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Absorption probability of De Sitter horizon for massless fields with spin

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Abstract

The evaluation of the absorption coefficients are important for particle emission caused by Hawking radiation. In the case of cosmological particle emission from the event horizon in De Sitter space, it is known that the scalar wave functions are solved in terms of Legendre functions. For fields with higher spin, the solution has been examined with low frequency approximation. We shows that the radial equations of the fields with spin $0, 1/2, 1$ and 2 can be solved analytically in terms of the hypergeometric functions. We calculate the absorption probability using asymptotic expansion for high frequency limit. It turns out that the absorption coefficients are universal to all bosonic fields; They depend only on the angular momentum and not spin. In the case of spin $1/2$ fermions, we can also find non-vanishing absorption probability in contrast to the previously known result.

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Since the celebrated discovery of the evaporation of the black holes[1] much effort has been given to get analytic expression of the evaporation rate[2]. The fact that the the corresponding field equations around black holes can not be expressed by the special functions prevent us from getting analytic expression for high-frequency mode. On the other hand, the Hawking radiation caused by the cosmological event horizon[3] seems to be much easier to handle because it is known that the scalar field equation in static coordinates of De Sitter space can be solved by Legendre functions[4]. We can therefore convert the solution at the event horizon to our observing world by analytic continuation for all the frequency. Also the Bogoliubov coefficients of the mode functions from the global coordinates has also been obtained[5]. As for the fields with spin, the solution was made only by low frequency approximation[6] where they did not found any radiation for spinor fields. Our aim of this short letter is to show that the the radial equation for spin fields can also be solved by special functions. We obtain the absorption probability using high frequency approximation. It turns out that the absorption probability for bosonic fields are universal and do not depend on the spin. For spinor, we find non-vanishing result for the absorption probability in contrast to the result given in ref.[6].

De Sitter metric in static coordinates takes the form;

$$ds^2 = [1-(\frac{r^2}{a^2})]dt^2 - [1-(\frac{r^2}{a^2})]^{-1}dr^2 - r^2(d\theta^2 + \sin\theta^2 d\varphi^2), \tag{1}$$

where a is related to the cosmological constant by $\Lambda = 3/a^2$. In the case of the type D space-time with vanishing electromagnetic fields, the equation of the first order variation of the scalar, spin 1/2 fermions, electromagnetic fields and gravity is universal and written by Teukolsky equations[7] and separable. In our case, the angular component can be solved in terms of the spin-weighted spherical harmonics ${}_sY_l^m(\theta)$ given in Ref.[8]. Then the radial equation is given in the variable $z \equiv r/a$ as[6]

$$\begin{aligned}
& z^2(1-z^2)^2(\frac{d}{dz})^2 R_p - [2(p+1)z^3 - 2(s+1)(1-z^2)z](1-z^2)\frac{d}{dz}R_p \\
& -\{(1-z^2)[(l-s)(l+s+1) + (s+p+1)(s+p+2)z^2] - (a\omega z)^2 + 2ia\omega zp\}R_p \\
= & \ 0,
\end{aligned} \tag{2}$$

where p takes the value $\pm s$. In the case of the scalar fields $s = p = 0$, by choosing variable as $y = z^2$, the singular point of the equation reduces to $0, 1$ and ∞ with definite singularity. The equation can then be solved in hypergeometric functions[4]. This choice of the variable is a natural choice because the metric depends only on z^2. For $s \neq 0$, however, this choice does not work. There is another natural choice of variable. Near horizon, a useful coordinate is given by $dz^* = dz/(1 - z^2)$ and therefore we choose a variable as

$$y = e^{-2z^*} = \frac{1-z}{1+z}. \tag{3}$$

With this choice of the variable, Eq.(2) can be written as

$$\begin{aligned}
&(\frac{d}{dy})^2 R_p + [(p+1)\frac{1}{y} - 2(s+1)\frac{1}{1-y} - 2(s+p+1)\frac{1}{y+1}]\frac{d}{dy}R_p \\
- \;& \{(l-s)(l+s+1)\frac{1}{(1-y)^2} - (s+p+1)(s+p+2)\frac{1}{(y+1)^2} + [-\frac{(a\omega)^2}{4} + \frac{ia\omega p}{2}] \\
+ \;& [-(l-s)(l+s+1) - ia\omega p]\frac{1}{y(y-1)} + (s+p+1)(s+p+2)\frac{1}{y(y+1)}\}R_p = 0. \quad (4)
\end{aligned}$$

