The Hierarchical ϕ^4 - Trajectory by Perturbation Theory in a Running Coupling and its Logarithm

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Abstract

We compute the hierarchical ϕ^4-trajectory in terms of perturbation theory in a running coupling. In the three dimensional case we resolve a singularity due to resonance of power counting factors in terms of logarithms of the running coupling. Numerical data is presented and the limits of validity explored. We also compute moving eigenvalues and eigenvectors on the trajectory as well as their fusion rules.

1 Introduction

In the block spin renormalization scheme of Wilson [?, ?] renormalized theories come as renormalized trajectories of effective actions. Departing from a bare action the renormalized trajectory is reached by an infinite iteration of block spin transformations. For this limit to exist the bare couplings have to be tuned as the number of block spin transformations is increased. Consider an asymptotically free model at weak coupling. There the point is to keep couplings under control which increase by value under a block spin transformation. Such couplings are called relevant. In weakly coupled models they can be identified by naive power counting. This renormalization scheme has been beautifully implemented both within and beyond perturbation theory. Let us mention the work of Polchinski [?], Gawedzki and Kupiainen [?] , Gallavotti [?], and Rivasseau [?] as a guide to the extensive literature. The underlying picture of an ultraviolet asymptotically free model is to think of the renormalized trajectory as unstable manifold of a trivial fixed point. Although this picture has been in mind behind block spin renormalization since the very beginning [?] it has not been formalized yet to an approach free of a bare action. This paper is a contribution to fill this gap. It extends the analysis begun in [?] and [?] in the context of renormalization group improved actions for the two dimensional $O(N)$-invariant nonlinear σ-model. Here we will work it out for the ϕ^4-trajectory in the hierarchical approximation. The hierarchical model was invented by Dyson [?] and Baker [?] and has enjoyed the attention of Bleher and Sinai [?], Collet and Eckmann [?], Koch and Wittwer [?], Felder [?], and Pordt [?], to mention a few. The ϕ^4-trajectory will be defined as a curve which departs the trivial fixed point in the ϕ^4-direction. Technically we perform a renormalized perturbation expansion in a running coupling. In the three dimensional case we perform a perturbation expansion in a running coupling and its logarithm. The dynamical principle which proves to be strong enough to determine the trajectory at least in perturbation theory is stability under the renormalization group. With stability we mean that the trajectory is left invariant under a transformation as a set in theory space. Recall that a renormalized action always comes together with a sequence of descendents generated by further block spin transformations. Even in the case of a discrete transformation this sequence will prove to consist of points on a continuous curve in theory space which is stable under the renormalization group. It is the computation of this curve we address. The result is an iterative form of renormalized perturbation theory. Its closest relatives in the literature are the renormalized tree expansions of Gallavotti and collaborators [?, ?, ?]. A pedagocial exposure can be found in [?]. Our expansion is however free of divergencies piled up in standard perturbation theory by infinitely iterated renormalization group transformations from the very beginning. Surprisingly we do not need to treat relevant and irrelevant couplings on a different footing. It will involve neither bare couplings nor renormalization conditions in the original sense. A renormalization group transformation in our approach translates to a transformation of the running coupling according to some β-function. We will consider in particular a choice of coordinate whose associated β-function is exactly linear. This idea has also appeared in [?] and references therein. Renormalized perturbation theory furthermore will surprise us with a sequence of discrete poles at special rational dimensions. These poles will be traced back to certain resonance conditions on the scaling dimensions of powers of fields. In particular the case of three dimensions will be shown to be resonant. We will resolve the associated singularity by a double expansion in both the running coupling and its logarithm. The expansion will then be extended to the computation of moving eigenvectors in the sense of [?] on the renormalized trajectory and their fusion rules. Finally we perform a numerical test of our renormalized actions. As expected they prove to work well in a small field region. The extension of our program to full models is under way. A prototype with momentum space regularization has been developed in [?].

