On Exactly Solvable Potentials

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Abstract

We investigate two methods of obtaining exactly solvable potentials with analytic forms.

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There are two methods of obtaining exactly solvable potentials in quantum mechanics. The first method was developed by applying the technique of supersymmetry (SUSY) to the Schrödinger equation and obtain two potentials with almost identical spectra. The two potentials can be considered to be superpartners of each other. It has been shown [1, 2] that if the two partners happen to be related by a simple relationship called shape invariance, the energy eigenvalues of the potential can be solved exactly. All the known solvable potentials with closed analytic forms can be shown to be shape invariant. In the literature, there are other solvable potentials that have not been shown to be shape invariant, however, they exist only complex numerical forms which we shall not consider in this article.

A second interesting method of obtaining solvable potential was proposed by Klein and Li[3] based on some special quantum commutation relationships. Li[4] has recently worked out the most general potential that can be obtained this way.

In this paper, we first investigate the relationship between the two approaches. We will show that the general solutions obtained by Li are special cases of the general solutions that can be obtained by solving shape invariance condition. Therefore, the solving shape invariance condition remains the most general method of obtaining exactly solvable potentials which can be given in analytic form. Unfortunately, there is no general method for getting analytic solutions of shape invariance condition. The best one can do seems to be starting from a guessing ansatz with an unknown function. The shape invariance is then enforced by demanding that the function satisfies an ordinary differential equation. The ansatz allows one to turn the difficult shape invariance condition into a problem of solving differential equation. It also allows one to associate each ansatz as defining a particular class of solutions. While the solutions obtained by Li correspond to those defined by a particular ansatz, in the literature, the largest classes of analytic solutions of the shape invariance condition was provided by Gendenshtein [1]. He proposed three ansatzs which define three classes of solutions which seem to cover all the known analytic exactly solvable potentials. We, therefore, proceed to solve the differential equations corresponding to these ansatzs and the energy eigenvalues of these three classes of solutions of shape invariance condition. In the process, we also demonstrate that all the known solutions in the literature are special cases of the one provided by Gendenshtein.

The three ansatzs of Gendenshtein and the forth one corresponding to Li's solutions also demonstrated that each potential can be simultaneously represented in many classes. It is also clear that one can in principle continue to invent more ansatz without obtaining any new potentials. Therefore, one is immediately faced with the intriguing question of what is the better way to classify these solvable potentials than using the ansatzs. One method seems to be starting from the n dependence of its spectrum. It is noted that only a few simple general forms of n dependence are allowed for the known exactly solvable potentials. We shall make some comments in this direction later and mention some interesting unsolved problems in the conclusion.

We first briefly review the procedure of constructing creation and annihilation operators of simple harmonic oscillator, now we can apply this method to the Schrödinger equation with an arbitrary potential $V(x)$.

$$(-\frac{1}{2}\frac{d^2}{dx^2}+V(x))\psi=E\psi \tag{1}$$

Assuming that we have already shifted the potential by a constant so that the ground state energy becomes zero, we shall denote this Hamiltonian and its potential with $(-)$ subscript and denote $\psi_n^{(-)}$ as its eigenfunctions. That is, $V_-(x)\equiv V(x)$, $\psi_0^{(-)}(x)\equiv\psi_0(x)$, $E_0^{(-)}=0$, with

$$H_-\psi_0^{(-)}=(-\frac{1}{2}\frac{d^2}{dx^2}+V_-(x))\psi_0^{(-)}=0. \tag{2}$$

The above equation, Eq.(2) is identical to

$$H_-=\frac{1}{2}(-\frac{d^2}{dx^2}+\frac{\psi_0^{(-)''}}{\psi_0^{(-)}}). \tag{3}$$

Then, the general creation and annihilation operators are:

$$\begin{aligned} A^+ &= \frac{1}{\sqrt{2}}(-\frac{d}{dx}-\frac{\psi_0^{(-)'}}{\psi_0^{(-)}}), \\ A &= \frac{1}{\sqrt{2}}(\frac{d}{dx}-\frac{\psi_0^{(-)'}}{\psi_0^{(-)}}). \end{aligned} \tag{4}$$

The Hamiltonian can be written as $H_-=A^+A$. Now we define a new Hamiltonian $H_+=AA^+$. which can be written as $H_+\equiv-\frac{1}{2}\frac{d^2}{dx^2}+V_+(x)$. The corresponding potential in the new Hamiltonian is:

$$V_+ = -V_- + (\frac{\psi_0^{(-)\prime}}{\psi_0^{(-)}})^2 = V_- - \frac{d^2}{dx^2}\ln\psi_0^{(-)}. \tag{5}$$

The V_+, V_- are called supersymmetric partner potentials.

