Matrix Models and Geometry of Moduli Spaces

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Abstract

We give the description of discretized moduli spaces (d.m.s.) $\overline{\mathcal{M}}_{g,n}^{disc}$ introduced in [1] in terms of discrete de Rham cohomologies for moduli spaces $\mathcal{M}_{g,n}$. The generating function for intersection indices (cohomological classes) of d.m.s. is found. Classes of highest degree coincide with the ones for the continuum moduli space $\overline{\mathcal{M}}_{g,n}$. To show it we use a matrix model technique. The Kontsevich matrix model is the generating function in the continuum case, and the matrix model with the potential $N\alpha\,\mathrm{tr}\left(-\frac{1}{4}\Lambda X\Lambda X-\frac{1}{2}\log(1-X)-\frac{1}{2}X\right)$ is the one for d.m.s. In the latest case the effects of Deligne–Mumford reductions become relevant, and we use the stratification procedure in order to express integrals over open spaces $\mathcal{M}_{g,n}^{disc}$ in terms of intersection indices, which are to be calculated on compactified spaces $\overline{\mathcal{M}}_{g,n}^{disc}$. We find and solve constraint equations on partition function $\mathcal{Z}$ of our matrix model expressed in times for d.m.s.: $t_m^{\pm}=\mathrm{tr}\,\frac{\partial^m}{\partial\lambda^m}\frac{1}{\mathrm{e}^{\lambda}-1}$. It appears that $\mathcal{Z}$ depends only on even times and $\mathcal{Z}[t_{\cdot}^{\pm}]=C(\alpha N)\,\mathrm{e}^{\mathcal{A}}\,\mathrm{e}^{F(\{t_{2n}^{-}\})+F(\{-t_{2n}^{+}\})}$, where $F(\{t_{2n}^{\pm}\})$ is a logarithm of the partition function of the Kontsevich model, $\mathcal{A}$ being a quadratic differential operator in $\frac{\partial}{\partial t_{2n}^{\pm}}$.

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0 Introduction.

It has been shown recently that there exists a close but still not properly understood connection between three items: geometrical invariants of moduli spaces of algebraic curves; matrix models; integrable systems related to these models. The first relation was established by M.Kontsevich in [2] who found a matrix model providing a generating function for intersection indices (integrals of the first Chern classes) on moduli spaces of algebraic curves.

In this paper we describe some newly found applications of matrix models to the description of geometrical properties of the moduli spaces of algebraic curves. Here we should first mention brilliant papers by Maxim Kontsevich [2], in which the Kontsevich matrix model was introduced as the generating function for intersection indices of the first Chern classes on the moduli (orbi)spaces $\overline{\mathcal{M}}_{g,n}$ of the surface of genus g and n punctures:

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \omega_i^{d_i}. \tag{0.1}$$

Here ω_i is a Chern class associated with ith puncture.

Kontsevich's papers were motivated by Witten's consideration [3] on two-dimensional (topological) gravity, or quantum gravity. Two different matrix model approaches were elaborated in order to describe such theory.

The first approach is due to [4] and it concerns a usual 1–matrix hermitian model with an arbitrary potential. In a "fat graph" technique starting with each graph we can construct the dual one corresponding to some Riemann surface with singularities of curvature concentrated in vertices of this dual graph. Then faces of this graph correspond to vertices in the matrix model graph and vice versa. If the initial potential contains only three valent vertices we can speak about "triangulation" of the Riemann surface. In what follows we shall deal with potentials of an arbitrary order, but we use the same term "triangulation". The model with an arbitrary potential was solved exactly in [4] in the double scaling limit when the number of triangles tends to infinity and these singular metrics approximate "random metrics" on the surface. This model was presented by a hermitian $N \times N$ one–matrix model

$$\int \exp\bigl(\operatorname{tr} P(X)\bigr) DX, \tag{0.2}$$

where $P(X) = \sum_n T_n \operatorname{tr} X^n$, T_n being times for the one–matrix model. For such system discrete Toda chain equations holds with an additional Virasoro symmetry imposed [5]. In the *double scaling limit* (d.s.l.) $N \to \infty$ and $P(X)$ transforms in a way to incorporate surfaces with infinitely growing number of partitions and, as a result, the Korteveg–de–Vries equation arises. The partition function of the two–dimensional gravity for this approach is a series in an infinite number of variables and coincides with the logarithm of some τ–function for KdV hierarchy.

The second approach is based on cohomological considerations. In two-dimensional quantum gravity we have to integrate over space of riemannian metrics on manifolds modulo diffeomorphisms. Therefore, a finite-dimensional moduli space of conformally nonequivalent metrics arises. Integrals over such spaces have a cohomological description as an intersection theory on the compactified moduli space of complex curves.

A fat graph technique was used in order to introduce coordinates on the moduli spaces. The coordinatization means that we assign lengths l_i to all edges of the fat graph and the number of punctures, n, is the number of faces of the graph. We call this space $\mathcal{M}_{g,n}^{comb}$.

The model proposed by Kontsevich is a generating function for intersection indices or integrals of first Chern classes on the corresponding moduli space. It was a proposition by Witten [3] that these integrals yield correlation functions for the two–dimensional gravity coupled to the matter. In the continuum case the relation (0.1) holds where the integral goes over properly compactified moduli space and ω_i are closed two-forms that are representatives of the first Chern classes of line bundles on $\overline{\mathcal{M}}_{g,n}$.

For a general oriented graph of genus g and number of faces n the total number of edges (for trivalent vertices of the general position) is $6g-6+3n$ which exceeds the dimension of $\mathcal{M}_{g,n}$ by n. So there are n extra parameters which are not related to the coordinates on the original moduli space itself. Namely, they are perimeters of the faces of the graph. In the continuum case, due to Strebel theorem [6], we have an isomorphism $\mathcal{M}_{g,n} \otimes \mathbf{R}_+^n \simeq \mathcal{M}_{g,n}^{comb}$ and we define a projection $\pi : \mathcal{M}_{g,n}^{comb} \to \mathbf{R}_+^n$ to the space of perimeters. The fibers $\pi^{-1}(p_*)$ of the inverse map are isomorphic to the initial moduli space $\mathcal{M}_{g,n}$ and hence they all are isomorphic to each other.

Intersection indices for continuum case are expressed via the Kontsevich integral $\int DX \exp\{ \operatorname{tr} \frac{1}{2}\Lambda X^2 + \frac{1}{6}X^3\}$ with an external (Hermitian) matrix Λ. It satisfies equation of KdV hierarchy in times $t_n = (2n-1)!! \operatorname{tr} \Lambda^{-2n-1}$. It is an asymptotic expansion of the string partition function

$$\tau(t) = \exp \sum_{g=0}^{\infty} \left\langle \exp \sum_n t_n \mathcal{O}_n \right\rangle_g , \tag{0.3}$$

and it is certainly a tau–function of the KdV hierarchy taken at a point of Grassmannian where it is invariant under the action of the set of the Virasoro constraints: $\mathcal{L}_n \tau(t) = 0$, $n \geq -1$ [7], [8], [9], [10]. One might say that the Kontsevich model is used to triangulate moduli space, whereas the original models triangulate Riemann surfaces (see e.g. [11]).

In our recent papers [1], [12] we have proposed and developed an approach to the discretization of an arbitrary moduli space of algebraic curve. The connection of these spaces to a matrix model was established and also it was demonstrated explicitly that in the limit where discretization parameter becomes small this matrix model goes to the Kontsevich one [2].

The discretization of $\mathcal{M}_{g,n}^{comb}$ is rather simple – we assume that all lengths of edges are to be integer numbers (probably zeros). Fixing perimeters we always have a finite number of admissible sets of edges and a finite number of possible base diagrams for fixed g and n. Putting together all these possibilities we get the union of points of the discretized moduli space $\overline{\mathcal{M}}_{g,n}^{disc}$.

We now are also able to define another projection $\widetilde{\pi} : \mathcal{M}_{g,n}^{disc} \to \mathbf{Z}_+^n|_{\sum p_i \in 2\mathbf{Z}_+}$ where all perimeters are strictly positive integers with even total sum and consider its fibers $\widetilde{\pi}^{-1}(p_*)$. They are, generally speaking, finite sets of points belonging to the initial moduli space $\overline{\mathcal{M}}_{g,n}$. These sets are no more isomorphic to each other. Moreover, among these points there are always points which correspond exactly to singular surfaces ("infinity points"). We assume that the space of singular surfaces is $\partial\mathcal{M}_{g,n} = \overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$. In the usual Teuchmüller picture all these points lie at the infinity, but in what follows we should include them explicitly into the game. We introduce an analogue of De Rham complex on these spaces using finite difference structures instead of differential ones. There are the spaces we call discretized moduli spaces (d.m.s.). Also instead of $U(1)$–bundles for continuum case we shall consider "$\mathbf{Z}_p$–bundles" over these spaces. Thus we can define cohomological classes for d.m.s. as well:

$$\langle\!\langle \tau_{d_1} \dots \tau_{d_n} \rangle\!\rangle_g = \int_{\widetilde{\pi}^{-1}(p_*)} \prod_{i=1}^{n} \widetilde{\omega}_i^{d_i}. \tag{0.4}$$

There is a unique (up to isomorphisms) closed moduli space $\overline{\mathcal{M}}_{g,n} = \pi^{-1}(p_*)[\mathcal{M}_{g,n}^{comb}]$ and an infinite series of nonisomorphic $\widetilde{\pi}^{-1}(p_*)[\overline{\mathcal{M}}_{g,n}^{disc}]$, but for all of them the relation

$$\langle\!\langle \tau_{d_1} \dots \tau_{d_n} \rangle\!\rangle_g = \langle \tau_{d_1} \dots \tau_{d_n} \rangle_g \tag{0.5}$$

holds true for higher-order integrals, $\sum d_i = d = 3g-3+n$. We shall present matrix model arguments in favour of this statement, but note that, due to possible nonzero curvature of covering manifold, $\langle\!\langle \tau_{d_1} \dots \tau_{d_n} \rangle\!\rangle_g$ may be nonzero even for $\sum d_i < d$, in contrast to the continuum case.

On the L.H.S. of (0.5) $\langle\!\langle \tau_{d_1} \dots \tau_{d_n} \rangle\!\rangle_g$ can be presented in a form similar to (0.1) but with all quantities being related to the discretized moduli space (d.m.s.).

One note about the structure of the simplicial complex described by the matrix model is in order. We denote this complex $\mathcal{T}_{g,n}$, $\mathcal{M}_{g,n}$ itself is a quotient of it by some symmetry group of a finite order:

$\mathcal{M}_{g,n} = \mathcal{T}_{g,n}/\Gamma_g$. All the integrals one can define on $\mathcal{T}_{g,n}$ instead of $\mathcal{M}_{g,n}$, moreover, we have strong arguments in favour of the assumption that $\mathcal{T}_{g,n}$ can be *manifold* (which is impossible in Teichmüller picture).

In the Kontsevich parameterization the evaluation of the integrals over $\pi^{-1}(p_*)$ and $\widetilde{\pi}^{-1}(p_*)$ can be reduced to the calculation of integrals over volume forms on above mentioned finite coverings $\mathcal{T}_{g,n}$. The discretization means in this language that we introduce an equidistant lattice on $\mathcal{T}_{g,n}$ and while calculating the volume we merely count a total number of sites in this lattice and divide it by some product of p_i^2: $p_1^{2a_1}\dots p_n^{a_n}$, where $\sum_{i=1}^n a_i = d = 3g-3+n$ is the total dimension of $\overline{\mathcal{M}}_{g,n}$. Note also that all $\mathcal{T}_{g,n}$ are compact spaces without boundaries. The sum over all points of a lattice is equivalent to the sum of unit cubes attached to each lattice point. Then we get the true volume of $\mathcal{T}_{g,n}$ only if all these points are nonsingular points, i.e., points of zero curvature. Since it seems not to be the case for every g and n (but holds for the case of torus with one puncture), some of indices $\langle\!\langle d_1\dots d_n\rangle\!\rangle_g$ may be nonzero for $\sum d_i < d$.

In order to find a connection between moduli spaces $\overline{\mathcal{M}}_{g,n}$ and d.m.s. we use a matrix model technique.

Generalization of the Kontsevich model — so-called Generalized Kontsevich Model (GKM) [13] is related to the two–dimensional Toda lattice hierarchy. It originated from the external field problem defined by the integral

$$Z[\Lambda;N] = \int DX \exp\left\{N\,\mathrm{tr}\left(\Lambda X - V_0(X)\right)\right\}, \tag{0.6}$$

where $V_0(X) = \sum_n t_n \,\mathrm{tr}\, X^n$ is some potential, t_n are related to times of the hierarchy. This model is equivalent to the Kontsevich integral for $V_0(X) \sim \mathrm{tr}\, X^3$. To solve the integral (0.6) one may use the Schwinger–Dyson equation technique [14] written in terms of eigenvalues of Λ. The Kontsevich model was solved in the genus expansion in the papers [9], [15] for genus zero (planar diagrams) and in [16] for higher genera.

Another explicitly solvable model was introduced [17]. The Lagrangian of this model has the following form:

$$\mathcal{Z}[\Lambda] = \int DX \exp\left(N\,\mathrm{tr}\left\{-\frac{1}{2}\Lambda X\Lambda X + \alpha\left[\log(1+X) - X\right]\right\}\right), \quad \Lambda = \mathrm{diag}(\mathrm{e}^{\lambda_1},\dots,\mathrm{e}^{\lambda_N}). \tag{0.7}$$

This model may be readily reduced to (0.6) with $V_0(X) = -X^2/2 + \alpha\log X$. It was solved in genus expansion in [17], [18]. It appears (see [19], [20]) that it is in fact equivalent to the one–matrix hermitian model (0.2) with the general potential

$$P(X) = \sum_{n=0}^{\infty} T_n \,\mathrm{tr}\, X^n,$$

where times are defined by the kind of Miwa transform ($\eta = \Lambda - \alpha\Lambda^{-1}$):

$$T_n = \frac{1}{n}\,\mathrm{tr}\,\eta^{-n} - \frac{N}{2}\delta_{n2} \;\text{ for }\; n \ge 1 \quad\text{and}\quad T_0 = \mathrm{tr}\,\log\eta^{-1}. \tag{0.8}$$

One of the motivations for considering such model was Penner's one-matrix model $\int DX \exp\,\mathrm{tr}\,\big(\log(1-X)+X\big)$ whose asymptotic expansion gives the so-called "virtual Euler characteristics" of moduli spaces of punctured Riemann surfaces. These are positive rational numbers, which may be non-integer due to the orbifold structure of moduli spaces.

It was demonstrated in [1] that (0.7) is a model that describes in a natural way the intersection indices for the case of d.m.s. The only complication is that this model does not present generating function for the indices (0.4) straightforwardly because of the contribution from reductions. Indeed, any matrix model can deal with only open strata of a moduli space. It was not essential for the case of the Kontsevich model since there the integration went over cells of the highest dimension in the simplicial complex partition of the moduli space $\overline{\mathcal{M}}_{g,n}$. All singular points are simplices of lower dimensions in

$\overline{\mathcal{M}}_{g,n}$ and give no contribution to the integral. But in the case of d.m.s. integrals over simplices of all dimensions are relevant due to the total discretization, so the integrals over reduced surfaces give nonzero contribution, which we should exclude in order to compare with the matrix model. The way to do it is to use a stratification procedure [21], which permits to express open moduli space $\mathcal{M}_{g,n}$ via $\overline{\mathcal{M}}_{g,n}$ and moduli spaces of lower genera.

In paper [12] the explicit solution to the model (0.7), or, equivalently, to the general one–matrix model was found in genus expansion. The key role in this consideration was played by the so-called "momenta" of the potential resembling in many details "momenta" that appeared in the genus expansion solution to the Kontsevich model [16]. We shall use some proper reexpansion of these momenta in terms of new times $T^{\pm}_{2n} = \mathrm{tr}\,\frac{\partial^{2n}}{\partial\lambda^{2n}}\frac{1}{\mathrm{e}^{\lambda}\pm 1}$ which stand just by the intersection indices (0.1), (0.4).

In the present paper we succeeded in finding and solving constraint equations for the model (1.C) in terms of times $T^{\pm}_{2n}$. It appears that the model is readily expressed as a *product* of two Kontsevich models in times $t^{\pm}_{2n+1} = \pm(2n+1)!!T^{\pm}_{2n}$ intertwined by a mixing operator $\mathcal{A}$ that has the form of a canonical transformation expressed in terms of free fields of some conformal field theory. No limiting procedure is needed.

The paper is organized as follows:

Section 1 contains main notations and assertions of the paper. Also in Section 1 we solve constraint equations arising from Schwinger–Dyson equations in terms of times for d.m.s. We discuss the algebra of constraint equations and find the relation between the Kontsevich and the matrix model for d.m.s. For technical details of the proof, see Appendix A. In Appendix B one can find solutions of the constraint equations to few lowest orders of perturbation theory. A short review of the geometric approach to the Kontsevich model is given in Section 2. The definition of the discretized moduli spaces and the corresponding matrix model are presented in Section 3. Review of the previous results on explicit solutions of the Kontsevich, one-matrix model, and the model for d.m.s. is contained in Section 4. A detailed description of the simplest modular space, $\mathcal{M}_{1,1}$, is contained in Section 5. Eventually, Section 6 contains a short summary of results and perspectives.

1 Main results.

1.1 Notations.

Let g and n be integers satisfying the conditions

$$g \geq 0, \quad n > 0, \quad 2 - 2g - n < 0.$$

Denote by $\mathcal{M}_{g,n}$ the moduli space of smooth complete complex curves C of genus g with n distinct marked points $x_1, \ldots, x_n$ and by $\overline{\mathcal{M}}_{g,n}$ — a smooth compactification of Deligne–Mumford type. (The concrete scheme of this compactification will be discuss below.)

Let $\mathcal{L}_i$, $i = 1, \ldots, n$ be line bundles on $\overline{\mathcal{M}}_{g,n}$. The fiber of $\mathcal{L}_i$ at $(C, x_1, \ldots, x_n)$ is the cotangent space $T^*_{x_i}C$.

Let $d_1, \ldots, d_n$ be non-negative integers satisfying

$$\sum_{i=1}^{n} d_i = \dim_{\mathbf{C}} \overline{\mathcal{M}}_{g,n} = 3g - 3 + n,$$

and denote by $\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g$ the intersection index:

$$\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} c_1(\mathcal{L}_i)^{d_i},$$

where $c_1(\mathcal{L}_i)$ are first Chern classes of the corresponding line bundles taken in the moduli space $\overline{\mathcal{M}}_{g,n}$.

All matrix integrals are assumed to be integrals over Hermitian $N \times N$ matrices with the standard measure $DX = \prod_{i<j}^{N} d\Re X_{ij} d\Im X_{ij} \prod_{i=1}^{N} dX_{ii}$.

1.2 Main results.

The main result by M.Kontsevich is the following

Theorem 1.1. (M. Kontsevich [2]) *Considering matrix integrals over Hermitian matrices $N \times N$ as asymptotic expansions in times $T_n = (n-1)!!\,\mathrm{tr}\,\frac{1}{\lambda^{n+1}}$ we obtain*

$$\sum_{g=0}^{\infty} \sum_{\substack{n=1,\ g>0\\ n=3,\ g=0}}^{\infty} \frac{1}{(\alpha N)^{2g-2+n}} \sum_{\Sigma\, d_i = 3g-3+n} \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g \frac{1}{n!} \prod_{i=1}^{n} T_{2d_i+1} =$$
$$= \log \frac{\int DX\, \mathrm{e}^{-\alpha N\,\mathrm{tr}\left(\frac{X^2\Lambda}{2} + \frac{X^3}{6}\right)}}{\int DX\, \mathrm{e}^{-\alpha N\,\mathrm{tr}\,\frac{X^2\Lambda}{2}}} = \mathcal{F}_K(T_1, T_3, \ldots),$$
$$\Lambda = diag\{\lambda_1, \ldots, \lambda_N\}. \qquad (1.1)$$

It was E.Witten who first proposed that these intersection indices are nothing but correlators of 2D topological gravity. More, it was conjectured in [3] and proved in [22], [2] that these integrals satisfy equations of KdV hierarchy for times T_n.

Consequently, in paper [1], we introduced a discretization of moduli spaces $\overline{\mathcal{M}}_{g,n}$ and corresponding intersection indices

$$\langle\!\langle \tau_{d_1} \ldots \tau_{d_n} \rangle\!\rangle_g \simeq \int_{\overline{\mathcal{M}}_{g,n}^{disc}} \prod_{i=1}^{n} \tilde{c}_1(\mathcal{L}_i)^{d_i}$$

which can be non-zero even for $\sum_i d_i < d \equiv 3g-3+n$ due to the discrete nature of this integral. One more complication is due to the necessity to integrate over the proper *closure* of the moduli space. In the continuum case it does not lead to any trouble since we may not concretize a compactification procedure (one can choose the one due to Deligne and Mumford [21]). The situation is different in the discrete case where singular curves give non-zero contributions to the integrals. Since there are no matrix model graphs corresponding to these singular curves we have to eliminate them using the stratification procedure which permits to "imitate" integrals over "open" part of discrete moduli spaces.

