Threshold Corrections and Gauge Symmetry in Twisted Superstring Models

David M. Pierce

Institute of Field Physics

Department of Physics and Astronomy

University of North Carolina

Chapel Hill, NC 27599-3255, USA

Threshold corrections to the running of gauge couplings are calculated for superstring models with free complex world sheet fermions. For two N=1 $SU(2) \times U(1)^5$ models, the threshold corrections lead to a small increase in the unification scale. Examples are given to illustrate how a given particle spectrum can be described by models with different boundary conditions on the internal fermions. We also discuss how complex twisted fermions can enhance the symmetry group of an N=4 $SU(3) \times U(1) \times U(1)$ model to the gauge group $SU(3) \times SU(2) \times U(1)$. It is then shown how a mixing angle analogous to the Weinberg angle depends on the boundary conditions of the internal fermions.

1. Introduction

The unification of gauge coupling constants is a necessary consequence of string theory. At tree level, the gauge couplings have simple relations to the string coupling constant. In higher orders of perturbation theory, this relation holds only at the Planck mass. Below this energy, the gauge couplings evolve as determined by the renormalization group equations. Threshold effects [1] can also modify the tree level relation and shift the unification scale. Although the effect of thresholds is small in grand unified field theories, the threshold corrections can be quite large in string theories [2-7] because there is an infinite tower of massive states, all of which contribute.

In this paper, we calculate threshold corrections in four-dimensional critical superstring models written in terms of free fermions with twisted boundary conditions. Complex fermions are useful for studying theories with chiral space-time fermions and for probing the structure of gauge symmetries. Here, we adapt the background field method [5,8] for calculating string thresholds to twisted models in the framework of type II theories. Although a phenomenologically realistic type II model has not yet been found, it provides a more economical construction for using the techniques of low energy string phenomenology. Calculations of thresholds in heterotic models[5-13] can be made large enough to lower the unification scale to an acceptable energy. This is achieved by fine tuning the many free moduli parameters of the theory. A desirable feature of type II strings is that there is less freedom to adjust the parameters of the theory.

In sect. 2, we give a general discussion of the background field method for running couplings in twisted models, emphasizing models with higher level Kac-Moody currents. In sect. 3, we calculate the threshold corrections for two twisted chiral models with $SU(2) \times U(1)^5$ gauge symmetry. These two models have the same massless particle spectrum but different boundary conditions on the internal fermions. This shows how different boundary conditions on the internal fermions affect the thresholds. In sect. 4, we investigate the relationship between the boundary conditions on the internal fermions and the ratio of the field theory couplings at the Plank mass[14]. Specifically, it will be shown how the twisted boundary conditions can enhance the symmetry group of an N=4 $SU(3) \times U(1) \times U(1)$ model to $SU(3) \times SU(2) \times U(1)$. We then determine a mixing angle analogous to the Weinberg angle of the standard model.

2. Background field calculation

The tree level relation between gauge couplings is [15]:

$$\frac{4\pi}{g_a^2} = 2x_a \frac{4\pi}{g_{str}^2} \tag{2.1}$$

where x_a is the level of Kac-Moody Algebra = N for SU(N). The factor of 2 is present because we choose a field theory normalization for the longest roots equal to 1. This relation is determined by comparing the scattering amplitudes (e.g. three-point gauge boson vertex) for the low energy string theory (massless modes) to field theory. The tree amplitudes are identical if one makes this identification, which holds at the Planck mass. We now want to calculate how this relation changes when higher order corrections and threshold effects are considered. This involves using the background field method [5,8,17] to find the threshold corrections. Since we are interested in models with chiral space-time fermions, we consider models with complex twisted fermions.

