.(A2.3) has now a very simple form:

$$P(\nu|B;C) d\nu(x) = \frac{1}{\sqrt{2\pi \langle \widetilde{\nu}^2 \rangle}} e^{-\frac{\widetilde{\nu}^2}{2\langle \widetilde{\nu}^2 \rangle}} d\nu \tag{A2.7}$$

As it is seen from (A2.6) and (A2.7), ν has the mean value:

$$\langle \nu \rangle = \sum_i \frac{\langle \nu y_i \rangle}{\langle y_i^2 \rangle} y_i \tag{A2.8}$$

and the variance:

$$\langle \nu^2 \rangle = 1 - \sum_i \frac{\langle \nu y_i \rangle^2}{\langle y_i^2 \rangle^2} y_i \ . \tag{A2.9}$$

The values in (A2.8) and (A2.9) are as follows:

$$\langle \nu y_1 \rangle = \Psi_1 - \Psi_2; \qquad \langle \nu y_2 \rangle = \Psi_1 + \Psi_2;$$

$$\langle \nu y_3 \rangle = -\delta_1 \frac{x - x_1}{r_1} + \delta_2 \frac{x - x_2}{r_2} + \frac{\delta}{1 - \Psi} \langle \nu y_2 \rangle;$$

$$\langle \nu y_4 \rangle = -\delta_1 \frac{x - x_1}{r_1} - \delta_2 \frac{x - x_2}{r_2} - \frac{\delta}{1 + \Psi} \langle \nu y_1 \rangle;$$

$$\langle \nu y_5 \rangle = \gamma(\varphi_1 - \varphi_2) + \gamma \frac{1 + 2\varphi}{2(1 - \Psi)} \langle \nu y_1 \rangle - \frac{2\lambda + \frac{\delta}{1-\Psi}\gamma(1 + 2\varphi)}{1 - 2\varphi - 2\frac{\delta^2}{1-\Psi}} \langle \nu y_4 \rangle;$$

$$\langle \nu y_6 \rangle = \gamma(\varphi_1 + \varphi_2) + \gamma \frac{1 - 2\varphi}{2(1 + \Psi)} \langle \nu y_2 \rangle + \frac{2\lambda + \frac{\delta}{1+\Psi}\gamma(1 - 2\varphi)}{1 + 2\varphi - 2\frac{\delta^2}{1+\Psi}} \langle \nu y_3 \rangle;$$

$$\langle y_1^2 \rangle = 2(1 - \Psi); \qquad \langle y_2^2 \rangle = 2(1 + \Psi);$$

$$\langle y_3^2 \rangle = 1 + 2\varphi - 2\frac{\delta^2}{1 + \Psi}; \qquad \langle y_4^2 \rangle = 1 - 2\varphi - 2\frac{\delta^2}{1 - \Psi};$$

$$\langle y_5^2 \rangle = \frac{3}{4} - 2f - \frac{\gamma^2(1 + 2\varphi)^2}{2(1 - \Psi)} - \frac{\left(2\lambda + \frac{\delta}{1-\Psi}\gamma(1 + 2\varphi)\right)^2}{1 - 2\varphi - 2\frac{\delta^2}{1-\Psi}};$$

$$\langle y_6^2 \rangle = \frac{3}{4} + 2f - \frac{\gamma^2(1 - 2\varphi)^2}{2(1 + \Psi)} - \frac{\left(2\lambda + \frac{\delta}{1+\Psi}\gamma(1 - 2\varphi)\right)^2}{1 + 2\varphi - 2\frac{\delta^2}{1+\Psi}} \ .$$

In order to construct maxima in x_1 and x_2, it is necessary to take into account that $\eta_{1,2} = 0$ and $\xi_{1,2} < 0$.