2 Hierarchical renormalization group

The hierarchical renormalization group in the form advocated by Gawedzki and Kupiainen [?] is a theory of the non linear transformation

m(1) $$\mathcal{R}Z(\psi) = \Big(\int d\mu_\gamma(\zeta)\ Z(L^{1-\frac{D}{2}}\psi + \zeta)\Big)^{L^D}$$

on some space of Boltzmann factors $Z(\phi)$. In the scalar theory ϕ is a single real field variable.

m(2) $$d\mu_\gamma(\zeta) = (2\pi\gamma)^{-\frac{1}{2}} \exp\Big(-\frac{\zeta^2}{2\gamma}\Big) d\zeta$$

is the Gaussian measure on $\mathbb{R}$ with mean zero and covariance γ. The parameters of (1) are the Euclidean dimension D and the block scale L. The subspace of even Boltzmann factors $Z(-\phi) = Z(\phi)$ is stable under (1). We will restrict our attention to this subspace. Let the potential be given by $Z(\phi) = \exp(-V(\phi))$. The transformation for the potential is

m(3) $$\mathcal{R}V(\psi) = -L^D \log\left(\int d\mu_\gamma(\zeta)\ \exp\left(-V(L^{1-\frac{D}{2}}\psi + \zeta)\right)\right) .$$

The below analysis will be done in terms of the potential. The method will be perturbation theory. The question of stability bounds will not be addressed. Regarding mathematical aspects of (1) and (3) we refer to the work of Collet and Eckmann [?], Gawedzki and Kupiainen [?] and of Koch and Wittwer [?].

3 The trivial fixed point

(3) has a trivial fixed point $V_*(\phi) = 0$. This fixed point is the hierarchical massless free field. The linearization of (3) at this trivial fixed point is given by

m(4) $$\mathcal{L}_{V_*}\mathcal{R}\mathcal{O}(\psi) = L^D \int d\mu_\gamma(\zeta)\ \mathcal{O}\left(L^{1-\frac{D}{2}}\psi + \zeta\right) .$$

This linearization is diagonalizable. The eigenvectors are normal ordered products

m(5) $$:\phi^n:_{\gamma\prime} = \left.\frac{\partial^n}{\partial j^n}\right|_{j=0} \exp\left(j\phi - \frac{j^2\gamma\prime}{2}\right)$$

with normal ordering covariance $\gamma\prime = \left(1 - L^{2-D}\right)^{-1}\gamma$. The normal ordering covariance has been chosen in order to be invariant with respect to integration with $d\mu_\gamma$. Its singularity at $D = 2$ is an infrared singularity of the hierarchical massless free field in two dimensions. The eigenvalues are

m(6) $$\lambda_n = L^{D+n\left(1-\frac{D}{2}\right)} .$$

The eigenvalue of $:\phi^4:_{\gamma\prime}$ is $\lambda_4 = L^{4-D}$. The eigenvector $:\phi^4:_{\gamma\prime}$ is therefore relevant for $D < 4$, marginal for $D = 4$, and irrelevant for $D > 4$ dimensions. Perturbation theory can be used to compute corrections to (4) in a neighbourhood of $V_*(\phi)$.

4 The ϕ^4 - trajectory

Let us define a curve $V(\phi,g)$ in the space of potentials parametrized by a local coordinate g. We call it the ϕ^4 - trajetory. We expand the potential

m(7) $$V(\phi,g)=\sum_{n=0}^{\infty}V_{2n}(g):\phi^{2n}:_{\gamma'}$$

in the base of eigenvectors (5). A natural coordinate in a vicinity of $V_*(\phi)$ is the ϕ^4 - coupling defined by $V_4(g)=g$. Let us use it for a moment. Let the ϕ^4 - trajectory then be the curve $V(\phi,g)$ defined by the following two conditions:

1) $V(\phi,g)$ is stable under $\mathcal{R}$. Then there exists a function $\beta(g)$ such that

   m(8) $$\mathcal{R}V(\phi,g)=V(\phi,\beta(g))\ .$$

   The function $\beta(g)$ is of course coordinate dependent. With the ϕ^4 - coupling as coordinate it is called β - function.

2) $V(\phi,g)$ visits the trivial fixed point $V_*(\phi)$ at $g=0$. The tangent to $V(\phi,g)$ at $V_*(\phi)$ is given by

   m(9) $$\left.\frac{\partial}{\partial g}\right|_{g=0}V(\phi,g)=\ :\phi^4:_{\gamma'}$$

   This condition is equivalent with $V_4(g)=g+O(g^2)$ together with $V_{2n}(g)=O(g^2)$, $n\neq 2$.

The ϕ^4 - trajectory is the object of principal interest in massless ϕ^4 - theory at weak coupling.