The background field method involves describing the effective action of the quantum gauge field as an effective action of a string propagating in a classical background gauge field:

$$\Gamma[A_\mu^a] = \Gamma[X^\mu = 0, \Psi^\mu = 0, A_\mu^a]\,. \tag{2.2}$$

The effective action will now include contributions from the massive modes of the string. X^μ, $\Psi^\mu = 0$ means that there are no external string states. In addition, since the gauge fields are classical, they do not circulate in loops. In other words, the classical gauge fields only exist as external states. Polchinski[16] derived a formula for the one-loop correction to (2.1):

$$\Gamma[X^\mu = 0, \Psi^\mu = 0, A_\mu^a] = \int d^4x \left(-\frac{1}{4g_a^2} F_{\mu\nu}^a F^{a\mu\nu}\right) + \int \frac{d^2\tau}{\tau_2} \mathbf{Z} + ... \tag{2.3}$$

where $\mathbf{Z}$ is the partition function (one loop with no external states) of the string in the presence of a background gauge field. $\tau = \tau_1 + i\tau_2$ are the coordinates on the torus. The first term in (2.3) is simply the classical action of the gauge field and $-\frac{1}{4g_a^2}$ is the tree approximation for the gauge coupling. The second term, which is the one-loop correction to the tree level result, can be shown to be equivalent to a one-loop two point string amplitude where the string vertex operator for the emission of a gauge boson is modified by making the substitution $\epsilon_\mu e^{ik\cdot X(z,\bar{z})} \to A_\mu = -\frac{1}{2}F_{\mu\nu}X^\nu$ provided that A_μ satisfies the classical equations of motion. The first order correction to the field theory coupling constants is then given by the the coefficient of the $-\frac{1}{4}F_{\mu\nu}^2$term in the one-loop two point background gauge field amplitude. We now outline this method for models containing twisted fermions.

Type II 4-dimensional string models can contain chiral space-time fermions [18-21] if some of the internal coordinates take values on a shifted lattice $\sqrt{2\alpha'}p \in Z + \nu$. These complex twisted fermions satisfy the following boundary conditions:

$$\psi(e^{2\pi i}z) = -e^{2\pi i\nu}\psi(z) \quad ; \quad \tilde{\psi}(e^{2\pi i}z) = -e^{-2\pi i\nu}\tilde{\psi}(z) \tag{2.4}$$

where ν is real. The space-time fermions are real, either Neveu-Schwarz or Ramond. They are given by:

$$\psi^{\mu}(z) = \sum_{r\in Z+\frac{1}{2}} \psi^{\mu}_{r} z^{-r-\frac{1}{2}} \quad ; \quad \psi^{\mu}(z) = \sum_{n\in Z} \psi^{\mu}_{n} z^{-n-\frac{1}{2}} \tag{2.5}$$

for Neveu-Schwarz and Ramond respectively. Twisted models differ from untwisted models in that internal fermions can be defined by the following for any value of λ

$$\psi^{i}(z) = \sum_{r\in Z+\lambda} \psi^{i}_{r} z^{-r-\frac{1}{2}} \quad ; \quad \tilde{\psi}^{i}(z) = \sum_{r\in Z-\lambda} \tilde{\psi}^{i}_{r} z^{-r-\frac{1}{2}} \tag{2.6}$$

where $\lambda = \frac{1}{2} - \nu$ and $f^{\dagger}_{r} = \tilde{f}_{-r}$. $\nu = 1/2$ is the Ramond case and $\nu = 0$ the Neveu-Schwarz case.

We consider 4-dimensional type II models in the light-cone description. Here, the left and right movers can each be described by two bosonic and twenty fermionic fields. The partition function without the zero modes p is given by