10 Appendix B1

In this appendix, we derive the differential and the integrated density of the saddle points only, since the density of maxima and minima was obtained by (B.E.). Here we present their result and compare it with the result for the saddle points, which can be obtained in a similar fashion. Following (B.E.), the covariance matrix M for the values ν, η_1 η_2, ξ_{11}, ξ_{22} and ξ_{12} can be written in the following form:

$$M = \begin{pmatrix} 1 & 0 & 0 & -\gamma/2 & -\gamma/2 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ -\gamma/2 & 0 & 0 & 3/8 & 1/8 & 0 \\ -\gamma/2 & 0 & 0 & 1/8 & 3/8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/8 \end{pmatrix} . \qquad (B1.1)$$

The joint probability is:

$$P(\nu, \eta_1, \eta_2, \xi_{11}, \xi_{22}, \xi_{12}) d\nu d\eta_1 d\eta_2 d\xi_{11} d\xi_{22} d\xi_{12} =$$

$$= \frac{2}{\pi^3} \frac{1}{\sqrt{1-\gamma^2}} e^{-Q} d\nu d\eta_1 d\eta_2 d\xi_{11} d\xi_{22} d\xi_{12} \qquad (B1.2)$$

$$Q = \frac{\nu^2}{2} + \eta_1^2 + \eta_2^2 + (\xi_{11} - \xi_{22})^2 + \frac{(\xi_{11} + \xi_{22} + \gamma\nu)^2}{2(1-\gamma^2)} + 4\xi_{12}^2.$$

The differential density of extrema is:

$$N_{ext}(\nu) = \int |\det \xi| P \cdot \delta(\eta_1)\delta(\eta_2) \cdot d\eta_1 ... d\xi_{12}.$$

We rotate the coordinate system by the angle $\varphi = \frac{1}{2} arctg \frac{2\xi_{12}}{\xi_{11} - \xi_{22}}$ to align with the principal axes of ξ_{ij}. Thus, we get the diagonal form — $\lambda_1 \lambda_2$, ordered by $\lambda_1 \geq \lambda_2$, and introduce variables λ_1, λ_2, φ:

$$\begin{aligned} \xi_{11} &= \lambda_1 \cos^2 \varphi + \lambda_2 \sin^2 \varphi, \\ \xi_{22} &= \lambda_1 \sin^2 \varphi + \lambda_2 \cos^2 \varphi, \\ \xi_{12} &= (\lambda_1 - \lambda_2) \sin \varphi \cos \varphi. \end{aligned} \qquad (B1.3)$$

Thus, the transformation of the volume element is:

$$d\xi_{11} d\xi_{22} d\xi_{12} = (\lambda_1 - \lambda_2) d\lambda_1 d\lambda_2 d\varphi. \qquad (B1.4)$$

The orientation angle φ is randomly distributed over $0 \leq \varphi \leq \pi$. The differential density of extrema is:

$$N_{ext}(\nu) d\nu = \frac{1}{4\pi^2} \frac{1}{\sqrt{1-\gamma^2}} \frac{\sigma_2^2}{\sigma_1^2} e^{-\frac{\nu^2}{2}} d\nu \int_{sad} e^{-\frac{(4+\gamma\nu)^2}{2(1-\gamma^2)}} x |y^2 - x^2| e^{-x^2} dx dy. \qquad (B1.5)$$

(B1.5) was obtained from (B1.1)-(B1.4) by means of substituting:

$$\lambda_1 = \frac{x+y}{2} , \qquad \lambda_2 = \frac{y-x}{2} . \qquad (B1.6)$$

The limits of the integration over S depends on the type of the extremal points, namely

$$S := \begin{cases} 0 > \lambda_1 \geq \lambda_2 & - \quad maxima \\ \lambda_1 \geq 0, \lambda_2 \leq 0 & - \quad saddle \quad points \\ 0 < \lambda_2 \leq \lambda_2 & - \quad minima \end{cases} \qquad (B1.7)$$