Now the equation have four defenite singularity at points $0, -1, +1$ and ∞. It turns out that the singularity at $y = -1$ can be factorized as

$$R_p = y^{-p-\frac{ia\omega}{2}}(1-y)^{l-s}(1+y)^{s+p+1} f_p \tag{5}$$

and f_p satisfies the hypergeometric equation

$$y(1-y)f_p'' + [1 - p - ia\omega - (2l + 3 - p - ia\omega)y]f_p' - (l + 1 - p)(l + 1 - ia\omega)f_p = 0, \tag{6}$$

which has two independent solutions. Out of two solutions, we take the outgoing solution at the future horizon $y \sim 0 (z \sim 1)$ which behaves as $\exp(2p + ia\omega)z^*$ where $z^* = -(\log y)/2$. The solution is

$$R_p = y^{-\frac{ia\omega}{2}-p}(1-y)^{l-s}(y+1)^{s+p+1}F(a,b;c;y), \tag{7}$$

where $F(a, b; c; y)$ is the hypergeometric function with

$$a = l + 1 - p, \qquad b = l + 1 - ia\omega, \qquad c = 1 - p - ia\omega. \tag{8}$$

By using the recursion relation of the hypergeometric functions, we can derive the recursion relations which relate the solutions with different spins;

$$\frac{d}{dy}[y^{-\frac{ia\omega}{2}}(1-y)^{2s+1}(1+y)^{-2s-1}R_s]$$
$$= -(s+ia\omega)y^{-\frac{ia\omega}{2}}(1-y)^{2s+1}(1+y)^{-2s-3}R_{s+1}, \tag{9}$$
$$\frac{d}{dy}[y^{\frac{ia\omega}{2}+s+1}(1-y)(1+y)^{-2s-3}R_{s+1}]$$
$$= -\frac{(l-s)(l+s+1)}{s+ia\omega}y^{\frac{ia\omega}{2+s}}(1-y)^{-1}(1+y)^{-2s-1}R_s. \tag{10}$$

We can also obtain the operations which change the angular momentum.

The behavior of this function near us ($y \simeq 1, z \sim 0$) can be obtained by the analytic continuation. Near $z=0$, the variable y can be written as $y \simeq e^{-2z}, \quad 1-y \simeq 2z$. Since the parameters in the hypergeometric function satisfies a relation $c-a-b = -(2l+1)$, the analytic continuation of the function produces logarithmic terms and the analysis becomes very complicated. Therefore, we will analytically continue the angular momentum to avoid such complication.(Strictly speaking, we cannot justify this procedure, although this technique is also used in the case of Black hole[2].) Around $z=0$, R_p behaves as

$$\begin{aligned} R_p \quad &= (\frac{1-z}{1+z})^{-p-\frac{ia\omega}{2}}(\frac{2z}{1+z})^{l-s}(\frac{2}{1+z})^{s+p+1} \\ &[\frac{\Gamma(1-p-ia\omega)\Gamma(-(2l+1))}{\Gamma(-l-ia\omega)\Gamma(-l-p)}F(l+1-p, l+1-ia\omega; 2l+2; \frac{2z}{1+z}) \\ &+\frac{\Gamma(1-p-ia\omega)\Gamma(2l+1)}{\Gamma(l-p+1)\Gamma(l+1-ia\omega)}(\frac{2z}{1+z})^{-(2l+1)}F(-l-p, -l-ia\omega; -2l; \frac{2z}{1+z})] \end{aligned} \tag{11}$$

We are going to consider the high frequency approximation $a\omega >> 1$. By using a formula

$$F(a,b;c;x) \sim F(a;c;bx), \tag{12}$$

when $b \to \infty$, and also considering the region where $|awz| >> 1$ and $|\ z\ | << 1$. Then we can use an asymptotic expansion of the confluent hypergeometric functions;

$$F(a;c;x) \sim \frac{\Gamma(c)}{\Gamma(c-a)}\Big(\frac{-1}{x}\Big)^a + \frac{\Gamma(c)}{\Gamma(a)}e^x x^{a-c} \tag{13}$$

and find the following asymptotic expansion of R_p;

$$
\begin{aligned}
R_p &\sim (-1)^{l+p}\frac{\Gamma(1-p-ia\omega)}{\Gamma(l+1-ia\omega)} \\
&\{\frac{e^{ia\omega z}}{z^{s-p+1}}4^p(ia\omega)^{l+p}[1-(-1)^{2p}(ia\omega)^{-(2l+1)}\frac{\Gamma(l+1-ia\omega)}{\Gamma(-l-ia\omega)}] \\
&+\frac{e^{-ia\omega z}}{z^{s+p+1}}(-ia\omega)^{l-p}\frac{\Gamma(-l+p)}{\Gamma(-l-p)}[1+(-1)^{2l}(ia\omega)^{-(2l+1)}\frac{\Gamma(l+1-ia\omega)}{\Gamma(-l-ia\omega)}]\}. \quad (14)
\end{aligned}
$$

We observe that R_p is not singular in the limit $2l$ =integer. In order to calculate the absorption coefficients, we use a trick used in Refs.[9, 2, 6]. When we write the asymptotic expansion of R_s as

$$R_s \sim Y^s_{in}e^{-ia\omega z}/z^{1+2s} + Y^s_{out}e^{ia\omega z}/z, \quad (15)$$

the absorption probability Γ can be calculated as[9]

$$1-\Gamma = |\frac{Y^s_{in}Y^{-s}_{in}}{Y^s_{out}Y^{-s}_{out}}|. \quad (16)$$

From Eq.(14), we find

$$
\begin{aligned}
Y^s_{out} &= A_s 4^p(ia\omega)^{l+p}[1-(-1)^{2p}(ia\omega)^{-(2l+1)}\frac{\Gamma(l+1-ia\omega)}{\Gamma(-l-ia\omega)}], \\
Y^s_{in} &= A_s(-ia\omega)^{l-p}\frac{\Gamma(-l+s)}{\Gamma(-l-s)}[1+(-1)^{2l}(ia\omega)^{-(2l+1)}\frac{\Gamma(l+1-ia\omega)}{\Gamma(-l-ia\omega)}], \quad (17)
\end{aligned}
$$

where

$$A_s = (-1)^{l+s}\frac{\Gamma(1-s-ia\omega)}{\Gamma(l+1-ia\omega)}. \quad (18)$$

By using Eqs.(16) and (17), we obtain

$$\Gamma = \frac{4\Delta_l}{(1+\Delta_l)^2}, \quad (19)$$

where

$$\Delta_{l:\text{integer}} = \prod_{n=1}^{l}(1+\frac{n^2}{(a\omega)^2}) \qquad (l \geq s \quad \text{for} \quad \text{bosons}), \quad (20)$$

$$\Delta_{l:\text{half}-\text{integer}} = \prod_{n=1}^{l+1/2}(1+\frac{(n-\frac{1}{2})^2}{(a\omega)^2}) \qquad (l \geq s \quad \text{for} \quad \text{fermions}). \tag{21}$$

It is amazing to observe that the absorption probablility is independent of the values of spins and depends on spices (fermion or boson) and the total angular momentum. For spin 0 case, our result in Eqs.(19) and (20) agrees with the one derived by Lohiya and Panchapakesan[4] and the absorption probability vanishes for $l = 0$. For $s = 1, 2$ cases, we have non-vanishig probabilities for all l and our formula are different from those by Lohiya and Panchapakesan[6]. In particular, we found non-vanishing probability for $s = 1/2$ in Eqs.(19) and (21) which is in contrast to the result in Ref.[6] who argued that the absorption probability is zero in their approximation.

For $s = 0$, the functions R_p is the same as R_{-p} so that the trick used to derive Γ is not applicable. Lohiya and Panchapakesan[4] derived Γ and showed that the result agreed with ours in Eqs.(19) and (20). As for $s = 1/2$, R_{-p} is expressed in terms of R_p so that another method using currents may be useful. For this purpose, we can use the conserved current to obtain the absorption coefficients as follows. We define

$$g_p = y^{(p+1)/2}(1-y)^{(s+1)}(1+y)^{-(s+p+1)}R_p \tag{22}$$

and then we find that

$$W = g_{-p}(\frac{d}{dy}g_p^*) - g_p^*(\frac{d}{dy}g_{-p}) = constant, \tag{23}$$

which is the Wronskian. For the solution in Eq.(7) for spin 1/2 case, we have

$$\begin{aligned} g_{1/2} &= y^{-\frac{1}{2}(ia\omega-\frac{1}{2})}(1-y)^{l+1}F(l+\frac{1}{2}, l+1-ia\omega, -ia\omega+\frac{1}{2}; y), \\ g_{-1/2} &= y^{-\frac{1}{2}(ia\omega-\frac{3}{2})}(1-y)^{l+1}F(l+\frac{3}{2}, l+1-ia\omega, -ia\omega+\frac{3}{2}; y). \end{aligned} \tag{24}$$

Since the functions $g_{1/2}$ and $g_{-1/2}$ are not independent, we can rewrite the conservation of current in the following form;

$$W = y^{-1/2}(1-y)^{-1}[g_{1/2}^* g_{1/2} - \frac{(l+\frac{1}{2})^2}{a^2\omega^2+\frac{1}{4}} g_{-1/2}^* g_{-1/2}] = \frac{1}{2} + ia\omega, \tag{25}$$

where the value of W is fixed by evaluation near the horizon ($y \sim 1$). We evaluate W by using the outgoing flux at the region $z << 1$ and $a\omega z >> 1$ and can obtain the absorption coefficient.

We have used the analytic continuation of the angular momentum. But we cannot justify the procedure of the analytic continuation whereas the justification can be achieved for the evaluation of the absorption coefficients of black holes[10]. At present, we cannot obtain the asymptotic expansion without using the analytic continuation. We hope that more rigorous approach can be done.

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