$$
\begin{aligned}
Z &= \prod_{l=0}^{k-1} \frac{1}{\mathcal{N}_l} \sum_{\alpha,\beta} c(\alpha,\beta) Tr_\alpha [q^{L'_0-\frac{1}{2}} \bar{q}^{\bar{L}'_0-\frac{1}{2}} (e^{-i\pi})^{\rho_\beta \cdot F}] \\
&= \prod_l \frac{1}{\mathcal{N}_l} \sum_{\alpha,\beta} c(\alpha,\beta) |\eta(q)|^{-24} \prod_{i=1}^{n} \left(\vartheta \begin{bmatrix} \rho^i_\alpha \\ \rho^i_\beta \end{bmatrix} (0|q) \right)^{1/2} \prod_{i=1}^{n'} \left(\vartheta \begin{bmatrix} \bar{\rho}^i_\alpha \\ \bar{\rho}^i_\beta \end{bmatrix} (0|\bar{q}) \right)^{1/2} \\
&\times \prod_{i=1}^{m} \vartheta \begin{bmatrix} \rho^i_\alpha \\ \rho^i_\beta \end{bmatrix} (0|q) \prod_{i=1}^{m'} \bar{\vartheta} \begin{bmatrix} \rho^i_\alpha \\ \rho^i_\beta \end{bmatrix} (0|\bar{q})
\end{aligned}
\tag{2.7}
$$

where the prime on the Hamiltonian denotes the omission of the bosonic zero modes. In this formalism, the partition function is a sum over the sectors generated by $\rho_\alpha \equiv (\rho_\alpha; \bar{\rho}_\alpha) \in \Omega$. Each sector α contains $n + n'$ real fermions and $m + m'$ complex fermions. ρ_α is a $(n + n' + m + m')$ dimensional vector which describes the boundary conditions of the fermions for each sector:

$$
\rho_\alpha = 2(\nu_1, ...\nu_n; \nu_1, ...\nu_m; \nu_1, ...\nu_{n'}; \nu_1, ...\nu_{m'})\,. \tag{2.8}
$$

$c(\alpha, \beta) = \delta_\alpha \epsilon(\alpha, \beta)$ are phases for the $(e^{-i\pi})^{\rho_\beta \cdot F}$ projections. F is a vector whose components are the operators $F_j = \sum_{r \in z+\lambda} : f^j_r \tilde{f}^j_{-r} :$ for complex fermions and $\sum_{s=1/2}^{\infty} b^j_{-s} b^j_s$ or $\sum_1^\infty d^j_{-n} d^j_n$ for real NS or R fermions respectively.

$$
\rho_\beta \cdot F = 2 \sum_{j=1}^{n} \nu_j F^L_j + 2 \sum_{j=1}^{m} \nu_j F^L_j - 2 \sum_{j=1}^{n\prime} \nu_j F^R_j - 2 \sum_{j=1}^{m\prime} \nu_j F^R_j\,. \tag{2.9}
$$

In addition, the factor $\prod_{l=0}^{k-1} \mathcal{N}_l$ (k= the number of generators) is the number of sectors where the order, $\mathcal{N}_\alpha$ is defined by $\alpha^{\mathcal{N}_\alpha} = \phi = ((1)^n; (1)^m; (1)^{n'}; (1)^{m'})$ and α is a vector whose components are given by

$$
\alpha = (e^{2\pi i \nu}, ...; e^{2\pi i \nu}, ...; e^{2\pi i \nu}, ...; e^{2\pi i \nu}, ...) \tag{2.10}
$$

The generalized Jacobi theta functions are given by

$$\vartheta\begin{bmatrix}\rho\\ \mu\end{bmatrix}(\nu|\tau) = \sum_{n\in Z} e^{i\pi\tau(n+\rho/2)^2} e^{-i2\pi(n+\rho/2)(\nu+\mu/2)} e^{i\pi\rho\mu/2}$$

$$\bar\vartheta\begin{bmatrix}\bar\rho\\ \bar\mu\end{bmatrix}(\nu|\bar\tau) = \sum_{n\in Z} e^{-i\pi\bar\tau(n+\bar\rho/2)^2} e^{i2\pi(n+\bar\rho/2)(\nu+\bar\mu/2)} e^{-i\pi\bar\rho\bar\mu/2} \qquad (2.11)$$

with $q = e^{2\pi i\tau}$, $\bar q = e^{-2\pi i\bar\tau}$,$\tau = \tau_1 + i\tau_2$, and $\bar\tau = \tau_1 - i\tau_2$. Note that this differs from that given in [18] because the left movers are now functions of q rather that $\bar q$.