If we are interested in the density of the saddle points, then

$$N_{sad}(\nu)d\nu = \qquad (B1.8)$$

$$= \frac{1}{4\pi^2} \frac{1}{\sqrt{1-\gamma^2}} \frac{\sigma_2^2}{\sigma_1^2} e^{-\frac{\nu^2}{2}} d\nu \int_0^{+\infty} x e^{-x^2} dx \int_{-x}^{x} e^{-\frac{(y+\gamma\nu)^2}{2(1-\gamma^2)}} (x^2 - y^2) dy.$$

After integrating, we get a very simple result:

$$N_{sad}(\nu)d\nu = \frac{1}{\sqrt{32\pi}} \frac{1}{\pi} \frac{\sigma_1^2}{\sigma_1^2} \frac{1}{\sqrt{3-2\gamma^2}} e^{-\frac{3\gamma^2}{2(3-2\gamma^2)}} d\nu \ . \qquad (B1.9)$$

The integrated density is

$$n_{sad}(\nu_t) = \int_{\nu_t}^{\infty} N_{sad}(\nu)d\nu = \frac{1}{8\pi\sqrt{3}} \frac{\sigma_2^2}{\sigma_1^2} \left[1 - \Phi\left(\frac{\nu_t \sqrt{3}}{\sqrt{2(3-2\gamma^2)}} \right) \right], \qquad (B1.10)$$

where $\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ is the probability integral (Gradshteyn and Ryzhik, 1980). The density of the saddle points of arbitrary height is

$$n_{sad}(-\infty) = \frac{1}{4\pi\sqrt{3}} \frac{\sigma_2^2}{\sigma_1^2} \ . \qquad (B1.11)$$

Analogous results were obtained by (B.E.) but for differential and integrated densities of the maxima and minima. The differential density of maxima is

$$N_{max}(\nu)d\nu = \frac{\sqrt{2}}{4\pi\sqrt{\pi}} \frac{\sigma_2^2}{\sigma_1^2} \cdot \left\{ \frac{1}{4\sqrt{3-2\gamma^2}} e^{-\frac{3\nu^2}{2(3-2\gamma^2)}} \left[1 + \Phi\left(\frac{\gamma\nu}{\sqrt{(2-2\gamma^2)(3-2\gamma^2)}} \right) \right] \right.$$

$$\left. + \frac{\gamma(\nu^2-1)}{4} e^{-\frac{\nu^2}{2}} \left[1 + \Phi\left(\frac{\gamma\nu}{\sqrt{2(1-\gamma^2)}} \right) \right] + \frac{\gamma\nu\sqrt{1-\gamma^2}}{\sqrt{2\pi}} e^{-\frac{\nu^2}{2(1-\gamma^2)}} \right\}. \qquad (B1.12)$$

Integrated number density must be evaluated numerically,

$$n_{max}(\nu_t) = \int_{\nu_t}^{\infty} N_{max}(\nu)d\nu. \qquad (B1.13)$$

The density of maxima of arbitrary height is

$$n_{max}(-\infty, \gamma) = \frac{1}{8\pi\sqrt{3}} \frac{\sigma_2^2}{\sigma_1^2} \ . \qquad (B1.14)$$

Note the following important statement:

$$n_{max}(\nu_t) + n_{min}(\nu_t) - n_{sad}(\nu_t) = \qquad (B1.15)$$

$$\frac{1}{8\pi\sqrt{2\pi}} \frac{\sigma_2^2}{\sigma_1^2} \int_0^{+\infty} e^{-\frac{y^2}{2}} (y^2 - 1) \cdot \left[2 - \Phi\left(\frac{\nu_t + \gamma\nu}{\sqrt{2(1-\gamma^2)}} \right) - \Phi\left(\frac{\nu_t - \gamma\nu}{\sqrt{2(1-\gamma^2)}} \right) \right] dy \ .$$

