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Dyonic membranes

J.M. Izquierdo, N.D. Lambert, G. Papadopoulos and P.K. Townsend

DAMTP, Silver St.,
Cambridge CB3 9EW, U.K.

ABSTRACT

We present dyonic multi-membrane solutions of the N=2 D=8 supergravity theory that serves as the effective field theory of the T^2-compactified type II superstring theory. The 'electric' charge is fractional for generic asymptotic values of an axion field, as for D=4 dyons. These membrane solutions are supersymmetric, saturate a Bogomolnyi bound, fill out orbits of an $Sl(2;\mathbb{Z})$ subgroup of the type II D=8 T-duality group, and are non-singular when considered as solutions of T^3-compactified D=11 supergravity. On K_3 compactification to D=4, the conjectured type II/heterotic equivalence allows the $Sl(2;\mathbb{Z})$ group to be reinterpreted as the S-duality group of the toroidally compactified heterotic string and the dyonic membranes wrapped around homology two-cycles of K_3 as S-duals of perturbative heterotic string states.

1. Introduction

A feature of recent developments in superstring theory is the emerging importance for a variety of non-perturbative phenomena of extended object, or 'p-brane', solutions of the classical string theory. In particular, these solutions are crucial for an understanding of the various conjectured duality symmetries of both the heterotic and type II superstrings (see [1] for a recent review). It is customary to call a p-brane 'electric' if it is the source for a $(p+1)$-form potential in the effective field theory Lagrangian and 'magnetic' if it is the source for the dual $(D-p-3)$-form potential. The word 'source' may need some explanation here: one first solves the source-free equations of motion of the effective field theory; it is necessary to introduce an actual, 'fundamental', source only if the analytic continuation of the source-free solution meets with a (timelike) singularity. Otherwise, no source is needed, but here one can interpret the extended object solution as an effective source on length scales that are long compared to the size of the object's core.

In a D-dimensional spacetime the magnetic dual of an electric p-brane is a $\tilde{p}$-brane, where $\tilde{p}$ is related to p by [2]

$$\tilde{p} = D - p - 4 \ . \tag{1.1}$$

It follows that a p-brane can carry *both* electric and magnetic charge only if

$$D = 2p + 4 \ , \qquad p = 0, 1, 2, \ldots \tag{1.2}$$

The simplest case is $D = 4$ for which there arises the possibility of particles carrying both electric and magnetic charge, i.e. dyons. The next simplest case is $D = 6$ for which there exists the possibility of dyonic strings. In fact, one can find a self-dual string in $D = 6$, which is intrinsically dyonic because the two-form potential to which it couples has a self-dual field strength [3]. This feature makes the D=6 case rather different from D=4, so we shall move on to the next case which is that of membranes, i.e. $p = 2$, in D=8 *. In this case we need a Lagrangian with a three-form potential. The unique D=8 supersymmetric field theory with this property is N=2 D=8 supergravity [5], which is the effective field theory for the T^2-compactified type II superstring. This article reports on the construction of dyonic multi-membrane solutions of this effective field theory.

The N=2 D=8 supergravity theory has an $Sl(3;\mathbb{R}) \times Sl(2;\mathbb{R})$ symmetry of the equations of motion. The discrete subgroup $Sl(3;\mathbb{Z}) \times Sl(2;\mathbb{Z})$ was conjectured in [6] to extend to a U-duality of the D=8 type II superstring theory; this group contains the T-duality group $SO(2,2;\mathbb{Z}) \equiv [Sl(2;\mathbb{Z}) \times Sl(2;\mathbb{Z})]/Z_2$. There is a consistent truncation of the N=2 D=8 supergravity in which the only surviving fields are the spacetime metric, $g_{\mu\nu}$, a scalar, σ, a pseudoscalar ρ and a three-form gauge potential, A, with four-form field strength $F = dA$. The Lagrangian of this truncated theory is

$$\begin{aligned}\mathcal{L} = N\Big\{ &\sqrt{-g}\big[R - 2\partial_\mu\sigma\partial^\mu\sigma - 2e^{4\sigma}\partial_\mu\rho\partial^\mu\rho - \frac{1}{12}e^{-2\sigma}F_{\alpha\beta\gamma\delta}F^{\alpha\beta\gamma\delta}\big] \\ &- \frac{1}{144}\varepsilon^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}\rho F_{\mu\nu\rho\sigma}F_{\alpha\beta\gamma\delta}\Big\}\ ,\end{aligned} \tag{1.3}$$

where N is a normalization factor, which we can choose at our convenience. The coefficient of the $\varepsilon\rho FF$ term is crucial to the results to follow so we should point out that we disagree by a factor of three with the coefficient of this term given in [5]. The coefficient can be simply determined by dimensional reduction of the D=11 supergravity theory, which was the method used in [5], but this leads to the coefficient used here rather than that of [5].

The σ and ρ kinetic terms of (1.3) constitute a sigma model with target space

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$\star$ During the preparation of this paper an article, chiefly about D=6 dyonic strings and their D=10 interpretation, appeared in which the possibility of dyonic membranes in $D = 8$ was noted [4].

$Sl(2;\mathbb{R})/U(1)$. It is convenient to introduce the complex field

$$\lambda = 2\rho + ie^{-2\sigma} \ , \tag{1.4}$$

taking values in the upper half complex plane, since the $Sl(2;\mathbb{R})$ group acts on λ by fractional linear transformations. Since the asymptotic value of λ is undetermined by the equations of motion, the possible vacua correspond to points in the upper half plane. However, T-duality of the type II D=8 superstring theory implies that points that lie in an orbit of an $Sl(2;\mathbb{Z})$ subgroup of $Sl(2;\mathbb{R})$ correspond to equivalent vacua. Thus, the moduli space of vacua in the string theory context is, assuming T-duality, the fundamental domain of $SL(2;\mathbb{Z})$ in the upper half complex plane.

We shall be interested in infinite planar membrane solutions of the equations of motion of (1.3) that are asymptotically flat as one approaches spatial infinity in non-coplanar directions; we shall call this 'transverse spatial infinity', which is topologically $S^4 \times \mathbb{R}^2$. Membrane solutions can be characterised by their electric and magnetic number densities

$$q = \frac{N}{e}\oint G \qquad p = \frac{e}{2\pi}\oint F \tag{1.5}$$

where the integral is over a 4-sphere cross-section of transverse spatial infinity, e is an arbitrary unit of 'electric' charge, and the two-form G is related to the Hodge dual $\tilde{F}$ of F by

$$G \equiv e^{-2\sigma}\tilde{F} - 2\rho F \ . \tag{1.6}$$

We shall require an asymptotic translational invariance in directions coplanar with the membrane so that these number densities are actually constant; we shall refer to these constants as the membrane 'charges'. Their conservation follows from the fact that the combined equations of motion and Bianchi identities of the field-strength four-form F can be written as $d\mathcal{F} = 0$ where $\mathcal{F}$ is the $Sl(2;\mathbb{R})$ doublet

$$\mathcal{F} = (F, G)\ . \tag{1.7}$$

We shall choose the constants N and e such that

$$q = \frac{1}{\Omega_4}\oint G \qquad p = \frac{1}{\Omega_4}\oint F \tag{1.8}$$

where $\Omega_4 = 2\pi^2$ is the volume of the unit 4-sphere. With this choice, the charges (p, q) form an $Sl(2;\mathbb{R})$ doublet.

As shown in [2], the electric and magnetic charges of extended objects are subject to a generalization of the Dirac quantization condition. However, just as the Dirac quantization condition must be replaced, in the context of dyons, by the Schwinger-Zwanziger quantization condition so, in the context of dyonic extended objects, the Nepomechie-Teitelboim (N-T) quantization condition must be replaced by an extended object analogue of the Schwinger-Zwanziger quantization condition. With the above choice of normalization constant, N, and electric charge unit, e, this generalized N-T quantization condition for two dyonic membranes with charges (p, q) and (p', q') takes the simple (manifestly $Sl(2;\mathbb{R})$ invariant) form

$$qp' - q'p \ \in \ \mathbb{Z}\ . \tag{1.9}$$

As for dyons in D=4 [7], this formula allows fractional q for dyonic membranes, but the consequences for dyonic membranes are not quite the same as those for dyons because one cannot take for granted the existence of purely electric membranes in the quantum theory.

In [8] it was shown how an analogue of the Bogomolnyi-Gibbons-Hull bound for particle-like solutions of Maxwell/Einstein theory can be derived for p-brane solutions of certain antisymmetric tensor generalizations of Maxwell/Einstein theory. The precise interactions of the antisymmetric tensor field, e.g. the coefficient of possible Chern-Simons terms was crucial to this result. In all cases, the interactions were precisely those for which the bosonic field theory could be interpreted as a consistent truncation of a supergravity theory. Since this condition is satisfied by the Lagrangian (1.3) one would expect to be able to derive a similar bound on the tension of membrane solutions of its equations of motion; this case is not covered by the results of [8] because Lagrangians with scalar fields were not considered there. This expectation is correct; we shall show that the tension, M, of membrane solutions of (1.3) satisfies the $Sl(2;\mathbb{R})$ invariant bound

$$M^2 \geq \frac{1}{4}\Big[e^{2\langle\sigma\rangle}\big(q+2\langle\rho\rangle p\big)^2 + e^{-2\langle\sigma\rangle}p^2\Big] \ , \tag{1.10}$$

where $\langle\rho\rangle$ and $\langle\sigma\rangle$ are the asymptotic values of ρ and σ.

Solutions which saturate the bound are 'supersymmetric' in that they admit Killing spinors. The purely electric and magnetic D=8 supersymmetric membrane solutions, with $\rho \equiv 0$, have been given previously [3]. The supersymmetric membrane solutions we construct here differ in that they have non-constant axion field and carry both electric and magnetic charge, i.e. they are 'dyonic'. There is a U(1) parameter family of these solutions for each value of the asymptotic values of σ and ρ, corresponding to the U(1) stability subgroup of $Sl(2;\mathbb{R})$ acting on the upper-half plane by fractional linear transformations. Although only a Z_2 family of these will survive quantization, the identification of vacua related by a transformation in the $Sl(2;\mathbb{Z})$ T-duality subgroup of $Sl(2;\mathbb{R})$ allows us to find $Sl(2;\mathbb{Z})$ orbits of membrane solutions about equivalent vacua, as has been done previously for particle-like solutions in D=4 [9]. Almost all such solutions are dyonic.

One motivation for our work derives from a recently suggested D=8 membrane/membrane duality [10]. The point here is, firstly, that while the purely electric membrane solution of N=2 D=8 supergravity theory can be interpreted as the membrane solution of D=11 supergravity in a T^3 compactified spacetime, the purely magnetic one can be interpreted as a double dimension reduction of the fivebrane solution of D=11 supergravity*. Secondly, the worldvolume action of this magnetic membrane is that of a D=11 supermembrane in a T^3 compactified spacetime (and not that of a D=8 supermembrane, as one might have guessed; the extra three coordinates come from the antisymmetric tensor in the fivebrane's worldvolume action). This suggests a complete non-perturbative equivalence between the electric and magnetic membranes. This equivalence would be guaranteed in string theory by non-perturbative T-duality. Unfortunately, this cannot be established in string perturbation theory, but one can reverse the logic and use the evidence of membrane/membrane duality given in [10] and the results presented here as evidence for the non-perturbative validity of T-duality.

Another motivation comes from the conjectured non-perturbative equivalence of the $K_3 \times T^2$ compactified type II superstring theory with the toroidally compactified heterotic string theory [6], for which there is now considerable evidence. Many recent papers dedicated to tests of this conjecture have taken as their starting point the related conjecture that the D=6 string theories obtained by compactification of the type IIA superstring on K_3 and the heterotic string on T^4 are non-perturbatively equivalent [11]. Given this D=6 equivalence, the equivalence in D=4 follows upon further compactification on T^2. S-duality of the heterotic string [12,13] can then be re-interpreted as T-duality of the type II superstring [14,11]. This approach to understanding D=4 S-duality via the heterotic/type II equivalence can be characterised by the motto "10 to 6 and then to 4". Our work can be viewed as a first step towards an understanding of heterotic S-duality via the alternative "10 to 8 and then to 4" approach. The first step is a T^2 compactification of both the type II and the heterotic string to D=8. A subsequent compactification of the D=8 type II superstring on K_3 is then expected to lead to a D=4 string theory that is equivalent to the T^4 compactified D=8 heterotic string. Because K_3 has no isometries, the full T-duality group of the D=4 type II superstring theory obtained in this way must be already apparent in D=8. Type II/heterotic duality implies

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⋆ This was stated in [10]; here we verify it.

that some $Sl(2;\mathbb{Z})$ subgroup of this $Sl(2;\mathbb{Z}) \times Sl(2;\mathbb{Z})$ T-duality group should be identified with the S-duality group of the heterotic string, and the relevant subgroup is precisely the $Sl(2;\mathbb{Z})$ T-duality subgroup that acts on dyonic membrane solutions of (1.3).

This can be seen (assuming the type II/heterotic equivalence) from the origin in the D=8 type II superstring of the 28 vector potentials of the effective D=4 Maxwell/Einstein N=4 supergravity theory. Six of them come from the six vector potentials that are already present in D=8; these were discarded in the truncation leading to (1.3). The remaining 22 come from the three-form potential A of (1.3) or, equivalently, its field-strength four-form F, via the ansatz

$$F(x,y) = F^I(x) \wedge \omega_I(y) \ , \tag{1.11}$$

where ω_I span the 22-dimensional space of harmonic two-forms on K_3 (coordinates y) and F^I are the 22 Maxwell field-strength two-forms of the D=4 spacetime (coordinates x). Since we discarded the six D=8 vector potentials, the D=8 Lagrangian (1.3) yields a D=4 Lagrangian with only 22 of the 28 vector potentials of the full effective supergravity theory and an $Sl(2;\mathbb{R}) \times SO(3,19)$ symmetry group of the equations of motion. When this D=4 theory is viewed as a truncation of the effective supergravity theory of the D=4 heterotic string the $SO(3,19)$ group is clearly the subgroup of the full $SO(6,22)$ 'classical' T-duality group that survives the truncation of the six vector potentials, so the $Sl(2;\mathbb{R})$ subgroup can only be identified with the 'classical' S-duality group of the heterotic string.

In the following, we begin with a presentation of the dyonic membrane solutions of the field equations of the Lagrangian (1.3). We then explain how these solutions were found and why their tension saturates a Bogomolnyi-Gibbons-Hull type bound. We also exhibit the Killing spinors admitted by these solutions, thereby establishing their supersymmetry. We then discuss the global structure of the dyonic membranes and their interpretation as solutions of D=11 supergravity. We conclude with some further comments on the significance of our results.

2. D=8 dyonic membranes

The field equations of the Lagrangian (1.3) are

$$\begin{aligned} G_{\mu\nu} &= 2T_{\mu\nu} \\ \partial_\mu\big(\sqrt{-g}\; e^{-2\sigma} F^{\mu\nu\rho\sigma}\big) &= -2\big(\partial_\mu\rho\big)\tilde F^{\mu\nu\rho\sigma} \\ \partial_\mu\big(\sqrt{-g}\; e^{4\sigma}\partial^\mu\rho\big) &= \frac{1}{24}F_{\mu\nu\rho\sigma}\tilde F^{\mu\nu\rho\sigma} \\ \partial_\mu\big(\sqrt{-g}\;\partial^\mu\sigma\big) &= \sqrt{-g}\big[2e^{4\sigma}(\partial\rho)^2 - \frac{1}{24}e^{-2\sigma}F^2\big]\ , \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} T_{\mu\nu} =& \big[\partial_\mu\sigma\partial_\nu\sigma - \frac{1}{2}g_{\mu\nu}(\partial\sigma)^2\big] + e^{4\sigma}\big[\partial_\mu\rho\partial_\nu\rho - \frac{1}{2}g_{\mu\nu}(\partial\rho)^2\big] \\ &+ \frac{1}{6}e^{-2\sigma}\big[F_{\mu\alpha\beta\gamma}F_\nu{}^{\alpha\beta\gamma} - \frac{1}{8}g_{\mu\nu}F^2\big]\ , \end{aligned} \tag{2.2}$$

and

$$\tilde F^{\mu\nu\rho\sigma} \equiv \frac{1}{24}\varepsilon^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}F_{\alpha\beta\gamma\delta}\ . \tag{2.3}$$

We shall consider field configurations representing an infinite planar membrane and choose coordinates such that it is aligned with the $x^1 \equiv y$ and $x^2 \equiv z$ axes. We shall look for product metrics in which the metric of the five-dimensional 'transverse' space is conformally flat and may therefore be parameterised by the coordinates $\mathbf{x} \equiv (x^3, \ldots, x^7)$ of an associated five-dimensional Euclidean space, $\mathbb{E}^5$. There are certainly many solutions of the field equations (1.3) within this class of field configurations, but we shall concentrate on those that admit Killing spinors. We shall first present these solutions. Then, in the following section, we shall explain how they were obtained and why they are supersymmetric. We shall present the solutions in terms of the complex field λ defined in (1.4). If we fix boundary conditions such that the spacetime is asymptotically flat as $|\mathbf{x}| \to \infty$, and such that

$$\lambda \to i\ , \tag{2.4}$$

then the following multi-membrane field configurations solve (2.1) for arbitrary angular parameter ξ:

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}}[-dt^2 + dy^2 + dz^2] + H^{\frac{1}{2}} d\mathbf{x}\cdot d\mathbf{x} \\ F &= \frac{1}{2}\cos\xi\,(\star dH) + \frac{1}{2}\sin\xi\, dH^{-1}\wedge dt\wedge dy\wedge dz \\ \lambda &= \frac{\sin 2\xi\,(1-H) + 2iH^{\frac{1}{2}}}{2(\sin^2\xi + H\cos^2\xi)}\ . \end{aligned} \tag{2.5}$$

Here, the symbol $\star$ indicates the Hodge dual in $\mathbb{E}^5$ and

$$H = 1 + \sum_{n=1}^{N} \frac{\mu_n}{|\mathbf{x}-\mathbf{x}_n|^3} \tag{2.6}$$

for n arbitrary constants μ_n associated with the N points $\mathbf{x}=\mathbf{x}_n$, for any finite value of n. That is, $H(\mathbf{x})$ solves the Laplacian on $\mathbb{E}^5$ with an arbitrary number of point sources and such that $H\to 1$ as $|\mathbf{x}|\to\infty$. The constants μ_n are proportional to the ADM tension of each membrane solution. Specifically, for a one membrane solution with parameter μ the ADM tension is

$$M = \frac{3}{4}\mu\ . \tag{2.7}$$

We have presented the solutions for a specially chosen asymptotic value of λ because a solution with any other asymptotic value of λ can be found by making use of the $Sl(2;\mathbb{R})$ invariance of the field equations. As stated earlier, this $Sl(2;\mathbb{R})$ group acts on λ by fractional linear transformations:

$$\lambda \to \frac{a\lambda + b}{c\lambda + d}\ , \tag{2.8}$$

where a,b,c,d are real numbers such that $ad-bc=1$. The $Sl(2;\mathbb{R})$ group acts on the four-form doublet $\mathcal{F}=(F,G)$ by a generalization of electromagnetic duality.

Specifically, if λ is transformed as in (2.8), then the associated transformation of $\mathcal{F}$ is

$$\mathcal{F} \to (F,G)\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \ . \tag{2.9}$$

Since there is a $U(1)$ isotropy subgroup of $Sl(2;\mathbb{R})$ that does not change the asymptotic value, $\langle\lambda\rangle$, of λ, there must be a $U(1)$ family of solutions for each choice of $\langle\lambda\rangle$. This is the significance of the angular parameter ξ in (2.5). This $U(1)$ group is an analogue of the electromagnetic duality group since it takes a purely electric or purely magnetic solution into a dyonic one. Thus, the general solution of (2.5) can be obtained by a $U(1)$ transformation of the purely magnetic solution

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}}[-dt^2 + dy^2 + dz^2] + H^{\frac{1}{2}} d\mathbf{x}\cdot d\mathbf{x} \\ F &= \frac{1}{2} \star dH \\ \lambda &= iH^{-\frac{1}{2}} \ . \end{aligned} \tag{2.10}$$

However, because of charge quantization, this classical $U(1)$ symmetry will be broken to Z_2 in the quantum theory; there will be some 'preferred' value of $\langle\lambda\rangle$ for which only the purely electric or purely magnetic solutions survive (by analogy with D=4 dyons one might suppose that $\langle\lambda\rangle = i$ is the 'preferred' value; we shall examine this hypothesis in more detail later). It might therefore appear that the more general dyonic membrane solutions of (2.5) are irrelevant to the type II string theory, at least for the 'preferred' value of $\langle\lambda\rangle$. However, the sigma-model target space of (1.3) is only required by supersymmetry to be *locally* isometric to the coset space $SL(2;\mathbb{R})/U(1)$. It may differ globally since it is possible to identify points on this space that differ by the action of $Sl(2;\mathbb{Z})$. Thus, the true sigma-model space could be

$$\mathcal{M} = Sl(2;\mathbb{Z})\backslash Sl(2;\mathbb{R})/U(1) \ . \tag{2.11}$$

In this case the true moduli space is not the entire upper-half λ-plane but rather the fundamental domain of $Sl(2;\mathbb{Z})$ in the upper half plane. In the context of the D=8 type II superstring theory, T-duality implies that this is indeed the true moduli space of vacua, so vacua which differ by the action of $Sl(2;\mathbb{Z})$ should be identified. Thus an $Sl(2;\mathbb{Z})$ transformation of the purely magnetic membrane solution (2.10) will produce a new solution with a different, but *equivalent*, value of λ, and this solution will have an effective non-zero value of ξ, i.e. it will be dyonic.

Actually, we shall find a more general class of dyon solutions by applying this procedure to the dyonic solutions (2.5) rather than to the purely magnetic solution (2.10), i.e. we allow for an arbitrary initial value of the angular parameter ξ. First we make an $Sl(2;\mathbb{R})$ transform of the solution (2.5)to arrive at

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}}[-dt^2 + dy^2 + dz^2] + H^{\frac{1}{2}} d\mathbf{x}\cdot d\mathbf{x} \\ F &= \frac{1}{2} e^{2\langle\sigma\rangle}\Big(\cos\psi \star dH + \sin\psi\ dH^{-1}\wedge dt\wedge dy\wedge dz\Big) \\ \lambda &= 2\langle\rho\rangle + e^{-2\langle\sigma\rangle}\cdot \frac{(1-H)\sin 2\psi + 2iH^{\frac{1}{2}}}{2(H\cos^2\psi + \sin^2\psi)}\ , \end{aligned} \tag{2.12}$$

where

$$e^{-2\langle\sigma\rangle} = \frac{1}{c^2+d^2}\ , \qquad 2\langle\rho\rangle = \frac{bd+ac}{c^2+d^2}\ , \tag{2.13}$$

and the new angular parameter ψ is given by

$$\tan\psi = \frac{d\sin\xi + c\cos\xi}{d\cos\xi - c\sin\xi}\ . \tag{2.14}$$

Then, we restrict a,b,c,d to be integers to obtain the dyon solutions with $\langle\lambda\rangle \cong i$. By construction, these solutions form a representation of $Sl(2;\mathbb{Z})$. Note that the set of dyon solutions obtained in this way will contain a purely magnetic solution if and only if $\tan\xi$ is rational. If this condition is satisfied then there will also be a purely electric solution.

Clearly, a similar set of dyonic membrane solutions can be found for any other initial choice of $\langle\lambda\rangle$. However, if initially $\langle\lambda\rangle \neq i$, then the $Sl(2;\mathbb{Z})$ subgroup is not found by simply restricting a,b,c,d to be integers. Rather, the elements of the $Sl(2;\mathbb{Z})$ subgroup are similarity transforms of matrices with integer entries.

3. Killing spinors and the Bogomol'nyi Bound

We have claimed that the dyonic membrane solutions presented above are supersymmetric, i.e. that they admit Killing spinors. We shall now elaborate on this point. A Killing spinor is a spinor field, ϵ, that is in the kernel of a first-order Lorentz-covariant Dirac-type operator $\hat{\mathcal{D}}$, i.e. $\hat{\mathcal{D}}\epsilon = 0$, where a minimal condition on $\hat{\mathcal{D}}$ is that the vector field $\bar{\epsilon}\gamma^\mu\epsilon$ is Killing if ϵ is. In the context of field theories with scalar and vector fields, this condition limits, but does not define, $\hat{\mathcal{D}}$. Within the context of a supergravity theory, $\hat{\mathcal{D}}$ is defined by the gravitini transformation laws, but an alternative intrinsic definition is possible in the context of an *a priori* arbitrary bosonic Lagrangian via the modified Nester tensor

$$\hat{E}^{\mu\nu} = \frac{1}{2}\bar{\epsilon}\Gamma^{\mu\nu\rho}\hat{\mathcal{D}}_\rho\epsilon + c.c. \ . \tag{3.1}$$

This is because the operator $\hat{\mathcal{D}}$ is fixed, if it exists, by the requirement that

$$\mathcal{D}_\nu \hat{E}^{\mu\nu} = \overline{\hat{\mathcal{D}}_\nu\epsilon}\, \Gamma^{\mu\nu\rho}\hat{\mathcal{D}}_\rho\epsilon - \frac{1}{2}\bar{\chi}\Gamma^\mu\chi \ , \tag{3.2}$$

as a consequence of the field equations, for some complex spinor χ. This requirement also fixes χ. The significance of the relation (3.2) is that it allows the derivation of a bound on the mass per unit p-volume, i.e. the tension, of configurations that are subject only to the boundary conditions at transverse spatial infinity satisfied by p-brane solutions of the equations of motion [8]. It can happen that the field equations of a given Lagrangian are such that (3.2) is not satisfied by any operator $\hat{\mathcal{D}}$ for any spinor χ. In this case a bound on the tension cannot be derived by this method. Conversely, requiring that such a bound be derivable in a Lagrangian whose interactions are parameterised by arbitrary functions of the scalar fields can fix these functions. For example, allowing arbitrary interactions of σ consistent with the requirement that the field equations be of second order, and an arbitrary coefficient of the $\rho F\tilde{F}$ term, one finds that the only Lagrangian in this class for which an energy bound on the membrane tension can be derived is precisely the Lagrangian of (1.3).

For the case in hand, one finds that

$$\hat{\mathcal{D}}_\mu \epsilon \equiv \mathcal{D}_\mu \epsilon - \frac{1}{2}\gamma_9 \epsilon \, e^{2\sigma}\partial_\mu \rho + \frac{1}{96}\Gamma^{\alpha\beta\gamma\delta}\Gamma_\mu \epsilon \, e^{-\sigma} F_{\alpha\beta\gamma\delta} \ , \tag{3.3}$$

and

$$\chi = \Gamma^\mu \epsilon \, \partial_\mu \sigma - \gamma_9 \Gamma^\mu \epsilon \, e^{2\sigma}\partial_\mu \rho - \frac{1}{48}\Gamma^{\alpha\beta\gamma\delta}\epsilon \, e^{-\sigma} F_{\alpha\beta\gamma\delta} \ . \tag{3.4}$$

The matrix γ_9 is defined by

$$\gamma_9 = \Gamma^{\underline{0}}\Gamma^{\underline{1}} \cdots \Gamma^{\underline{7}} \tag{3.5}$$

where the underlining indicates a flat space Dirac matrix. It follows from (3.3) that

$$\hat{E}^{\mu\nu} = E^{\mu\nu} - \frac{1}{2}e^{2\sigma}(\bar{\epsilon}\Gamma^{\mu\nu\alpha}\gamma_9 \epsilon)\partial_\alpha \rho - \frac{1}{4}e^{-\sigma}\bar{\epsilon}(F^{\mu\nu\alpha\beta}\Gamma_{\alpha\beta} - \tilde{F}^{\mu\nu\alpha\beta}\Gamma_{\alpha\beta}\gamma_9)\epsilon \tag{3.6}$$

where $E^{\mu\nu}$ is the standard Nester tensor. Note that the Dirac conjugate $\bar{\psi}$ of a spinor ψ is defined by

$$\bar{\psi} = \psi^\dagger \Gamma^{\underline{0}} \ , \tag{3.7}$$

so that $\bar{\psi}\Gamma^{\underline{0}}\psi$ is negative definite. Note also that the Lorentz invariant $\bar{\psi}\psi$ is pure imaginary; this follows from the fact that this invariant vanishes identically when ψ is Majorana*.

As explained in the introduction, the relevant concept for defining membrane charges is transverse spatial infinity, which has topology $S^4 \times \mathbb{R}^2$. It is convenient to choose periodic boundary conditions to convert this to $S^4 \times T^2$, i.e. we consider the membrane to be wrapped around a large two-torus. The energy per unit area,

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$\star$ To see this, choose the Majorana basis in which the matrices Γ^μ are pure imaginary (in D=4, with the same metric convention they would be all real). But $\Gamma^{\underline{0}}$ is anti-Hermitian, because it is unitary and squares to one, so $\Gamma^{\underline{0}}$ is symmetric in the Majorana basis (in fact, equal to i times the charge conjugation matrix, which is symmetric in D=8). But, in this basis a Majorana spinor is real, so that $\bar{\psi}\psi = \psi^T \Gamma^{\underline{0}}\psi$ for real spinor ψ, which vanishes by symmetry.

M, is then the, now finite, total energy divided by the volume, V_2, of the two-torus. This energy can expressed as an integral over the $S^4 \times T^2$ surface at spatial infinity. Specifically, if $\mathbf{P}$ is the total transverse 6-momentum per unit area, such that $M = \sqrt{-|\mathbf{P}|^2}$, then [8]

$$\bar{\epsilon}_\infty \mathbf{\Gamma} \cdot \mathbf{P} \epsilon_\infty = \frac{1}{2V_2\Omega_4} \oint_\infty dS_{\mu\nu} E^{\mu\nu} \ , \tag{3.8}$$

where Ω_4 is the volume of the unit 4-sphere. With appropriate asymptotic fall off conditions on the metric, and assuming that

$$\epsilon \longrightarrow \epsilon_\infty \tag{3.9}$$

as $|\mathbf{x}| \to \infty$, for some constant spinor ϵ_∞, (3.8) can be rewritten as

$$\bar{\epsilon}_\infty \mathbf{\Gamma} \cdot \mathbf{P} \epsilon_\infty = \frac{1}{2\Omega_4} \oint_\infty dS_{ij} E^{ij} \ , \tag{3.10}$$

where the integral is now over the 4-sphere at spatial infinity and the index i is associated with the coordinates $\mathbf{x}$ of the transverse space.

Assuming that the only components of F that are non-vanishing at transverse spatial infinity are F_{ijkl} and F_{tyzi}, and that these components depend asymptotically only on x^i, one has that

$$\begin{aligned} \frac{1}{2V_2\Omega_4} \oint_\infty dS_{\mu\nu} \hat{E}^{\mu\nu} &= \frac{1}{2\Omega_4} \oint_\infty dS_{ij} \hat{E}^{ij} \\ &= \bar{\epsilon}_\infty \Big[\mathbf{\Gamma} \cdot \mathbf{P} - \frac{1}{8\Omega_4} e^{-\langle\sigma\rangle} \Gamma_{kl} \oint_\infty dS_{ij} \Big(F^{ijkl} - \tilde{F}^{ijkl} \gamma_9 \Big) \Big] \epsilon_\infty \ , \end{aligned} \tag{3.11}$$

since the $\partial\rho$ term in (3.6) does not contribute to the integral. From the definitions

(1.8) of the charges (p, q) one then finds that

$$\frac{1}{2V_2\Omega_4}\oint_{\infty} dS_{\mu\nu}\hat{E}^{\mu\nu} = \bar{\epsilon}_{\infty} K \epsilon_{\infty} \tag{3.12}$$

where

$$K = \mathbf{\Gamma}\cdot\mathbf{P} - \frac{1}{2}\Big[e^{\langle\sigma\rangle}(q + 2\langle\rho\rangle p)\Gamma_{yz} - e^{-\langle\sigma\rangle}p\ \Gamma_{yz}\gamma_9\Big]\ . \tag{3.13}$$

Using Gauss's law, the relation (3.2), and choosing ϵ to satisfy a 'modified Witten condition', one can prove that the integral on the left hand side of (3.12) is positive, subject to the usual assumptions. It follows that the Dirac matrix K is positive semi-definite, which implies the bound (3.11) quoted in the introduction.

This bound is saturated by solutions of the equations of motion for which there exists a spinor ϵ such that

$$\hat{\mathcal{D}}_\mu\epsilon = 0\ , \qquad\qquad \chi = 0\ . \tag{3.14}$$

Non-trivial solutions of these relations, i.e. those for which $M \neq 0$, require ϵ to satisfy a condition of the form

$$\big[\alpha(\mathbf{x})\Gamma_* + \beta(\mathbf{x})\Gamma_*\gamma_9\big]\epsilon(\mathbf{x}) = \epsilon(\mathbf{x})\ , \tag{3.15}$$

where

$$\Gamma_* = \Gamma^{\underline{0}}\Gamma^{\underline{1}}\Gamma^{\underline{2}}\ . \tag{3.16}$$

and α and β are functions such that

$$\alpha^2 + \beta^2 = 1\ . \tag{3.17}$$

This can be seen from the fact that the spinor ϵ must be an eigen-spinor of the matrix K with zero eigenvalue. The angular parameter ξ enters into the solutions

(2.5) as the limit of the ratio of the functions α and β, i.e.

$$\lim_{|\mathbf{x}|\to\infty}\left(\frac{\alpha}{\beta}\right) = \tan\xi \ . \tag{3.18}$$

The multi-dyon solutions (2.5) were obtained by substituting an appropriate ansatz into the relations (3.14). The constraint (3.15) reduces the dimension of the space of Killing spinors to half that of the constant Killing spinors of the vacuum. Thus, the solutions we find in this way will break half the supersymmetry. Furthermore, they saturate the bound (3.11) by construction, so their membrane tension is given by the formula

$$M^2 = \frac{1}{4}\left[e^{2\langle\sigma\rangle}(q+2\langle\rho\rangle p)^2 + e^{-2\langle\sigma\rangle}p^2\right] \ . \tag{3.19}$$

where M is related to the constants μ_n appearing in the solutions by

$$M = \frac{3}{4}\sum_{n=1}^{N}\mu_n \ . \tag{3.20}$$

Since this applies for any value of N we may suppose that each μ_n satisfies a similar bound so that, in particular, $\mu_n \geq 0$. In this case, the only singularities of the metric are at the 'centres' $\mathbf{x} = \mathbf{x}_n$. The question whether these are real singularities or merely coordinate singularities will be addressed in the following section. From (2.9) we see that the $Sl(2;\mathbb{R})$ transformation of (p,q) is

$$(p,q) \to (p,q)\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \ . \tag{3.21}$$

Given that $\langle\sigma\rangle$ and $\langle\rho\rangle$ are also transformed according to (2.8), the $SL(2;\mathbb{R})$ invariance of the formula (3.19) is easily verified.

The above procedure has the advantage that it not only yields the solutions admitting Killing spinors, for given boundary conditions, but also the Killing spinors. For the solutions (2.5) one finds that

$$\epsilon = \frac{1}{\sqrt{2}} H^{-\frac{1}{8}} (H\cos^2\xi + \sin^2\xi)^{-\frac{1}{4}} \Big\{ \big[(\sin^2\xi + H\cos^2\xi)^{\frac{1}{2}} + H^{\frac{1}{2}}\cos\xi\big]^{\frac{1}{2}} + \big[(\sin^2\xi + H\cos^2\xi)^{\frac{1}{2}} - H^{\frac{1}{2}}\cos\xi\big]^{\frac{1}{2}}\gamma_9 \Big\}\epsilon_0 \ , \tag{3.22}$$

where the constant spinor ϵ_0 must satisfy

$$\Gamma_*\gamma_9\epsilon_0 = \epsilon_0 \ , \tag{3.23}$$

in order that ϵ satisfy the constraint (3.15). It follows that the space of Killing spinors is half that of the vacuum solution, as anticipated. The expression (3.22) for the Killing spinor ϵ can be rewritten as

$$\epsilon = e^{\frac{1}{2}\theta\gamma_9} H^{-\frac{1}{8}}\epsilon_0 \ , \tag{3.24}$$

where

$$\tan\theta = H^{-\frac{1}{2}}\tan\xi \ . \tag{3.25}$$

Note that these spinors vanish at the zeros of H^{-1}.

In order to show that the $SL(2;\mathbb{R})$ transform of the solutions (2.5), for which $\langle\lambda\rangle \neq i$, are also supersymmetric it suffices to show that the conditions (3.14) are $SL(2;\mathbb{R})$ invariant. Let us denote by $\hat{\mathcal{D}}(\lambda,\mathcal{F})$ the covariant derivative $\hat{\mathcal{D}}$ in (3.3), thereby making explicit the dependence of this differential operator on the fields. Under the $SL(2;\mathbb{R})$ transformation of these fields, $\lambda \to \lambda'$ and $\mathcal{F} \to \mathcal{F}'$ (given explicitly in (2.8) and (2.9)), one can show that

$$\hat{\mathcal{D}}(\lambda',\mathcal{F}') = e^{\frac{1}{2}\phi\gamma_9}\hat{\mathcal{D}}(\lambda,\mathcal{F})e^{-\frac{1}{2}\phi\gamma_9} \tag{3.26}$$

where

$$\tan\phi = \frac{-ic(\lambda-\bar{\lambda})}{2d + c(\lambda+\bar{\lambda})} \ ; \tag{3.27}$$

i.e. $\hat{\mathcal{D}}(\lambda,\mathcal{F})$ is an $Sl(2;\mathbb{R})$-invariant covariant derivatve. If we take the $Sl(2;\mathbb{R})$

transform of ϵ to be

$$\epsilon' = e^{\frac{1}{2}\phi\gamma_9}\epsilon \ , \tag{3.28}$$

then

$$\hat{\mathcal{D}}(\lambda', \mathcal{F}')\epsilon' = e^{\frac{1}{2}\phi\gamma_9}\hat{\mathcal{D}}(\lambda, \mathcal{F})\epsilon \ , \tag{3.29}$$

Similarly, if $\chi(\lambda, \mathcal{F})$ is the spinor of (3.4) then one can show that

$$\chi(\lambda', \mathcal{F}') = e^{-\frac{1}{2}\phi\gamma_9}\chi(\lambda, \mathcal{F}) \ . \tag{3.30}$$

It follows that given background fields and a Killing spinor ϵ satisfying the conditions (3.14) for $\langle\lambda\rangle = i$, then the spinor

$$\epsilon' = e^{\frac{1}{2}(\theta+\phi)\gamma_9} H^{-\frac{1}{8}}\epsilon_0 \tag{3.31}$$

satisfies the same conditions for the $Sl(2;\mathbb{R})$ transformed solution with new asymptotic value $\langle\lambda'\rangle \neq i$. Incidentally, this result establishes the $Sl(2;\mathbb{R})$ invariance of the modified Nester tensor $\hat{E}^{\mu\nu}$ (assuming the above transformation property of ϵ) and the invariance of the Bogomolnyi bound is an immediate consequence of this.

4. Singularity structure

We now turn to the singularity structure of the dyonic membrane solutions (2.5). Near a zero of H^{-1} we have

$$H \sim \frac{\mu}{r^3} \tag{4.1}$$

where

$$r \equiv |\mathbf{x} - \mathbf{x}_n| \ . \tag{4.2}$$

The asymptotic metric is

$$r^{\frac{3}{2}}(-dt^2 + dy^2 + dz^2) + \frac{dr^2}{r^{\frac{3}{2}}} + r^{\frac{1}{2}}d\Omega_4^2 \tag{4.3}$$

where $d\Omega_4^2$ is the metric on the unit 4-sphere. One sees from this result that the proper distance to $r=0$ on a surface of constant t, y, z is finite, and that the radius of the four-sphere of constant r on this surface shrinks to zero as $r \to 0$. It follows that the 'lines' of force of F must end on a singularity at $r=0$.

It is instructive to consider the membrane spacetime in the metric

$$d\tilde{s}^2 = e^{2\sigma} ds^2 \ , \tag{4.4}$$

for which

$$d\tilde{s}^2 = (\cos^2\xi + H^{-1}\sin^2\xi)[-dt^2 + dy^2 + dz^2] + (\sin^2\xi + H\cos^2\xi)\ d\mathbf{x}\cdot d\mathbf{x}\ . \tag{4.5}$$

The purely electric case now has a timelike naked singularity at zeros of H^{-1}, i.e. at a membrane core, so it would have to be identified with a fundamental membrane. For this reason, one might choose to call the metric $d\tilde{s}^2$ the 'membrane metric'. Note that it would be the 'string metric' if σ were the dilaton, but σ is *not* the dilaton. In this 'membrane metric' the metric for a membrane carrying magnetic charge approaches the asymptotic metric

$$d\tilde{s}^2 \sim \cos^2\xi \Big\{[-dt^2 + dy^2 + dz^2] + H\ d\mathbf{x}\cdot d\mathbf{x}\Big\} \tag{4.6}$$

near any of the membrane cores. Since $H \sim \frac{\mu}{r^3}$ in this limit, we now find that the proper distance to $r=0$ is infinite on a hypersurface of constant t, y, z. Moreover, this remains true for timelike and null geodesics. Thus, the dyonic multi-membrane solutions are geodesically complete *in the 'membrane' metric* provided that the magnetic charge is non-zero.

Because σ is not the dilaton, the interpretation of the above result within string theory is unclear*. Moreover, the fact that the D=8 dyonic membrane solutions are singular in the Einstein metric, which coincides with the string metric in the type

$\star$ However, σ *can* be interpreted as the dilaton of the equivalent heterotic theory after compactification to D=4 on K_3.

II superstring theory context, must be considered a difficulty. Fortunately, this difficulty has a simple resolution if one considers the dyonic solutions as solutions of D=11 supergravity, which can be viewed as an effective action for the strongly coupled type IIA superstring [15,11]. Consider the following 11-metric and four-form

$$
\begin{aligned}
ds_{11}^2 &= e^{\frac{2}{3}\sigma} ds_8^2 + e^{-\frac{4}{3}\sigma}\, d\mathbf{u}\cdot d\mathbf{u} \\
F_{11} &= F + 6 du_1 \wedge du_2 \wedge du_3 \wedge d\rho \ ,
\end{aligned}
\tag{4.7}
$$

where $\mathbf{u}$ are the coordinates of T^3 and F is a field strength four-form (F=dA) of the eight-dimensional spacetime. This field configuration solves the equations of D=11 supergravity if the 8-metric, four-form F, and scalar fields σ and ρ solve the D=8 field equations (2.1). This allows us to lift the D=8 dyonic membrane solutions (2.5) to D=11. The result is

$$
\begin{aligned}
ds_{11}^2 &= H^{-\frac{2}{3}}\Big[\sin^2\xi + H\cos^2\xi\Big]^{\frac{1}{3}}(-dt^2+dy^2+dz^2) \\
&\quad + H^{\frac{1}{3}}\Big[\sin^2\xi + H\cos^2\xi\Big]^{\frac{1}{3}} d\mathbf{x}\cdot d\mathbf{x} + H^{\frac{1}{3}}\Big[\sin^2\xi + H\cos^2\xi\Big]^{-\frac{2}{3}} d\mathbf{u}\cdot d\mathbf{u} \\
F_{11} &= \frac{1}{2}\cos\xi(\star dH) + \frac{1}{2}\sin\xi\ dH^{-1}\wedge dt\wedge dy\wedge dz \\
&\quad - \frac{3\sin 2\xi}{2[\sin^2\xi + H\cos^2\xi]^2} du_1\wedge du_2\wedge du_3\wedge dH\ .
\end{aligned}
\tag{4.8}
$$

In the purely electric case, $\cos\xi = 0$, we have

$$
\begin{aligned}
ds_{11}^2 &= H^{-\frac{2}{3}}(-dt^2+dy^2+dz^2) + H^{\frac{1}{3}}(d\mathbf{x}\cdot d\mathbf{x} + d\mathbf{u}\cdot d\mathbf{u}) \\
F_{11} &= \frac{1}{2} dH^{-1}\wedge dt\wedge dy\wedge dz\ .
\end{aligned}
\tag{4.9}
$$

The harmonic function $H(\mathbf{x})$ can now be interpreted as a harmonic function on $\mathbb{E}^5 \times T^3$. The only difference between this solution of D=11 supergravity and the multi-membrane solution found in [16] is that there H was a harmonic function on $\mathbb{E}^8$. Thus, the solution (4.9) can be interpreted as a D=11 membrane in a background spacetime of topology $M_6 \times T^3$ instead of M_{11}, where M_k indicates a k-dimensional Minkowski spacetime.

In the purely magnetic case, $\sin\xi = 0$, we have

$$\begin{aligned} ds_{11}^2 &= H^{-\frac{1}{3}}(-dt^2 + dy^2 + dz^2 + d\mathbf{u}\cdot d\mathbf{u}) + H^{\frac{2}{3}} d\mathbf{x}\cdot d\mathbf{x} \\ F_{11} &= \frac{1}{2}\star dH \ , \end{aligned} \tag{4.10}$$

which is the fivebrane solution of D=11 supergravity [17], except for the periodic identification of the T^3 coordinates. We can therefore interpret the purely magnetic D=8 membrane as a D=11 fivebrane wrapped around a three-torus. The D=11 multi-fivebrane solution of [17] is geodesically complete [8], the singularities of H being degenerate Killing horizons, so the singularity of the magnetic D=8 membrane solution is resolved by its interpretation in D=11, apart from mild singularities introduced by the periodic identification of the T^3 coordinates.

These results for the purely electric and purely magnetic D=8 membranes confirm the assumption made in [10] concerning their D=11 origin. Now we find that the more general dyonic membrane solution also has a D=11 interpretation. Although the D=11 solution does not have an obvious p-brane interpretation, it is non-singular, as we now show. Provided the magnetic charge is non-zero, i.e. $\cos\xi \neq 0$, the asymptotic form of the metric ds_{11}^2 of (4.8) near any zero of H^{-1} is

$$ds_{11}^2 \sim (\cos\xi)^{\frac{2}{3}}\left\{ H^{-\frac{1}{3}}(-dt^2 + dy^2 + dz^2 + d\mathbf{v}\cdot d\mathbf{v}) + H^{\frac{2}{3}} d\mathbf{x}\cdot d\mathbf{x} \right\} , \tag{4.11}$$

where we have set $\mathbf{u} = (\cos\xi)\mathbf{v}$. Apart from the overall factor the result is independent ofξ. That is, the structure of the dyonic membrane near the singularities of H is the same as for the purely magnetic case. We conclude that the singularities of the dyonic membranes are equally resolved in D=11.

5. Comments

In this paper we have obtained a bound on the tension of membrane solutions of N=2 D=8 supergravity, and we have found the supersymmetric membrane solutions that saturate this bound. In general these solutions are dyonic. Since N=2 D=8 supergravity is obtained by a T^3 compactification of $D = 11$ supergravity, followed by a consistent truncation of the massive modes, the D=8 dyonic membranes can be interpreted as solutions of D=11 supergravity. The purely electric and purely magnetic D=8 membranes become the D=11 membrane and fivebrane respectively. The dyonic membranes have no obvious p-brane interpretation but they are new solutions of D=11 supergravity which are non-singular if the periodic identification of the T^3 coordinates ($\mathbf{u}$) is relaxed. These new solutions are intermediate between the D=11 membrane and fivebrane solutions. They might therefore be expected to play a role in the conjectured D=11 membrane/fivebrane duality [6,10].

The dyonic membrane solutions were given initially for a particular choice of the asymptotic values of the scalar fields that parameterise the possible vacua, but they can then be found for any choice of vacuum by means of an $Sl(2;\mathbb{R})$ transformation. In the context of type II string theory, an infinite set of dyonic membrane solutions can be found, in equivalent vacua, by the action of an $Sl(2;\mathbb{Z})$ subgroup of $Sl(2;\mathbb{R})$ since this is a subgroup of the $SO(2,2;\mathbb{Z})$ T-duality group. As explained in the introduction, this group can be re-interpreted as the S-duality group of the equivalent heterotic string theory after a compactification of the D=8 type II superstring to D=4 on K_3. Some D=4 dyon solutions of the heterotic string will thereby acquire an interpretation as D=8 dyonic membranes wrapped around the homology two-cycles of K_3. These dyons *all* correspond to non-perturbative R-R states in the type II D=4 superstring but, according to the type II/ heterotic equivalence conjecture, correspond to perturbative states of the heterotic string and their non-perturbative S-duals. In fact, they must include the dyons that can become massless at special points in the K_3 moduli space [18], as expected from the known symmetry restoration of the heterotic string at special points in its moduli space.

Dyonic membranes have many features in common with dyons. For example, let us suppose that there is a purely magnetic membrane with charges $(p, q) = (1, 0)$ when $\langle\lambda\rangle = i$; this amounts to the assumption that the choice of $\xi = 0$ in (2.5) is admissable in the quantum theory. Now consider a new vacuum related to the original one by an $Sl(2;\mathbb{R})$ transformation with the element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} . \tag{5.1}$$

One finds that $\langle\lambda'\rangle = b+i$, or equivalently $\langle\sigma\rangle = 0$, $2\langle\rho\rangle = b$, in the new vacuum and that the membrane solution in this vacuum has charges $(p, q) = (1, b) = (1, 2\langle\rho\rangle)$. Thus, a dyonic membrane with unit magnetic charge has a fractional electric charge given by

$$q = 2\langle\rho\rangle \tag{5.2}$$

This is just the generalization to dyonic membranes of the Witten effect for dyons [7]. The identification of vacua related by an $Sl(2;\mathbb{Z})$ tansformation implies, in particular, that $2\rho \cong 2\rho + 1$, so the value of q for a dyon with unit magnetic charge will change by one as the asymptotic value of 2ρ is smoothly continued from $2\langle\rho\rangle$ to $2\langle\rho\rangle + 1$. In the D=4 dyon case, this continuation of $\langle\rho\rangle$ can be realized physically by transport around an axion string. In the D=8 dyonic membrane case it could be achieved by transport around an axionic fivebrane.

There is, however, a new feature of dyonic membranes not shared by dyons. To see this, we note that given the existence of a particle with charges $(0, 1)$ in the vacuum with $\lambda = i$, the DSZ quantization condition implies that for any other particle with charges (p, q), necessarily $p \in \mathbb{Z}$, i.e. while electric charge can be fractional, magnetic charge cannot be. Had we assumed the existence of a particle with charges $(1, 0)$ we would have instead deduced that $q \in \mathbb{Z}$ and p could be fractional. The DSZ quantization condition does not distinguish between these possibilities, but perturbation theory does: in string perturbation theory there exist particles with only electric charge and all semi-classical dyons have integer magnetic charge. A similar conclusion can be made for any of the vacua in the same equivalence class of $\lambda = i$; as we saw earlier for dyonic membranes, the assumption that there exist purely electric solutions is equivalent to the assumption that $\tan\xi$ is rational. It seems, therefore, that for dyons the appeal to perturbation theory allows us to restrict the allowed values of the angular parameter analogous to ξ, but the same does not apply to dyonic membranes, at least in the context of type II superstring theory, because all membrane solutions, electric, magnetic or dyonic, are non-perturbative.

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