The one-loop two point amplitude contribution to the effective lagrangian for A^a_μ background is :

$$\mathcal{L}'(A^a_\mu) = \prod_l \frac{1}{\mathcal{N}}_l \sum_{\alpha,\beta} c(\alpha,\beta) \int \frac{d^4p}{(2\pi)^4} Tr_\alpha[\Delta V^a(1,1)\Delta V^a(1,1)(e^{-i\pi})^{\rho_\beta\cdot F}] \qquad (2.12)$$

where the sum over sectors corresponds to a generalized GSO projection[18]. The closed string propagator is

$$\Delta = \frac{1}{4\pi}\int_{|z|\le 1} \frac{dz d\bar z}{|z|^2} z^{L_0-\frac{1}{2}} \bar z^{\bar L_0 - \frac{1}{2}} . \qquad (2.13)$$

The Hamiltonian L_0 and associated Virasoro generators are given by:

$$L_n = \sum_{r\in Z+\lambda} (r - \frac{n}{2}) : \tilde f_{n-r} f_r : + \frac{1}{4}\sum (\lambda - \frac{1}{2})^2 \delta_{n.0} . \qquad (2.14)$$

Recall that the background field method involves making the substitution $\epsilon_\mu e^{ik\cdot X(z,\bar z)} \to A_\mu(x)$ in the vertex operator for a gauge boson provided that $A_\mu(x)$ satisfies the equation of motion $\partial_\mu F^{\mu\nu} = 0$. The vertex operator for a gauge boson, $b^{La}_{-\frac{1}{2}} \epsilon\cdot b^R_{-\frac{1}{2}}|0>$, is constructed in part from the Kac-Moody currents of (2.15) and (2.16). In models with complex fermions, such as the two chiral models that are discussed in sect. 3, the affine algebra is constructed for a particular gauge group. For a model with complex fermions and an $SU(2) \times U(1)^5$ gauge symmetry, the currents are given by:

$$J^a(z) = \sum_{n \in Z} J^a_n z^{-z} = -\frac{i}{2} f_{abc} \psi^b(z) \psi^c(z) \tag{2.15}$$

where f_{abc} are the structure constants of $SU(2)$ and $3 \leq a, b, c \leq 5$.

The U(1) currents are:

$$J^a(z) =: f^j(z) \tilde{f}^j(z) : +\nu_j \tag{2.16}$$

for $6 \leq a \leq 10$, $1 \leq j \leq 5$, and no sum on j. The zero modes of these currents generate the gauge symmetry. The vertex operator for an $SU(2)$ gauge boson is:

$$\begin{aligned} V^a(k, \epsilon, z, \bar{z}) =& [\tfrac{1}{2} k \cdot \psi^L(z) \psi^{La}(z) - \tfrac{i}{2} f_{abc} \psi^{Lb}(z) \psi^{Lc}(z)] \\ & \epsilon \cdot [i \bar{z} \bar{\partial} X_{\mathrm{R}}(\bar{z}) - \tfrac{1}{2} \psi^R(\bar{z}) k \cdot \psi^R(\bar{z})] e^{ik \cdot X(z, \bar{z})} \end{aligned} \tag{2.17}$$

Here, all the fermionic oscillators are real. On the other hand, the vertex operators for the U(1) gauge bosons are constructed from the corresponding Kac-Moody currents :

$$\begin{aligned} V^a(k, \epsilon, z, \bar{z}) =& [\tfrac{1}{2} k \cdot \psi^L(z) \psi^{La}(z) + : \psi^{Lj}(z) \tilde{\psi}^{Lj}(z) :] \\ & \epsilon \cdot [i \bar{z} \bar{\partial} X_{\mathrm{R}}(\bar{z}) - \tfrac{1}{2} \psi^R(\bar{z}) k \cdot \psi^R(\bar{z})] e^{ik \cdot X(z, \bar{z})} \end{aligned} \tag{2.18}$$