Using the property $\Phi(x) = -\Phi(-x)$ and (B1.15), we obtain

$$n_{max}(0) + n_{min}(0) - n_{sad}(0) = 0. \qquad (B1.16)$$

11 Appendix B2

Similar to one-dimension, we define events B, C as the presence of two maxima at the points 1 and 2 respectively with parameters: ν^1; η_1^1; η_2^1; ξ_{11}^1; ξ_{22}^1; ξ_{12}^1 - for the first maximum and ν^2; η_1^2; η_2^2; ξ_{11}^2; ξ_{22}^2; ξ_{12}^2 for the second one. We introduce the coordinate system x, y as following. Let the first maximum is located at the point $(0, 0)$ and the second maximum is located at the point $\overline{r}_0 = (r_0, 0)$. The conditional probability for $\nu(\overline{r})$ to be in the range from $\nu(\overline{r})$ to $\nu(\overline{r}) + d\nu(\overline{r})$ is

$$P(\nu|B;C)d\nu(\overline{r}) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\det M_2}{\det M_1}} e^{-(Q_1 - Q_2)} d\nu(\overline{r}) \ , \qquad (B2.1)$$

where M_1 and M_2 are the 13×13 and 12×12 covariance matrices respectively, Q_1 and Q_2 are the quadratic forms. Elements of these matrices can be written in terms of the functions:

$$\psi = \frac{c}{\sigma_0^2}; \quad \delta = \frac{c'}{\sigma_1\sigma_0}; \quad \varphi = \frac{c''}{\sigma_1^2}; \quad \widetilde{\varphi} = \frac{c_1}{\sigma_1^2}; \quad \lambda = \frac{c'''}{\sigma_1\sigma_2}; \quad \widetilde{\lambda} = \frac{c_1'}{\sigma_1\sigma_2}$$

$$f = \frac{c''''}{\sigma_2^2}; \quad \widetilde{f} = \frac{c''}{\sigma_2^2}; \quad \widetilde{\widetilde{f}} = \frac{c_2}{\sigma_2^2}, \qquad (B2.2)$$

where

$$c = \pi \int kC(k)J_0(kr)dk,$$

is the correlation function and $c^{(n)}$ for $n = 1, ..., 4$ its derivatives,

$$c_1 = -\pi \int k^3 C(k)J_0(kr)dk,$$

$c_1^{(n)}$ for $n = 1, 2$ is its derivatives, and

$$c_2 = \pi \int k^5 C(k)J_0(kr)dk.$$

The matrices M_1 and M_2 can be written in a convenient and simple diagonal forms:

$$M_{1ij} = \langle z_i z_j \rangle \delta_{ij}, \quad i, j, = 1, ..., 13,$$

$$M_{2ij} = \langle z_i z_j \rangle \delta_{ij}, \quad i, j, = 1, ..., 12.$$

The difference between two quadratic forms is then $Q_1 - Q_2 = \frac{1}{2} z_{13} / \langle z_{13}^2 \rangle$. Also $\det M_1 = \langle z_{13}^2 \rangle \det M_2$ and for conditional probability one can write:

$$P(\nu | B; C) d\nu(\overline{r}) = \frac{1}{\sqrt{2\pi \langle z_{13}^2 \rangle}} e^{-\frac{z_{13}^2}{2\langle z_{13}^2 \rangle}} d\nu \ , \qquad (B2.3).$$

The variable z_{13} is the linear combination of $\nu = \nu(x, y)$ and z_i, $i = 1, ..., 13$:

$$z_{13} = \nu - \sum_i \frac{\langle \nu z_i \rangle}{\langle z_i^2 \rangle} z_i. \qquad (B2.4)$$

As it is seen from Eqs.(B2.3) and (B2.4), that ν has the mean value:

$$\langle \nu \rangle = \sum_i \frac{\langle \nu z_i \rangle}{\langle z_i^2 \rangle} z_i, \qquad (B2.5)$$

and the variance:

$$\langle \nu^2 \rangle = \sum_i \frac{\langle \nu z_i \rangle^2}{\langle z_i^2 \rangle}. \qquad (B2.6)$$