One can define a new function, $W(x)$, called superpotential,

$$W(x) = -\frac{1}{\sqrt{2}}\frac{\psi_0^{(-)\prime}}{\psi_0^{(-)}}. \tag{6}$$

Solving the differential equation we get

$$\psi_0^{(-)}(x) = exp(-\sqrt{2}\int^x W(x')dx'), \tag{7}$$

and $A^+ = -\frac{1}{\sqrt{2}}\frac{d}{dx} + W(x)$, $A = \frac{1}{\sqrt{2}}\frac{d}{dx} + W(x)$. The two partner potentials are made from superpotential $W(x)$,

$$V_\pm = W^2 \pm \frac{1}{\sqrt{2}}\frac{d}{dx}W(x). \tag{8}$$

A^+ and A do not commute with each other and satisfy $[A, A^+] = \sqrt{2}W(x)'$. To compare the spectra of these two partners, denote $\psi^{(-)}$ as the eigenfunction of H_- and $\psi^{(+)}$ as that of H_+. Then, $A\psi^{(-)}$ is an eigenfunction of H_+ because

$$H_+(A\psi_n^{(-)}) = AA^+A\psi_n^{(-)} = AH_-\psi_n^{(-)} = E_n^{(-)}(A\psi_n^{(-)}). \tag{9}$$

Similarly, $A^+\psi^{(+)}$ is an eigenfunction of H_-, $H_-(A^+\psi_n^{(+)}) = E_n^{(+)}(A^+\psi_n^{(-)})$. Therefore A, A^+ can transform an eigenfunction of one potential into a eigenfunction of it's partner with the same energy. However, note that the ground state of V_- is annihilated by A and have no partner state of V_+. Therefore the ground state of V_- has no superpartner.

Relations of energy eigenvalues and wavefunctions are:

$$\begin{aligned} E_n^{(+)} &= E_{n+1}^{(-)}, \\ \psi_n^{(+)} &= \frac{1}{\sqrt{E_{n+1}^{(-)}}}A\psi_{n+1}^{(-)}, \quad n = 0, 1, 2, \ldots, \\ \psi_{n+1}^{(-)} &= \frac{1}{\sqrt{E_n^{(+)}}}A^+\psi_n^{(+)}, \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{10}$$

In 1983, Gendenshtein [1] suggested that all exactly solvable potentials are “shape invariant” (he used “form invariant” instead). Shape invariance means that the superpartner potentials have the same $x-$dependence modulo the changes in values of a set of parameters that define the potential. Mathematically, it means the superpartner of $V_-(x;a_0)$, $V_+(x;a_0)$ can be written as

$$V_+(x;a_0) = V_-(x;a_1) + R(a_1) \tag{11}$$

and $a_1 = f(a_0)$, where a_0 are the set of parameters in V_+, and f is the transformation function that maps a_0 into a_1. The remainder $R(a_1)$ is independent of x.

To show that the eigenvalues can be obtained easily from the above condition, we construct a series of Hamiltonian $H^{(k)}$, $k = 0, 1, 2, \ldots$, with $H^{(0)} \equiv H_-$, $H^{(1)} \equiv H_+$,

$$H^{(k)} = -\frac{1}{2}\frac{d^2}{dx^2} + V_-(x;a_k) + \sum_{s=1}^{k} R(a_s), \tag{12}$$

where $a_s = f^s(a_0)$, i.e., f mapped s times. Furthermore, we have From Eq.(12), we see that the ground state of $H^{(k)}$ has the energy:

$$E_0^{(k)} = \sum_{s=1}^{k} R(a_s). \tag{13}$$

Since $(n+1)$th energy eigenvalue of $H^{(0)}$ $(=H_-)$, whose ground state energy is zero, is coincident with the ground state energy of Hamiltonian $H^{(n)}$, the complete eigenvalues of H_- are:

$$E_n^{(-)} = \sum_{k=1}^{n} R(a_k) \quad , \quad E_0^{(-)} = 0. \tag{14}$$

If a potential is shape invariant, we can also get the bound state wavefunctions $\psi_n^{(-)}$ easily. This is because A and A^+ can link up the wavefunctions of the superpartners with the same energy. Starting from $H_n^{(-)}$, it's ground state $\psi_0^{(-)}(x;a_n)$ corresponds to the first excited state $\psi_1^{(-)}(x;a_{n-1})$ of $H_{n-1}^{(-)}$. In the same manner, eventually, it will correspond to the nth state of $H_0^{(-)}$. Recalling Eq.(10), one obtains

$$\psi_n^{(-)} \propto A^+(x;a_0)A^+(x;a_1)\ldots A^+(x;a_{n-1})\psi_0^{(-)}(x;a_n). \tag{15}$$

To compare the supersymmetry approach with the second approach proposed by Klein and Li[3, 4], we shall first give a brief review. Consider an one dimensional quantum system with the Hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x). \tag{16}$$

For an arbitrary function of position, $f = f(x)$, one can easily derive

$$[[f, H], H] = -(f''H + Hf'') + 2f''V + f'V' - \frac{f^{(4)}}{4}. \tag{17}$$

To obtain solvable potential Klein and Li proposed to impose a so-called linear double commutator relation demanding that for some f and V, the right hand side become a linear functional of f. In order to make the operator equation linear in f, one imposes

$$2f''V + f'V' = \alpha f + \beta, \tag{18}$$

and

$$f'' = \mu f + \nu. \tag{19}$$

Eq.(18) relates V to f and can be solved to give the potential $V(x)$

$$V(x) = \frac{\alpha f^2 + 2\beta f + \gamma}{2(f')^2}. \tag{20}$$

Eq.(19) can be solved for f as

$$f = ax^2 + bx + c \qquad (\ \mu = 0\ ), \tag{21}$$

or

$$f = Ae^{\sqrt{\mu}x} + Be^{-\sqrt{\mu}x} + C \qquad (\ \mu \neq 0\ ). \tag{22}$$

For $\mu = 0$, the resulting potentials are just those for one- or three-dimensional harmonic oscillator problems. For $\mu \neq 0$, one can get the general potential

$$V(x) = \frac{\beta(Ae^{\sqrt{\mu}x} + Be^{-\sqrt{\mu}x}) + \frac{\gamma}{2}}{\mu(Ae^{\sqrt{\mu}x} - Be^{-\sqrt{\mu}x})^2}, \tag{23}$$

which are special cases of Morse and Pőschl-Teller potentials. Li also solved the energy eigenvalues by the method of Heisenberg's matrix mechanics[4] under some general assumptions. The eigenvalue is proportional to n^2, where n is a quantum number. Also, from Eqs. (19) and (20), the corresponding superpotential, with linear double commutator relation, can be written as[4]

$$W(x) = \frac{af(x) + b}{f'(x)}. \tag{24}$$

Here we shall first show that the superpotential is indeed shape invariance and then solve the eigenvalues using the supersymmetric methods outlined earlier.

Given, Eqs.(24) and (19), one can easily check the shape invariance condition by working out the potentials $V_\pm$ as

$$V_\pm = \frac{a(a\mp\frac{\mu}{\sqrt{2}})f^2 + (a(b\mp\frac{\nu}{\sqrt{2}}) + b(a\mp\frac{\mu}{\sqrt{2}}))f + b(b\mp\frac{\nu}{\sqrt{2}})}{(f')^2} \pm \frac{a}{\sqrt{2}}. \tag{25}$$

Therefore, $V_\pm$ are related by shape invariance condition

$$V_+(a,b;x) = V_-(a-\frac{\mu}{\sqrt{2}}, b-\frac{\nu}{\sqrt{2}}; x) + \sqrt{2}a - \frac{\mu}{2}. \tag{26}$$

The energy eigenvalues can be straight-forwardly worked out to be

$$E_n = \sqrt{2}na - \frac{\mu n^2}{2}, \tag{27}$$

where the ground state energy has been shifted to zero. SUSY method and Li's method gives the same results as expected.