where ν_j in the current from (2.16) is zero. The vertex operator also contains the bosonic fields given by

$$\begin{aligned} X^\mu(z, \bar{z}) &= x^\mu + \frac{p^\mu}{4i} (\ln z + \ln \bar{z}) + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}^\mu_n z^{-n} + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_n \bar{z}^{-n} \\ &= \frac{1}{2} \left(X^\mu_{\mathrm{L}}(z) + X^\mu_{\mathrm{R}}(\bar{z}) \right) \end{aligned} \tag{2.19}$$

It is only necessary to evaluate one component of a non-abelian subgroup at a time. For example, for $SU(2)$, one would look at only one of the three gauge bosons . Due to gauge invariance, it doesn't matter which one is selected. For a constant $F_{\mu\nu}$ corresponding to a given component of subgroup a, the resulting background field vertex is

$$\begin{aligned} V^a[F_{\mu\nu}](z,\bar{z}) = \tfrac{i}{4}F_{\mu\nu}\{J^a(z)[2X^\mu(z,\bar{z})\bar{z}\bar{\partial}X^\nu_{\mathrm{R}}(\bar{z}) - \psi^{R\mu}(\bar{z})\psi^{R\nu}(\bar{z})] \\ - i[\psi^{L\mu}(z)\psi^{La}(z)][\bar{z}\bar{\partial}X^\nu_{\mathrm{R}}(\bar{z})]\} \end{aligned} \tag{2.20}$$

where the gauge currents are given in (2.15) and (2.16). Using operator methods one can rewrite (2.12) as:

$$\begin{aligned} \mathcal{L}'(A^a_\mu) = \prod_l \frac{1}{\mathcal{N}_l}\sum_{\alpha,\beta} c(\alpha,\beta)\pi^2 \int \frac{d^4p}{(2\pi)^4}\int_\Gamma d^2\tau \int_{0\leq Im\nu\leq Im\tau} d^2\nu \\ \times Tr_\alpha[V^a(z,\bar{z})V^a(1,1)q^{L_0-1/2}\bar{q}^{\bar{L}_0-1/2}(e^{-i\pi})^{\rho_\beta\cdot F}] \end{aligned} \tag{2.21}$$

where z=$e^{2\pi i\nu}$, $q = e^{2\pi i\tau}$, and the integration is restricted to the fundamental region Γ. Performing the trace:

$$\begin{aligned} & Tr_\alpha[V^a(z,\bar{z})V^a(1,1)q^{L_0-1/2}\bar{q}^{\bar{L}_0-1/2}(e^{-i\pi})^{\rho_\beta\cdot F}] = \\ & -\tfrac{1}{16}F_{\mu\nu}F_{\rho\sigma}Tr_\alpha[q^{L_0-1/2}\bar{q}^{\bar{L}_0-1/2}(e^{-i\pi})^{\rho_\beta\cdot F}] \\ & \times\{\langle J^a(z)J^a(1)\rangle[\langle 2X^\mu(z,\bar{z})\bar{z}\bar{\partial}X^{R\nu}(\bar{z})2X^\rho(1,1)\bar{\partial}X^{R\sigma}(1)\rangle + \langle\psi^\mu(\bar{z})\psi^\nu(\bar{z})\psi^\rho(1)\psi^\sigma(1)\rangle] \\ & -\langle\psi^{L\mu}(z)\psi^{La}(z)\psi^{L\rho}(1)\psi^{La}(1)\rangle\langle\bar{z}\bar{\partial}X^{R\nu}(\bar{z})\bar{\partial}X^{R\sigma}(1)\rangle\} \end{aligned} \tag{2.22}$$

where the two point correlation function on a torus is defined by:

$$\langle A(z,\bar{z})B(w,\bar{w})\rangle \equiv \frac{Tr_\alpha[A(z,\bar{z})B(w,\bar{w})q^{L_0-1/2}\bar{q}^{\bar{L}_0-1/2}(e^{-i\pi})^{\rho_\beta\cdot F}]}{Tr_\alpha[q^{L_0-1/2}\bar{q}^{\bar{L}_0-1/2}(e^{-i\pi})^{\rho_\beta\cdot F}]} \tag{2.23}$$