The variables z_i and $\langle z_i^2 \rangle$ are as follows:

$$z_1 = \nu^1 - \nu^2; \quad z_1^2 = 2(1 - \psi); \quad z_2 = \nu_1 + \nu_2; \quad z_2^2 = 2(1 + \psi);$$

$$z_3 = \eta_1^1 - \eta_1^2 + \frac{\delta}{1 + \psi} z_2; \quad \langle z_3^2 \rangle = 1 + 2\varphi - 2\frac{\delta^2}{1 + \psi};$$

$$z_4 = \eta_1^1 + \eta_1^2 - \frac{\delta}{1 - \psi} z_1; \quad \langle z_4^2 \rangle = 1 - 2\varphi - 2\frac{\delta^2}{1 - \psi}$$

$$z_5 = \eta_2^2 - \eta_2^2; \quad \langle z_5^2 \rangle = 1 - 2(\varphi - \widetilde{\varphi}); \quad z_6 = \eta_2^2 + \eta_2^2; \quad \langle z_6^2 \rangle = 1 + 2(\varphi - \widetilde{\varphi});$$

$$z_7 = \xi_{11}^1 - \xi_{11}^2 + \frac{\gamma(1 + 2(\widetilde{\varphi} - \varphi))}{\langle z_1^2 \rangle} z_1 - \frac{2\lambda + \frac{\gamma\delta((1+2(\widetilde{\varphi}-\varphi))}{1-\psi}}{\langle z_4^2 \rangle} z_4;$$

$$\langle z_7^2 \rangle = \frac{3}{4} - 2f - \frac{\gamma^2(1 + 2(\widetilde{\varphi} - \varphi))^2}{\langle z_1^2 \rangle} - \frac{\left(2\lambda + \frac{\gamma\delta((1+2(\widetilde{\varphi}-\varphi))}{1-\psi}\right)^2}{\langle z_4^2 \rangle};$$

$$z_8 = \xi_{11}^1 + \xi_{11}^2 + \frac{\gamma(1-2(\widetilde{\varphi}-\varphi))}{\langle z_2^2\rangle} z_1 + \frac{2\lambda + \frac{\gamma\delta((1-2(\widetilde{\varphi}-\varphi))}{1+\psi}}{\langle z_3^2\rangle} z_3;$$

$$\langle z_8^2\rangle = \frac{3}{4} + 2f - \frac{\gamma^2(1-2(\widetilde{\varphi}-\varphi))^2}{\langle z_2^2\rangle} - \frac{\left(2\lambda + \frac{\gamma\delta((1-2(\widetilde{\varphi}-\varphi))}{1+\psi}\right)^2}{\langle z_3^2\rangle};$$

$$z_9 = \xi_{22}^1 - \xi_{22}^2 - \xi_{11}^1 + \xi_{11}^2 - \frac{2(\widetilde{\lambda}-2\lambda)}{\langle z_4^2\rangle} z_4 -$$

$$\left(\frac{4f - 2\widetilde{f} - \frac{1}{2}}{\langle z_7^2\rangle} - \frac{2(\widetilde{\lambda}-2\lambda)(2\lambda + \frac{\gamma\delta((1+2(\widetilde{\varphi}-\varphi))}{1-\psi}}{\langle z_4^2\rangle\langle z_7^2\rangle}\right) z_7;$$

$$\langle z_9^2\rangle = \frac{3}{4} - 2(f - 2\widetilde{f} + \widetilde{\widetilde{f}}) - \frac{\gamma^2(1+2(\widetilde{\varphi}-\varphi))^2}{\langle z_1^2\rangle} - \frac{\left(2(\widetilde{\lambda}-\lambda) + \frac{\gamma\delta((1+2(\widetilde{\varphi}-\varphi))}{1-\psi}\right)^2}{\langle z_4^2\rangle} -$$

$$\frac{\left\{\frac{1}{4} + 2(f-\widetilde{f}) - \frac{\gamma^2(1+2(\widetilde{\varphi}-\varphi))^2}{\langle z_1^2\rangle} - \frac{\left[2\lambda + \frac{\gamma\delta((1+2(\widetilde{\varphi}-\varphi))}{1-\psi}\right]\left[2(\widetilde{\lambda}-\lambda) + \frac{\gamma\delta((1+2(\widetilde{\varphi}-\varphi))}{1-\psi}\right]}{\langle z_4^2\rangle}\right\}^2}{\langle z_7^2\rangle}$$