In the SUSY approach, shape invariance requires that the superpotential $W(x)$ satisfies the functional differential equation:

$$W^2(a,x) + \frac{1}{\sqrt{2}}W'(a,x) = W^2(a_1,x) - \frac{1}{\sqrt{2}}W'(a_1,x) + R(a_1), \tag{28}$$

where a_0 represents a set of parameters, called the shape invariance relation(SIR). The values of the set, a_1, on the right-hand side, depend on the value of a_0, that is, $a_1 = f(a_0)$ for some function f. The only known way of solving SIR is to impose an educated ansatz to turn it into a differential equation. In this direction, Gendenshtein [1] had proposed three ansatzs which provide some general classes of solutions. There is no claim that the three classes should encompass the most general solutions of SIR. However, it is interesting to note that all the known solutions can be shown to be special cases of one of the three classes as we shall demonstrate later. Also, the three classes are not mutually exclusive. That is, some solutions can be represented in more than one classes.

We shall discuss the three classes in order, solve the corresponding nonlinear differential equations and work out the potentials and energy eigenvalues for each class. The three classes can be described as follow:

(I).Class one:

The ansatz for the superpotential is of the form

$$W = af_1 + b. \tag{29}$$

The SIR then requires f_1 to satisfy

$$f_1' = pf_1^2 + qf_1 + r. \tag{30}$$

$V_\pm$ can be worked out to be

$$V_\pm = a(a \pm \frac{p}{\sqrt{2}})f_1^2 + 2a(b \pm \frac{q}{2\sqrt{2}})f_1 + b^2 \pm \frac{ar}{\sqrt{2}}, \tag{31}$$

with the parameters transforming as

$$\begin{aligned} a_n &= a + \frac{np}{\sqrt{2}}, \\ b_n &= \frac{ab}{a + \frac{np}{\sqrt{2}}} + \frac{na\frac{q}{\sqrt{2}} + \frac{n^2pq}{4}}{(a + \frac{np}{\sqrt{2}})}, \\ R &= b_0^2 - b_1^2 + \frac{r}{\sqrt{2}}(a_0 + a_1). \end{aligned} \tag{32}$$

For the case when $p \neq 0$, the eigenvalues can be worked out to be

$$\begin{aligned} E_n &= \sum_{k=1}^{n} R(a_k, b_k) \\ &= \sum_{k=1}^{n} [(b_{k-1}^2 - b_k^2) + \frac{r}{\sqrt{2}}(a_{k-1} + a_k)] \\ &= b^2 - [\frac{ab}{a + \frac{np}{\sqrt{2}}} + \frac{na\frac{q}{\sqrt{2}} + \frac{n^2pq}{4}}{(a + \frac{np}{\sqrt{2}})}]^2 + \sqrt{2}arn + \frac{pr}{2}n^2. \end{aligned} \tag{33}$$

Ordering the terms by power of n, E_n can be written as

$$E_n = (b - \frac{aq}{2p})^2 - (b - \frac{aq}{2p})^2 \frac{a^2}{(a + n\frac{p}{\sqrt{2}})^2} + \sqrt{2}(ar - \frac{aq^2}{4p})n + (\frac{pr}{2} - \frac{q^2}{8})n^2. \tag{34}$$

For the case when $p = 0$, the transformation of parameter and the eigenvalues can be obtained as

$$\begin{aligned} b_n &= b + \frac{nq}{\sqrt{2}}, \\ R &= b_0^2 - b_1^2 + \sqrt{2}ar, \\ E_n &= \sum_{k=1}^{n} R_k = \sqrt{2}(ar - bq)n - \frac{q^2}{2}n^2. \end{aligned} \tag{35}$$

The known exactly solvable potentials in this class and their corresponding parameters are listed in Table 1.

Table 1: Class One $W = af(x) + b$; $f' = pf^2 + qf + r$

Potential	$W(x)$	$f(x)$	a	b	p	q	r
Shifted Oscillator	$\sqrt{\frac{1}{2}}\omega x - b$	x	$\sqrt{\frac{1}{2}}\omega$	$-b$	0	0	1
Coulomb	$\sqrt{\frac{1}{2}}\frac{e^2}{l+1} - \frac{l+1}{\sqrt{2}r}$	r^{-1}	$-\frac{l+1}{\sqrt{2}}$	$\sqrt{\frac{1}{2}}\frac{e^2}{l+1}$	-1	0	0
Morse	$A - Be^{-\alpha x}$	$e^{-\alpha x}$	$-B$	A	0	$-\alpha$	0
Rosen-Morse	$A\tanh\alpha x + \frac{B}{A}$	$\tanh\alpha x$	A	$\frac{B}{A}$	$-\alpha$	0	α
Eckart	$-A\coth\alpha r + \frac{B}{A}$ $(B > A^2)$	$\coth\alpha r$	$-A$	$\frac{B}{A}$	$-\alpha$	0	α

The most general solutions for function f in this case can be summarized as

$$f = rx + c \quad ,(p = q = 0); \tag{36}$$

$$f = -\frac{1}{px + c} \quad ,(p \neq 0, q = r = 0); \tag{37}$$

$$f = ce^{qx} - \frac{r}{q} \quad ,(p = 0, q \neq 0); \tag{38}$$

$$f = \frac{\sqrt{4pr - q^2}}{2p}\tan(\frac{\sqrt{4pr - q^2}}{2}x + c) - \frac{q}{2p} \quad ,(p \neq 0, q \text{ or } r \neq 0), \tag{39}$$

where c is an integration constant.

(II). Class two:

The superpotential in this ansatz is assumed to be of the form

$$W = af_2 + \frac{b}{f_2},$$
$$f_2' = pf_2^2 + q. \tag{40}$$

$V_\pm$ can be worked out to be

$$V_\pm = a(a \pm \frac{p}{\sqrt{2}})f_2^2 + b(b \mp \frac{q}{\sqrt{2}})f_2^{-2} + 2ab \pm \frac{1}{\sqrt{2}}(aq - pb), \tag{41}$$

with the parameters transforming as

$$a_n = a + \frac{np}{\sqrt{2}},$$
$$b_n = b - \frac{nq}{\sqrt{2}},$$
$$R_n = 2\sqrt{2}(aq - bp) + 2(2n-1)pq. \tag{42}$$

The eigenvalues are

$$E_n = 2\sqrt{2}(aq - bp)n + 2pqn^2. \tag{43}$$

The well-known examples in the class and their corresponding parameters are listed in Table 2 for illustration.

Table 2: Class Two $W = af + \frac{b}{f}$; $f' = pf^2 + q$

Potential	$W(x)$	$f(x)$	a	b	p	q
3-D Oscillator	$\sqrt{\frac{1}{2}}\omega r - \frac{l+1}{\sqrt{2}r}$	r	$\sqrt{\frac{1}{2}}\omega$	$-\frac{l+1}{\sqrt{2}}$	0	1
Pöschl-Teller I	$A\tan\alpha x - B\cot\alpha x$ $(0 < \alpha x < \frac{\pi}{2})$	$\tan\alpha x$	A	$-B$	α	α
Pöschl-Teller II	$A\tanh\alpha r - B\coth\alpha r$ $(B < A)$	$\tanh\alpha r$	A	$-B$	$-\alpha$	α

The general solutions for function f in this case can be written as

$$f = qx + c \quad (p = 0), \tag{44}$$
$$f = \sqrt{\frac{q}{p}}\tan(\sqrt{pq}x + c) \quad (p \neq 0, q \neq 0), \tag{45}$$
$$f = -\frac{1}{px + c} \quad (p \neq 0, q = 0), \tag{46}$$

where c is an integration constant.

(III).Class three:

The superpotential is assumed to be of the form

$$W = \frac{a + b\sqrt{pf_3^2 + q}}{f_3},$$
$$f_3' = \sqrt{pf_3^2 + q}. \tag{47}$$

$V_\pm$ can be worked out to be

$$V_\pm = \frac{1}{f_3^2}[a^2 + bq(b \mp \sqrt{\frac{1}{2}}) + 2a(b \mp \frac{1}{2\sqrt{2}})\sqrt{pf_3^2 + q}] + b^2 p, \tag{48}$$

with the parameters transforming as

$$\begin{aligned} b_n &= b - \frac{n}{\sqrt{2}}, \\ R_n &= p(\sqrt{2}b + \frac{1}{2}) - pn. \end{aligned} \tag{49}$$

The eigenvalues are

$$E_n = \sqrt{2}npb - \frac{pn^2}{2}. \tag{50}$$

The well-known exactly solvable potentials in this class are listed in Table 3.

The general solutions for function f are

$$f = \sqrt{q}x + c \quad , (p = 0); \tag{51}$$

$$f = ce^{\sqrt{p}x} \quad , (p \neq 0, q = 0); \tag{52}$$

$$f = \frac{1}{2}\sqrt{\frac{q}{p}}(e^{\sqrt{p}(x+c)} - e^{-\sqrt{p}(x+c)}) \quad , (p, q \neq 0), \tag{53}$$

c is an integration constant.

Table 3: Class Three $W = (a + b\sqrt{pf^2 + q})/f$,$f' = \sqrt{pf^2 + q}$

Potential	$W(x)$	$f(x)$	a	b	p	q
Morse	$A - Be^{-\alpha x}$	$e^{\alpha x}$	$-B$	$\frac{A}{\alpha}$	α^2	0
	$A \tanh \alpha x + B\,\text{sech}\alpha x$	$\cosh \alpha x$	B	$\frac{A}{\alpha}$	α^2	$-\alpha^2$
Rosen-Morse	$A \coth \alpha r - B\,\text{csch}\alpha r$ $(A < B)$	$\sinh \alpha r$	$-B$	$\frac{A}{\alpha}$	α^2	α^2
Eckart	$-A \cot \alpha x + B \csc \alpha x$ $(0 < \alpha x < \pi, A > B)$	$\sin \alpha x$	B	$-\frac{A}{\alpha}$	$-\alpha^2$	α^2

The tables showed that all known exact solvable potentials with analytic forms can be put into one of these three classes. Whatever has spectrum $1/n^2$ can be classified as the first case. However, only the class two solutions can produce three dimensional oscillator, type-I Pőshl-Teller or type-II Pőshl-Teller potentials. But two potentials, Morse and Eckart, can be considered both as case one and as case three. As mentioned before, Li's results are just the special case of shape invariance solutions. However it cannot be so easily fit into one of the three classes given by Gendenshtein. In fact one can consider Eq.(19) and Eq.(24) to be the equations that define a fourth ansatz for solutions of SIR. The set of solutions overlaps with those of the other three classes provided by Gendenshtein but does not generate a new one.

From this point of view, it is clear that the ansatz does not provide a very precise classification of the solutions of shape invariance condition. It is not too hard to propose new ansatz, however, it is much harder to generate new solutions. Typically one can not be sure whether an ansatz generates any new solution or not until they are solved completely. Therefore it seems that a better classifying solutions may be to use the energy spectrum and its quantum number dependence instead.

In conclusion, we have discussed the two methods of obtaining exactly solvable potentials in quantum mechanics. One of them requires the shape invariance between the superpartners of the potentials in the supersymmetric formulation. The other one imposes a so-called "linear double commutator relations". We have shown that the second case only produces solutions which are special solutions of the first approach. From this point of view, the shape invariance approach is still the most general method of producing the analytic, exactly solvable potentials. We also argued that the n dependence of the energy spectrum may be a better way of telling the difference between different classes of potentials. In addition, we work out the energy eigenvalues of the most general classes of potentials in the literature.

Unfortunately, it is still not possible to obtain the most general solutions to SIR. From the table of [2], one observes that all the known shape invariant potentials have basically only one parameter changing under the SUSY transformation. In particular, for Pöschl-Teller I potentials, only the combination $A + B$ is changing. The parameter $A - B$ is invariant. For Pöschl-Teller potentials II, on the other hand, the combination $A-B$ is changing and the parameter $A+B$ is invariant. It is not surprising because as long as the transformation property of the parameters are linear for a proper choice of parameters, one can always make linear combinations such that only one of them is changing during the transformation. Since, in general, there is no reason for the transformation of the parameters to be linear, one would expect to have a lot more interesting solutions of shape invariance condition waiting to be discovered.

Another interesting observation is that all the shape invariant solutions has the spectrum of one of the following the forms (modulo a constant that sets the ground state energy to zero): (1) an, $(a>0)$; (2) $-\frac{b}{(n+a)^2}$, $(b>0)$; (3) $\pm(a+bn)^2$ or their linear combination. The harmonic oscillator is an example for the first form. The hydrogen atom is an example for the second form and, the square well is the simplest example for the third form. Also note that the general spectra of all three classes of solutions suggested by Gendenshtein are all linear combinations of the above three forms. One may wonder if there is something fundamental about these kinds of spectra that made them represent the spectra of all the shape invariant, exactly solvable, potentials.

Finally, regarding the approach proposed by Klein and Li, since Li's solutions in [4] produce only part of the solutions of SIR, it suggests that it may be possible to generalize their approach within its' framework. We have made some attempts in this direction, however, so far without success.

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