5 Perturbation theory

The ϕ^4 - trajectory can be computed by perturbation theory in g as solution to (8) and (9). Potentials on the ϕ^4 - trajectory are said to scale. A potential $V(\phi,g)$ is said to scale to order s in g if there exists a function

m(10) $$\beta(g)=\beta^{(s)}(g)+O(g^{s+1})\ ,\qquad \beta^{(s)}(g)=\sum_{r=1}^{s}b_r g^r\ ,$$

such that

$$V(\phi,g) = V^{(s)}(\phi,g) + O(g^{s+1}) \ ,$$

m(11) $$\mathcal{R}V^{(s)}(\phi,g) = V^{(s)}(\phi,\beta(g)) + O(g^{s+1}) \ ,$$

and

m(12) $$V^{(1)}(\phi,g) = g : \phi^4 :_{\gamma\prime} \ .$$

The scheme is to compute $\beta^{(s+1)}(g)$ and $V^{(s+1)}(\phi,g)$ given $\beta^{(s)}(g)$ and $V^{(s)}(\phi,g)$ to some order s. Let us explain it in some detail at the case of $D=4$ dimensions, block scale $L=2$, and covariance $\gamma=1$. Then the normal ordering covariance is $\gamma\prime = \frac{4}{3}$. Computing a block spin transformation (3), we speak of $V(\phi)$ as bare potential and of $\mathcal{R}V(\phi)$ as effective potential. The point of departure is (12). Anticipating the terms generated in $\mathcal{R}V^{(1)}(\phi,g)$ to second order in g we make the ansatz

m(13) $$V^{(2)}(\phi,g) = c_0 g^2 + c_2 g^2 : \phi^2 : + g : \phi^4 : + c_6 g^2 : \phi^6 : \ .$$

The coefficients are determined by the condition that (11) be fulfilled to second order. (13) is mapped to

m(14) $$\begin{aligned}\mathcal{R}V^{(2)}(\phi,g(g')) &= (16c_0 - \frac{5440}{9})g'^2 + (4c_2 - 448)g'^2 : \phi^2 : \\ &+ g' : \phi^4 : + (\frac{c_6}{4} - 2)g'^2 : \phi^6 : + O(g'^3) \ .\end{aligned}$$

Here the effective coupling defined as the coefficient of $:\phi^4:$ in the effective potential is given by

m(15) $$g'(g) = g - 60g^2 + O(g^3) \ .$$

Comparing the effective potential as a function of the effective coupling with the bare potential as a function of the bare coupling we conclude that

m(16) $$c_0 = \frac{1088}{27} \ , \ c_2 = \frac{448}{3} \ , \ c_6 = -\frac{8}{3}$$

on the ϕ^4 - trajectory. The coefficients of the β - function (10) to this order are

m(17) $$b_1 = 1 \ , \ b_2 = -60 \ .$$

It follows that g is marginally irrelevant in four dimensions. This completes the first step. It is iterated in the obvious manner. The general form of the order s approximation is

$$V^{(s)}(\phi, g) = \sum_{n=0}^{s+1} c_{2n}^{(s)}(g) : \phi^{2n} : ,$$

$$c_{2n}^{(s)}(g) = \sum_{r=2}^{s} c_{2n,r} g^r , \quad n \leqslant 1 ,$$

$$c_4^{(s)}(g) = g ,$$

m(18)
$$c_{2n}^{(s)}(g) = \sum_{r=n-1}^{s} c_{2n,r} g^r , \quad n \geqslant 3 .$$

It includes all normal ordered products generated in the effective potential by (3) from (12) to order s in g. The iteration proceeds as above with the order $s+1$ ansatz of the form (18). The condition (11) yields a system of linear equations for the order $s+1$ coefficients. (To highest order the coefficients have no other choice.) This system has a unique solution: the ϕ^4 - trajectory. Note that the coefficient b_{s+1} of the β - function is already determined by $V^{(s)}(\phi, g)$. For instance (15) does not contain any of the coefficients in (13). The expansion can be computed to higher orders using computer algebra. To third order we find

$$c_0^{(3)}(g) = \frac{1088}{27} g^2 - \frac{54784}{27} g^3 ,$$

$$c_2^{(3)}(g) = \frac{448}{3} g^2 - \frac{497408}{27} g^3 ,$$

$$c_6^{(3)}(g) = -\frac{8}{3} g^2 + 352 g^3 ,$$

m(19)
$$c_8^{(3)}(g) = \frac{32}{3} g^3$$

together with

m(20)
$$\beta^{(3)}(g) = g - 60 g^2 + 8880 g^3 .$$

Let us remark that the perturbation coefficients (10) and (11) come with alternating signs. The coefficients show a frightening increase in absolute value with the order in g. The full series is not expected to converge. Note that the coefficients look better when g is replaced by $g/4!$.