The last term is a total derivative and drops out after using the generalized Gauss's theorem. After performing the p integration we have

$$\mathcal{L}'(A_\mu^a) = \tfrac{1}{4}F_{\mu\nu}^2 \frac{1}{16\pi^2}\prod_l \frac{1}{\mathcal{N}_l}\sum_{\alpha,\beta} c(\alpha,\beta)\int_\Gamma \frac{d^2\tau}{\tau_2}\int \frac{d^2\nu}{\tau_2}\langle J^a(z)J^a(1)\rangle_{\alpha,\beta} \\ \times 2[\langle\Psi(\bar z)\Psi(1)\rangle_{\alpha,\beta}^2 - \langle X^R(\bar z)\bar z\bar\partial X^R(1)\rangle_{\alpha,\beta}^2] Tr_\alpha[q^{L_0'-1/2}\bar q^{\bar L_0'-1/2}(e^{-i\pi})^{\rho_\beta\cdot F}] \quad . \tag{2.24}$$

The fermionic current gives rise to a gauge independent(apart from k_i) part and a gauge dependent part:

$$\langle J^a(z)J^a(1)\rangle_{\alpha,\beta} = -k_a(z\frac{\partial}{\partial z})^2 \log\theta_1(z,q) + \langle J_0^a J_0^a\rangle\,. \tag{2.25}$$

Since k_a is one, the first part will shift all the groups by the same amount and can thus be absorbed into a redefinition of the string coupling constant. Using the explicit expressions for the currents, their correlation functions are found to be:

$$\langle J_0^a J_0^a\rangle_{\alpha,\beta} = \frac{1}{2}f_{cd}^a f_{cd}^a 2q\log\theta\begin{bmatrix}\rho_\alpha^c\\ \rho_\beta^c\end{bmatrix}(0|q) \tag{2.26}$$

for the SU(2) currents(no sum on a) and

$$\langle J_0^a J_0^a\rangle_{\alpha,\beta} = 2q\log\theta\begin{bmatrix}\rho_\alpha^j\\ \rho_\beta^j\end{bmatrix}(0|q) \tag{2.27}$$

for the U(1) currents($1\le j\le 5$).

In the $SU(3)\times U(1)\times U(1)$ model presented in sect. 4, the currents are given by:

$$J^a(z) = \bar J^a(z) + q^a(z)$$

$$\bar J^a(z) = -\frac{i}{2}f_{abc}b^b(z)b^c(z) \quad ; \quad q^q(z) = i\lambda_{ij}^a : f^i(z)\tilde f^j(z) : \tag{2.28}$$

$$J^{11}(z) = \frac{1}{\sqrt3}[: f^i(z)\tilde f^i(z) : +3\nu_2] \quad ; \quad J^{12}(z) = [: f^1(z)\tilde f^1(z) : +\nu_1]$$

where $3 \leq a,b,c \leq 10$, $2 \leq i,j \leq 4$, and the strucure constants of SU(3) are normalized so $C_\psi = 2$.