$$z_{10} = \xi_{22}^1 + \xi_{22}^2 - \xi_{11}^1 - \xi_{11}^2 + \frac{2(\widetilde{\lambda}-2\lambda)}{\langle z_3^2\rangle} z_3 -$$

$$\left[\frac{2\widetilde{f} - 4f - \frac{1}{2}}{\langle z_8^2\rangle} - \frac{2(\widetilde{\lambda}-2\lambda)(2\lambda + \frac{\gamma\delta((1-2(\widetilde{\varphi}-\varphi))}{1+\psi}}{\langle z_3^2\rangle\langle z_8^2\rangle}\right] z_8;$$

$$\langle z_{10}^2\rangle = \frac{3}{4} + 2(f - 2\widetilde{f} + \widetilde{\widetilde{f}}) - \frac{\gamma^2(1-2(\widetilde{\varphi}-\varphi))^2}{\langle z_2^2\rangle} - \frac{\left[2\lambda - (\widetilde{\lambda}) - \frac{\gamma\delta((1-2(\widetilde{\varphi}-\varphi))}{1+\psi}\right]^2}{\langle z_3^2\rangle} -$$

$$\frac{\left\{\frac{1}{4} + 2(\widetilde{f}-f) - \frac{\gamma^2(1-2(\widetilde{\varphi}-\varphi))^2}{\langle z_1^2\rangle} - \frac{\left[2\lambda + \frac{\gamma\delta((1-2(\widetilde{\varphi}-\varphi))}{1+\psi}\right]\left[2(\widetilde{\lambda}-\lambda) + \frac{\gamma\delta((1-2(\widetilde{\varphi}-\varphi))}{1+\psi}\right]}{\langle z_4^2\rangle}\right\}^2}{\langle z_8^2\rangle};$$

$$z_{11} = \xi_{12}^1 - \xi_{12}^2 - \frac{2(\widetilde{\lambda}-\lambda)}{\langle z_6^2\rangle} z_6;$$

$$\langle z_{11}^2\rangle = \frac{1}{4} + 2(f-\widetilde{f}) - \frac{4(\widetilde{\lambda}-\lambda)}{\langle z_6^2\rangle} z_6;$$

$$z_{12} = \xi^1_{12} + \xi^2_{12} - \frac{2(\widetilde{\lambda} - \lambda)}{\langle z_5^2\rangle} z_5;$$

$$\langle z_{12}^2\rangle = \frac{1}{4} + 2(f - \widetilde{f}) - \frac{4(\widetilde{\lambda} - \lambda)}{\langle z_5^2\rangle} z_5.$$

For values $\langle \nu z_i\rangle$, $i = 1, ..., 12$ we obtained:

$$\langle \nu z_1\rangle = \psi_1 - \psi_2; \quad \langle \nu z_2\rangle = \psi_1 + \psi_2;$$

$$\langle \nu z_3\rangle = -\frac{x}{r_1}\delta_1 + \frac{x - r_0}{r_2}\delta_2 + \frac{\delta}{1+\psi}\langle \nu z_2\rangle;$$

$$\langle \nu z_4\rangle = -\frac{x}{r_1}\delta_1 - \frac{x - r_0}{r_2}\delta_2 - \frac{\delta}{1-\psi}\langle \nu z_1\rangle;$$

$$\langle \nu z_5\rangle = -\frac{y}{r_1}\delta_1 + \frac{y}{r_2}\delta_2; \quad \langle \nu z_6\rangle = -\frac{y}{r_1}\delta_1 - \frac{y}{r_2}\delta_2;$$

$$\langle \nu z_7\rangle = \frac{x^2 - y^2}{r_2^2}\varphi_1 - \frac{(x - r_0)^2 - y^2}{r_2^2}\varphi_2 + \frac{y^2}{r_1^2}\widetilde{\varphi}_1 - \frac{y^2}{r_2^2}\widetilde{\varphi}_2 +$$

$$\frac{\gamma(1 + 2(\widetilde{\varphi} - \varphi))}{\langle z_1^2\rangle}\langle \nu z_1\rangle - \frac{2\lambda + \frac{\gamma\delta((1+2(\widetilde{\varphi}-\varphi))}{1-\psi}}{\langle z_4^2\rangle}\langle \nu z_4\rangle;$$

$$\langle \nu z_8\rangle = \frac{x^2 - y^2}{r_2^2}\varphi_1 - \frac{(x - r_0)^2 - y^2}{r_2^2}\varphi_2 + \frac{y^2}{r_1^2}\widetilde{\varphi}_1 + \frac{y^2}{r_2^2}\widetilde{\varphi}_1 +$$

$$\frac{\gamma(1 - 2(\widetilde{\varphi} - \varphi))}{\langle z_2^2\rangle}\langle \nu z_2\rangle + \frac{\left[2\lambda + \frac{\gamma\delta((1+2(\widetilde{\varphi}-\varphi))}{1-\psi}\right]}{\langle z_3^2\rangle}\langle \nu z_3\rangle;$$

$$\langle \nu z_9\rangle = \frac{y^2 - x^2}{r_1^2}(2\varphi_1 - \widetilde{\varphi}_1) - \frac{y^2 - (x - r_0)^2}{r_2^2}(2\varphi_2 - \widetilde{\varphi}_2) - \frac{2(\widetilde{\lambda} - 2\lambda)}{\langle z_4^2\rangle}\langle \nu z_4\rangle -$$

$$\left[\frac{4f - 2\widetilde{f} - \frac{1}{2}}{\langle z_7^2\rangle} - \frac{2(\widetilde{\lambda} - 2\lambda)(2\lambda + \frac{\gamma\delta((1+2(\widetilde{\varphi}-\varphi))}{1-\psi}}{\langle z_4^2\rangle\langle z_7^2\rangle}\right]\langle \nu z_7\rangle;$$

$$\langle \nu z_{10}\rangle = \frac{y^2 - x^2}{r_1^2}(2\varphi_1 - \widetilde{\varphi}_1) + \frac{y^2 - (x - r_0)^2}{r_2^2}(2\varphi_2 - \widetilde{\varphi}_2) + \frac{2(\widetilde{\lambda} - 2\lambda)}{\langle z_4^2\rangle}\langle \nu z_4\rangle -$$

$$\left[\frac{2\widetilde{f} - 4f - \frac{1}{2}}{\langle z_8^2\rangle} - \frac{2(\widetilde{\lambda} - 2\lambda)(2\lambda + \frac{\gamma\delta((1+2(\widetilde{\varphi}-\varphi))}{1+\psi}}{\langle z_3^2\rangle\langle z_8^2\rangle}\right]\langle \nu z_8\rangle;$$

$$\langle \nu z_{11}\rangle = \frac{xy}{r_1^2}(2\varphi_1 - \widetilde{\varphi}_1) - \frac{(x - r_0)y}{r_2^2}(2\varphi_2 - \widetilde{\varphi}_2) - \frac{2(\widetilde{\lambda} - \lambda)}{\langle z_6^2\rangle}\langle \nu z_6\rangle;$$

$$\langle \nu z_{12}\rangle = \frac{xy}{r_1^2}(2\varphi_1 - \widetilde{\varphi}_1) + \frac{(x - r_0)y}{r_2^2}(2\varphi_2 - \widetilde{\varphi}_2) - \frac{2(\widetilde{\lambda} - \lambda)}{\langle z_5^2\rangle}\langle \nu z_5\rangle;$$

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Figure captions

Fig.1 Correlation function for CDM models with Harrison-Zel'dovich spectrum of the initial perturbations, $\Omega_0 = 1$, $h = 0.5$. (a) Filtered correlation function with filters H_0 and H_2. Dashed line: $\Omega_b = 0.1$. Dotted line: $\Omega_b = 0.03$. Solid line: models without a Doppler-peak and $C_l = \frac{1}{l(l+1)}$. (b) The same as Fig.1a, but for filtered correlation functions with filter $H_*(\delta_{max}, \delta_{min}$,r). 1: for $\delta_{max} \sim l = 10$, $\delta_{min} \sim l = 150$; 2: for $\delta_{max} \sim l = 30$, $\delta_{min} \sim l = 150$. (c) Filtered correlation functions for the filter H_2 with the following width δ (curves from the left to the right):(1) $\delta \sim l = 660$, (2) $\delta \sim l = 350$, (3) $\delta = 0.67^o \sim l = 150$ corresponding to the resonance filter, (4) $\delta \sim l = 100$, (5) $\delta \sim l = 75$, (6) $\delta \sim l = 50$.

Fig.2 Correlation functions for the model $\Omega_b = 0.03$ (ensemble average) shown as solid lines and their dispersions $C(r) \pm \sqrt{D(r)}$ as dashed lines; realizations are shown as *. The field size is $10^o \times 10^o$. The upper part of the figure, marked by H_o, corresponds to the unfiltered Map 1a. The lower part, marked by H_2, corresponds to the filtered Map 1b.

Fig.3 The dependence of the spectral parameter γ on the value of the filter width δ. (a) For the filters H_0 and H_2. The solid line is without a Doppler-peak; the dotted line corresponds to $\Omega_b = 0.03$; the dashed line to $\Omega_b = 0.1$. (b) The same as Fig.3a, but for the filter H_*. 1: corresponds to $\delta_{max} \sim l = 10$, 2: corresponds to $\delta_{max} \sim l = 30$.

Fig.4 (a) A one-dimensional cut of the simulated map $10^o \times 10^o$. Arrows indicate two maxima, which are tested in Fig.4b. (b) The solid line is an expected value of the field in the vicinity of two neighbouring maxima indicated in Fig.4a. The dashed lines correspond to expected values $\pm\sqrt{variance}$.

Fig.5 Clusterisation of Maxima in one-dimension for different values of γ. (a) Differential densities of maxima and minima, solid line for $\gamma = 0.1$, dashed line for $\gamma = 0.5$, dotted line for $\gamma = 0.9$. (b) the mean length of the clusters: solid line for $\gamma = 0.1$, dashed line for $\gamma = 0.5$, dotted line for $\gamma = 0.9$.

Fig.6 $\Delta T/T$ as a two-dimensional surface in a three-dimensional space.

Fig.7 Clusterisation of peaks in the two-dimensional case. (a) Differential density of maxima shown as a dotted line; saddle points as a solid line; minima as a dashed line. (b) Mean length of clusters of maxima (right-hand side) and minima (left-hand side) for different γ: solid line for $\gamma = 0.1$, dashed line for $\gamma = 0.5$, dotted line for $\gamma = 0.9$.

Fig.8 Different slices of the $\Delta T/T$ maps in the vicinity of two maxima. (a) a model with $\Omega_b = 0.1$, (b) a model without a Doppler-peak.

Map 1 (a) Simulated map of $\Delta T/T$ for a region of $10^o \times 10^o$ for a model with $\Omega_b = 0.03$. (b) The same as for (a) but with the filter H_2, $\delta = 0.67^o$.

Map 2 Simulated map of $\Delta T/T$ for a region of $10^o \times 10^o$, for a model with $\Omega_b = 0.03$ and filter H_2, $\delta = 0.67^o$. *=maxima, X=saddle points; contours correspond to the level $\nu_t = 1$.

Map 3 Percolation through the black zone ($\nu_t > 0$).

Map 4 Simulated maps of $10^o \times 10^o$ for two different models (see text) (a) $\Omega_b = 0.1$, (b) Model without a Doppler-peak.