Note, however, that apart from three-valent graphs, as in the Kontsevich case, we take into account explicitly graphs with vertices of *arbitrary* valence since they all contribute in the discrete case. These curves are non-singular. In this paper we prove the following

Theorem 1.2. *In the compactification scheme consistent with the Strebel parametrization of the moduli space $\mathcal{M}_{g,n}$ the following asymptotic expansion in terms of times*

$$T_k^{\pm} = \tfrac{1}{(k+1)!}\,\mathrm{tr}\,\frac{\partial^k}{\partial \lambda^k} \frac{1}{\mathrm{e}^{\lambda} \pm 1} \qquad (1.2a)$$

is valid:

$$\sum_{g=0}^{\infty} \sum_{\substack{n=1,\ g>0\\ n=3,\ g=0}}^{\infty} \frac{1}{(\alpha N)^{2g-2+n}} \sum_{\substack{reductions\\ q-component}} c_{g,n,r_q} \left(-\frac{1}{2}\right)^{|r_q|} 2^{1-q} \prod_{j=1}^{q} \Bigg\{ \sum_{\sum d_\xi = 3g_j - 3 + n_j + k_j} \frac{1}{n_j!} \times$$
$$\times \langle\!\langle \underbrace{\tau_{d_1} \ldots \tau_{d_{n_j}}}_{n_j} \underbrace{\tau_0 \ldots \tau_0}_{k_j} \rangle\!\rangle_{g_j} \left(\prod_{k=1}^{n_j} (2d_j+1)!!\, T_{2d_j}^{-} + (-1)^{n_j} \prod_{k=1}^{n_j} (2d_j+1)!!\, T_{2d_j}^{+} \right) \Bigg\} =$$
$$= \log \frac{\int DX\, \mathrm{e}^{-\alpha N\,\mathrm{tr}\left(\frac{1}{4}\Lambda X \Lambda X + \frac{1}{2}\log(1-X) + \frac{1}{2}X\right)}}{\int DX\, \mathrm{e}^{-\alpha N\,\mathrm{tr}\left(\frac{1}{4}\Lambda X \Lambda X - \frac{1}{4}X^2\right)}} = \mathcal{F}_{KP}(\{T_{2n}^{\pm}\}),$$

$$\Lambda = diag\{\, e^{\lambda_1}, \ldots, e^{\lambda_N}\}. \tag{1.2}$$

Here the sum runs over all reductions, c_{g,n,r_q} are positive rational numbers – coefficients of reductions, $|r_q|$ is non-negative integer, the power of reduction, $0 \le |r_q| \le 3g-3+n$, and, eventually, q is the number of components of the singular curve, q varies from 1 to $2g-2+n$. Insertions of k_j additional τ_0 in the correlation function are due to reductions.

The first term in (1.2), $|r_q| = 0$, $c_{g,n,r_q} = 1$ and $q = 1$ corresponds to the "highest" non-reduced modular space $\overline{\mathcal{M}}^{disc}_{g,n}$ from which we subtract integrals over reductions of the first power, which are again some closed moduli spaces of lower overall dimension, from which we are to subtract integrals obtained in points of intersections of these singular curves, et cetera.

Our picture differs at this point from the one by the Deligne–Mumford compactification, where singular curve subspaces have symmetry groups of infinite orders. Therefore, in the Deligne–Mumford closure of the moduli space, these curves give no contribution to (1.2) since reduction coefficients c_{g,n,r_q} are inversely proportional to the orders of the corresponding symmetry groups. It is not the case for the model (1.2) corresponding to the Penner–Strebel coordinatization picture.

The matrix model on the R.H.S. of (1.2) was introduced in [17] where it was shown that it is exactly solvable in $1/N$ expansion. It turns out [19], [20] that with changed normalization factor it is equivalent to the Hermitian one-matrix model with general potential:

$$\begin{aligned}
&\int DX \exp\left\{-\alpha N \,\mathrm{tr}\,\left(\tfrac{1}{4}\Lambda X \Lambda X - \tfrac{1}{2}\log(1-X) - \tfrac{X}{2}\right)\right\} = \\
&\quad = (\det\Lambda)^{-N-\alpha N/2}\, e^{-\frac{\alpha N}{4}\,\mathrm{tr}\,\Lambda^2}\, e^{N\,\mathrm{tr}\,(\Lambda+\Lambda^{-1})^2} \int_{\frac{\alpha N}{2}\times\frac{\alpha N}{2}} DY\; e^{-U(Y)}, \\
&U(Y) = \sum_{n=1}^{\infty} \xi_n \,\mathrm{tr}\, Y^n, \quad \xi_n = \frac{1}{k}\,\mathrm{tr}\,\frac{1}{(\Lambda+\Lambda^{-1})^n} - N\delta_{n,2},
\end{aligned} \tag{1.A}$$

where integral on the R.H.S. is done over Hermitian matrices of modified size $\frac{\alpha N}{2} \times \frac{\alpha N}{2}$ and we had to change the sign standing by the logarithmic term in order to keep this dimension positive. (In asymptotic expansion in $1/N$ one may easily make a transition back to "negative" dimensions by replacing $N \to -N$.)

As we deal with the general one-matrix model, we know that it obeys the equations of discrete Toda chain hierarchy in terms of times ξ_n [5].

Note, however, that these times ξ_n are of no use if we deal with times $T^{\pm}_{2n}$ because their singularities are at different values of λ. For ξ_n they are $\lambda = i\pi/2 + i\pi k,\ \ k \in \mathbf{Z}$, and for $T^{\pm}_n$ they are $i\pi(2k+1)$ for T^+ and $2i\pi k$ for $T^-,\ \ k \in \mathbf{Z}$.

Now we are going to discuss relations between the Kontsevich matrix model (1.1) and the model introduced in (1.2). We assume in what follows that the asymptotic expansion of (1.2) will be done in proper times $T^{\pm}_{2n}$. An intermediate lemma states

Lemma 1.2a. *Partition function of the matrix model* (1.2) $\mathcal{F}_{KP}(\{T^{\pm}_{\cdot}\})$ *depends only on even times* $T^{\pm}_{2n} = \frac{1}{(2n+1)!}\,\mathrm{tr}\,\frac{\partial^{2n}}{\partial\lambda^{2n}}\frac{1}{e^{\lambda}\pm 1}$

The procedure of the *double scaling limit* (d.s.l.) permits to obtain the Kontsevich matrix model (1.1) starting from one-matrix model (1.A). Moreover, it is much easier to get (1.1) from the model(1.2) than from the one-matrix model.

Namely, let us rescale integration variables in (1.2) as follows:

$$\begin{aligned}
\alpha &\to \alpha\varepsilon^{-3} \\
X &\to \varepsilon X \\
e^{\lambda} &\to e^{\varepsilon\lambda}
\end{aligned} \tag{1.B}$$

In the limit $\varepsilon \to 0$ only those terms survive in the action of model (1.2) that give the Kontsevich action (1.1). It corresponds to the d.s.l. of one-matrix model with asymmetric potential. If we are going to consider the d.s.l. in the model with the symmetric potential where all odd ξ_n are zero, then we have to choose a block–diagonal form of the external field matrix $\Lambda = \mathrm{diag}\,\left(\mathrm{e}^{\lambda_1}, \dots, \mathrm{e}^{\lambda_{N/2}}, \mathrm{e}^{-\lambda_1}, \dots, \mathrm{e}^{-\lambda_{N/2}}\right)$. Then in the limit $\varepsilon \to 0$ we get two independent Kontsevich integrals over half-dimensional Hermitian matrices $N/2 \times N/2$ [12] taken with the *same* external field matrix $\widetilde{\Lambda} = \mathrm{diag}\,\left(\mathrm{e}^{\lambda_1}, \dots, \mathrm{e}^{\lambda_{N/2}}\right)$.

These relations were due to some *limiting procedure* which lead to some loss of information encoded in the model (1.2). It turns out that there exists a *exact relation* between models (1.1) and (1.2). In the present paper we prove the following theorem exactly solving the set of constraint equations:

Theorem 1.3. *Partition function of* (1.2) *and the Kontsevich model* (1.1) *satisfy an exact relation:*

$$\mathrm{e}^{\mathcal{F}_{KP}(\{T^{\pm}_{2n}\})} = \mathrm{e}^{C(\alpha N)}\,\mathrm{e}^{-\mathcal{A}}\,\mathrm{e}^{\mathcal{F}_K(\{\xi^+_{2n+1}\})+\mathcal{F}_K(\{\xi^-_{2n+1}\})} \tag{1.3}$$

where $\xi^{\pm}_{2n+1} = \pm(2n+1)!!T^{\pm}_{2n}$ *and* $\mathcal{A}$ *is a quadratic differential operator in* $\partial/\partial T^{\pm}_{\cdot}$

$$\begin{aligned}\mathcal{A} \;=\; & \sum_{m,n=0}^{\infty} \frac{B_{2(n+m+1)}}{4(n+m+1)}\frac{1}{(2n+1)!\,(2m+1)!}\times \\ & \times\left\{\frac{\partial}{\partial T^+_{2m}}\frac{\partial}{\partial T^+_{2n}} + \frac{\partial}{\partial T^-_{2m}}\frac{\partial}{\partial T^-_{2n}} - 2(2^{2(n+m+1)}-1)\frac{\partial}{\partial T^+_{2m}}\frac{\partial}{\partial T^-_{2n}}\right\} \\ & +\sum_{n=2}^{\infty}\alpha\,\frac{2^{2n-1}}{(2n+1)!}\left(\frac{\partial}{\partial T^-_{2n}} - \frac{\partial}{\partial T^+_{2n}}\right).\end{aligned}$$

$C(\alpha N)$ is a function depending only on αN that ensures that $\mathcal{F}_{KP}(\{T^{\pm}_{2n}\}) = 0$, where $T^{\pm}_{2n} \equiv 0$. B_{2k} are Bernoulli numbers.

1.3 Constraint equations for the model (1.2)

In this subsection we present a sketch of the proof for Theorem 1.3 together with the algebra of constraints to the matrix model (1.2). (The complete proof is contained in Appendix A.)

In what follows we treat all times (1.2a) as *independent* variables. Just like in the Kontsevich model, times $\{T^+_k\}$ (respectively, $\{T^-_k\}$ become independent as $N \to \infty$. However, there are mixing relations for these two sets that are valid (at least, formally) for all N. For instance, taking into account the pole structure of times (1.2a) in λ variables we have

$$t^+_0 = \mathrm{tr}\,\frac{1}{\mathrm{e}^{\lambda}+1} = -\,\mathrm{tr}\,\frac{1}{\mathrm{e}^{\lambda+i\pi}-1} = -\sum_{k=0}^{\infty}(k+1)(i\pi)^k T^-_k.$$

These expressions always include infinite sums and, since the answer in (1.2) for finite g and n has finite polynomial structure, it is unique in terms of times $\{T^{\pm}_{2k}\}$. Therefore, in asymptotic expansion over N and α we can treat all these times as independent variables.

We begin with the matrix integral

$$w(\mathrm{e}^{\lambda}) = \log\left(\frac{\int DX\,\exp -\alpha N\,\mathrm{tr}\,\left(\frac14\Lambda X\Lambda X + \frac12\log(1-X)+\frac12 X\right)}{\int DX\,\exp -\alpha N\,\mathrm{tr}\,\left(\frac14\Lambda X\Lambda X - \frac14 X^2\right)}\right), \quad \Lambda \equiv \mathrm{e}^{\lambda}. \tag{1.C}$$

Integrating out angular variables we remain with the integral over eigenvalues x_i of the matrix X, for which we can write down the Schwinger–Dyson equations in terms of λ_i (A.13). After a subtle algebra these equations can be reformulating in terms of times $\{T^{\pm}_{2k}\}$ alone. The constraints acquire the form:

$$\sum_{k=0}^{\infty}\widetilde{t}^+_k(\lambda_j)\{\widetilde{L}^+_{k-1}\,\mathrm{e}^{w(\lambda)}\} + \sum_{k=0}^{\infty}\widetilde{t}^-_k(\lambda_j)\{\widetilde{L}^-_{k-1}\,\mathrm{e}^{w(\lambda)}\} = 0. \tag{1.D}$$

Here

$$\widetilde{t}_n^{\pm}(\lambda) = \frac{2^{2n+1}}{(2n+1)!}\left[(2n+1)\frac{\partial}{2\partial\lambda}B_{2n}\left(\frac{\partial}{2\partial\lambda}\right) - 2nB_{2n+1}\left(\frac{\partial}{2\partial\lambda}\right)\right]\frac{1}{\mathrm{e}^{\lambda}\pm 1},$$

where $B_n(x) = \sum_{s=0}^{n}\binom{s}{n}B_s x^{n-s}$ are Bernoulli polynomials, B_s being Bernoulli numbers. As we treat all times as independent, we get from (1.D) two independent sets of constraints on $\mathrm{e}^{w(\lambda)}$:

$$\widetilde{L}_k^{\pm}\,\mathrm{e}^{w(\lambda)} = 0, \quad k \geq -1.$$

Here $\widetilde{L}_{-1}^{\pm}$ is given by (A.36) and $\widetilde{L}_s^{\pm}$, $s \geq 0$, by (A.34).

The constraint operators $\widetilde{L}_k^{\pm}$ satisfy two halves of Virasoro algebra:

$$\begin{aligned}
&[\widetilde{L}_s^{\pm}, \widetilde{L}_t^{\pm}] = \frac{4}{\alpha^2}(s-t)\widetilde{L}_{s+t}^{\pm}, \quad s,t \geq -1,\\
&[\widetilde{L}_s^{+}, \widetilde{L}_t^{-}] = 0.
\end{aligned}$$

Let us introduce creation–annihilation operators as follows (omitting $\pm$ signs):

$$\begin{aligned}
a_{-m-\frac{1}{2}} &= \frac{1}{2}\frac{\partial}{\partial T_{2m}}, \quad m \geq 0,\\
a_{m+\frac{1}{2}} &= \left(m+\frac{1}{2}\right)T_{2m}, \quad m \geq 0,
\end{aligned}$$

with corresponding commutation relations (for half-integer $\mu, \nu \in \frac{1}{2} + \mathbf{Z}$):

$$[a_\mu, a_\nu] = -\tfrac{1}{2}\mu\delta_{\mu+\nu,0}.$$

The vacuum $|0\rangle$ is annihilated by a_s with $s < 0$.

We find the operator $\mathcal{A}$ (A.39) of canonical transformation:

$$\begin{aligned}
\widehat{a}_\mu^{\pm} &= \mathrm{e}^{-\mathcal{A}}a_\mu^{\pm}\,\mathrm{e}^{\mathcal{A}}\\
\mathcal{L}_s^{\pm} &= \mathrm{e}^{-\mathcal{A}}\widetilde{L}_s^{\pm}\,\mathrm{e}^{\mathcal{A}}, \quad s \geq -1,
\end{aligned}$$

which completely split the dependence on "left" and "right" times in the Virasoro generators $\mathcal{L}_s^{\pm}$

$$\mathcal{L}_s^{\pm} = \sum_{m=-\infty}^{\infty} :\widehat{a}^{\pm}_{m+\frac{1}{2}}\widehat{a}^{\pm}_{-m-s-\frac{1}{2}}: + \frac{\delta_{s,0}}{16} - (\alpha N)\widehat{a}^{\pm}_{-\frac{3}{2}-s} + \frac{(\alpha N)^2}{2}\delta_{s,-3}. \tag{1.E}$$

Here the normal ordering $:\cdot:$ is defined w.r.t. the vacuum $|0\rangle$.

These generators are nothing but Virasoro generators in the Kontsevich model (1.1). Therefore we get the assertion of the Theorem 1.3.

One can interpret $\mathrm{e}^{\mathcal{F}_K(\{\xi^+_{2n+1}\})+\mathcal{F}_K(\{\xi^-_{2n+1}\})}|0\rangle$ as a conformal vacuum $|\,\mathrm{conf}\,\rangle$ of some $c = 1$ theory since it satisfies Virasoro conditions of the form:

$$\mathcal{L}_s^{\pm}|\,\mathrm{conf}\,\rangle = 0, \quad s \geq -1, \tag{1.F}$$

where $\mathcal{L}_s^{\pm}$ obey two independent Virasoro algebra relations,

$$\begin{aligned}
&[\mathcal{L}_s^{\pm}, \mathcal{L}_t^{\pm}] = (s-t)\mathcal{L}_{s+t}^{\pm} + \frac{t(t^2-1)}{12}\delta_{s+t,0},\\
&[\mathcal{L}_s^{+}, \mathcal{L}_t^{-}] = 0.
\end{aligned}$$

$\pm$ are now related to the left and right moving sectors, which split completely on the level of Virasoro algebra. The Kontsevich integral makes a transition from a constant vacuum field $|0\rangle$, which is annihilated by a_μ with $\mu < 0$, ($\mathcal{L}_s|0\rangle = \frac{1}{16}\delta_{s,0}$ for $s \geq 0$), to a conformal vacuum $|\,\mathrm{conf}\,\rangle$, which satisfies left and right Virasoro conditions (1.F).

The operator $\mathcal{A}$ can be presented in an integral form. We assume that the numbers $\pm(\alpha N)$ are related to eigenvalues of the momentum operators:

$$p_{\pm}|0\rangle = \pm\frac{\alpha N}{2}|0\rangle.$$

Let us introduce a two-component bosonic field $\Phi(T,\lambda)$

$$\Phi(T,\lambda) = \begin{pmatrix} \phi(T^{+}_{\cdot},\lambda) \\ \phi(T^{-}_{\cdot},\lambda) \end{pmatrix},$$

where

$$\phi(T^{\pm}_{\cdot},\lambda) = \sum_{n=0}^{\infty} T^{\pm}_{n}\lambda^{n+1} + x_{\pm} + p_{\pm}\log\lambda + \sum_{n=0}^{\infty}\frac{\lambda^{-n-1}}{n+1}\frac{\partial}{\partial T^{\pm}_{n}}.$$

Here the sum runs over all times, not necessarily even, derivatives act on the right. The corresponding currents, $\partial\phi(T^{\pm},\lambda)$ have normal ordering relations $\langle\partial\phi(\lambda)\partial\phi(\mu)\rangle \sim \frac{1}{(\lambda-\mu)^2}$.

Then the operator $\mathcal{A}$ has the form:

$$\mathcal{A} = \oint\frac{d\lambda}{2\pi i}\oint\frac{d\mu}{2\pi i}\Phi^{T}(T,\lambda)\mathbf{A}(\lambda+\mu)\Phi(T,\mu), \tag{1.G}$$

where $\mathbf{A}(\lambda+\mu)$ is the following 2×2 matrix:

$$\mathbf{A}(y) = \begin{bmatrix} \log\frac{1-\mathrm{e}^{-y}}{y} + \frac{1}{2}\sinh 2y - \frac{2}{3}y^3 & \log(1+\mathrm{e}^{-y}) \\ \log(1+\mathrm{e}^{-y}) & \log\frac{1-\mathrm{e}^{-y}}{y} + \frac{1}{2}\sinh 2y - \frac{2}{3}y^3 \end{bmatrix} \tag{1.H}$$

This expression contains ambiguities. In fact, only symmetrical with respect to the change of variables $y \to -y$ part of the matrix $\mathbf{A}(y)$ is rigidly fixed. The antisymmetrical part mixes odd and even time derivatives, and, therefore, gives zero when it acts on $\mathrm{e}^{\mathcal{F}_K(\{T^{\pm}_{2n}\})}|0\rangle$. The only non-zero contribution arises when mixing of $\frac{\partial}{\partial T^{\pm}_{2n}}$ and $p^{\pm}$ occurs. (Two terms in the diagonal part, $\frac{1}{2}\sinh 2y - \frac{2}{3}y^3$, and linear in y part of $\log\frac{1-\mathrm{e}^{-y}}{y}$ are combined in such a way that linear in derivatives term of (1.3) appears.

2 Continuum Moduli Space $\mathcal{M}_{g,n}$.

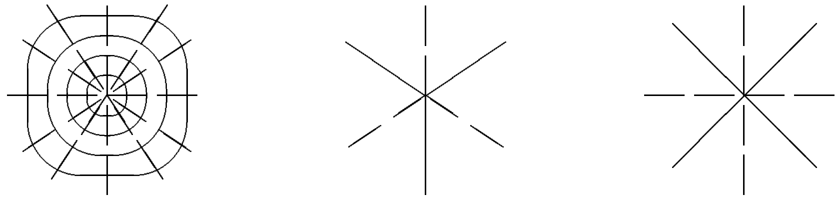
2.1 Moduli space of algebraic curves and its parametrization in terms of Strebel–Jenkins differentials.

We begin with an explicit coordinatization of the moduli space $\mathcal{M}_{g,n}$ using the results of K.Strebel. He established the equivalence between "decorated" moduli spaces of algebraic curves and moduli spaces of ribbon ("fat", "oriented") graphs (R.Penner, J.Harer, D.Mumford and W.Thurston).

A *quadratic differential* φ on a Riemann surface C is a holomorphic section of the line bundle $(T^{\star})^{\otimes 2}$. In local coordinates it defines a flat metric on the complement of the discrete set of its zeros and poles:

$$|\varphi(z)|\cdot|dz|^2, \quad \text{where } \varphi = \varphi(z)dz^2. \tag{2.1}$$

All poles of the Strebel differentials are double poles in points of punctures. More, the quadratic residues in double poles are strictly positive real numbers. Since by the Riemann theorem for quadratic differentials $\#\,\text{zeros} - \#\,\text{poles} = 4(g-1)$ (each is counted with its order), then for a general position point of $\mathcal{M}_{g,n}$ for the surface with n punctures there are $4g-4+2n$ simple zeros.



1a 1b 1c

Fig.1. Horizontal and vertical lines of Strebel metric in poles and zeros.

A *horizontal trajectory* (geodesic) of a quadratic differential is a curve, along which $\varphi(z)dz^2$ is real and positive. A *vertical trajectory* (geodesic), to the contrary, is the one along which $\varphi(z)dz^2$ is real and negative. Let us consider the system of horizontal and vertical lines in the vicinity of the double pole, where $\varphi(z) = \frac{p_i^2}{(2\pi)^2(z-z_i)^2}$ (Fig. 1a). It is easy to see that horizontal lines are concentric circles around the point z_i while vertical geodesics are half-lines with the summit at the same point. If we take a kth-order zero of $\varphi(z)$, then the situation is the following: there are exactly $k+2$ horizontal and $k+2$ vertical half-lines with the endpoint z_j (Fig. 1b for $k=1$ and Fig. 1c for $k=2$). For the general choice of the differential φ, horizontal trajectories are not closed, but *Jenkins–Strebel differentials* are those for which the union of non–closed horizontal trajectories has zero measure. These trajectories are just those that connect zeros of the differential. Moreover, they decompose the surface into simply connected pieces (faces) with exactly one puncture in each. And, eventually, the lengths of all horizontal trajectories belonging to one face are the same – they are equal p_i. K.Strebel proved the following:

Theorem 2.1. *For any connected Riemann surface C and n distinct points $x_1, \ldots, x_n \in C$, $n > 0$, $n > \xi(C)$ and n positive real numbers $p_1, \ldots, p_n$ there exists a unique JS quadratic differential on $C \backslash \{x_1, \ldots, x_n\}$ whose maximal ring domains are n punctured discs D_i, $x_i \in D_i$, with circumference p_i.*

The collection of these non–closed horizontal lines is an oriented graph corresponding to the Riemann surface. Orientation means that one may think about the edges of these graphs as strips with two sides belonging to two faces of the surface separated by this edge. (Looking forward, in matrix model technique, the index i will run along boundary line of a face.)

Thus, in the Strebel metric each face converts into half–infinite cylinder with the boundary consisting of borders of strips of the fat graph, i.e., the boundary is a polygon with the perimeter p_i for ith face.

What are the coordinates on the moduli space $\mathcal{M}_{g,n}$ in this picture? Let us supply the fat graph with additional data: we assign to each edge a positive real number $l_s \in \mathbf{R}^+$, s being the number of the edge. These l_s have the meaning of lengths of edges of the fat graph Γ_φ of a genus g. l_s define coordinates on $\mathcal{M}_{g,n}^{comb}$ – $6g-6+3n$-dimensional linear space of graphs. Then p_i are sums of lengths of edges incident to ith cycle.

Zeros of JS differential correspond to the vertices of the graph, the valence of the vertex being equal to $k+2$ for a kth-order zero. For a general case when all vertices are three valent the number of edges is equal to $6g-6+3n$, which is the real dimension of the moduli space $\mathcal{M}_{g,n}$ plus n additional parameters – the perimeters of the faces. In order to find coordinates on $\overline{\mathcal{M}}_{g,n}$ itself we must get rid of the dependence of these perimeters.

Now we consider a set of *all* graphs with fixed g and n endowed with metric described above. To any JS differential we associate some graph. The inverse statement is also valid: having a graph one can construct the Riemann surface endowed with JS differential structure whose residues are squares of perimeters of cycles on the graph. The set of all graphs modulo symmetry groups of graphs is a combinatorial moduli space $\mathcal{M}_{g,n}^{comb}$. The following statement holds true:

Theorem 2.2. *Let $\mathcal{M}_{g,n}^{comb}$ denote the set of equivalence classes of connected fat graphs with metric and with valency of each vertex greater or equal 3. The map $\mathcal{M}_{g,n} \otimes \mathbf{R}_+^n \to \mathcal{M}_{g,n}^{comb}$, which associates to the surface C and positive numbers $p_1, \ldots, p_n$ the critical graph of the canonical JS–differential, is* *one–to–one.*

Thus considering all graphs of the fixed g and n we obtain a stratification on $\mathcal{M}_{g,n}^{comb}$ with the dimensions of strata equal to the numbers of edges. The open strata correspond to three–valent graphs and have the dimension $6g-6+3n$.

We conclude this subsection with some notations from the graph theory necessary for what follows. For a fat graph Γ let X_0 denote the set of vertices, X_1 – the set of edges together with orientations defined for each edge, and X_2 – the set of faces of the graph. Let s_0 and s_1 be two permutations of X_1: s_1 changes the orientation of all edges simultaneously and is an involution, $s_1^2 = \mathrm{id}$. s_0 is defined as a rotation (clockwise, due to orientation) of edges incident to a vertex.

Note: These transformations s_0 and s_1 are generators of a cartographic group corresponding to a chosen graph. The complete cartographic group can be represented as follows: let all edges be divided into two halves, all these halves being numerated. Therefore, we have a finite set of $2 \times \#\,\mathrm{edges}$ elements. The transformations s_i, $i = 1, 2$ define a permutation group on this set. These permutations do not necessarily preserve the edges of the graph as a whole, thus the authomorphism group of the graph is always a subgroup of the cartographic group and only in very few cases these groups coincide.

2.2 Geometry of fiber bundles on $\overline{\mathcal{M}}_{g,n}$ and matrix integral.

M.Kontsevich proposed a procedure for finding intersection indices (or, equivalently, integrals of the first Chern classes) on the moduli spaces.

Let us consider a set of line bundles $\mathcal{L}_i$ whose fiber at a point $\Sigma \in \mathcal{M}_{g,n}$ is a cotangent space to the puncture point x_i on the surface Σ. The first Chern class $c_1(\mathcal{L}_i)$ of the line bundle $\mathcal{L}_i$ admits a representation in terms of lengths of the edges l_j. The perimeter of the boundary component is $p_i = \sum_{l_\alpha \in I_i} l_\alpha$.

The first step in constructing $c_1(\mathcal{L}_i)$ is to determine α_i, which is $U(1)$–connection on the boundary component corresponding to the ith puncture. Since we already have an explicit coordinatization of the moduli space, we need only to make a proper choice of this connection in terms of l_j. It is convenient to introduce "polygon bundles" for each face – $BU(1)_{(i)}^{comb}$ in Kontsevich's notations. These polygon bundles are sets of equivalent classes of all sequences of positive real numbers $l_1, \ldots, l_k$ modulo cyclic permutations.

$BU(1)^{comb}$ is the moduli (orbi)space of numbered ribbon graphs with metric whose underlying graphs are homeomorphic to the circle. There is an S^1–bundle over this orbispace whose total space $EU(1)^{comb}$ is an ordinary space. The fiber of the bundle over the equivalence class of sequences $l_1, \ldots, l_k$ is a union of intervals of lengths $l_1 \ldots, l_k$ with pairwise glued ends, i.e. a polygon. The inverse images of S^1–bundles are naturally isomorphic to the circle bundles associated with the complex line bundles $\mathcal{L}_i$.

Let us now compute the first Chern class of the circle bundle on $BU(1)^{comb}$. The points of $EU(1)^{comb}$ can be identified with pairs (p, S) where p is the perimeter and S is a nonempty finite subset (vertices) of the circle $\mathbf{R}/p\mathbf{Z}$. Let ϕ_i, $0 \le \phi_1 < \ldots < \phi_k < p$, be representatives of points of S. The lengths of the edges of the polygon are

$$l_i = \phi_{i+1} - \phi_i \quad (i = 1, \ldots, k-1), \quad l_k = p + \phi_1 - \phi_k. \tag{2.2}$$

A convenient form for S^1–connections on these polygon bundles is provided by 1-form α on $EU(1)^{comb}$:

$$\alpha = \sum_{i=1}^{k} \frac{l_i}{p} \times d\left(\frac{\phi_i}{p}\right). \tag{2.3}$$

α is well–defined and the integral of it over each fiber of the universal bundle $EU(1)^{comb} \to BU(1)^{comb}$

is equal to -1. The differential $d\alpha$ is the pullback of a 2–form ω on the base $BU(1)^{comb}$,

$$\omega = \sum_{1\le i<j\le k-1} d\left(\frac{l_i}{p}\right)\wedge d\left(\frac{l_j}{p}\right). \tag{2.4}$$

Extrapolating these results to the compactified moduli spaces we obtain that the pullback ω_i of the form ω under the ith map $\overline{\mathcal{M}}_{g,n}\times \mathbf{R}_+^n \to BU(1)^{comb}$ represents the class $c_1(\mathcal{L}_i)$.

Let us denote by $\pi : \mathcal{M}_{g,n}^{comb} \to \mathbf{R}_+^n$ the projection to the space of perimeters. Intersection indices are given by the formula:

$$\langle \tau_{d_1}\dots\tau_{d_n}\rangle = \int_{\pi^{-1}(p_*)}\prod_{i=1}^{n}\omega_i^{d_i}, \tag{2.5}$$

where $p_* = (p_1,\dots,p_n)$ is an arbitrary sequence of positive real numbers and $\pi^{-1}(p_*)$ is a fiber of $\overline{\mathcal{M}}_{g,n}$ in $\mathcal{M}_{g,n}^{comb}$.

We denote by Ω the two–form on open strata of $\mathcal{M}_{g,n}^{comb}$:

$$\Omega = \sum_{i=1}^{n} p_i^2 c_1(\mathcal{L}_i), \tag{2.6}$$

whose restriction to the fibers of π has constant coefficients in coordinates $\big(l(e)\big)$. Denote by d the complex dimension of $\mathcal{M}_{g,n}$, $d = 3g-3+n$. The volume of the fiber of π with respect to Ω is

$$\begin{aligned}\mathrm{vol}\big(\pi^{-1}(p_1\dots,p_n)\big) &= \int_{\pi^{-1}(p_*)}\frac{\Omega^d}{d!} = \frac{1}{d!}\int_{\pi^{-1}(p_*)}\big(p_1^2c_1(\mathcal{L}_1)+\dots+p_n^2c_1(\mathcal{L}_n)\big)^d = \\ &= \sum_{\sum d_i = d}\prod_{i=1}^{n}\frac{p_i^{2d_i}}{d_i!}\langle\tau_{d_1}\dots\tau_{d_n}\rangle_g. \end{aligned}\tag{2.7}$$

One important note is in order. It is a theorem by Kontsevich that these integrations extend continuously to the closure of the moduli space $\overline{\mathcal{M}}_{g,n}$, following the procedure by Deligne and Mumford [21]. (It means that we deal with a stable cohomological class of curves.) It is obligatory to consider a closure of the moduli space because all intersections can be consistently defined only on compact spaces. But it does not change insomuch our consideration until we integrate over continuum moduli space where nonzero contributions are given only by integrations over higher dimensional cells. All additional simplices which we add in order to close $\mathcal{M}_{g,n}$ are of lower dimensions. However, they will play a crucial role in the case of discrete moduli spaces.

In order to compare with a matrix model we should take the Laplace transform over variables p_i of volumes of fibers of π:

$$\int_0^\infty dp_i\,\mathrm{e}^{-p_i\lambda_i}p_i^{2d_i} = (2d_i)!\lambda_i^{-2d_i-1}, \tag{2.8}$$

for the quantities standing on the right–hand side of (2.7). On the left–hand side we have

$$\int_0^\infty\dots\int_0^\infty dp_1\wedge\dots\wedge dp_n\,\mathrm{e}^{-\sum p_i\lambda_i}\int_{\overline{\mathcal{M}}_{g,n}}\mathrm{e}^{\Omega}, \tag{2.9}$$

and, due to cancellations of all p_i^2 multipliers with p_i standing in denominators of the form Ω, we get:

$$\mathrm{e}^{\Omega}dp_1\wedge\dots\wedge dp_n = \rho\prod_{e\in X_1}dl_e. \tag{2.10}$$

We use standard notations: X_q is the total number of q–dimensional cells of a simplicial complex. (X_1 is the number of edges, X_0 – the number of vertices, etc). ρ is a positive function defined on open cells:

$$\rho = \left(\prod_{i=1}^{n}|dp_i|\times\frac{\Omega^d}{d!}\right) : \prod_{e\in X_1}|dl(e)|. \tag{2.11}$$

Surprisingly, the constant ρ in fact depends only on Euler characteristic of the graph Γ, $\rho = 2^{-\kappa}$,

$$\rho = 2^{d+\#X_1-\#X_0}. \tag{2.12}$$

The integral

$$I_g(\lambda_*) := \int_{\mathcal{M}_{g,n}^{comb}} \exp\left(-\sum \lambda_i p_i\right) \prod_{e\in X_1} |dl(e)| \tag{2.13}$$

is equal to the sum of integrals over all open strata in $\mathcal{M}_{g,n}^{comb}$. These open strata are in one–to–one correspondence with a complete set of three–valent graphs contributing to this order in g and n. It is also necessary to take into account internal automorphisms of the graph (their number, in fact, shows how many replica of moduli space one may find in the cell). The last step is to present the sum $\sum \lambda_i p_i$ in a form dependent on l_e:

$$\sum_{i=1}^{n} \lambda_i p_i = \sum_{e\in X_1} l_e(\lambda_e^{(1)} + \lambda_e^{(2)}) \tag{2.14}$$

for each graph. Here $\lambda_e^{(1)}$ and $\lambda_e^{(2)}$ are variables of two cycles divided by eth edge. Performing now the Laplace transform we get the celebrated relation [2] (Theorem 1.1):

$$\sum_{d_1,\ldots,d_n=0}^{\infty} <\tau_{d_1},\tau_{d_2},\ldots,\tau_{d_n}> \prod_{i=1}^{n}(2d_i-1)!!\lambda_i^{-(2d_i+1)} = \sum_{\Gamma} \frac{2^{-\#X_0}}{\#\,\mathrm{Aut}\,(\Gamma)} \prod_{\{ij\}} \frac{2}{\lambda_i+\lambda_j}, \tag{2.15}$$

where the objects standing in angular brackets on the left–hand side are (rational) numbers describing intersection indices, and on the right–hand side the sum runs over all oriented connected trivalent "fat-graphs" Γ with n labeled boundary components, regardless of the genus, $\#X_0$ is the number of vertices of Γ, the product runs over all the edges in the graph and $\#\,\mathrm{Aut}$ is the volume of discrete symmetry group of the graph Γ.

The amazing result by Kontsevich is that the quantity on the right hand side of (2.15) is equal to the free energy in the following matrix model:

$$\mathrm{e}^{\,F_N(\Lambda)} = \frac{\int dX\ \exp\alpha N\left(-\frac{1}{2}\,\mathrm{tr}\,\Lambda X^2 + \frac{1}{6}\,\mathrm{tr}\,X^3\right)}{\int dX\ \exp\alpha N\left(-\frac{1}{2}\,\mathrm{tr}\,\Lambda X^2\right)}, \tag{2.16}$$

where X is an $N\times N$ hermitian matrix and $\Lambda = \mathrm{diag}\,(\lambda_1,\ldots,\lambda_N)$. α is an additional parameter enumerating the boundary components. Though each selected diagram has quantities $(\lambda_i+\lambda_j)$ in the denominator, when taking a sum over all diagrams of the same genus and the same number of boundary components all these factors are canceled with the ones from nominator.

Feynman rules for the Kontsevich matrix model are the following: as in usual matrix models, we deal with the so-called "fat graphs" or "ribbon graphs" with propagators having two sides, each carries corresponding index. The Kontsevich model varies from the standard one–matrix hermitian model by additional variables λ_i associated with index loops in the diagram, the propagator being equal to $2/(\lambda_i+\lambda_j)$, where λ_i and λ_j are variables of two cycles (perhaps the same cycle) incident to two sides of the propagator. There are also trivalent vertices presenting the cell decomposition of the moduli space.

Let us consider the simplest example of genus zero and three boundary components which we symbolically label λ_1, λ_2 and λ_3. There are two kinds of diagrams giving contribution into this order (Fig.2). The contribution to the free energy arising from this sum is

$$\begin{aligned}
& \frac{1}{6(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)} + \frac{1}{3}\left\{\frac{1}{4\lambda_1(\lambda_2+\lambda_1)(\lambda_3+\lambda_1)} + \right.\\
+ & \left.(1\to 2, 2\to 3, 3\to 1) + (1\to 3, 3\to 2, 2\to 1)\right\}\\
= & \frac{2\lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3(\lambda_2+\lambda_3) + \lambda_1\lambda_3(\lambda_1+\lambda_3) + \lambda_1\lambda_2(\lambda_1+\lambda_2)}{12\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}\\
= & \frac{1}{12\lambda_1\lambda_2\lambda_3}.
\end{aligned} \tag{2.17}$$

This example demonstrates the cancellations of $(\lambda_i + \lambda_j)$–terms in the denominator.

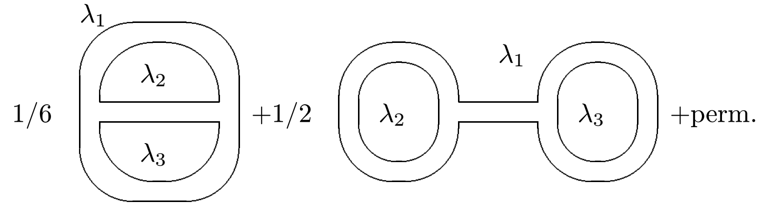


Fig.2. The g=0, s=3 contribution to the Kontsevich model

The quantity standing in the R.H.S. of (2.15) is nothing but a term from $1/N$ expansion of the Kontsevich matrix model. Eventually, we have:

$$\sum_{\substack{g=0\\n=1}}^{\infty} N^{2-2g}\alpha^{2-2g-n} \sum_{s_1+2s_2+\ldots+ks_k=d} \langle(\tau_0)^{s_0}\ldots(\tau_k)^{s_k}\rangle_g \frac{1}{s_0!\ldots s_k!}\prod_{i=1}^{n} \mathrm{tr}\,\frac{(2d_i-1)!!}{\Lambda^{2d_i+1}}$$

$$= \log\frac{\int_{N\times N} DX \exp\left\{N\alpha\,\mathrm{tr}\left(-\frac{X^2\Lambda}{2}+\frac{X^3}{6}\right)\right\}}{\int_{N\times N} DX \exp\left\{N\alpha\,\mathrm{tr}\left(-\frac{X^2\Lambda}{2}\right)\right\}} \tag{2.18}$$

Thus, the Kontsevich matrix model provides a generating function for the intersection indices of the first Chern classes on the moduli (orbi)spaces.

Let us introduce a new important object – the 2–vector β which is defined on the higher dimension cells of $\mathcal{M}_{g,n}$:

$$\beta = \frac{1}{2}\sum_{x\in X_1}\frac{\partial}{\partial l(x)}\wedge\frac{\partial}{\partial l(s_0(x))}. \tag{2.19}$$

Here s_0 is an authomorphism of $\mathcal{M}_{g,n}^{comb}$ which "rotates" each edge x by $\frac{2}{3}\pi$ clockwise over a vertex. This 2–vector defines a Poisson structure on the cells of the higher dimension in $\mathcal{M}_{g,n}^{comb}$. In order to see it let us calculate its kernel:

Proposition 2.1 $\mathrm{Ker}\,\beta = \pi^* T^* \mathbf{R}_+^n$.

Proof. $\mathrm{Ker}\,\beta$ is a space of functions on the edges of the graph such that the following relation holds:

$$f_1 + f_3 = f_2 + f_4. \tag{2.20}$$

If we take all four edges neighbour to the edge f_0, the order is clockwise. (If we combine all terms from (2.19) with $\partial_5 \equiv \frac{\partial}{\partial l_5}$, we just get $\partial_5 \wedge (\partial_1 + \partial_3 - \partial_2 - \partial_4)$. Hereafter we shall denote the derivative over l_i by ∂_i. There exists a unique function g defined on the set of faces (boundary components) X_2 such that on each edge $f_i = g_i^{(1)} + g_i^{(2)}$ – the sum of g-variables of faces incident to this edge. In the neighborhood of each vertex g can be reconstructed inambiguously: $g_1 = (f_2 + f_3 - f_1)/2$. Condition (2.20) ensures that moving from vertex to vertex this number is preserved. This g–function does not coincide with perimeters p_i for each face, but one may find the relation between them. (In order to construct p_i from the set of g–variables one may simply take the sum of $f_i = g_i^{(1)} + g_i^{(2)}$ over all edges surrounding the face.

Thus $\overline{\mathcal{M}}_{g,n}$ is a Poisson manifold whose symplectic leaves are fibers of the projection π. The following proposition again by M.Kontsevich establishes the relation between β and Chern classes ω_i (2.4):

Proposition 2.2. *On* $\pi^{-1}(p_*)$ $4\beta^{-1} = \sum_{i=1}^{n} p_i^2 \times \omega_i$.

3 Discrete Moduli Space $\overline{\mathcal{M}}_{g,n}^{disc}$.

We describe a discretization of the moduli spaces in this section. We hope that these discrete moduli spaces would be helpful when taking a quantum deformation of the Poisson structures on the moduli spaces. Also, these discrete spaces have their proper meaning because, as we shall show, they admit a nice description in terms of another explicitly solvable matrix model. We shall start with a description of this model, merely for being acquainted with it.

3.1 The matrix model for d.m.s.

Let us consider the matrix model (1.C) [17], [19], where we denote, for simplicity, $\mu_i \equiv \mathrm{e}^{\lambda_i}$. It includes, in contrast to the Kontsevich model, all powers of X^n in the potential since it describes the partition of moduli space into cells of a simplicial complex, the sum runs over all simplices with different dimensions. (In the language of the Kontsevich model the lower the dimension is, the more and more edges of the fat graph are reduced).

The logarithmic potential makes this model similar to the Penner matrix model [23] $\int DX \exp \operatorname{tr} \big(\log(1-X)+X\big)$ counting virtual Euler characteristics of the moduli spaces.

We find the Feynman rules for the theory (1.C). First, as in the standard Penner model, we have vertices of all orders in X. Due to rotational symmetry, the factor $1/n$ standing with each X^n cancels, and only symmetrical factor $1/\#\mathrm{Aut}\,\Gamma$ survives. Also there is a factor $(\alpha/2)$ standing with each vertex. As in the Kontsevich model, there are variables μ_i associated with each cycle. But the form of propagator changes — instead of $2/(\lambda_i+\lambda_j)$, we have $2/\alpha(\mu_i\mu_j-1)$.

Let us consider the same case ($g=0$, $n=3$) as for Kontsevich model. One additional diagram resulting from vertex X^4 arises (Fig.3).

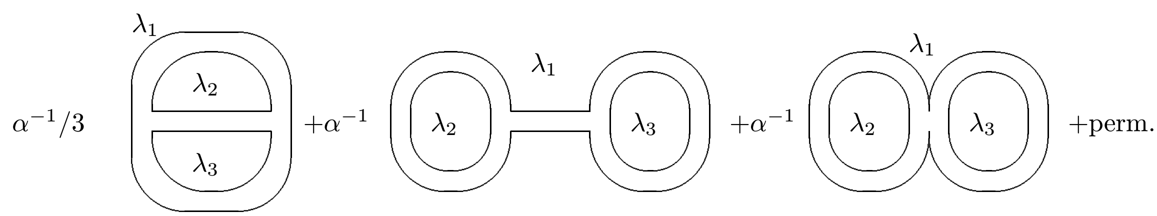


Fig.3. g=0, s=3 contribution to the model (1.C).

Symmetrization over μ_1, μ_2 and μ_3 gives:

$$
\begin{aligned}
&- \frac{N^2}{3}\left\{\frac{2\alpha^{-1}}{2(\mu_1\mu_2-1)(\mu_1\mu_3-1)} + \text{ perm.}\right\} + \frac{2\alpha^{-1}N^2}{6(\mu_1\mu_2-1)(\mu_1\mu_3-1)(\mu_2\mu_3-1)} \\
&+ \frac{1}{3}\left\{\frac{2\alpha^{-1}N^2}{2(\mu_1^2-1)(\mu_1\mu_2-1)(\mu_1\mu_3-1)} + \text{ perm.}\right\}
\end{aligned}
\tag{3.1}
$$

Again, collecting all terms we get:

$$
\frac{2\alpha^{-1}N^2}{6\prod_{i<j}(\mu_i\mu_j-1)}\left\{\sum_{i<j}\mu_i\mu_j - 2 + \left(\frac{\mu_2\mu_3-1}{\mu_1^2-1}+\frac{\mu_1\mu_2-1}{\mu_3^2-1}+\frac{\mu_1\mu_3-1}{\mu_2^2-1}\right)\right\}, \tag{3.2}
$$

and after a little algebra we obtain the answer:

$$
F_{0,3} = N^2\alpha^{-1}\frac{\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3+1}{3(\mu_1^2-1)(\mu_2^2-1)(\mu_3^2-1)}. \tag{3.3}
$$

We see that here, just as in the standard Kontsevich model, the cancellation of intertwining terms in the denominator occurs that leads to the factorization of the answer over $1/(\mu_i^2 - 1)$–terms. This simplest example shows that there should be some underlying geometric structures in this case as well.

Note that technically the reason why (1.C) depends only on $\operatorname{tr}\Lambda^k$ $(k \le 0)$ is the following: This model, as well as the Kontsevich one, belongs to the class of Generalized Kontsevich Models [13]. It means that after some simple transformation we get from (1.C) the model with the potential $\Lambda X + V(X)$, which depends only on Miwa's times.

3.2 Moduli Spaces and discrete De Rham cohomologies.

Let us now consider a *discretization* of the moduli spaces $\overline{\mathcal{M}}_{g,n}$ and $\mathcal{M}_{g,n}^{comb}$. We shall consider the following set of parameters l_i:

$$l_i \in \mathbf{Z}_+ \cup \{0\}, \quad p_i \in \mathbf{Z}_+, \quad \sum_{i=1}^{n} p_i \in 2\mathbf{Z}_+. \tag{3.4}$$

So all l_i and p_i are now integers, but some of l_i can be zeros while all perimeters are strictly positive. The sum of all perimeters is even because each edge contributes twice into it. We call this (combinatorial) space $\overline{\mathcal{M}}_{g,n}^{disc}$. It is worth mentioning that now we explicitly include into play such points of the original $\overline{\mathcal{M}}_{g,n}$ which are points of reductions (singular curves) and curvature (orbifold points that are stable under the action of some non-unit subgroup of the symmetry group in the Teichmüller space). While keeping all p_i fixed in a general case we can put a number of l_j exactly equal zero. Some of these configurations belong to the interior of $\mathcal{M}_{g,n}$, but not all — it means that among the points of $\overline{\mathcal{M}}_{g,n}^{disc}$ there are points that lie on the boundary $\partial\mathcal{M}_{g,n}$. Such points correspond to the reductions of the algebraic curve. Also we shall use the notation $\mathcal{M}_{g,n}^{disc}$ for such subset of $\overline{\mathcal{M}}_{g,n}^{disc}$ where all points of reduction are excluded.

It turns out that this choice for d.m.s. is rather natural since all the quantities (2.3-2.6) have corresponding contemplates in this discrete case.

First we define the action of the external derivative d and the integration over these (orbi)spaces. We shall write the d-action on functions, the extrapolation to the space of skew symmetric forms is obvious:

$$df(l_1,\dots,l_k) = \sum_{i=1}^{k} \big(f(l_1,\dots,l_i+1,\dots,l_k) - f(l_1,\dots,l_k)\big)dl_i \tag{3.5}$$

As for the integral over domain Ω, there is again a proper generalization of it to this discrete case:

$$\int_\Omega f(l_1,\dots,l_k)dl_1\dots dl_k := \sum_{\substack{l_i\in\mathbf{Z}_+\cup\{0\}\\ \{l_1,\dots,l_k\}\in\Omega}} f(l_1,\dots,l_k). \tag{3.6}$$

Instead of $BU(1)^{comb}$ we have (orbi)space of equivalence classes of all sequences of non-negative integers $l_1,\dots,l_k$ modulo cyclic permutations. An analog of S^1–bundle is now a kind of $\mathbf{Z}_p$–"bundle" over this new discrete orbispace whose total space $E\mathbf{Z}_p^{comb}$ is an ordinary rectangular lattice. The fiber of the bundle over the equivalence class of sequences $l_1,\dots,l_k$ is again the polygon with integer lengths of edges $l_1\dots,l_k$.

Let ϕ_i be coordinates on $E\mathbf{Z}_p$, just as in (2.2):

$$l_i = \phi_{i+1} - \phi_i \quad (i=1,\dots,k-1), \quad l_k = p + \phi_1 - \phi_k. \tag{3.7}$$

Due to the linearity of (2.3) in l_i and ϕ_j it can be straightforwardly generalized to our case. Denote by $\widetilde{\alpha}$ the 1–form on $E\mathbf{Z}_p^{comb}$, which is equal to

$$\widetilde{\alpha} = \sum_{i=1}^{k} \frac{l_i}{p} \times \frac{d\phi_i}{p}. \tag{3.8}$$

The integral of $\widetilde{\alpha}$ over each fiber of the universal bundle $E\mathbf{Z}_p^{comb} \to BU(1)^{comb}$ is equal to -1. The differential $d\widetilde{\alpha}$ is the pullback of a 2–form $\widetilde{\omega}$ on the base $B\mathbf{Z}_p^{comb}$,

$$\widetilde{\omega} = \sum_{1\le i<j\le k-1} \frac{dl_i}{p} \wedge \frac{dl_j}{p}. \tag{3.9}$$

Extrapolating these results to the whole discrete moduli space we obtain that the pullback $\widetilde{\omega}_i$ of the form $\widetilde{\omega}$ under the ith map $\overline{\mathcal{M}}_{g,n}^{disc} \to B\mathbf{Z}_p^{comb}$ represents the class $\widetilde{c}_1(\mathcal{L}_i)$.

Let us denote by $\widetilde{\pi} : \overline{\mathcal{M}}_{g,n}^{disc} \to \left[\mathbf{Z}_+^n\right]_{even}$ the projection to the space of perimeters with the restriction $\sum_i p_i \in 2\cdot \mathbf{Z}_+$. Intersection indices are given again by the formula:

$$\langle\!\langle \tau_{d_1} \ldots \tau_{d_n} \rangle\!\rangle_g = \int_{\widetilde{\pi}^{-1}(p_*)} \prod_{i=1}^{n} \widetilde{\omega}_i^{d_i}, \tag{3.10}$$

where $p_* = (p_1, \ldots, p_n)$ is an arbitrary sequence of positive integer numbers, $\sum_{i=1}^n p_i$ being necessarily even, and $\pi^{-1}(p_*)$ is an analogue of the fiber of $\overline{\mathcal{M}}_{g,n}$ in $\overline{\mathcal{M}}_{g,n}^{disc}$. Note, however, that now the volume of $\widetilde{\pi}^{-1}(p_*)$ may depend on p_i in a non-monomial way. Therefore, these indices may be non-zero for $\sum_i d_i \le d \equiv 3g-3+n$.

One important note is in order. Each fiber $\widetilde{\pi}^{-1}(p_*)$ contains finite number of points. Thus, these fibers are not isomorphic. But they all are analogues of the initial moduli space $\overline{\mathcal{M}}_{g,n}$ taken with different perimeters. For this reason we call them *discretized moduli spaces*. It appears that relation (3.10) remains valid independently of how many points from the initial $\overline{\mathcal{M}}_{g,n}$ give contribution to the fiber $\widetilde{\pi}^{-1}(p_*)$ (it can be even only one point of reduction, as we shall see for $\overline{\mathcal{M}}_{1,1}$). Values of these intersection indices are some rational numbers due to the orbifold nature of the initial moduli space $\overline{\mathcal{M}}_{g,n}$. This nature reveals itself as symmetries of the graphs. But these symmetry properties are the same, whatever case – continuum or discrete, and whatever values of perimeters we choose. Thus, preserving symmetry properties we preserve the values of cohomological classes on both continuum and discrete moduli spaces.

The difference appears when we consider $\langle\!\langle \tau_{d_1} \ldots \tau_{d_n} \rangle\!\rangle_g$ with $\sum d_i < 3g-3+n$. In the continuum case such quantities vanish in contrast to the discrete one where they may be non-zero due to the possible non-zero curvature of the *covering manifold*.

3.3 Matrix Integral for Discretized Moduli Space.

We denote by $\widetilde{\Omega}$ the two–form on $\overline{\mathcal{M}}_{g,n}^{disc}$:

$$\widetilde{\Omega} = \sum_{i=1}^{n} p_i^2 \widetilde{\omega}_i, \tag{3.11}$$

whose restriction to the fibers of $\widetilde{\pi}$ has constant coefficients in coordinates $\bigl(l(e)\bigr)$. d is again the complex dimension of $\mathcal{M}_{g,n}$, $d = 3g-3+n$. The volume of the fiber of $\widetilde{\pi}$ with respect to $\widetilde{\Omega}$ is

$$\begin{aligned} \mathrm{vol}\bigl(\widetilde{\pi}^{-1}(p_1 \ldots, p_n)\bigr) &= \int_{\widetilde{\pi}^{-1}(p_*)} \frac{\widetilde{\Omega}^d}{d!} = \frac{1}{d!}\int_{\widetilde{\pi}^{-1}(p_*)} \bigl(p_1^2\widetilde{\omega}_1 + \ldots + p_n^2\widetilde{\omega}_n\bigr)^d = \\ &= \sum_{\sum d_i \le d} \prod_{i=1}^{n} \frac{p_i^{2d_i}}{d_i!} \langle\!\langle \tau_{d_1} \ldots \tau_{d_n} \rangle\!\rangle_g. \end{aligned} \tag{3.12}$$

Next step is to do a Laplace transform in both sides of (3.12). Of course, now we should replace continuum Laplace transform by the discrete one and also explicitly take into account that the sum of all p_i is even. On the R.H.S. we have:

$$\sum_{\substack{p_i \in \mathbf{Z}_+,\\ \sum p_i \in 2\mathbf{Z}_+}} \mathrm{e}^{-\sum_i p_i\lambda_i} p_1^{2d_1} \ldots p_n^{2d_n} = \prod_{i=1}^{n} \left(\frac{\partial}{\partial\lambda_i}\right)^{2d_i} \times \frac{1}{2}\left\{ \prod_{i=1}^{n} \frac{1}{\mathrm{e}^{\lambda_i}-1} + (-1)^n \prod_{i=1}^{n} \frac{1}{\mathrm{e}^{\lambda_i}+1} \right\}. \tag{3.13}$$

On the L.H.S. of (3.12) we again substitute

$$\mathrm{e}^{\widetilde{\Omega}} dp_1 \wedge \ldots dp_n \big|_{\sum p_i \in 2\mathbf{Z}_+} = \widetilde{\rho} \prod_{e \in X_1} dl_e. \tag{3.14}$$

Here the constant $\widetilde{\rho}$ is the ratio of measures similar to (2.11) and we only need to take into account the restriction that the sum of all p_i is even. It leads to the renormalization of ρ for the case of d.m.s.:

$$\widetilde{\rho} = \rho/2, \tag{3.15}$$

where ρ is given by (2.12).

Now we give a matrix model description for these "new" intersection indices. Here we immediately encounter some troubles. Let us consider the correspondence between graphs and different points of $\widetilde{\pi}^{-1}(p_*)$. First, there are points of a general position with all l_i greater than zero, which correspond to graphs with only trivalent vertices. Second, there are such points of $\widetilde{\pi}^{-1}(p_*)$ where some l_i are zeros, but these points still do not correspond to reductions. For example, see Fig.4, where for the torus case Fig.4a represents a point of a general position, $l_i > 0$, $i = 1, 2, 3$, and Fig.4b gives an example of the graph for which one (and only one) of l_i is zero. Such graphs do not correspond necessarily to singular curves. The one depicted in Fig.4b determines the curve in $\mathcal{M}_{1,1}$ (in Teichmüller parameterization) with purely imaginary modular parameter τ. Of certain, if we want to include such graphs into consideration we should take not only trivalent vertices, but vertices of arbitrary order. In the continuum limit we did not take into account such graphs, since they correspond to subdomains of lower dimensions in the interior of the moduli space, the integration measure being continuous and we may neglect them. Here the situation is different and we should take into account all such diagrams as well.

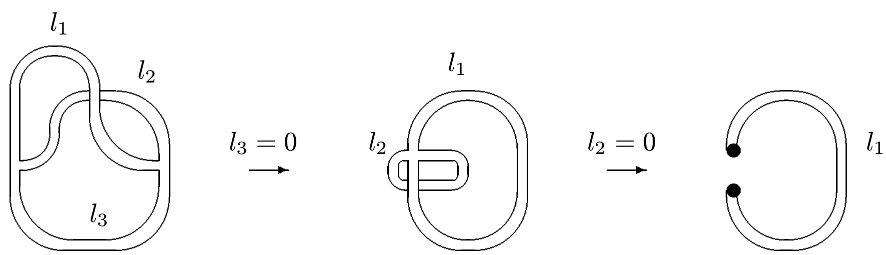


Fig.4a Fig.4b Fig.4c

Fig.4. Diagrams for different regions of $\overline{\mathcal{M}}_{1,1}$

But in each $\widetilde{\pi}^{-1}(p_*)$ there always are (except for the case $\mathcal{M}_{0,3}$) true points of reduction (see, for example, Fig.4c when two of l_i are zeros). We are not able to give a matrix model description to such curves. At first sight it would mean that all the construction fails since we still do not discuss how to "exclude" such reduction points from $\widetilde{\pi}^{-1}(p_*)$ by modifying somehow the relation (3.10). Let $\mathcal{M}_{g,n}^{disc}$ be such subset of $\overline{\mathcal{M}}_{g,n}^{disc}$ where all points of reduction are excluded. Thus we need to release the integration over open $\mathcal{M}_{g,n}$ from the total integration over $\overline{\mathcal{M}}_{g,n}$. In order to do it we use a stratification procedure a'la Deligne and Mumford [21]. The idea is to present the open moduli space $\mathcal{M}_{g,n}$ as a combination of $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g_j,n_j}$ of lower dimensions. The description of this procedure one can find in [22], [11].

Geometrical meaning of the reduction procedure is that we subsequently pinch the handles of the surface (Fig.5). One can see that there are two types of such reduction: for the first one, by pinching a handle we result in the surface of genus $g-1$ and two additional punctures. Thus, from the space $\overline{\mathcal{M}}_{g,n}$ we get after such reduction $\overline{\mathcal{M}}_{g-1,n+2}$ (Fig.5a). In the second case by pinching an intermediate cylinder we get two surfaces of the same total genus and two new punctures: one per each new component. It means that the initial moduli space $\overline{\mathcal{M}}_{g,n}$ splits into the product $\overline{\mathcal{M}}_{g_1,n_1+1} \otimes \overline{\mathcal{M}}_{g_2,n_2+1}$, $g_1 + g_2 = g$, $n_1 + n_2 = n$ (Fig.5b).

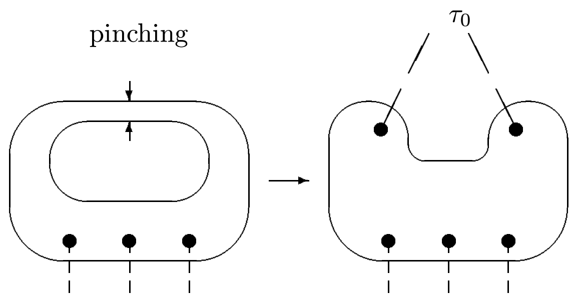


Fig.5a. One–component type of the reduction.

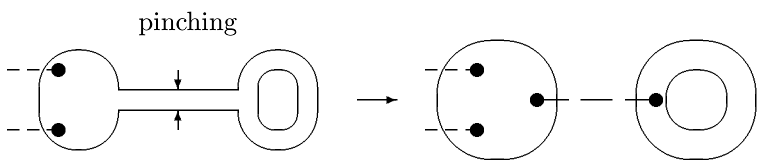


Fig.5b. Two–component type of the reduction.

One may easily check that total complex dimension of the resulting spaces is in both cases $d-1$, where $d = 3g-3+n$ is the dimension of $\overline{\mathcal{M}}_{g,n}$. So the general receipt of how to express $\mathcal{M}_{g,n}$ via closed moduli spaces is to construct an alternative sum over reductions:

$$\mathcal{M}_{g,n} = \sum_{\substack{reductions \\ r_q=0}}^{3g-3+n} (-1)^{r_q} \otimes_{j=1}^{q} \overline{\mathcal{M}}_{g_j,n_j+k_j}, \tag{3.16}$$

where the sum runs over all q–component reductions, r_q is the reduction degree and k_j being the number of the additional punctures due to reductions. The dimension of $\overline{\mathcal{M}}_{g_j,n_j+k_j}$ is $d_j = 3g_j - 3 + n_j + k_j$,

$$\sum_{j=1}^{q} n_j = n, \quad \sum_{j=1}^{q} d_j = d - r_q. \tag{3.17}$$

Thus we have:

$$\int_{\mathcal{M}_{g,n}^{disc}} \mathrm{e}^{\widetilde{\Omega}} \times \mathrm{e}^{\sum_i \lambda_i p_i} dp_1 \wedge \ldots \wedge dp_n = \frac{1}{d!} \int_{\overline{\mathcal{M}}_{g,n}^{disc}} \left(\sum_{i=1}^{n} p_i^2 \widetilde{\omega}_i \right)^d \mathrm{e}^{\sum_i \lambda_i p_i} dp_1 \wedge \ldots \wedge dp_n +$$
$$+ \sum_{\substack{reductions \\ r_q=1}}^{3g-3+n} (-1)^{r_q} \bigotimes_{j=1}^{q} \int_{\overline{\mathcal{M}}_{g_j,n_j+k_j}} \left(\sum_{a=1}^{n_j} p_a^2 \widetilde{\omega}_a \right)^{d_j} \mathrm{e}^{\sum_i \lambda_i p_i} dp_1 \wedge \ldots \wedge dp_{n_j}. \tag{3.18}$$

Now we can find, using (3.14), a matrix model description for the L.H.S. of (3.18). Just as in the continuum case we have:

$$\text{L.H.S.} = \int_{\mathcal{M}_{g,n}^{disc}} \exp\left\{-\sum_{e\in X_1} l_e(\lambda_e^{(1)}+\lambda_e^{(2)})\right\} \times \widetilde{\rho} \times \prod_{e\in X_1} |dl(e)|, \tag{3.19}$$

where $\widetilde{\rho} = 2^{d+\#X_1-\#X_0-1}$.

This last expression can be presented as a sum over all possible "fat graphs" Γ with vertices of all possible valences for given genus g and number of faces n. We should again take into account the volume of the authomorphism group for the graph. This volume coincides with the number of copies of equivalent domains of the moduli space $\mathcal{M}_{g,n}$, which constitute this cell of the combinatorial simplicial complex. Finally, we "integrate" over each $l(e)$, *i.e.* , take the sum over all positive integer values of $l(e)$ (because we already took into account all zero values of $l(e)$ doing the sum over *all* graphs). Eventually, we have:

$$\text{L.H.S.} = 2^{d-1} \sum_{\substack{all \\ Graphs\ \Gamma}} \frac{1}{\#\mathrm{Aut}\,(\Gamma)} \times 2^{-\#X_0} \times \prod_{e\in X_1} \frac{2}{\mathrm{e}^{\lambda_e^{(1)}+\lambda_e^{(2)}}-1}. \tag{3.20}$$

It is nothing but a term from the genus expansion of the matrix model (1.C) with

$$\Lambda = \mathrm{diag}\,\{\,\mathrm{e}^{\lambda_1},\ldots,\,\mathrm{e}^{\lambda_N}\}. \tag{3.21}$$

Then $\log\mathcal{Z}[\Lambda]$ has the following genus expansion:

$$\log\mathcal{Z}[\Lambda] = \sum_{g=0}^{\infty}\sum_{n=1}^{\infty}(N\alpha)^{2-2g}\alpha^{-n}w_g(\lambda_1,\ldots,\lambda_n). \tag{3.22}$$

Let us use the relations (3.13) in order to express the R.H.S. of (3.18) via intersection indices.

$$\begin{aligned} w_g(\lambda_1,\ldots,\lambda_n) = \frac{1}{2^{d-1}} \sum_{\substack{reductions \\ q-component}} (-1)^{r_q} \prod_{j=1}^{q} \Bigg\{ \sum_{\sum d_\xi = 3g_j-3+n_j+k_j} \frac{1}{n_j!} \langle\!\langle \underbrace{\tau_{d_1}\ldots\tau_{d_{n_j}}}_{n_j} \underbrace{\tau_0\ldots\tau_0}_{k_j} \rangle\!\rangle_{g_j} \\ \times\ \mathrm{tr}\left[\prod_{k=1}^{n_j}\left(\frac{\partial}{\partial\lambda_k}\right)^{2d_k}\right] \frac{1}{(d_k)!}\cdot\frac{1}{2}\left(\prod_{k=1}^{n_j}\frac{1}{\mathrm{e}^{\lambda_k}-1} + (-1)^{n_j}\prod_{k=1}^{n_j}\frac{1}{\mathrm{e}^{\lambda_k}+1}\right)\Bigg\}. \end{aligned} \tag{3.23}$$

Theorem 1.2 is proved.

Formula (3.23) is our main result. For practical reasons it is sometimes convenient to rewrite

$$\begin{aligned} \sum_{\sum d_\xi = 3g_j-3+n_j+k_j} \frac{1}{n_j!} \langle\!\langle \overbrace{\underbrace{\tau_{d_1}\ldots\tau_{d_{n_j}}}_{n_j} \underbrace{\tau_0\ldots\tau_0}_{k_j}}^{s_j} \rangle\!\rangle_{g_j} = \\ \sum_{\substack{b_0+b_1+\ldots+b_k=n_j \\ 0\cdot b_0+1\cdot b_1+\ldots+k\cdot b_k = 3g_j-3+s_j}} \frac{1}{b_0!\ldots b_k!} \langle\!\langle (\tau_0)^{b_0}\ldots(\tau_k)^{b_k} \underbrace{\tau_0\ldots\tau_0}_{k_j} \rangle\!\rangle_{g_j}. \end{aligned} \tag{3.24}$$

Taking this expression for the case $\mathcal{M}_{0,3}$ (without reductions) and recalling that $\langle\tau_0^3\rangle_0 = 1$ we immediately get the answer (3.3) after a substitution $\mu_i = \mathrm{e}^{\lambda_i}$.

Since the matrix model (1.C) is equivalent to the hermitian one–matrix model with an arbitrary potential, formulae (3.22)–(3.24) above give the solution to such models in geometric invariants of the d.m.s.

There are two complications in the final relation (3.23) that make it qualitatively different from the Kontsevich model. First of them is the "sum over reductions". Another is the "new" averaging $\langle\!\langle\ldots\rangle\!\rangle_g$. The sum over reductions turns to be rather involved for the following reasons: When we considered the orbits $\widetilde{\pi}^{-1}(p_*)$ we assumed that they belonged to *one* copy of the moduli space $\overline{\mathcal{M}}_{g,n}$. But when we deal with the cell decomposition it is much more convenient first to consider the total simplicial complex (which we shall denote $\mathcal{T}_{g,n}$) and only afterwards take into account internal automorphisms of $\mathcal{T}_{g,n}$, which eventually produce $\overline{\mathcal{M}}_{g,n}$ as a coset over a symmetry group $G_{g,n}$. One may consider, instead of π and $\widetilde{\pi}$, mappings β and $\widetilde{\beta}$, respectively, $\beta : \mathcal{T}_{g,n} \otimes \mathbf{R}_{+}^{n} \to \mathcal{M}_{g,n}^{comb}$ and $\widetilde{\beta} : \mathcal{T}_{g,n}^{disc}[p_*] \times [\mathbf{Z}_{+}^{n}]_{even} \to \overline{\mathcal{M}}_{g,n}^{disc}$, where $\mathcal{T}_{g,n}^{disc}$ are again finite (nonisomorphic) sets of points of $\mathcal{T}_{g,n}$ supplied with the discrete de Rham cohomology structure.

For these spaces $\mathcal{T}_{g,n}$ an analogue of the formula (3.23) exists. The only difference is that we should multiply all indices $\langle\!\langle \tau_{d_1} \dots \tau_{d_n} \rangle\!\rangle_g$ by the order of the symmetry group $G_{g,n}$. This number of copies multiplied by the order of the group G_{g_j,n_j+k_j} is not necessarily divisible by the order of $G_{g,n}$. Thus when we write in formula (3.23) “the sum over reductions” we bear in mind that the coefficients in this sum are not necessarily integers! (See Section 6 for an example).

The next section is devoted to the comparison of the matrix integrals in the Kontsevich and the matrix model for d.m.s. using exclusively matrix model tools, which permits to prove the coincidence of $\langle \tau_{d_1} \dots \tau_{d_n} \rangle_g$ and $\langle\!\langle \tau_{d_1} \dots \tau_{d_n} \rangle\!\rangle_g$ in the highest dimension $\sum_{i=1}^{n} d_i = d \equiv 3g - 3 + n$.

4 Comparison of two matrix models

This section is based on the results of papers [18] and [12]. It was explicitly demonstrated in [19] [20] that the matrix model (1.C) is equivalent to the standard hermitian one–matrix model

$$\mathcal{Z}[g, \widetilde{N}] = \int_{\widetilde{N} \times \widetilde{N}} d\phi \exp\bigl(-\widetilde{N} \operatorname{tr} V(\phi)\bigr), \tag{4.1}$$

where the integration goes over hermitian $\widetilde{N} \times \widetilde{N}$ matrices and

$$V(\phi) = \sum_{j=1}^{\infty} \frac{g_j}{j} \phi^j \tag{4.2}$$

is a general potential. Then the following relation holds:

$$\mathcal{Z}[g, \widetilde{N}(\alpha)] = \mathrm{e}^{-N \operatorname{tr} \eta^2/2} \mathcal{Z}_P[\eta, N], \quad \widetilde{N}(\alpha) = -\alpha N. \tag{4.3}$$

Here the partition function $\mathcal{Z}_P[\eta, N]$ is

$$\mathcal{Z}_P[\eta, N] = \int_{N \times N} dX \exp\left[N \operatorname{tr}\left(-\eta X - \frac{1}{2} X^2 - \alpha \log X\right)\right], \tag{4.4}$$

the integral being done over hermitian matrices of *another* dimension $N \times N$ and the set of the coupling constants (4.2) being related to the matrix η by the Miwa transformation

$$g_k = \frac{1}{N} \operatorname{tr} \eta^{-k} - \delta_{k,2} \text{ for } k \geq 1, \quad g_0 = \frac{1}{N} \operatorname{tr} \log \eta^{-1}. \tag{4.5}$$

(Note the changing of sign in front of the logarithmic term.) Now after the substitution

$$\eta = \sqrt{\alpha}(\Lambda + \Lambda^{-1}) \tag{4.6}$$

and after the change of variables $X \to (X-1)\Lambda\sqrt{\alpha}$ we reconstruct the integral (1.C) (with α multiplied by two).

Note that we can do a limiting procedure (which is a sort of the double scaling limit for the standard model (4.1)) resulting in the Kontsevich integral (2.2) starting from the model (1.C). It looks even more natural in terms of this model than in terms of the one–matrix integral (4.1). Namely, let us take in (1.C)

$$\Lambda = \mathrm{e}^{\varepsilon \lambda}, \quad \alpha = \frac{1}{\varepsilon^3}. \tag{4.7}$$

Then after rescaling $X \to \varepsilon X$ in the limit $\varepsilon \to 0$ we explicitly reproduce (2.2) from (1.C). During this procedure we can keep the size N of matrices of (1.C) constant, but the size $\widetilde{N}(\alpha)$ of the matrices of hermitian model goes to infinity in the limit $\varepsilon \to \infty$.

4.1 Review of the solutions to Kontsevich model and model (1.C).

Since the models (1.C) and (4.1) are equivalent, we can use the explicit answers for (4.1) found in [12] in order to check the validity of our formulae (3.23) and to compare the values of intersection indices in both Kontsevich model (2.16) and the model (1.C). Both these models were solved in genus expansion in terms of the corresponding momenta. For the Kontsevich model this solution was presented in [16] and for (4.4) or, equivalently, (4.1) — in [12]. Here we present the results. (Throughout this section the expansion parameter α should be replaced by -2α in order to compare with the results of [12].)

1. The solution to the Kontsevich model is

$$\log \mathcal{Z}_K[N,\Lambda] = \sum_{g=0}^{\infty} N^{2-2g} F_g^{Kont}. \tag{4.8}$$

For the genus expansion coefficients we have

$$F_g^{Kont} = \sum_{\substack{\alpha_j>1 \\ \sum_{j=1}^{n}(\alpha_j-1)=3g-3}} \langle \tau_{\alpha_1}\dots\tau_{\alpha_n}\rangle_g \frac{I_{\alpha_1}\dots I_{\alpha_n}}{(I_1-1)^{\alpha}} \quad \text{for} \quad g\ge 1, \tag{4.9}$$

where $\langle\cdot\rangle_g$ are just intersection indices and the moments I_k's depending on the external field M are defined by

$$I_k(M) = \frac{1}{(2k-1)!!}\frac{1}{N}\sum_{j=1}^{N}\frac{1}{(m_j^2-2u_0)^{k+1/2}} \qquad k\ge 0, \tag{4.10}$$

and $u_0(M)$ is determined from the equation

$$u_0 = I_0(u_0, M). \tag{4.11}$$

2. The solution to the model (4.4) can be written as

$$\log \mathcal{Z}_P[N,\eta] = \sum_{g=0}^{\infty} N^{2-2g} F_g, \tag{4.12}$$

where

$$F_g = \sum_{\alpha_j>1,\ \beta_i>1} \langle \alpha_1\dots\alpha_s;\beta_1\dots\beta_l|\alpha,\beta,\gamma\rangle_g \frac{M_{\alpha_1}\dots M_{\alpha_s}J_{\beta_1}\dots J_{\beta_l}}{M_1^{\alpha}J_1^{\beta}d^{\gamma}} \qquad g>1. \tag{4.13}$$

This solution originated from the one–cut solution to the loop equations in the hermitian one–matrix model, x and y being endpoints of this cut, $d=x-y$, and for momenta M_k, J_k we have

$$M_k = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{(\eta_j-x)^{k+1/2}(\eta_j-y)^{1/2}} - \delta_{k,1} \quad k\ge 0, \tag{4.14}$$

$$J_k = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{(\eta_j-x)^{1/2}(\eta_j-y)^{k+1/2}} - \delta_{k,1} \quad k\ge 0. \tag{4.15}$$

The brackets $\langle\cdot\rangle_g$ denote rational numbers, the sum is finite in each order in g, while the following restrictions are fulfilled: If we denote by N_M and N_J the total powers of M's and J's, respectively, i.e.

$$N_M = s-\alpha, \quad N_J = l-\beta, \tag{4.16}$$

then it holds that $N_M\le 0$, $N_J\le 0$, and

$$\begin{aligned} F_g:&\quad N_M+N_J = 2-2g,\\ F_g:&\quad \sum_{i=1}^{s}(\alpha_i-1)+\sum_{j=1}^{l}(\beta_j-1)+\gamma = 4g-4\\ F_g:&\quad \sum_{i=1}^{s}(\alpha_i-1)+\sum_{j=1}^{l}(\beta_j-1)+\gamma \le 3g-3 \end{aligned} \tag{4.17}$$

We again have nonlinear functional equations determining the positions of the endpoints x and y:

$$\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\sqrt{(\eta_i-x)(\eta_i-y)}}-\frac{x+y}{2} = 0, \tag{4.18}$$

$$\frac{1}{N}\sum_{i=1}^{N}\frac{\eta_i-\frac{x+y}{2}}{\sqrt{(\eta_i-x)(\eta_i-y)}}-\frac{(x-y)^2}{8} = -2\alpha+1. \tag{4.19}$$

The solutions to the first two genera have, as usual, some peculiarities. For $g=1$ we have

$$F_1=-\frac{1}{24}\log M_1 J_1 d^4, \tag{4.20}$$

and for zero genus we have, after taking a double derivative in α in order to exclude divergent parts,

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}F_0=4\log d. \tag{4.21}$$

The last property of the expression (4.13), which we want to notice here, is its symmetry under interchanging x and y, or equivalently, M_i and J_i:

$$\langle\alpha_1\ldots\alpha_s;\beta_1\ldots\beta_l|\alpha,\beta,\gamma\rangle_g=(-1)^\gamma\langle\beta_1\ldots\beta_l;\alpha_1\ldots\alpha_s|\beta,\alpha,\gamma\rangle_g. \tag{4.22}$$

This relation is equivalent to the symmetrization $\mathrm{e}^{\lambda}\to-\mathrm{e}^{\lambda}$ in the formula (3.23).

3. In the d.s.l. $\varepsilon\to 0$ we may put

$$y=-\frac{\sqrt{2}}{\varepsilon^{3/2}},\quad x=\frac{\sqrt{2}}{\varepsilon^{3/2}}+\sqrt{2}\,u_0+\ldots, \tag{4.23}$$

and the equation (4.11) arises. The scaling behaviours of the momenta M_k, J_k and d are

$$\begin{aligned} &J_k\to-2^{-(3k/2+1)}\varepsilon^{(3k+1)/2}I_0+\delta_{k1},\\ &M_k\to-2^{(k-1)/2}\varepsilon^{-(k-1)/2}((2k-1)!!I_k-\delta_{k1}),\\ &d\to 2^{3/2}\varepsilon^{-3/2}\end{aligned} \tag{4.24}$$

Thus, only terms of the highest order in α_i that are independent on J_k survive in the d.s.l. when the expression (4.13) converts into the answer for the Kontsevich model (4.9). Then the coefficients $\langle\alpha_1\ldots\alpha_s;\{\text{nothing}\}\,|\alpha,0,\gamma\rangle_g$ coincide with the Kontsevich intersection indices $\langle\tau_{\alpha_1}\ldots\tau_{\alpha_n}\rangle_g$. In [12] an iterative procedure was proposed for finding coefficients of the expansion (4.13); all these coefficients were found in the genus 2 (for $g=0,1$ see [18]). It was proved that coefficients of the highest order in α_k coincide in a proper normalization with the Kontsevich indices.

4.2 Relation between momenta and d.m.s. variables.

We are going to express (4.13) in terms of the quantities standing in the R.H.S. of (3.23).

At first, let us expand both momenta M_k, J_k and the restriction equations (4.18, 4.19) in terms of λ–variables, where $\eta=\sqrt{\alpha}(\mathrm{e}^{\lambda}+\mathrm{e}^{-\lambda})$. Then, for the endpoints of the cut, we have:

$$x=2\sqrt{\alpha}+\xi,\quad y=-2\sqrt{\alpha}+\beta, \tag{4.25}$$

where ξ and β themselves are some polynomials in the higher momenta M_i and J_i with $i,j\ge 0$. Thus, after a little algebra we shall obtain, say, for the moment M_k:

$$M_k=\frac{1}{N}\,\mathrm{tr}\,\frac{(\mathrm{e}^{\lambda})^{k+1}}{\sqrt{\alpha}\Big((\mathrm{e}^{\lambda}-1)^2-\frac{\xi}{\sqrt{\alpha}}\mathrm{e}^{\lambda}\Big)^{k+1/2}\Big((\mathrm{e}^{\lambda}+1)^2-\frac{\beta}{\sqrt{\alpha}}\mathrm{e}^{\lambda}\Big)^{1/2}}-\delta_{k,1}. \tag{4.26}$$

(For J_k the expression is just the same with interchanging the powers $k+1/2$ and $1/2$ for the two terms in the denominator.)

It is convenient now to introduce new momenta:

$$\begin{aligned}\widetilde{M}_k &= \frac{1}{N}\,\mathrm{tr}\,\frac{\sqrt{\eta-y}}{(\eta-x)^{k+1/2}},\\ \widetilde{J}_k &= \frac{1}{N}\,\mathrm{tr}\,\frac{\sqrt{\eta-x}}{(\eta-y)^{k+1/2}},\end{aligned} \tag{4.27}$$

that are related to the initial ones via the following relations:

$$\begin{aligned}\widetilde{M}_k &= M_{k-1}+\delta_{k,2}+d(M_k+\delta_{k,1}),\\ \widetilde{J}_k &= J_{k-1}+\delta_{k,2}-d(J_k+\delta_{k,1}),\\ M_0=J_0 &= (\widetilde{M}_0-\widetilde{J}_0)/d.\end{aligned} \tag{4.28}$$

Then for these new $\widetilde{M}_k$ we have

$$\widetilde{M}_k=\frac{1}{N}\,\mathrm{tr}\,\frac{1}{\sqrt{\alpha}^k}\,\frac{(\,\mathrm{e}^{\,\lambda}+1)\,\mathrm{e}^{\,\lambda k}}{(\,\mathrm{e}^{\,\lambda}-1)^{2k+1}}\,\frac{\left[1-\frac{\beta}{\sqrt{\alpha}}\frac{\mathrm{e}^{\,\lambda}}{(\,\mathrm{e}^{\,\lambda}+1)^2}\right]^{1/2}}{\left[1-\frac{\xi}{\sqrt{\alpha}}\frac{\mathrm{e}^{\,\lambda}}{(\,\mathrm{e}^{\,\lambda}-1)^2}\right]^{k+1/2}}. \tag{4.29}$$

The expansion in (4.29) goes over the terms

$$H_{ab}=\frac{1}{N}\,\mathrm{tr}\,\frac{(\,\mathrm{e}^{\,\lambda}+1)\,\mathrm{e}^{\,a\lambda}}{(\,\mathrm{e}^{\,\lambda}-1)^{2a+1}}\cdot\frac{\mathrm{e}^{\,b\lambda}}{(\,\mathrm{e}^{\,\lambda}+1)^{2b}}, \tag{4.30}$$

where $b\geq 0$, $a\geq k$.

Let us prove now that H_{ab} can be presented as a linear sum of

$$\begin{aligned}L_{2a} &= \frac{1}{N}\,\mathrm{tr}\,\frac{\partial^{2a}}{\partial\lambda^{2a}}\,\frac{1}{\mathrm{e}^{\,\lambda}-1},\\ R_{2b} &= \frac{1}{N}\,\mathrm{tr}\,\frac{\partial^{2b}}{\partial\lambda^{2b}}\,\frac{1}{\mathrm{e}^{\,\lambda}+1},\end{aligned} \tag{4.31}$$

i.e., the sum goes only over even powers of derivatives in λ:

$$H_{ab}=\sum_{i=0}^{a}\alpha^i_{ab}L_{2i}+\sum_{j=0}^{b-1}\beta^j_{ab}R_{2j}. \tag{4.32}$$

This assertion follows directly from the symmetry properties of H_{ab}:

$$\begin{aligned}H_{ab}(-\lambda) &= -H_{ab}(\lambda),\\ L_i(-\lambda) &= (-1)^{i+1}L_i(\lambda)-\delta_{i,0},\\ R_i(-\lambda) &= (-1)^{i+1}R_i(\lambda)+\delta_{i,0}.\end{aligned} \tag{4.33}$$

Thus, Lemma 1.2a is proved.

Keeping only terms of zero and first orders in traces of λ in the expressions for momenta we get:

$$\begin{aligned}M_k &\sim \frac{1}{\sqrt{\alpha}^{k+1}}\frac{1}{N}\,\mathrm{tr}\,\frac{\mathrm{e}^{\,\lambda(k+1)}}{(\,\mathrm{e}^{\,\lambda}-1)^{2k+1}(\,\mathrm{e}^{\,\lambda}+1)}+\delta_{k1},\\ J_k &\sim \frac{1}{\sqrt{\alpha}^{k+1}}\frac{1}{N}\,\mathrm{tr}\,\frac{\mathrm{e}^{\,\lambda(k+1)}}{(\,\mathrm{e}^{\,\lambda}-1)(\,\mathrm{e}^{\,\lambda}+1)^{2k+1}}+\delta_{k1},\\ d &\sim \sqrt{\alpha}\left\{4-\frac{1}{\alpha}\cdot\frac{1}{N}\,\mathrm{tr}\,\frac{2}{(\,\mathrm{e}^{\,\lambda}-1)(\,\mathrm{e}^{\,\lambda}+1)}\right\}.\end{aligned} \tag{4.34}$$

The terms surviving in the d.s.l. are just the ones arising from the term without reductions on the L.H.S. of (3.23). Therefore, we eventually prove (0.5):

$$\langle\langle\tau_{d_1}\ldots\tau_{d_n}\rangle\rangle_g = \langle\tau_{d_1}\ldots\tau_{d_n}\rangle_g \quad \text{for } d_1+\ldots+d_n = 3g-3+n. \tag{4.35}$$

Note that L_a and R_a are just analogues of the Kontsevich's times $T_n = (2n-1)!!\Lambda^{2n+1}$. They can be transformed into $T_n \cdot 2^n(n-1)!$ in the d.s.l. and in both cases there is no dependence on odd derivatives in Λ. As we have mentioned in Introduction there are two possible ways to do d.s.l. in this model. In the first scenario (4.23–4.24) only one set of times $\{L_n\}$ survives and it turns into the set $\{T_n\}$ in the limit $\varepsilon \to 0$. But if we choose Λ to be symmetrical, $\Lambda = \mathrm{diag}\{\lambda_1,\ldots,\lambda_{N/2},-\lambda_1,\ldots,-\lambda_{N/2}\}$, then another limiting procedure is possible where $L_i = R_i$, and each of these sets generates $\{T_n\}$ thus producing a square of the integral (2.16).

The last note on the reduction procedure concerns the sum over multicomponent reductions on the L.H.S. of (3.23). Using the matrix model technique we have an opportunity to distinguish between different types of reductions mostly due to the remarkable fact that symmetrization $\mathrm{e}^{\lambda} \to -\mathrm{e}^{\lambda}$ goes in each component *separately*. Only this property makes λ–dependent terms different for, say, $\langle\tau_1(\tau_0)^3\rangle_0 \cdot \langle\tau_2\tau_0\rangle_1$ and $\langle\tau_2\tau_1(\tau_0)^4\rangle_0$ (see Fig.6) — both these terms appear in the reduction procedure of the genus two surface with two punctures. But one of them is due to the two–component reduction and another is of one–component type. Evidently, while fixing the number of punctures, n, only terms containing products of exactly n traces of λ contribute to the L.H.S. of (3.23).

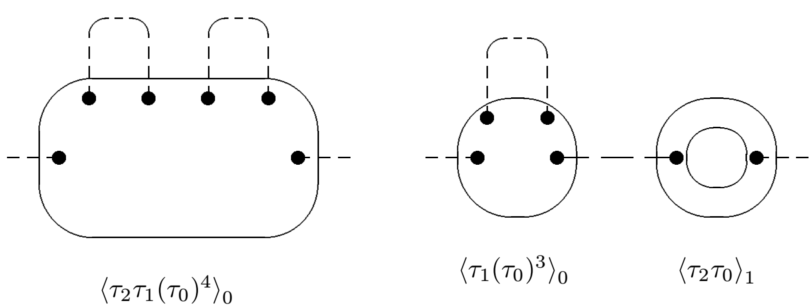


Fig.6. Two examples of one– and two-component reduction for $\mathcal{M}_{2,2}$

5 The moduli space $\overline{\mathcal{M}}_{1,1}$.

Let us consider an example of modular space $\mathcal{M}_{1,1}$, i.e. the torus with one puncture. One can immediately imagine the copy of the modular figure in Teichmüller upper half–plane — a strip from $-1/2$ to $1/2$ along the imaginary axis bounded from below by a segment of a semicircle of the radius 1 with the origin at zero point. In order to get the modular space itself we should identify both sides of the strip as well as two halves of this segment being correspondingly on the left and on the right of the imaginary axis $\Re z = 0$. There are three points where the metric on the moduli space is not conformally flat, namely, $z = i$ (square point), $z = \mathrm{e}^{i\pi/3}$ (or, the same, $\mathrm{e}^{2i\pi/3}$) (triple point), and $z = i\infty$ (infinity point). All these points also have a property that each of them is stable under the action of some operator from the modular transformation group. For triple point the subgroup of such operators has the third order, for square point it is of order 2, and for the infinity point – of an infinite order.

Thus, the modular space $\mathcal{M}_{1,1}$ is an orbifold of the (open) upper half–plane. It was namely this way how Harer and Zagier [24] introduced virtual Euler characteristics for such spaces. And it was Penner [23] who found a simple one–matrix hermitian model with the potential $\log(1+X) - X$ which generated these characteristics. In the Penner approach [23], [25] a factor 1 is assigned to each edge (instead of an arbitrary length in the Kontsevich case). Then for an arbitrary $\mathcal{M}_{g,n}$ there is one–to–one correspondence between the cells of the simplicial decomposition of the *open* moduli space $\mathcal{M}_{g,n}$ and the graphs of the Penner model. Symmetrical coefficients for cells and corresponding graphs coincide. Then the virtual Euler characteristic $\kappa_{g,s}$ can be calculated using the formula:

$$\kappa_{g,s} = \sum_{\substack{cells\\(Graphs)}} \frac{(-1)^{n_G}}{\#\mathrm{Aut}\, G}, \tag{5.36}$$

where n_G is the codimension of the cell in the simplicial complex.

In the case of $\mathcal{M}_{1,1}$ the triple point graph corresponds to the higher dimensional cell, the square point graph – to the cell of codimension 1. In the complex there is also an infinity point of the lowest dimension, but there is no graph corresponding to it. Thus, for the virtual Euler characteristic we get

$$\kappa_{1,1} = \frac{1}{3}\cdot(-1)^0 + \frac{1}{2}\cdot(-1)^1 + \frac{1}{\infty}\cdot(-1)^2 = -\frac{1}{6}. \tag{5.37}$$

Let us now consider the same case, but already in the Kontsevich–Strebel parameterization. We know that there are three types of diagrams depicted in Fig.4a–c. The case of Fig.4a where all l_i are greater than zero corresponds to the cell of the higher dimension. In $\overline{\mathcal{M}}_{1,1}$ it is a domain where $\sum_{i=1}^{3} l_i = p/2$, i.e., it is an interior of the equilateral triangle. Note that due to two possible choices of orientation there are two such congruent cells. The next case is when one of l_i is equal zero (Fig.4b). Taking various l_i to be zero we drive to the boundary of the previous case, *i.e.* , we get open intervals lying on the boundary of the triangles. But it is not the whole boundary as yet — there remains one point at the summit of the triangles and it corresponds to the last case, Fig.4c, where two of l_i are equal zero, that is a point of reduction. The unique reduction of the torus with one puncture is the sphere with three punctures whose modular space $\mathcal{M}_{0,3}$ consists from only one point.

Now let us draw this simplicial complex $\mathcal{T}_{1,1}$ graphically (Fig.7). We are to identify the opposite edges of the parallelogram thus obtaining the torus. This torus complex consists from two open triangles (Fig.4a), three edges separating these triangles (Fig.4b) and the unique point of reduction – the vertex (Fig.4c). The centers of the triangles marked by small discs correspond to two copies of the triple point and centers of edges – to three copies of the square point (small circles). We have six copies of the original modular figure on this torus, one of them is hatched.

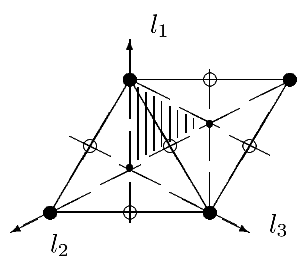


Fig. 7. A simplicial complex for $\overline{\mathcal{M}}_{1,1}$ in the Kontsevich picture.

Thus we result in the conclusion that in the Kontsevich's parameterization the modular space $\overline{\mathcal{M}}_{1,1}$ is the orbifold of a torus $\mathbf{T}^1$ with parameters $(1,\, \mathrm{e}^{i\pi/3})$ which possesses an internal symmetry group $G_{1,1}$ of the sixth order:

$$\overline{\mathcal{M}}_{1,1} = \mathbf{T}^1/G_{1,1}. \tag{5.38}$$

This torus is a totally flat compact space. And here is a point which is different from the Penner construction of orbifolds of the upper half–plane, because there all infinity points are of *infinite* order, and here the order of this point is obviously finite! It means that for this case formula (5.37) will change, and using (5.36) we should add 1/6 to (5.37) thus obtaining zero for our new "virtual Euler characteristic" in the Kontsevich picture.

To complete this geometric part, note that we can think about the torus $\mathbf{T}^1$ as a fundamental domain of the subgroup Γ_2 of the modular group. This domain is depicted on Fig.8 and it again contains six copies of the modular figure. Black discs mark the positions of triple points and small circles – the ones of square points. If we identify the left half–line with the right half–circle and vice versa we shall obtain the torus (if we do not care about conformal properties of this transformation at the infinity point).

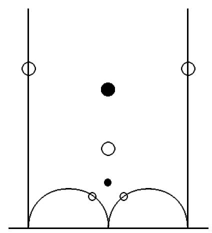


Fig.8. The fundamental domain for subgroup Γ_2 of the modular group.

Let us turn now to our basic formula (3.23). First, using diagram technique for the matrix model (1.C) it is easy to get the answer (after substitution $\Lambda = \mathrm{e}^{\lambda}$). Combining all terms we obtain:

$$F_{1,1} = \alpha^{-1}\frac{3\,\mathrm{e}^{2\lambda}-1}{6(\mathrm{e}^{2\lambda}-1)^3}, \tag{5.39}$$

and we need to express it in terms of derivatives (4.31). Note that formula (5.39) can be obtained from the expansions (4.20) and (4.34) substituting $\alpha \to -\alpha/2$. After a little algebra we get an answer:

$$F_{1,1} = \frac{1}{48\alpha}\cdot\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{\mathrm{e}^{\lambda}-1}-\frac{1}{\mathrm{e}^{\lambda}+1}\right]-\frac{1}{12\alpha}\left[\frac{1}{\mathrm{e}^{\lambda}-1}-\frac{1}{\mathrm{e}^{\lambda}+1}\right]. \tag{5.40}$$

The first term gives us the proper value of $\langle\tau_1\rangle_1 = 1/24$. As for the second term, it originated from the "sum over reductions" and the only reduction of the torus is the sphere with three punctures, for which $\langle\tau_0^3\rangle_0 = 1$. We see that the sum over reductions gives an additional fractional factor 1/6, but now we know the nature of it. In the simplicial complex (Fig.7) there are six copies of the modular space $\overline{\mathcal{M}}_{1,1}$ and only one of the infinity point. So we see that we just have "one sixth" of this point contributing to the expression (3.23) in this order in g and n.

Now we are able to understand the structure of d.m.s. for $\overline{\mathcal{M}}_{1,1}$. In the Kontsevich parameterization we use the form Ω (2.6) in order to evaluate the volume of the corresponding modular space. Since the intersection indices coincide for both continuum and discrete cases, it does not matter how we calculate the total volume of the torus $\mathbf{T}^1$: either by standard continuum integration or by doing a sum over points of integer lattice, each taken with unit measure. For the torus with the perimeter equal p (which is always even) there are exactly $(p/2)^2$ points from d.m.s. lying in $\mathcal{T}_{1,1} = \mathbf{T}^1$. Thus the total volume per one copy of the initial moduli space is $(p/2)^2$ divided by the number of copies, i.e., $p^2/24$ in our case.

6 Conclusions

Let us summarize the obtained results and discuss some unresolved problems.

1. We hope that the established correspondence between the model (1.C) and the discretized moduli spaces (Theorem 1.2) may be useful for the understanding of the structure of $\overline{\mathcal{M}}_{g,n}$. Here we can select the following topics:

First, in the Kontsevich–Strebel parameterization it seems that the compactification of the moduli space $\mathcal{M}_{g,n}$ is not by Deligne–Mumford, since (as the example of $\mathcal{M}_{1,1}$ demonstrates) all points of singular curves have symmetry groups of finite orders. In the standard Teichmüller picture all such points have infinite order symmetry groups thus giving zero contribution to the corresponding virtual Euler characteristics. It seems also true that in the Kontsevich–Strebel picture each moduli space $\overline{\mathcal{M}}_{g,n}$ possesses a covering manifold, $T_{g,n}$, i.e., $\overline{\mathcal{M}}_{g,n} = T_{g,n}/\Gamma_g$, where Γ_g is a symmetry group of a finite order.

Second, due to a possible nonzero curvature of the covering manifold intersection indices $\langle\langle \ldots \rangle\rangle_g$ may differ from the original $\langle \ldots \rangle_g$ for lower orders $\sum d_i < d$. Therefore, these curvature points (submanifolds) could be *singular* points for the Poisson structures on $\overline{\mathcal{M}}_{g,n}$. Here the problem of extracting of symplectic leaves appears [26], [27]. This problem is closely related to the problem of quantization of these structures; an attempt in this direction was made by the author [28], where a quantum group structure was proposed in case of $\mathcal{M}_{1,1}$. There the exceptional representations played an important role. However, all these questions are still on the stage of formulation rather than resolving.

2. A remarkable but a little bit mysterious relation (1.3) establishes a direct bridge from the model (1.C) and, therefore, a one-matrix model, to the Kontsevich matrix model. We stress that there is *no limiting procedure* and one can in principle invert the relations (1.3) and (1.A). Moreover, the intertwining operator $\mathcal{A}$ (1.3) is similar to the free field representation operator in the conformal field theory and both one-matrix model and the Kontsevich model are obviously interacting models. At present we are unable to give any reasonable interpretation for the very existence of such relation.

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Appendix A Proof of Theorem 1.3.

In this Appendix we find and solve constraints on the partition function (1.C) in terms of times corresponding to d.m.s.

A.1 Algebra of times.

We express partition function of (1.C) in times

$$t_k^{\pm} = \frac{1}{N}\,\mathrm{tr}\,\frac{1}{(k+1)!}\frac{\partial^k}{\partial\lambda^k}\frac{1}{\mathrm{e}^{\lambda}\pm 1}. \tag{A.1}$$

We shall find "fusion rules" for times $t_k^{\pm}(\lambda)$:

$$t_k^{\pm}(\lambda_j) = \frac{1}{k!}\frac{\partial^k}{\partial{\lambda_j}^k}\frac{1}{\mathrm{e}^{\lambda_j}\pm 1}. \tag{A.2}$$

An expansion formula for $\frac{1}{\mathrm{e}^{\lambda}-1}$ is

$$\frac{1}{\mathrm{e}^{\lambda}-1} = \frac{1}{\lambda} - \frac{1}{2} + \sum_{m=0}^{\infty}\frac{B_{2m+2}}{(2m+2)!}\lambda^{2m+1} \tag{A.3}$$

where B_m are Bernoulli numbers. Expansion of $\frac{1}{\mathrm{e}^{\lambda}+1}$ is quite the same in the vicinity of the pole $\lambda = i\pi$, but we also need its expansion in λ around origin. Using $\frac{1}{\mathrm{e}^{\lambda}+1} = \frac{1}{\mathrm{e}^{\lambda}-1} - \frac{2}{\mathrm{e}^{2\lambda}-1}$ we have

$$\frac{1}{\mathrm{e}^{\lambda}+1} = \frac{1}{2} + \sum_{m=0}^{\infty} \frac{B_{2m+2}}{(2m+2)!}(1-2^{2m+2})\lambda^{2m+1}. \tag{A.4}$$

Taking derivatives for both expressions we get that all odd times are strictly symmetrical under changing the sign of λ: $\lambda \to -\lambda$:

$$t_k^{\pm}(-\lambda) = t_k^{\pm}(\lambda)(-1)^{k+1} \pm \delta_{k,0}.$$

We find "merging relations" for $t_k^{\pm}(\lambda)$ observing that "negative" times t_k^{-} contain only pure poles of $k+1$-th order in λ. From here we can deduce for odd times:

$$\begin{aligned} t_{2n+1}^{\pm}(\lambda)t_{2m+1}^{\pm}(\lambda) = \; & \pm \; t_{2(n+m)+3}^{\pm}(\lambda) \mp \sum_{k=0}^{m} \frac{B_{2(n+m-k+1)}}{2(n+m-k+1)} \frac{t_{2k+1}^{\pm}(\lambda)}{(2m-2k)!\,(2n+1)!} \\ & \mp \sum_{p=0}^{n} \frac{B_{2(n+m-p+1)}}{2(n+m-p+1)} \frac{t_{2p+1}^{\pm}(\lambda)}{(2n-2p)!\,(2m+1)!}, \end{aligned} \tag{A.5}$$

and for mixing relation:

$$\begin{aligned} t_{2n+1}^{-}(\lambda)t_{2m+1}^{+}(\lambda) = \; & - \sum_{k=0}^{n} \frac{B_{2(n+m-k+1)}}{2(n+m-k+1)} \frac{\left(2^{2(n+m-k+1)}-1\right)}{(2n-2k)!\,(2m+1)!} t_{2k+1}^{-}(\lambda) \\ & + \sum_{k=0}^{m} \frac{B_{2(n+m-k+1)}}{2(n+m-k+1)} \frac{\left(2^{2(n+m-k+1)}-1\right)}{(2m-2k)!\,(2n+1)!} t_{2k+1}^{+}(\lambda). \end{aligned} \tag{A.6}$$

A.2 Constraints on partition function (1.C).

In this chapter we derive an algebra of constraints imposed on the partition function of the model (1.C) in terms of times $t_{2k}^{\pm}$. We start with the matrix integral over hermitian $N \times N$ matrices X:

$$w(\mathrm{e}^{\lambda}) = \log\left(\frac{\int DX\, \exp -\alpha N \operatorname{tr}\left(\frac{1}{4}\Lambda X \Lambda X + \frac{1}{2}\log(1-X) + \frac{1}{2}X\right)}{\int DX\, \exp -\alpha N \operatorname{tr}\left(\frac{1}{4}\Lambda X \Lambda X - \frac{1}{4}X^2\right)}\right), \quad \Lambda \equiv \mathrm{e}^{\lambda}. \tag{A.7}$$

Without loss of generality we may suppose matrix Λ to be diagonal, $\Lambda = \operatorname{diag}\{\Lambda_1, \ldots, \Lambda_N\}$. Changing the variable $X \to -\Lambda^{-1/2}X\Lambda^{-1/2} - 1$ we reduce the upper integral in (A.7) to a standard integral with an external field of GKM type. The integral in the denominator of (A.7) can be easily done, as a result we have

$$\begin{aligned} w(\Lambda) \;=\; & \log\left(\frac{(-1)^{-\frac{\alpha N^2}{2}}(\det\Lambda)^{-N+\frac{\alpha N}{2}}\,\mathrm{e}^{-\frac{\alpha N^2}{2}-\frac{\alpha N}{4}\operatorname{tr}\Lambda^2}}{\prod_{i,j=1}^{N}(\Lambda_i\Lambda_j-1)^{-1/2}\left(\frac{2\pi}{\alpha N}\right)^{N^2/2}}\right. \\ & \left.\times \int DX\, \exp -\alpha N \operatorname{tr}\left(\frac{1}{4}X^2 + \frac{1}{2}\log X + \frac{1}{2}(\Lambda+\Lambda^{-1})X\right)\right). \end{aligned} \tag{A.8}$$

Doing Itzykson–Zuber integration we get rid of the angular variables, and only eigenvalue integration remains (we omit irrelevant numerical factors):

$$\begin{aligned} w(\mathrm{e}^{\lambda}) \;=\; & \log\left(\frac{\mathrm{e}^{\left(\frac{\alpha N}{2}-N\right)\sum_i \lambda_i}\,\mathrm{e}^{-\frac{\alpha N}{4}\sum_i \mathrm{e}^{2\lambda_i}}}{\prod_{i=1}^{N}(\mathrm{e}^{2\lambda_i}-1)^{-1/2}\prod_{i<j}^{N}(\mathrm{e}^{-\lambda_i}-\mathrm{e}^{-\lambda_j})}\right. \\ & \left.\times \int \prod_{i=1}^{N} dx_i\, \mathrm{e}^{-\frac{\alpha N}{2}\sum_{i=1}^{N}\left(\frac{1}{2}x_i^2+\log x_i+(\mathrm{e}^{\lambda_i}+\mathrm{e}^{-\lambda_i})x_i\right)} \prod_{i<j}^{N}(x_i-x_j)\right). \end{aligned} \tag{A.9}$$

Let $\eta_{ij} = (\mathrm{e}^{\lambda} + \mathrm{e}^{-\lambda})_{ij}$. Then we can derive explicitly the Schwinger–Dyson (SD) equations for

$$\mathcal{F}(\mathrm{e}^{\lambda}) = \int DX_{ij}\, \exp\left\{-\frac{\alpha N}{2}\,\mathrm{tr}\left[X^2/2 + \log X + X(\mathrm{e}^{\lambda} + \mathrm{e}^{-\lambda})\right]\right\} \tag{A.10}$$

in terms of η_{ij}. Let $\langle\cdot\rangle$ mean averaging with the exponential measure taken from (A.10). We have

$$\begin{aligned}&\int [DX]\,\frac{\partial}{\partial x_{ij}}\exp\left\{-\frac{\alpha N}{2}\,\mathrm{tr}\left[X^2/2 + \log X + X(\mathrm{e}^{\lambda} + \mathrm{e}^{-\lambda})\right]\right\}\\ &= -\frac{\alpha N}{2}\,\langle x_{ij} + [x^{-1}]_{ij} + \eta_{ij}\rangle = 0.\end{aligned} \tag{A.11}$$

Taking into account that $\langle x_{ij}\rangle = -\frac{2}{\alpha N}\frac{\partial}{\partial\eta_{ij}}\mathcal{F}(\mathrm{e}^{\lambda})$ and doing one additional external derivative in $\frac{\partial}{\partial\eta_{jk}}$ we obtain SD equation for $\mathcal{F}(\Lambda)$ ([17]):

$$\left(\frac{1}{(\alpha N/2)^2}\frac{\partial}{\partial\eta_{ij}}\frac{\partial}{\partial\eta_{jk}} + \delta_{ik}\left(1-\frac{2}{\alpha}\right) - \frac{1}{\alpha N/2}\eta_{ij}\frac{\partial}{\partial\eta_{jk}}\right)\mathcal{F}(\mathrm{e}^{\lambda}) = 0. \tag{A.12}$$

Using the method described in [17] we reduce (A.12) to the equation in terms of eigenvalues η_i of η that are equal to $\mathrm{e}^{\lambda_i} + \mathrm{e}^{-\lambda_i}$, $\frac{\partial}{\partial\eta} \equiv \frac{1}{\mathrm{e}^{\lambda}-\mathrm{e}^{-\lambda}}\frac{\partial}{\partial\lambda}$:

$$\left\{\frac{1}{(\alpha N/2)^2}\left[\frac{\partial^2}{\partial{\eta_j}^2} + \sum_{i\neq j}\frac{\partial/\partial\eta_j - \partial/\partial\eta_i}{\eta_j - \eta_i}\right] + (1-2/\alpha) - \frac{2}{\alpha N}\eta_j\frac{\partial}{\partial\eta_j}\right\}\mathcal{F}(\mathrm{e}^{\lambda}) = 0,$$

or, equivalently,

$$\begin{aligned}&\left\{\frac{1}{(\alpha N/2)^2}\left[\frac{1}{\mathrm{e}^{\lambda_j}-\mathrm{e}^{-\lambda_j}}\vec{\partial}_j\frac{1}{\mathrm{e}^{\lambda_j}-\mathrm{e}^{-\lambda_j}}\vec{\partial}_j + \sum_{i\neq j}\frac{\frac{1}{\mathrm{e}^{\lambda_j}-\mathrm{e}^{-\lambda_j}}\vec{\partial}_j - \frac{1}{\mathrm{e}^{\lambda_i}-\mathrm{e}^{-\lambda_i}}\vec{\partial}_i}{\mathrm{e}^{\lambda_j}+\mathrm{e}^{-\lambda_j} - (\mathrm{e}^{\lambda_i}+\mathrm{e}^{-\lambda_i})}\right]\right.\\ &\left.+(1-2/\alpha) - \frac{2}{\alpha N}[\mathrm{e}^{\lambda_j}+\mathrm{e}^{-\lambda_j}]\frac{1}{\mathrm{e}^{\lambda_j}-\mathrm{e}^{-\lambda_j}}\vec{\partial}_j\right\}\mathcal{F}(\mathrm{e}^{\lambda}) = 0,\quad \vec{\partial}_j \equiv \frac{\partial}{\partial\lambda_j}\end{aligned} \tag{A.13}$$

We are interested in the set of equations for $\mathrm{e}^{w(\lambda)}$ related to $\mathcal{F}(\lambda)$:

$$\mathcal{F}(\lambda) = \mathrm{e}^{w(\lambda)}\prod_{i,j=1}^{N}(\mathrm{e}^{\lambda_i+\lambda_j}-1)^{-1/2}e^{-N(\alpha/2-1)\sum_{i=1}^{N}\lambda_i + \alpha N/4\sum_{i=1}^{N}\mathrm{e}^{2\lambda_i}}. \tag{A.14}$$

Commuting these extra factors with the differentials in (A.13), we eventually get:

$$\begin{aligned}&\left\{\frac{1}{(\alpha N/2)^2}\left[\frac{1}{\mathrm{e}^{\lambda_j}-\mathrm{e}^{-\lambda_j}}\vec{\partial}_j\frac{1}{\mathrm{e}^{\lambda_j}-\mathrm{e}^{-\lambda_j}}\vec{\partial}_j + \sum_{i\neq j}\frac{\frac{1}{\mathrm{e}^{\lambda_j}-\mathrm{e}^{-\lambda_j}}\vec{\partial}_j - \frac{1}{\mathrm{e}^{\lambda_i}-\mathrm{e}^{-\lambda_i}}\vec{\partial}_i}{\mathrm{e}^{\lambda_j}+\mathrm{e}^{-\lambda_j} - (\mathrm{e}^{\lambda_i}+\mathrm{e}^{-\lambda_i})}\right]\right.\\ &+(1-2/\alpha) - \frac{2}{\alpha N}[\mathrm{e}^{\lambda_j}+\mathrm{e}^{-\lambda_j}]\frac{1}{\mathrm{e}^{\lambda_j}-\mathrm{e}^{-\lambda_j}}\vec{\partial}_j\\ &+\frac{1}{(\alpha N/2)^2}\left[\frac{2}{(\mathrm{e}^{\lambda_j}-\mathrm{e}^{-\lambda_j})^2}\left(-\sum_{i=1}^{N}\frac{1}{\mathrm{e}^{\lambda_i+\lambda_j}-1} - \frac{\alpha N}{2} + \frac{\alpha N}{2}\mathrm{e}^{2\lambda_j}\right)\vec{\partial}_j\right]\\ &+\frac{1}{(\alpha N/2)^2}\,\frac{1}{(\mathrm{e}^{\lambda_j}-\mathrm{e}^{-\lambda_j})^2}\left[\frac{\mathrm{e}^{\lambda_j}+\mathrm{e}^{-\lambda_j}}{\mathrm{e}^{\lambda_j}-\mathrm{e}^{-\lambda_j}}\left(\sum_{i=1}^{N}\frac{1}{\mathrm{e}^{\lambda_i+\lambda_j}-1} + \frac{\alpha N}{2}(1-\mathrm{e}^{2\lambda_j})\right)\right.\\ &+\left(\sum_{i\neq j}\frac{\mathrm{e}^{\lambda_i+\lambda_j}}{(\mathrm{e}^{\lambda_i+\lambda_j}-1)^2} + \frac{2\,\mathrm{e}^{2\lambda_j}}{(\mathrm{e}^{2\lambda_j}-1)^2} + \alpha N\,\mathrm{e}^{2\lambda_j}\right)\\ &\left.+\left(\sum_{i=1}^{N}\frac{1}{\mathrm{e}^{\lambda_i+\lambda_j}-1} + \frac{\alpha N}{2}(1-\mathrm{e}^{2\lambda_j})\right)\left(\sum_{k=1}^{N}\frac{1}{\mathrm{e}^{\lambda_k+\lambda_j}-1} + \frac{\alpha N}{2}(1-\mathrm{e}^{2\lambda_j})\right)\right]\end{aligned}$$

$$
+\frac{1}{(\alpha N/2)^2}\sum_{i\neq j}\frac{1}{(\,\mathrm{e}\,^{\lambda}_j-\,\mathrm{e}\,^{\lambda}_i)(1-\,\mathrm{e}\,^{-\lambda_i-\lambda_j})}\times
$$

$$
\times\left[\frac{1}{\,\mathrm{e}\,^{\lambda_j}-\,\mathrm{e}\,^{-\lambda_j}}\left(-\sum_{k=1}^{N}\frac{1}{\,\mathrm{e}\,^{\lambda_k+\lambda_j}-1}-\frac{\alpha N}{2}(1-\,\mathrm{e}\,^{2\lambda_j})\right)\right.
$$

$$
\left.-\frac{1}{\,\mathrm{e}\,^{\lambda_i}-\,\mathrm{e}\,^{-\lambda_i}}\left(-\sum_{k=1}^{N}\frac{1}{\,\mathrm{e}\,^{\lambda_k+\lambda_i}-1}-\frac{\alpha N}{2}(1-\,\mathrm{e}\,^{2\lambda_i})\right)\right]
$$

$$
+\frac{1}{(-\alpha N/2)}\frac{\,\mathrm{e}\,^{\lambda_j}+\,\mathrm{e}\,^{-\lambda_j}}{\,\mathrm{e}\,^{\lambda_j}-\,\mathrm{e}\,^{-\lambda_j}}\left(-\sum_{i=1}^{N}\frac{1}{\,\mathrm{e}\,^{\lambda_j+\lambda_i}-1}-\frac{\alpha N}{2}(1-\,\mathrm{e}\,^{2\lambda_j})\right)\Bigg\}\,\mathrm{e}\,^{w(\lambda)}=0. \tag{A.15}
$$

First, we know that there are no poles in $w(\lambda)$ of the form $\frac{1}{e^{\lambda_i+\lambda_j}-1}$, since the original expression (A.7) is nonsingular at these points. It means that all such terms should factorize into finite sums of times $t^{\pm}_k$ and $t^{\pm}_k(\lambda)$. So far, we deal first with the part of (A.15) which does not contain derivatives in λ. Tedious algebra demonstrates gentle cancellations of all unwanted terms and gives as a result a very simple answer:

$$
\frac{4}{\alpha^2}\left\{\frac{1}{16}\left(t^+_2(\lambda_j)-t^-_2(\lambda_j)-\frac{2}{3}\big(t^+_0(\lambda_j)-t^-_0(\lambda_j)\big)\right)+\frac{N^2}{4}\big((t^+_0)^2t^+_0(\lambda_j)-(t^-_0)^2t^-_0(\lambda_j)\big)\right\}
$$

$$
=\quad\frac{4}{\alpha^2}\frac{\partial}{\partial\lambda_j}\left\{\frac{1}{16}\left(t^+_2-t^-_2-\frac{2}{3}(t^+_0-t^-_0)\right)+N^2\frac{(t^+_0)^3-(t^-_0)^3}{12}\right\} \tag{A.16}
$$

Next, we already prove from other viewpoint in Chapter 4 (and are able to prove the same from explicit analysis of times dependence) that $w(\lambda)$ depends only on even times $t^{\pm}_{2k}$, $k\geq 0$. On an intermediate stage we get:

$$
\frac{1}{\alpha^2N^2}\big(t^+_1(\lambda_j)-t^-_1(\lambda_j)\big)\sum_{k,p=0}^{\infty}\sum_{(\pm)(\pm)}t^{\pm}_{2k+1}(\lambda_j)t^{\pm}_{2p+1}(\lambda_j)\left[\frac{\partial w(\lambda)}{\partial t^{\pm}_{2k}}\frac{\partial w(\lambda)}{\partial t^{\pm}_{2p}}+\frac{\partial^2 w(\lambda)}{\partial t^{\pm}_{2k}\partial t^{\pm}_{2p}}\right]
$$

$$
+\frac{2}{\alpha}\sum_{\substack{k=0\\ \pm}}^{\infty}t^{\pm}_{2k+1}(\lambda_j)\frac{\partial}{\partial t^{\pm}_{2k}}w(\lambda)
$$

$$
+\frac{4}{\alpha^2}\left\{\sum_{\substack{k=0\\ (\pm)}}^{\infty}t^{\pm}_{2k+1}(\lambda_j)\frac{1}{2}\big(t^+_1(\lambda_j)t^+_0+t^-_1(\lambda_j)t^-_0\big)\frac{\partial}{\partial t^{\pm}_{2k}}w(\lambda)\right.
$$

$$
-\sum_{\substack{k=0\\ (\pm)}}^{\infty}\sum_{i=1}^{N}\frac{1}{(2k+1)!\,(\,\mathrm{e}\,^{\lambda_i}-\,\mathrm{e}\,^{-\lambda_i})(1-\,\mathrm{e}\,^{-\lambda_i-\lambda_j})}\times
$$

$$
\left.\times\left[1+\frac{\partial}{\partial\lambda_i}+\frac{\partial}{\partial\lambda_j}\right]^{2k+1}\frac{1}{(\,\mathrm{e}\,^{\lambda_i}\pm1)(\,\mathrm{e}\,^{\lambda_j}\pm1)}\frac{\partial}{\partial t^{\pm}_{2k}}w(\lambda)\right\}
$$

$$
+\frac{4}{\alpha^2}\frac{\partial}{\partial\lambda_j}\left\{\frac{1}{16}\left(t^+_2-t^-_2-\frac{2}{3}(t^+_0-t^-_0)\right)+N^2\frac{(t^+_0)^3-(t^-_0)^3}{12}\right\}=0. \tag{A.17}
$$

From this expression we already can select a part standing by some chosen $t^{\pm}_{2k+1}(\lambda_j)$. Every such a part generates some linearly (but not algebraically) independent constraint on $\mathrm{e}^{w(\lambda)}$. The only trouble is with the middle term with one derivative originated from the "integral" term in SD equations. We treat it now in details, since it is the only one which needs some trick to deal with.

We start with an identity

$$
\frac{1}{1-\,\mathrm{e}\,^{-x-y}}(1+\partial_x+\partial_y)^{2k+1}\frac{1}{(\,\mathrm{e}\,^{x}\pm1)(\,\mathrm{e}\,^{y}\pm1)}
$$

$$
=\frac{1}{\,\mathrm{e}\,^{y}-\,\mathrm{e}\,^{x}}(\partial_y-\partial_x)^{2k+1}\frac{\pm\,\mathrm{e}\,^{x+y}}{(\,\mathrm{e}\,^{x}\pm1)(\,\mathrm{e}\,^{y}\pm1)} \tag{A.18}
$$

which is actually due to the symmetry of the expression in x and y both under transformation $x \to -x$ and $y \to -y$. For the L.H.S. of (A.18) we have:

$$\begin{aligned}
&\frac{1}{1-\mathrm{e}^{-x-y}}(1+\partial_x+\partial_y)^{2k+1}\frac{1}{(\mathrm{e}^{x}\pm 1)(\mathrm{e}^{y}\pm 1)} \\
=\ &\frac{1}{\mathrm{e}^{x}-\mathrm{e}^{-y}}(\partial_y+\partial_x)^{2k+1}(\pm)\frac{\mathrm{e}^{x-y}}{(\mathrm{e}^{x}\pm 1)(\mathrm{e}^{y}\pm 1)} \\
=\ &\sum_{n,m} d^k_{n,m} t^{\pm}_{2n+1}(x) t^{\pm}_{2m+1}(y),
\end{aligned} \tag{A.19}$$

where $d^k_{n,m} \in \mathbf{C}$ are some coefficients. We multiply the R.H.S. of (A.18) and the L.H.S. of (A.19) by $\mathrm{e}^{y}-\mathrm{e}^{x}$ and $\mathrm{e}^{y}-\mathrm{e}^{-x}$, respectively, in order to eliminate the prefactors in front of the derivative terms, and then subtract one expression from the other. It gives

$$\begin{aligned}
\sum_{n,m} d^k_{n,m}(\mathrm{e}^{y}-\mathrm{e}^{-y}) t^{\pm}_{2n+1}(y) t^{\pm}_{2m+1}(x) &= (\partial_y-\partial_x)^{2k+1}\left[-\frac{1}{\mathrm{e}^{x}\pm 1}-\frac{\mathrm{e}^{x}}{(\mathrm{e}^{x}\pm 1)(\mathrm{e}^{y}\pm 1)}\right] \\
&\quad +(\partial_y+\partial_x)^{2k+1}\frac{\mathrm{e}^{x}}{(\mathrm{e}^{x}\pm 1)(\mathrm{e}^{y}\pm 1)}.
\end{aligned} \tag{A.20}$$

From this relation it is already easy to find that

$$\begin{aligned}
&-\sum_{\substack{k=0\\(\pm)}}^{\infty}\sum_{i=1}^{N}\frac{1}{(\mathrm{e}^{\lambda_i}-\mathrm{e}^{-\lambda_i})(1-\mathrm{e}^{-\lambda_i-\lambda_j})}\left[1+\frac{\partial}{\partial\lambda_i}+\frac{\partial}{\partial\lambda_j}\right]^{2k+1}\frac{1}{(\mathrm{e}^{\lambda_i}\pm 1)(\mathrm{e}^{\lambda_j}\pm 1)} \\
=\ &\pm\sum_{i=1}^{N}\frac{2}{(\mathrm{e}^{\lambda_i}-\mathrm{e}^{-\lambda_i})^2}\sum_{n=1}^{k}\begin{pmatrix}2n\\2k+1\end{pmatrix}\left(\frac{\partial}{\partial\lambda_i}\right)^{2n}\frac{1}{\mathrm{e}^{\lambda_i}\pm 1}\left(\frac{\partial}{\partial\lambda_j}\right)^{2(k-n)+1}\frac{1}{\mathrm{e}^{\lambda_j}\pm 1} \\
&\pm\left(\frac{\partial}{\partial\lambda_j}\right)^{2k+1}\frac{1}{\mathrm{e}^{\lambda_j}\pm 1}\left(\frac{3}{2}t^{\pm}_2-\frac{1}{8}(t^{+}_0+t^{-}_0)\right)
\end{aligned} \tag{A.21}$$

Combining all terms from (A.17), we shall obtain a set of conditions on $\mathrm{e}^{w(\lambda)}$ of the form:

$$\sum_{k=0}^{\infty} t^{+}_{2k+1}(\lambda_j)\big(L^{+}_{2k+1}\,\mathrm{e}^{w(\lambda)}\big)+\sum_{k=0}^{\infty} t^{-}_{2k+1}(\lambda_j)\big(L^{-}_{2k+1}\,\mathrm{e}^{w(\lambda)}\big),$$

which are valid for any j. We assume that all traces of matrix Λ are independent. Note, however, that from this it does not follow that all times t^{+}_{2k+1} and t^{-}_{2k+1} are linearly independent. There exists a formula that connects these two sets of times by re-expansion of positive times via negative ones using the Taylor expansion and the shift relation $t^{+}_s(\lambda) = -t^{-}_s(\lambda+i\pi)$. But as far as we are looking for an expansion of $w(\lambda)$ over additional parameters N and α, and in each fixed order in N and α the time dependence is polynomial, then it follows that this expansion is unique (the connection formulas between positive and negative times are obviously non-polynomial). So, we treat all operators $L^{\pm}_{2k+1}$ as independent generators of the constraint algebra for $\mathrm{e}^{w(\lambda)}$. As usual, the first generator, $L^{\pm}_1$, is somehow selected from the whole set and we give it separately:

$$\begin{aligned}
L^{+}_1 = \frac{1}{\alpha^2N^2}\Bigg\{ &-\sum_{n,m=0}^{\infty}\frac{B_{2(n+m+2)}}{(2n+2m+4)!}\big(1+(2n+2m+3)2^{2(n+m+2)}\big)\frac{\partial}{\partial t^{+}_{2m}}\frac{\partial}{\partial t^{+}_{2n}} \\
&+2\sum_{n,m=0}^{\infty}\Bigg(\sum_{k=0}^{m}\frac{B_{2(n+m-k+1)}}{2(n+m-k+1)}\frac{1}{(2m-2k)!\,(2n+1)!}\times \\
&\qquad\times\frac{B_{2(k+1)}}{(2k+2)!}\big(1+(2k+1)2^{2(k+1)}\big)\Bigg)\frac{\partial}{\partial t^{+}_{2m}}\frac{\partial}{\partial t^{+}_{2n}} \\
&-\sum_{n,m=0}^{\infty}\frac{B_{2(n+m+2)}}{(2n+2m+4)!}\big(2^{2(n+m+2)}-1\big)\frac{\partial}{\partial t^{-}_{2m}}\frac{\partial}{\partial t^{-}_{2n}}
\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{n,m=0}^{\infty}\left(\sum_{k=0}^{m}\frac{B_{2(n+m-k+1)}}{2(n+m-k+1)}\frac{1}{(2m-2k)!\,(2n+1)!}\times\right.\\
&\qquad\left.\times\frac{B_{2(k+1)}}{(2k+2)!}\left(2^{2(k+1)}-1\right)\right)\frac{\partial}{\partial t^{-}_{2m}}\frac{\partial}{\partial t^{-}_{2n}}\\
&-2\sum_{n,m=0}^{\infty}\left(\sum_{k=0}^{m}\frac{B_{2(n+m-k+1)}}{2(n+m-k+1)}\frac{2^{2(n+m-k+1)}-1}{(2m-2k)!\,(2n+1)!}\times\right.\\
&\qquad\left.\times\frac{B_{2(k+1)}}{(2k+2)!}\left(1+(2k+1)2^{2(k+1)}\right)\right)\frac{\partial}{\partial t^{-}_{2n}}\frac{\partial}{\partial t^{+}_{2m}}\\
&-2\sum_{n,m=0}^{\infty}\left(\sum_{k=0}^{n}\frac{B_{2(n+m-k+1)}}{2(n+m-k+1)}\frac{2^{2(n+m-k+1)}-1}{(2n-2k)!\,(2m+1)!}\times\right.\\
&\qquad\left.\times\frac{B_{2(k+1)}}{(2k+2)!}\left(2^{2(k+1)}-1\right)\right)\frac{\partial}{\partial t^{-}_{2n}}\frac{\partial}{\partial t^{+}_{2m}}\Bigg\}\\
&+\frac{2}{\alpha}\frac{\partial}{\partial t^{+}_{0}}+\frac{2}{\alpha^2}\left[-\sum_{n=0}^{\infty}\frac{B_{2(n+1)}}{(2n+2)!}t^{+}_{0}\frac{\partial}{\partial t^{+}_{2n}}+\sum_{n=0}^{\infty}\frac{B_{2(n+1)}}{(2n+2)!}(2^{2n+2}-1)t^{+}_{0}\frac{\partial}{\partial t^{-}_{2n}}\right.\\
&\left.+\sum_{n=0}^{\infty}\left((2n+3)t^{+}_{2n+2}-\sum_{m=0}^{n}(2n-2m+1)\frac{B_{2m+2}}{2m+2}\frac{2^{2m+2}}{(2m)!}t^{+}_{2(n-m)}\right)\frac{\partial}{\partial t^{+}_{2n}}\right]\\
&+\frac{N^2}{\alpha^2}(t^{+}_{0})^2-\frac{1}{6\alpha^2},
\end{aligned}
\tag{A.22}
$$

and for $s>0$

$$
\begin{aligned}
L^{+}_{2s+1} \;=\;& \frac{1}{\alpha^2N^2}\left\{\sum_{m=0}^{s-2}\frac{\partial}{\partial t^{+}_{2m}}\frac{\partial}{\partial t^{+}_{2(s-m-2)}}\right.\\
&-\sum_{n+m\geq s-1}^{\infty}\frac{B_{2(n+m+2-s)}}{2(n+m+2-s)}\frac{2^{2(n+m+2-s)}}{(2n+2m+2-2s)!}\frac{\partial}{\partial t^{+}_{2m}}\frac{\partial}{\partial t^{+}_{2n}}\\
&-2\sum_{n=0}^{\infty}\sum_{m=s-1}^{\infty}\frac{B_{2(n+m+2-s)}}{2(n+m+2-s)}\frac{1}{(2m-2s+2)!\,(2n+1)!}\frac{\partial}{\partial t^{+}_{2n}}\frac{\partial}{\partial t^{+}_{2m}}\\
&+2\sum_{n=0}^{\infty}\sum_{m=s}^{\infty}\left(\sum_{k=0}^{m-s}\frac{B_{2(n+m-s-k+1)}}{2(n+m-s-k+1)}\frac{1}{(2m-2k-2s)!\,(2n+1)!}\times\right.\\
&\qquad\left.\times\frac{B_{2k+2}}{2k+2}\frac{2^{2k+2}}{(2k)!}\right)\frac{\partial}{\partial t^{+}_{2n}}\frac{\partial}{\partial t^{+}_{2m}}\\
&+2\sum_{n=0}^{\infty}\sum_{m=s-1}^{\infty}\frac{B_{2(n+m+2-s)}}{2(n+m+2-s)}\frac{2^{2(n+m+2-s)}-1}{(2m-2s+2)!\,(2n+1)!}\frac{\partial}{\partial t^{-}_{2n}}\frac{\partial}{\partial t^{+}_{2m}}\\
&-2\sum_{n=0}^{\infty}\sum_{m=s}^{\infty}\left(\sum_{k=0}^{m-s}\frac{B_{2(n+m-s-k+1)}}{2(n+m-s-k+1)}\frac{2^{2(n+m-s-k+1)}-1}{(2m-2k-2s)!\,(2n+1)!}\times\right.\\
&\qquad\left.\left.\times\frac{B_{2k+2}}{2k+2}\frac{2^{2k+2}}{(2k)!}\right)\frac{\partial}{\partial t^{-}_{2n}}\frac{\partial}{\partial t^{+}_{2m}}\right\}\\
&+\frac{2}{\alpha}\frac{\partial}{\partial t^{+}_{2s}}+\frac{2}{\alpha^2}\left[t^{+}_{0}\frac{\partial}{\partial t^{+}_{2(s-1)}}+\sum_{n=0}^{\infty}\left((2n+3)t^{+}_{2n+2}\right.\right.\\
&\left.\left.-\sum_{m=0}^{n}(2n-2m+1)\frac{B_{2m+2}}{2m+2}\frac{2^{2m+2}}{(2m)!}t^{+}_{2(n-m)}\right)\frac{\partial}{\partial t^{+}_{2(n+s)}}\right]+\frac{\delta_{s,1}}{4\alpha^2}.
\end{aligned}
\tag{A.23}
$$

Analogous formulas for L^{-}_{2s+1} can be obtained changing all $t^{+}_{2s}\leftrightarrow -t^{-}_{2s}$ and $\frac{\partial}{\partial t^{+}_{2s}}\leftrightarrow -\frac{\partial}{\partial t^{-}_{2s}}$.

A.3 Algebra of constraints $L^{\pm}_{2s+1}$.

We derive now commutation relations for generators $L^{\pm}_{2s+1}$. Tedious but again direct calculations show

$$[L^{+}_{\cdot}, L^{-}_{\cdot}] \equiv 0, \tag{A.24}$$

i.e. in spite of the fact that generators $L^{+}_{\cdot}$ and $L^{-}_{\cdot}$ contain derivatives in both positive and negative times, two halves of the algebra factorize.

Let us consider the algebra of $L^{+}_{\cdot}$. We have

$$[L^{+}_{2s+1}, L^{+}_{2t+1}] = \frac{4(s-t)}{\alpha^2}\left(L^{+}_{2s+2t-1} - \sum_{m=0}^{\infty}\frac{B_{2m+2}}{2m+2}\frac{2^{2m+2}}{(2m)!}L^{+}_{2(s+t+m)+1}\right). \tag{A.25}$$

After an upper triangular transformation of generators:

$$\widetilde{L}_s = \sum_{k=0}^{\infty}\frac{2^{2k+1}}{(2k+2)!}L^{+}_{2(s+k)+3}, \qquad s \geq -1, \tag{A.26}$$

we arrive to the standard Virasoro algebra :

$$[\widetilde{L}_s, \widetilde{L}_t] = \frac{4}{\alpha^2}(s-t)\widetilde{L}_{s+t}, \quad s,t \geq -1. \tag{A.27}$$

We are interested in the time transformation corresponding to the change of generators (A.26). In fact, by analysis in orders of α and N, we know that in order by order calculations the lowest term is the one from (A.23) that is equal to $\frac{2}{\alpha}\frac{\partial}{\partial t^{+}_{2s}}$. So we look for such deformed times $\widetilde{t}$, in which this term preserves its form in $\widetilde{L}_s$. It means that

$$\frac{\partial}{\partial \widetilde{t}_s} = \sum_{k=0}^{\infty}\frac{2^{2k+1}}{(2k+2)!}\frac{\partial}{\partial t^{+}_{2(s+k)}}, \quad s \geq 0.$$

A solution to this set of equations is provided by lower triangular transformed times:

$$\widetilde{t}^{+}_s = t^{+}_{2s} - \sum_{m=0}^{s-1}\frac{B_{2m+2}}{2m+2}\frac{2^{2m+2}}{(2m)!}t^{+}_{2(s-m-1)}, \quad s \geq 0. \tag{A.28}$$

(Note that most of these transformations are based on an identity:

$$\left(\frac{1}{x^2} - \sum_{m=0}^{\infty}\frac{B_{2m+2}}{2m+2}\frac{2^{2m+2}}{(2m)!}x^{2m}\right)\left(\sum_{k=0}^{\infty}\frac{2^{2k+1}}{(2k+2)!}x^{2k+2}\right) \equiv 1,$$

which also generates famous relations for Bernoulli numbers.)

In fact, there is no great simplification of the formulas for $\widetilde{L}_s$ in terms of these new times. We present here only the expression for the part of $\widetilde{L}_s$, $s \geq 0$, which is linear in derivatives:

$$\text{linear in derivatives part of } \widetilde{L}_s = \frac{2}{\alpha}\frac{\partial}{\partial \widetilde{t}_{s+1}} + \frac{2}{\alpha^2}\left\{\widetilde{t}_0\frac{\partial}{\partial \widetilde{t}_s} + \sum_{n=0}^{\infty}(2n+3)\widetilde{t}_{n+1}\frac{\partial}{\partial \widetilde{t}_{n+s+1}}\right.$$
$$\left. + \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{B_{2k+2}}{(2k+2)!}2^{2k+3}\widetilde{t}_i\frac{\partial}{\partial \widetilde{t}_{k+i+s+1}}\right\}, \quad s \geq 0. \tag{A.29}$$

Eventually, L^{+}_{2s+1} were combined with times $t^{+}_{2s+1}(\lambda)$. $\widetilde{t}_n(\lambda)$ standing with $\widetilde{L}_s$ are

$$\widetilde{t}_n(\lambda) = t^{+}_{2n+1}(\lambda) - \sum_{k=0}^{n-1}\frac{B_{2k+2}}{2k+2}\frac{2^{2k+2}}{(2k)!}t^{+}_{2n-2k-1}(\lambda), \qquad n \geq 0,$$

or, in terms of the Bernoulli polynomials:

$$\widetilde{t}^{\pm}_n(\lambda) = \frac{2^{2n+1}}{(2n+1)!}\left[(2n+1)\frac{\partial}{2\partial\lambda}B_{2n}\left(\frac{\partial}{2\partial\lambda}\right) - 2nB_{2n+1}\left(\frac{\partial}{2\partial\lambda}\right)\right]\frac{1}{\mathrm{e}^{\lambda}\pm 1}, \tag{A.30}$$

where $B_n(x)=\sum_{s=0}^{n}\binom{s}{n}B_s x^{n-s}$, $B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_4=-1/30$, $B_6=1/42$, $B_8=-1/30$, $B_{10}=5/66$, etc, $B_{2n+1}=0$ for $n>0$.

We also present two formulas showing how "non-tilde" times are expressed via $\widetilde{t}_n$:

$$\frac{\partial}{\partial t^+_{2n}} = \frac{\partial}{\partial\widetilde{t}_n}-\sum_{p=0}^{\infty}\frac{B_{2p+2}}{2p+2}\frac{2^{2p+2}}{(2p)!}\frac{\partial}{\partial\widetilde{t}_{n+p+1}} \tag{A.31}$$

$$t^+_{2n} = \sum_{m=0}^{n}\frac{2^{2m+1}}{(2m+2)!}\widetilde{t}_{n-m}. \tag{A.32}$$

Let us now turn again to the "old" (non-tilde) times. We use expressions (A.5) and (A.6) in order to simplify expression (A.22). For example, we consider the term with two "minus" derivatives which occupies third to fifth lines in (A.22). It originates from the term proportional to $t^+_0(\lambda)$ in the expansion of $t^-_{2n+1}(\lambda)t^-_{2m+1}(\lambda)t^+_0(\lambda)$. If we first expand $t^-_{2m+1}(\lambda)t^+_0(\lambda)$ and after that merge it with $t^-_{2n+1}(\lambda)$, then it appears that it simplifies drastically and converts into

$$\sum_{m,n=0}^{\infty}\frac{B_{2n+2}}{(2n+2)!}(2^{2n+2}-1)\frac{B_{2m+2}}{(2m+2)!}(2^{2m+2}-1)\frac{\partial}{\partial t^-_{2n}}\frac{\partial}{\partial t^-_{2m}}.$$

It is also worth to note that after shift (A.26) most of "tails" in formulas (A.22) and (A.23) disappeared and as a result we have simplified expressions for $\widetilde{L}_{-1}$:

$$\begin{aligned}\widetilde{L}_{-1} &= \frac{1}{\alpha^2N^2}\Bigg\{-2\sum_{n,m=0}^{\infty}\frac{B_{2(n+m+2)}}{2(n+m+2)}\frac{1}{(2n+1)!\,(2m+2)!}\frac{\partial}{\partial t^+_{2n}}\frac{\partial}{\partial t^+_{2m}}\\ &\quad+2\sum_{n,m=0}^{\infty}\frac{B_{2(n+m+2)}}{2(n+m+2)}\frac{2^{2(n+m+2)}-1}{(2n+1)!\,(2m+2)!}\frac{\partial}{\partial t^-_{2n}}\frac{\partial}{\partial t^+_{2m}}\\ &\quad+\left(\sum_{n=0}^{\infty}\frac{B_{2n+2}}{(2n+2)!}\frac{\partial}{\partial t^+_{2n}}\right)^2-2\left(\sum_{n=0}^{\infty}\frac{B_{2n+2}}{(2n+2)!}\frac{\partial}{\partial t^+_{2n}}\right)\left(\sum_{m=0}^{\infty}\frac{B_{2m+2}}{(2m+2)!}(2^{2m+2}-1)\frac{\partial}{\partial t^-_{2m}}\right)\\ &\quad+\left(\sum_{n=0}^{\infty}\frac{B_{2n+2}}{(2n+2)!}(2^{2n+2}-1)\frac{\partial}{\partial t^-_{2n}}\right)^2\Bigg\}\\ &\quad+\frac{2}{\alpha}\sum_{k=0}^{\infty}\frac{2^{2k+1}}{(2k+2)!}\frac{\partial}{\partial t^+_{2k}}+\frac{2}{\alpha^2}\Bigg\{\sum_{n=0}^{\infty}(2n+3)t^+_{2n+2}\frac{\partial}{\partial t^+_{2n}}\\ &\quad-\sum_{n=0}^{\infty}\frac{B_{2n+2}}{(2n+2)!}t^+_0\left(\frac{\partial}{\partial t^+_{2n}}-(2^{2n+2}-1)\frac{\partial}{\partial t^-_{2n}}\right)\Bigg\}+\frac{N^2}{\alpha^2}\left(t^+_0\right)^2-\frac{1}{12\alpha^2},\end{aligned} \tag{A.33}$$

and for $\widetilde{L}_s$, $s\geq 0$:

$$\begin{aligned}\widetilde{L}_s &= \frac{1}{\alpha^2N^2}\Bigg\{\sum_{m=0}^{s-1}\frac{\partial}{\partial t^+_{2m}}\frac{\partial}{\partial t^+_{2(s-m-2)}}-2\sum_{n,m=0}^{\infty}\frac{B_{2(n+m+1)}}{2(n+m+1)}\frac{1}{(2n+1)!\,(2m)!}\frac{\partial}{\partial t^+_{2n}}\frac{\partial}{\partial t^+_{2(m+s)}}\\ &\quad+2\sum_{n,m=0}^{\infty}\frac{B_{2(n+m+1)}}{2(n+m+1)}\frac{2^{2(n+m+1)}-1}{(2n+1)!\,(2m)!}\frac{\partial}{\partial t^-_{2n}}\frac{\partial}{\partial t^+_{2(m+s)}}\Bigg\}\\ &\quad+\frac{2}{\alpha}\sum_{k=0}^{\infty}\frac{2^{2k+1}}{(2k+2)!}\frac{\partial}{\partial t^+_{2(s+k+1)}}+\frac{2}{\alpha^2}\sum_{n=0}^{\infty}(2n+1)t^+_{2n}\frac{\partial}{\partial t^+_{2(n+s)}}+\frac{\delta_{s,0}}{4\alpha^2},\qquad s\geq 0.\end{aligned} \tag{A.34}$$

We checked explicitly that these generators do satisfy two halves of Virasoro algebras Vir_+:

$$\begin{aligned}[\widetilde{L}^\pm_s,\widetilde{L}^\pm_t] &= \frac{4}{\alpha^2}(s-t)\widetilde{L}^\pm_{s+t},\quad s,t\geq -1,\\ \left[\widetilde{L}^+_s,\widetilde{L}^-_t\right] &= 0\ \text{ for all } s,t,\end{aligned} \tag{A.35}$$

where $\widetilde{L}_s \equiv \widetilde{L}_s^+$, and $\widetilde{L}_s^-$ are obtained from $\widetilde{L}_s$ by the interchange $t_{2s}^+ \leftrightarrow -t_{2s}^-$ and $\frac{\partial}{\partial t_{2s}^+} \leftrightarrow -\frac{\partial}{\partial t_{2s}^-}$.

It is worth noting that expression (A.33) can be rewritten in the form

$$
\begin{aligned}
\widetilde{L}_{-1} \;\; = \;\; & \frac{1}{\alpha^2 N^2}\Bigg\{-2\sum_{n,m=0}^{\infty}\frac{B_{2(n+m+2)}}{2(n+m+2)}\frac{1}{(2n+1)!\,(2m+2)!}\frac{\partial}{\partial t_{2n}^+}\frac{\partial}{\partial t_{2m}^+} \\
& +2\sum_{n,m=0}^{\infty}\frac{B_{2(n+m+2)}}{2(n+m+2)}\frac{2^{2(n+m+2)}-1}{(2n+1)!\,(2m+2)!}\frac{\partial}{\partial t_{2n}^-}\frac{\partial}{\partial t_{2m}^+}\Bigg\} \\
& +\frac{2}{\alpha}\sum_{k=0}^{\infty}\frac{2^{2k+1}}{(2k+2)!}\frac{\partial}{\partial t_{2k}^+}+\frac{2}{\alpha^2}\sum_{n=0}^{\infty}(2n+3)t_{2n+2}^+\frac{\partial}{\partial t_{2n}^+} \\
& +\frac{N^2}{\alpha^2}\left[t_0^+-\frac{1}{N^2}\sum_{n=0}^{\infty}\frac{B_{2n+2}}{(2n+2)!}\left(\frac{\partial}{\partial t_{2n}^+}-(2^{2n+2}-1)\frac{\partial}{\partial t_{2n}^-}\right)\right]^2, \qquad \text{(A.36)}
\end{aligned}
$$

where all derivatives are assumed to act on the right (that eliminates the constant term). This formula gives us a hint how to simplify further the expressions (A.33) and (A.34).

Let us look for such a canonical transformation of times and their derivatives that shifts times by some linear combination of derivatives plus constant terms and leaves time derivatives unchanged:

$$
\begin{aligned}
\widehat{t_{2n}^{\pm}} \;\; &= \;\; \mathrm{e}^{-\mathcal{A}} t_{2n}^{\pm}\,\mathrm{e}^{\mathcal{A}} = \\
&= \;\; t_{2n}^{\pm}-\frac{1}{N^2}\sum_{m=0}^{\infty}\frac{B_{2(n+m+1)}}{2(n+m+1)}\frac{1}{(2n+1)!\,(2m+1)!}\left(\frac{\partial}{\partial t_{2m}^{\pm}}-(2^{2(n+m+1)}-1)\frac{\partial}{\partial t_{2m}^{\mp}}\right) \\
& \quad \pm(1-\delta_{n,0}-\delta_{n,1})\alpha\frac{2^{2n-1}}{(2n+1)!}, \qquad \text{(A.37)} \\
\frac{\partial}{\partial \widehat{t_{2n}^{\pm}}} \;\; &= \;\; \mathrm{e}^{-\mathcal{A}}\frac{\partial}{\partial t_{2n}^{\pm}}\mathrm{e}^{\mathcal{A}} = \frac{\partial}{\partial t_{2n}^{\pm}}. \qquad \text{(A.38)}
\end{aligned}
$$

This immediately gives us

$$
\begin{aligned}
\mathcal{A} \;\; = \;\; & \frac{1}{N^2}\sum_{m,n=0}^{\infty}\frac{B_{2(n+m+1)}}{4(n+m+1)}\frac{1}{(2n+1)!\,(2m+1)!}\times \\
& \times\left\{\frac{\partial}{\partial t_{2m}^+}\frac{\partial}{\partial t_{2n}^+}+\frac{\partial}{\partial t_{2m}^-}\frac{\partial}{\partial t_{2n}^-}-2(2^{2(n+m+1)}-1)\frac{\partial}{\partial t_{2m}^+}\frac{\partial}{\partial t_{2n}^-}\right\} \\
& +\sum_{n=2}^{\infty}\alpha\frac{2^{2n-1}}{(2n+1)!}\left(\frac{\partial}{\partial t_{2n}^-}-\frac{\partial}{\partial t_{2n}^+}\right). \qquad \text{(A.39)}
\end{aligned}
$$

Making now the last substitution:

$$
\begin{aligned}
T_n^{\pm} \;\; &= \;\; N\widehat{t_{2n}^{\pm}} \\
\frac{\partial}{\partial T_n^{\pm}} \;\; &= \;\; N^{-1}\frac{\partial}{\partial \widehat{t_{2n}^{\pm}}}, \qquad \text{(A.40)}
\end{aligned}
$$

we eventually obtain a simple expression for the Virasoro generators (A.33) and (A.34) in terms of $T_{\cdot}^{\pm}$:

$$
\mathcal{L}_{-1}^{\pm}=\frac{\alpha^2}{4}\widetilde{L}_{-1}^{\pm} \;\; = \;\; \frac{1}{2}\sum_{n=0}^{\infty}(2n+3)T_{n+1}^{\pm}\frac{\partial}{\partial T_n^{\pm}}+\frac{T_0^{\pm\,2}}{4}+\frac{\alpha N}{2}\frac{\partial}{\partial T_0^{\pm}} \qquad \text{(A.41)}
$$

$$
\begin{aligned}
\mathcal{L}_s^{\pm}=\frac{\alpha^2}{4}\widetilde{L}_s^{\pm} \;\; = \;\; & \frac{1}{4}\sum_{m=0}^{s-1}\frac{\partial}{\partial T_m^{\pm}}\frac{\partial}{\partial T_{s-m-1}^{\pm}}+\frac{1}{2}\sum_{n=0}^{\infty}(2n+1)T_n^{\pm}\frac{\partial}{\partial T_{n+s}^{\pm}} \\
& +\frac{\alpha N}{2}\frac{\partial}{\partial T_{s+1}^{\pm}}+\frac{\delta_{s,0}}{16}, \quad s\geq 0. \qquad \text{(A.42)}
\end{aligned}
$$

But these Virasoro conditions are nothing but the Virasoro constraints of the Kontsevich matrix model! (see [5]). Their solution is well-known. It satisfies the KdV equations and was elaborated to the third and the forth orders in [16]. Therefore, there exists a canonical transformation of variables of the initial matrix model (A.7) that reduces it to two copies of the Kontsevich model. Taking into account that the vacuum is invariant under these transformations we obtain for (A.7):

$$\mathrm{e}^{\,w(t^+_\cdot,t^-_\cdot)} = \mathrm{e}^{\,C(\alpha N)}\,\mathrm{e}^{\,F(T^+_\cdot)+F(T^-_\cdot)}\cdot 1 = \mathrm{e}^{\,-\mathcal{A}}\,\mathrm{e}^{\,F(t^+_{2n})+F(t^-_{2n})}, \tag{A.43}$$

where $\mathcal{A}$ is found in (A.39), and $C(\alpha N)$ is a function independent of times, such that both sides of (A.43) are units when $T^\pm\equiv 0$,

$$F(t^\pm_{2n})\equiv F(\xi_{2n+1})\Big|_{\xi_{2n+1}=t^\pm_{2n}}, \tag{A.44}$$

where ξ_{2n+1} are odd times of the Kontsevich model and $F(\xi_{2n+1})$ is just the partition sum of the Kontsevich model. Therefore we have proved the main assertion of the Theorem 1.3.

A.4 Perturbative solution for $\langle\!\langle\tau_{d_1}\dots\tau_{d_n}\rangle\!\rangle$.

Let us consider an expansion of $w(\lambda)$ in α and N:

$$w(\lambda)=\sum_{n=3}^{\infty}N^2\alpha^{2-n}w_{0,n}(\lambda)+\sum_{g=1}^{\infty}\sum_{n=1}^{\infty}N^{2-2g}\alpha^{2-2g-n}w_{g,n}(\lambda). \tag{A.45}$$

In terms of times $T^\pm_n$ the expansion coefficients are $(\alpha N)^{2-2g-n}$ and we have an asymptotic expansion of the form:

$$w(\lambda)=\frac{1}{\alpha N}\mathcal{F}_1+\frac{1}{(\alpha N)^2}\mathcal{F}_2+\dots \tag{A.46}$$

Here, taking the times $\xi^\pm_{2n+1}=(2n+1)!!T^\pm_n$ in order to compare with the answer for the Kontsevich model ([16]), we have

$$\begin{aligned}
\mathcal{F}_1 &= \frac{(\xi_1^-)^3}{3!}-\frac{(\xi_1^+)^3}{3!}+\frac{1}{24}(\xi_3^- - \xi_3^+)-\frac{1}{12}(\xi_1^- - \xi_1^+),\\
\mathcal{F}_2 &= \xi_3^+\frac{(\xi_1^+)^3}{3!}+\xi_3^-\frac{(\xi_1^-)^3}{3!}-\frac{1}{2}\left[\frac{(\xi_1^-)^4}{4!}+\frac{(\xi_1^+)^4}{4!}\right]\\
&\quad -\frac{1}{8}\left[\frac{(\xi_1^+)^2}{2!}+\frac{(\xi_1^-)^2}{2!}\right]^2+\frac{1}{24}\left[\frac{(\xi_3^+)^2}{2!}+\frac{(\xi_3^-)^2}{2!}+\xi_5^+\xi_1^+ +\xi_5^-\xi_1^-\right]\\
&\quad -\frac{1}{8}[\xi_1^+\xi_3^+ +\xi_1^-\xi_3^-]+\frac{5}{48}\left[\frac{(\xi_1^+)^2}{2!}+\frac{(\xi_1^-)^2}{2!}\right]+\frac{1}{64}(\xi_1^+-\xi_1^-)^2\\
\mathcal{F}_3 &= \frac{1}{1152}(\xi_9^- - \xi_9^+)-\frac{13}{1920}(\xi_7^- - \xi_7^+)+\frac{1}{24}\left[\xi_7^-\frac{(\xi_1^-)^2}{2!}-\xi_7^+\frac{(\xi_1^+)^2}{2!}\right]\\
&\quad +f(\xi_1^\pm,\xi_3^\pm,\xi_5^\pm).
\end{aligned} \tag{A.47}$$

Appendix B The explicit solution to $\overline{\mathcal{M}}_{2,1}$

Here we shall present the form of formula (3.23) for the case of genus two moduli space with one puncture. In paper [12] the explicit form of genus two partition function in terms of momenta was found:

$$\begin{aligned}
F_2 &= -\frac{181}{480J_1^2d^4}-\frac{181}{480M_1^2d^4}-\frac{5}{16J_1M_1d^4}+\frac{181J_2}{480J_1^3d^3}-\frac{181M_2}{480M_1^3d^3}\\
&\quad +\frac{3J_2}{64J_1^2M_1d^3}-\frac{3M_2}{64J_1M_1^2d^3}-\frac{11J_2^2}{40J_1^4d^2}-\frac{11M_2^2}{40M_1^4d^2}\\
&\quad +\frac{J_2M_2}{64J_1^2M_1^2d^2}+\frac{43M_3}{192M_1^3d^2}+\frac{43J_3}{192J_1^3d^2}+\frac{21J_2^3}{160J_1^5d}-\frac{21M_2^3}{160M_1^5d}\\
&\quad -\frac{29J_2J_3}{128J_1^4d}+\frac{29M_2M_3}{128M_1^4d}+\frac{35J_4}{384J_1^3d}-\frac{35M_4}{384M_1^3d}.
\end{aligned} \tag{B.1}$$

In order to investigate the modular space $\overline{\mathcal{M}}_{2,1}$ it is enough to use the expansions (4.34), because we must keep only terms of the first order in traces. Then the only thing we need more is to express the quantities

$$\begin{aligned} p_k &= \frac{\mathrm{e}^{\lambda(k+1)}}{(\mathrm{e}^{\lambda}-1)^{2k+1}(\mathrm{e}^{\lambda}+1)}, \\ q_k &= \frac{\mathrm{e}^{\lambda(k+1)}}{(\mathrm{e}^{\lambda}-1)(\mathrm{e}^{\lambda}+1)^{2k+1}} \end{aligned} \tag{B.2}$$

via the derivatives L_a and R_a (4.31). We omit all lengthy calculations and present here only the final answer. After replacing $\alpha \to -\alpha/2$ we remain with

$$w_2(\lambda) = \frac{1}{2^d}\left\{\frac{L_4}{4!}\cdot\frac{1}{1152} - \frac{L_3}{3!}\cdot\frac{1}{24}\cdot\frac{13}{40}\right. \\ \left. + \frac{L_2}{2!}\cdot\frac{119}{1440} - L_1\cdot\frac{143}{180} + L_0\cdot\frac{11659}{15360} + (L_a \to R_a)\right\}. \tag{B.3}$$

Here $\frac{1}{1152} = \langle\tau_4\rangle_2$.

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