The correlation function of the $SU(3)$ current is(no sum on a):

$$\langle J_0^a J_0^a \rangle_{\alpha,\beta} = \frac{1}{2} f_{cd}^a f_{cd}^a 2q \log \theta \begin{bmatrix} \rho_\alpha^c \\ \rho_\beta^c \end{bmatrix} (0|q) + \lambda_{ij}^a \lambda_{ij}^{a\,*} 2q \log \theta \begin{bmatrix} \rho_\alpha^j \\ \rho_\beta^j \end{bmatrix} (0|q) \,. \tag{2.29}$$

Performing the ν integration on the gauge dependent part,we have:

$$\mathcal{L}'(A_\mu^a) = \tfrac{1}{4} F_{\mu\nu}^2 \frac{1}{16\pi^2} \prod_l \frac{1}{\mathcal{N}_l} \sum_{\alpha,\beta} c(\alpha,\beta) \int_\Gamma \frac{d^2\tau}{\tau_2} 2 \langle J_0^a J_0^a \rangle$$
$$\times\, 2\bar{q} \frac{d}{d\bar{q}} \log \left(\frac{\left(\bar{\vartheta} \begin{bmatrix} \bar{\rho}_\alpha^1 \\ \bar{\rho}_\beta^1 \end{bmatrix} (\bar{q}) \right)^{\frac{1}{2}} \left(\bar{\vartheta} \begin{bmatrix} \bar{\rho}_\alpha^2 \\ \bar{\rho}_\beta^2 \end{bmatrix} (\bar{q}) \right)^{\frac{1}{2}}}{\eta(\bar{q})} \right) Tr_\alpha [q^{L_0' - \frac{1}{2}} \bar{q}^{\bar{L}_0' - \frac{1}{2}} (e^{-i\pi})^{\rho_\beta \cdot F}] \tag{2.30}$$

Now using the expression for the partition function for twisted fermions (2.4), we see that the one-loop two point background gauge field contribution to the effective lagrangian is

$$\mathcal{L}'(A_\mu^a) = -\tfrac{1}{4} F_{\mu\nu}^2 \frac{1}{16\pi^2} \int_\Gamma \frac{d^2\tau}{\tau_2} [2B_a(q,\bar{q}) + Y'] \tag{2.31}$$

where Y' is the gauge independent part and

$$B_a(q,\bar{q}) = -\prod_l \frac{1}{\mathcal{N}_l} \sum_{\alpha,\beta} c(\alpha,\beta) \frac{|\eta(q)|^{-22}}{\eta(q)} 2\bar{q} \frac{d}{d\bar{q}} \left(\left(\bar{\vartheta} \begin{bmatrix} \bar{\rho}_\alpha^1 \\ \bar{\rho}_\beta^1 \end{bmatrix} (\bar{q}) \bar{\vartheta} \begin{bmatrix} \bar{\rho}_\alpha^2 \\ \bar{\rho}_\beta^2 \end{bmatrix} (\bar{q}) \right)^{\frac{1}{2}} \Big/ \eta(\bar{q}) \right) \langle J_0^a J_0^a \rangle$$
$$\times \prod_{j=1}^{n} \left(\vartheta \begin{bmatrix} \rho_\alpha^j \\ \rho_\beta^j \end{bmatrix} (q) \right)^{1/2} \prod_{j=3}^{n\prime} \left(\bar{\vartheta} \begin{bmatrix} \bar{\rho}_\alpha^j \\ \bar{\rho}_\beta^j \end{bmatrix} (\bar{q}) \right)^{1/2} \prod_{j=1}^{m} \vartheta \begin{bmatrix} \rho_\alpha^j \\ \rho_\beta^j \end{bmatrix} (q) \prod_{j=1}^{m\prime} \bar{\vartheta} \begin{bmatrix} \bar{\rho}_\alpha^j \\ \bar{\rho}_\beta^j \end{bmatrix} (\bar{q}) \tag{2.32}$$

The first order correction to the field theory coupling is given by the coefficient of $-\frac{1}{4}F_{\mu\nu}^2$ in the one-loop two point background gauge field amplitude (2.31). In this analysis, we use the type II string normalization of $C_\psi = 2$. To compare this result with field theory, which uses a normalization of $C_\psi = 1$, we must multiply this result by a factor $\Psi^2_{FT}/\Psi^2_{str} = x_a/2$. Here, Ψ^2 is the length squared of the longest root in the subgroup and x_a is the level of the Kac-Moody algebra. Then, following ref.[5], we get an equation for the $\overline{DR}$ couplings

$$\frac{1}{\alpha_i(\mu)} = \frac{x_a/2}{\alpha_{\rm GUT}} - \frac{b_a}{2\pi}\log\mu/M_{\rm str} + \frac{\Delta_a}{4\pi}\,. \tag{2.33}$$

The gauge independent part has been absorbed into a redefinition of the string coupling by

$$\frac{1}{\alpha_{\rm GUT}} = \frac{4\pi}{g^2_{str}} + \frac{Y}{4\pi} \tag{2.34}$$

where $Y = \int_\Gamma \frac{d^2\tau}{\tau_2} Y'$ and $\alpha_a = g_a^2/4\pi$. The massive string contributions are given by the thresholds Δ_a:

$$\Delta_a = \int_\Gamma \frac{d^2\tau}{\tau_2}[x_a B_a(q,\bar q) - b_a]\,. \tag{2.35}$$

The b_a are the field theory β functions given by[22]

$$b_a = -\frac{11}{3}Tr_{\rm V}(Q_i^2) + \frac{2}{3}Tr_{\rm F}(Q_i^2) + \frac{1}{6}Tr_{\rm S}(Q_i^2) \tag{2.36}$$

Here, the traces are over two-component fermions and real scalars(in field theory normalization). The massless contribution from the string is given by

$$b_a = \lim_{q\to 0} x_a B_a(q,\bar q) \tag{2.37}$$

and should equal the field theory result. This provides a consistency check for the string calculation. One can facilitate the numerical integration by making a change of variables:

$\tau_2 \equiv Im\tau = 1/\tau_2'$. Since $B_a(-\tau_1, \tau_2) = B(\tau_1, \tau_2)^*$, the imaginary part drops out leaving:

$$\Delta_a = -2Re \int_0^{.5} d\tau_1 \int_{1/\sqrt{1-\tau_1^2}}^{0} \frac{d\tau_2'}{\tau_2'} [x_a B_a(\tau_1, \tau_2') - b_a] \,. \tag{2.38}$$

3. Threshold calculation for two N=1 $SU(2)\times U(1)^5$ chiral models

We now calculate the threshold corrections for two twisted models with $SU(2)\times U(1)^5$ symmetry. In both cases, the thresholds increase the unification scale by a very small amount.

Example 1

A model with $SU(2)\times U(1)^5$ gauge symmetry [23,19] can be described by three generators $b_0, b_1, b_2 : \mathcal{N}_0 = \mathcal{N}_1 = 2 : \mathcal{N}_2$=4: K=2. The sixteen sectors of the model can be determined from the vectors ρ_{b_i} describing the generators:

$$\rho_{b_0} = ((1)^{12};(1)^4;(1)^{12};(1)^4)$$

$$\rho_{b_1} = ((0)^{12};(0)^4;(1)^4(0)^8;(0)^2(1)^2) \tag{3.1}$$

$$\rho_{b_2} = ((0)^{10}(1)^2;(1/2)^4;(0)^2(1)^2(0)^4(1)^4;(1/2)^4)$$

Figure 1 illustrates the boundary conditions for the sectors of the model. The massless states satisfy the following criteria:

(1): For the states to survive the projections, we must have:

$$e^{-i\pi\rho_{b_i}\cdot F}\alpha = \epsilon(\alpha, b_i)^*\alpha \tag{3.2}$$

(2): The left and right movers must each have a mass eigenvalue of 0:

$$\alpha' {m_L}^2|S>= (L_0^L - 1/2)|S>= 0 \quad ; \quad \alpha' {m_R}^2|S>= (L_0^R - 1/2)|S>= 0 \tag{3.3}$$

The massless states come from the sectors: $b_1,\ b_2,\ ,\ b_1b_2,\ b_2^2,\ b_1b_2^2,\ b_2^3,\ b_1b_2^3$. For $g = SU(2)\times U(1)^5$, they are: