Batalin–Vilkovisky Formalism and Odd Symplectic Geometry

O.M. Khudaverdian *

Department of Theoretical Physics, Yerevan State University
A. Manoukian St.1, 375049 Yerevan, Armenia
and
Laboratory of Theoretical Physics, Yerevan Physics Institute
Br. Alikhanian St.2, 375036, Yerevan Armenia
e–mail: "khudian@sc2a.unige.ch" or "khudian@vx1.yerphi.am"

August 1995

It is a review of some results in Odd symplectic geometry related to the Batalin–Vilkovisky Formalism

Contents

* Partially supported by Grant No. M3Z300 of International Science Foundation

1 Introduction

The last twenty years the methods dealing with constrained systems dynamics were essentially developed on the base of BRST method. BRST method was first introduced in [15,16] and [58] for treating the gauge theories and nowadays this method is the most powerful when dealing with the degenerated Lagrangians in Field Theory.

The BRST method got very elegant mathematical formulation in the Hamiltonian as well as in the Lagrangian frameworks in the series of remarkable works of Fradkin, Batalin, Vilkovisky [25,26,10,27] (see also the review [31]) and [11,12,13,14] . —It turns out that BRST method which in fact is highly developed Lagrangian multipliers method [42] received its mathematical formulation in terms of the Symplectic Geometry of Superspace. In general case where the algebra of symmetries of the Theory is not closed off–shell (i.e. the commutator of two infinitesimal symmetry transformations is symmetry transformation up to equations of motion) the

Superspace = Space of the initial fields + Odd Space of the ghosts fields corresponding to symmetries

(Superspace= Space of the fields + Odd Space of antifields)

provided with the Poisson bracket corresponding to Even (Odd) symplectic structure is the bag in which can be packed in a very compact and beautiful way all the stuff (constraints, structure functions, ghosts,...) arising during BRST procedure in Hamiltonian (Lagrangian) frameworks. In both approaches the application of the Symplectic Geometry is highly formal and technical. But there is an essential difference between Hamiltonian and Lagrangian cases. One cannot say that the necessity of application of Even symplectic geometry in Hamiltonian framework induced its development in mathematics. It is not the case for Odd symplectic geometry.

In the pioneer works of Batalin–Vilkovisky [11,12,13,14] the Lagrangian covariant formulation of BRST formalism was constructed. These works in fact contain the constructions which were the beginning of Odd symplectic geometry. The following mathematical constructions used in these works were proposed for mathematical investigations:

1) The master–equation of the Theory was formulated in terms of Odd Poisson Bracket

2) For formulating a Quantum Master–equation it was introduced the Delta–operator in the space of fields–antifields (Φ^A, Φ^*_A):

$$\Delta = \frac{\partial^2}{\partial\Phi^A \partial\Phi^*_A} \tag{1.1}$$

3) It was considered the group of canonical transformations preserving this operator— canonical transformations preserving canonical volume form in the space of fields–antifields. (Canonical transformations do not preserve volume form)

During the years it becomes clear that these mathematical constructions are very fruitful for mathematical investigations.—They indeed contain a rich and beautiful geometry.

This paper is mostly devoted to the geometrical problems arising from the constructions of Batalin–Vilkovisky (BV) formalism in the [11,12] and to the interpretation of the BV formalism in terms of this geometry.

We sketch here briefly the main properties of Odd symplectic geometry.

On the superspace one can consider Even or Odd symplectic structures given correspondingly by Even or Odd non–degenerated closed two form on it. The analogue of Darboux Theorem [1] states that there are (locally) the coordinates in which to Even structure corresponds Poisson bracket which conjugates half of bosonic coordinates to another half (as for usual symplectic structure on the underlying space) and fermionic coordinates to themselves. If the symplectic structure is Odd then there are coordinates in which Poisson bracket conjugates bosonic coordinates to fermionic ones (see [57]).

There is essential difference between Even and Odd symplectic structures. Even structure on a superspace can be considered as a natural prolongation of the usual symplectic structure from the underlying space. It is not the case for Odd one. Let us consider following basic example:

Let T^*M be cotangent bundle of M with canonically defined symplectic structure on it [1]. By changing the parity of covectors we come from T^*M to the superspace ST^*M associated with T^*M. The canonical symplectic structure transforms to Odd symplectic structure.(See for details Section 4). The natural correspondence between polyvectorial fields on M and the functions on ST^*M transforms Schoutten bracket to Odd Poisson bracket*

Indeed roughly speaking for physicists the supermathematics often is nothing but changing of small greek and latin letters on capital letters and putting in the suitable places the corresponding sign factors—powers of (-1). And very often it is the fact. (See for example the most part of the formulae in this paper). But there are cases where the constructions in supermathematics have the properties which radically differ from the properties of their ancestors (in a bosonic case). And it is the case when we deal with Odd symplectic structure.

Like for usual symplectic structure the group of transformations preserving Even (Odd) symplectic structure is infinite–dimensional: to every function corresponds vector field–infinitesimal transformation preserving symplectic structure.** That is why mechanics is meaningful and geometry is very poor. In the case of usual symplectic geometry canonical transformations "kill" all the invariants except the Liouville volume form (and corresponded Poincare–Cartan integral invariant). The same happens in supercase.

---

* It is the reason why one of the names of Odd bracket proposed by Leites [43,45] is Buttin bracket— In 1969 C. Buttin in [22] investigated the graded algebras of polyvectorial fields.

** Symplectic geometry is adequate language for Hamiltonian Mechanics. And more natural is application of Even and Odd symplectic geometry for formulation of Hamiltonian mechanics in superspace [43]. The formulation of Hamiltonian mechanics in terms of even bracket describes the classical mechanics of fermions (See for example [18]). In the middle of 80–th D.V.Volkov with collaborators proposed to consider odd symplectic structure as more fundamental for quantization. ([60,61], see also [36]. But till now there is no essential development in this direction.

Moreover (and here begins the essential difference between Even and Odd structures) the Odd canonical transformations on the contrary to Even ones do not preserve any volume form. (If bosonic coordinate x^1 is multiplied by 2 and conjugated fermionic one θ^1 is divided by 2 then the volume form $dx^1 d\theta^1$ is multiplied by 4). So at first sight the Odd symplectic structure have more poor geometry than Even one. But the fact that no volume form is preserved by the Odd canonical transformations makes meaningful to consider the superspace provided with Odd symplectic structure and a volume form simultaneously. One can consider as a group of transformations the group of Odd canonical transformations preserving this volume form. It turns out that non–trivial geometry is related with this structure. The geometrical objects depended on a higher derivatives appear [34,35]. Let we consider for example the second order operator with value on a function equal to the divergence (by the volume form) of the Hamiltonian vector field corresponding to this function via Odd symplectic structure. One can see that it is second–order differential operator which is the covariant expression of the Delta–operator (1.1) [34]. (The corresponding constructions for Even structure are trivial). In the special case where Delta– operator on ST^*M is generated by volume form on M one can see that its action on the function corresponds to the action of divergence operator on polyvector fields i.e. it is nilpotent:

$$\Delta^2 = 0. \tag{1.2}$$

In general it is not the case. It turns out that

The BV master–equation can be formulated as the nilpotency condition of the Delta–operator corresponding to the volume form (in the space of fields–antifields) related with the exponent of the master–action of the theory.

One has to note that in the physical examples of local field theories with an open algebra of the symmetries (such as supergravity Lagrangians) the Delta–operator governing BV– quantum master–equation has a pure academical interest. The known cases are treated by the procedure suggested in [33,51] which is a special case of BV–formalism. During the years its geometrical properties were not under the serious attention. Some problems of Odd symplectic geometry were considered in [34,35,38].

In a [70] Witten proposed a program for the construction of String Field Theory in the framework of the Batalin–Vilkovisky formalism and noted the necessity of its geometrical investigation. The properties of this geometry were investigated in [55,56], [38,39,40] and [30]. The most detailed analysis was performed by A.S.Schwarz [55,56].

The BV formalism is developing now in different directions.

The understanding of the meaning of the Delta–operator induces the activity for investigating the algebraical properties of Delta–operator and its application to Topological Field Theory. (See for example [52], [29]). We have to note also multilevel field–antifield formalism with the most general Lagrangian hypergauges developed by Batalin and Tyutin [7,8,9] and of course SP(2) BV–formalism (see [4,5], [6] and also [50]). It is interesting to note the problem of locality of the master–equation general solution and the approach to the BV formalism based on the Koszul–Tate resolution ([42], [23], [46,21] and [24,32,59]). There are also an interesting results of application of Odd symplectic geometry which are not strictly connected with BV formalism [38,47,48,49]. In this paper we do not consider these topics. Our aim is very restrictive: to give a description of the pioneer work of

Batalin–Vilkovisky on the basis of Odd symplectic geometry.(We even do not consider here so called case of reducible theories [13], [53]).

In the second section of this paper we give a survey of BV formalism making accent on its algebraico–geometrical meaning.

The content of the third section is devoted to the integration theory over surfaces in a superspace [40]. We consider densities—the objects which can be integrated over the surfaces and investigate the problem of defining the right generalization of the closed differential forms on the supercase. This problem indeed is strictly connected with a problem of reducing of partition function of degenerated theory on the surface of the constraints (gauge conditions).— From the geometrical point of view to the symmetries of a Theory correspond vectors fields on the space of fields which preserve the action. The reduced partition function, when gauge conditions are fixed is the integral of a non–local density constructed by means of these vector fields over the surface defined by the gauge conditions. The gauge independence means that this density has to be closed.

In the bosonic case differential forms are simultaneously linear functions on the tangent vectors and well defined integration objects. In the supercase it is not the case.— The role of the differential forms as integration objects are played by so called pseudodifferential and pseudointegral forms. (The investigations of these problems were started in a right direction in a works [19,20] then were continued in [28] and [2,3] and were considered in details in the series of papers [62–68].) Our considerations in this section are based on these works.

In the 4-th Section we deliver the main results in Odd symplectic geometry (described shortly above) related to BV formalism and give an interpretation of BV formalism in terms of this geometry.

Our considerations are based on the works [38,39,40,56] and on unpublished results of the author.

We use the definitions and notations in supermathematics following [17,44,45,54]. All the derivatives in this paper are left.

2. Batalin–Vilkovisky Formalism

In this section we give the description of BV formalism [11,12,14] making accents on its geometrical meaning.

2.1 Closed and open algebras of symmetries

Let $\mathcal{E}$ be the space of all field configurations and a theory be described by the action

$$S = S(\varphi^A),\ \varphi^A \in \mathcal{E}. \tag{2.1.1}$$

We use the language of de-Witt condensed notations. Index A runs over all discrete and continuous indices * ([69]).

$$\mathcal{F}_A = \frac{\partial S(\varphi)}{\partial \varphi^A} = 0. \tag{2.1.2}$$

* On this language the field $\varphi(x)$ is the point φ^A in $\mathcal{E}$. The action—field dependent func-

are classical equations of motion which define the space M_{st} of the stationary points (field configurations) of the function $S(\varphi^A)$ (functional $S\,(\varphi^a(x))$).

$$M_{st} = \left\{\varphi^A : \quad \mathcal{F}_A(\varphi) = 0\right\}. \tag{2.1.3}$$

The action $S(\varphi)$ is non–degenerated if

$$corank\,\frac{\partial \mathcal{F}_A(\varphi)}{\partial \varphi^B}|_{M_{st}} = 0 \quad \text{or} \quad Det\frac{\partial^2 S}{\partial \varphi^A \partial \varphi^B}|_{M_{st}} \neq 0. \tag{2.1.4}$$

In a general case if (2.1.4) does not hold the theory is degenerated.

Let R^A_α be a set of vector fields— symmetries of the theory

$$R^A_\alpha \mathcal{F}_A = 0 \tag{2.1.5}$$

$$\text{i.e. } S(\varphi^A + \delta\varphi^A) - S(\varphi^A) \approx 0 \quad \text{for infinitesimal variations} \quad \delta\varphi^A = \delta\xi^\alpha R^A_\alpha \tag{2.1.6}$$

which do not vanish "classically"

$$R^A_\alpha|_{M_{st}} \neq 0\,. \tag{2.1.7}$$

(2.1.5) are Noether identities of second kind. $(S(\varphi))$ is local functional:

$$S(\varphi^A) = \int \mathcal{L}(\varphi^a(x), \frac{\partial \varphi^a(x)}{\partial x^\mu}, \cdots) d^4x \quad (A = (a, x^\mu)) \tag{2.1.8}$$

The global symmetries (when $\delta\xi$ in (2.1.6) do not depend on x^μ) do not put identities (2.1.5) on the motion equations (2.1.2) (See in details [69])

The global symmetries are excluded out of consideration. If theory is not degenerated then (2.1.4) leads to

$$\dim M_{st} = 0 \tag{2.1.9}$$

for (2.1.3).

Of course (2.1.9) follows from (2.1.4) only if we consider the solutions of (2.1.2) obeying to the initial conditions which exclude the global symmetries. It is the case when we consider a continual integral

$$Z(J) = \int e^{\frac{1}{\hbar}S(\varphi) + J\varphi}\mathcal{D}\varphi \tag{2.1.10}$$

which yields the Green functions of the theory.

---

tional $S = S(\varphi^A)$ is (2.1.1). The variational derivative of the functional $\frac{\delta S(\varphi^a(x))}{\delta \varphi^a(y)}$ is $\frac{\partial S(\varphi)}{\partial \varphi^A}$. The expressions like $\int e^{S(\varphi^A)} D\varphi$ (continual integral) are formal. All our considerations below have exact meaning in the finite–dimensional case. In the real (infinite–dimensional) case they need a special interpretation which comes from a physical context. The serious drawback of this language is that the difference between local and non–local functionals is not explicit in these notations.

In the case if (2.1.4) ((2.1.9)) obeys, (2.1.10) can be calculated perturbatively in power series on $\hbar$ by extracting the quadratic part of the action and calculating the corresponding Gaussian integral—It corresponds to the expansion of the action $S(\varphi)$ around the set of stationary points—M_{st}.

It is easy to see that the vector fields:

$$R^A = E^{AB}\mathcal{F}_B \tag{2.1.12}$$

where E^{AB} is arbitrary antisymmetric tensor

$$E^{AB} = -E^{BA} \tag{2.1.13}$$

evidently obeys to (2.1.5) and do not obeys to (2.1.7)—it is the symmetries vanishing on classical level. *

One can see that if two vector fields–symmetries T^A and T'^A obey to (2.1.5) and coincide on M_{st}

$$T^A\mathcal{F}_A = T'^A\mathcal{F}_A = 0, \tag{2.1.14}$$

$$T^A \approx T'^A \quad \text{i.e.} \quad T^A|_{M_{st}} = T'^A|_{M_{st}} \tag{2.1.15}$$

then there exist E^{AB} obeying (2.1.13) such that

$$T^A - T^{A'} = E^{AB}\mathcal{F}_B\,. \tag{2.1.16}$$

We consider so called irreducible theories and assume that the set $\{\mathbf{R}_\alpha\}$ of the symmetries is complete:

$$\sum_\alpha \lambda^\alpha \mathbf{R}_\alpha \approx 0 \;\Rightarrow\; \forall\alpha\;\lambda^\alpha \approx 0 \tag{2.1.17}$$

and

$$\forall T^A: \quad \sum_A T^A\mathcal{F}_A \approx 0 \quad \Rightarrow T^A \approx \sum_\alpha \lambda^\alpha R^A_\alpha. \tag{2.1.18}$$

The set $\{R^A_\alpha\}$ obeying to the conditions (2.1.17) and (2.1.18) we call the basis of the symmetries of the theory.

The "number" of symmetries of irreducible theory is equal to dimension of M_{st}

It is useful to represent the considerations above in the following exact sequence:

$$0 \to F \to E \to B \to 0 \tag{2.1.19}$$

where F is the space of symmetries vanishing on the M_{st} (2.1.12) ("on–shell"vanishing symmetries), E is the space of all vector fields obeying to (2.1.5) (symmetries) and

$$B = E/F \tag{2.1.20}$$

---

* We often omit the sign factor in the formulae—i.e. the corresponding expressions are exact in the case where the space $\mathcal{E}$ of the fields is bosonic. For example in (2.1.13) one have to add the sign factor $(-1)^{p(A)p(B)}$

B is the space of the symmetries of classical theory.

E and F are the moduli on the algebra of the functions on $\mathcal{E}$. We have to note that the sequence

$$0 \to F \to E \to B \to 0$$

is typical for the theories of constrained systems. The fact that the "physical space" is B and on other hand the space E is preferable to work in, is the source of arising the ghosts in the formalisms of these theories (see [42,23,24,32]).

The set of equivalence classes $\{[\mathbf{R}_\alpha]\}$ consist the basis in B and $\{\mathbf{R}_\alpha\}$ are the representatives of this basis in E. (The basis of symmetries $\{\mathbf{R}_\alpha\}$) defined above is the set of representatives of the basis $\{[\mathbf{R}_\alpha]\}$ in B.

It is easy to see using (2.1.5) that commutator of two symmetries $\mathbf{R}_\alpha$, $\mathbf{R}_\beta$ $[\mathbf{R}_\alpha, \mathbf{R}_\beta]$ is the symmetry too. So comparing (2.1.5), (2.1.12) and (2.1.16) we see that

$$[\mathbf{R}_\alpha, \mathbf{R}_\beta] = t^\gamma_{\alpha\beta} R_\gamma + E^{AB}_{\alpha\beta} \mathcal{F}_B \tag{2.1.21}$$

Where $E^{AB}_{\alpha\beta}$ are obeyed to (2.1.13). In the case if

$$E^{AB}_{\alpha\beta} = 0 \tag{2.1.22}$$

the algebra of symmetries of the theory in physics is called "closed algebra"("off–shell algebra of symmetries"). In the case if (2.1.22) does not hold the algebra of symmetries of theory is called "open algebra" ("on–shell algebra of symmetries").

Of course these definitions are $\mathbf{R}_\alpha$–basis dependence. The space B defined by (2.1.20) is in usual sense the algebra Lie, because F is ideal in E as algebra of vector fields. It is easy to see that the transformation

$$\mathbf{R}_\alpha \to \lambda^\beta_\alpha \mathbf{R}_\beta + E^{AB}_\alpha \mathcal{F}_B \tag{2.1.23}$$

where E^{AB}_α is antisymmetric (See (2.1.13)) changes the basis of symmetries to another one. In principal by this transformation one can construct the basis of symmetries for which $E^{AB}_{\alpha\beta}$ in (2.1.21) and even $t^\gamma_{\alpha\beta}$ is vanished— so called *abelian basis of symmetries* (See subsection 2.3).

But in field theory we are restricted in a choosing arbitrary basis $\mathbf{R}_\alpha$ of symmetries (the representatives $\mathbf{R}_\alpha$) for the basis $[\mathbf{R}_\alpha]$) in B. These restrictions are locality conditions on $\mathbf{R}_\alpha$.

2.2 BV prescription

For calculating the (2.1.10)—the generating functional for Green functions in the case if theory is degenerated ($dim M|_{st} \neq 0$) one have to exclude the degrees of freedom connected with the symmetries (2.1.5), (2.1.7).

If the basis of symmetries $\mathbf{R}_\alpha$ is local and abelian the gauge degrees of freedom are easily extracted from (2.1.10). If the basis of symmetries $\{\mathbf{R}_\alpha\}$ consist the Lie algebra ($t^\gamma_{\alpha\beta} \equiv const$, $E = 0$ in (2.1.21)) then we come to well–known Faddeev–Popov trick.

The BV–prescription for calculating the generating functional (2.1.10) works in a most general case (2.1.21). We recall here briefly this prescription and give in the next subsection the arguments explaining it.

For the degenerated theory with action $S(\varphi)$ and with basis of symmetries $\{\mathbf{R}_\alpha\}$ let equations

$$\Psi^\alpha = 0 \tag{2.2.1}$$

define the surface Ω in the space $\mathcal{E}$ of fields which defines gauge conditions corresponding to the symmetries $\{\mathbf{R}_\alpha\}$ $(dim(M|_{st} \cap \Omega) = 0)$. To reduce the continual integral

$$Z = \int e^{\frac{S(\varphi)}{\hbar}} \mathcal{D}\varphi \quad (\mathcal{D}\varphi = \prod_A d\varphi^A) \tag{2.2.2}$$

to the integral defined on this surface (the eliminating the gauge degrees of freedom) one have consider the following construction [11]:

Let $\mathcal{E}^e$ be a space with coordinates

$$\Phi^A = (\varphi^A, c^\alpha, \nu_\beta, \lambda_\sigma) \tag{2.2.3}$$

where auxiliary coordinates c^α, ν_β are ghosts corresponding to the symmetries $\mathbf{R}_\alpha$, λ_α– Lagrange multipliers corresponding to constraints (gauge conditions) Ψ^α. The parity of Lagrange multipliers coincide and the parity of ghosts is opposite to the parity of corresponding symmetry:

$$p(c^\alpha) = p(\nu_\alpha) = p(\lambda_\alpha) + 1 = p(\mathbf{R}_\alpha) + 1. \tag{2.2.4}$$

We introduce a space of fields and antifields $S\mathcal{E}^e$ with coordinates (Φ^A, Φ^*_A) where Φ^* have opposite parity to Φ:

$$p(\Phi^*_A) = p(\Phi^A) + 1. \tag{2.2.4a}$$

It is convenient to consider the subspace $\mathcal{E}^e_{min}$ of $\mathcal{E}^e$ containing the fields $\Phi^A = (\varphi^A, c^\alpha)$ and correspondingly a subspace $S\mathcal{E}^e_{min}$ of $S\mathcal{E}^e$— the space of $(\Phi^A_{min}, \Phi^*_{Amin}) - (\varphi^A, c^\alpha, \varphi^*_A, c^*_\alpha)$.

In the space of fields antifields one have to define the odd symplectic structure (see for details the Section 4) by Poisson bracket

$$\{F, G\} = \frac{\partial F}{\partial \Phi^A} \frac{\partial G}{\partial \Phi^*_A} + (-1)^{p(F)} \frac{\partial F}{\partial \Phi^*_A} \frac{\partial G}{\partial \Phi^A} \tag{2.2.5}$$

and Delta–operator*

$$\Delta F = \frac{\partial^2 F}{\partial \Phi^A \partial \Phi^*_A} \tag{2.2.6}$$

Then one have to define the master–action—-the function $\mathcal{S}(\Phi, \Phi^*)$ obeying to equation

$$\Delta e^{\frac{\mathcal{S}}{\hbar}} = 0 \Leftrightarrow \hbar \Delta \mathcal{S} + \frac{1}{2}\{\mathcal{S}, \mathcal{S}\} = 0 \tag{2.2.7}$$

---

* all the derivatives are left

or classically

$$\{\mathcal{S},\mathcal{S}\}=0 \tag{2.2.7a}$$

(the term proportional to $\hbar$ in (2.2.7) is responsible to measure factor.)
and to initial conditions which are defined by the action $S(\varphi)$ and symmetries $\mathbf{R}_\alpha$:

$$\mathcal{S}|_{\Phi^*=0}=S(\varphi),\quad \frac{\partial^2\mathcal{S}}{\partial c^\alpha\partial\varphi^*_A}|_{\Phi^*_A=0}=R^A_\alpha,\ \mathcal{S}(\Phi,\Phi^*)=\nu^{*\beta}\lambda_\beta+\mathcal{S}(\Phi_{min},\Phi^*_{min}) \tag{2.2.8}$$

i.e.

$$\mathcal{S}=\mathcal{S}(\varphi,c,\varphi^*,c^*)+\nu^{*\alpha}\lambda_\alpha=S(\varphi^A)+c^\alpha R^A_\alpha\varphi_A+\ldots+\nu^{*\alpha}\lambda_\alpha\,. \tag{2.2.8a}$$

(The dependence of $\mathcal{S}(\Phi,\Phi^*)$ on the fields $(\lambda,\nu,\lambda^*,\nu^*)$ is trivial) The equation (2.2.7) is called "master–equation". It can be proved that the master–equation with boundary conditions (2.2.8) have unique solution [14].

To gauge fixing conditions corresponds gauge fermion

$$\Psi=\Psi^\alpha\nu_\alpha \tag{2.2.9}$$

The partition function (2.2.2) is reduced to integral

$$Z'=\int e^{\mathcal{S}(\Phi.\Phi^*)}\delta\left(\Phi^*_A-\frac{\partial\Psi}{\partial\Phi^A}\right)\prod_A d\Phi^A d\Phi^*_A \tag{2.2.10}$$

To the changing of gauge (2.2.1) corresponds the changing of Ψ in (2.2.9). The integral (2.2.10) does not depend on the choice of Ψ. (Later we will discuss the geometrical meaning of this construction).

In the case if basis of symmetries $\mathbf{R}_\alpha$ consists Lie algebra one can show that

$$\mathcal{S}=S(\varphi)+c^\alpha R^A_\alpha\varphi^*_A+\frac{1}{2}t^\alpha_{\beta\gamma}c^*_\alpha c^\beta c^\gamma+\nu^{*\alpha}\lambda_\alpha \tag{2.2.11}$$

and (2.2.10) reduces to well–known Faddeev–Popov trick.

In the next section we deliver arguments explaining these constructions.

2.3 Abelization of Gauge Symmetries and BV prescription

"Make straight the way of the Lord"
(St John **1**: 23)

In this subsection we will give motivation for BV prescription and will see how the odd symplectic structure arise in this procedure. Our considerations in this subsection are based on [12]. In 4–th Section we will study this problem on the background of odd symplectic geometry.

Let us consider first a simplest case where $\{\mathbf{R}_\alpha\}$ is abelian basis of symmetries.

$$[\mathbf{R}_\alpha,\mathbf{R}_\beta]=0. \tag{2.3.1}$$

We will show below that in this case the eliminating of gauge degrees of freedom reduces the partition function (2.2.2) to the

$$Z' = \int e^{S(\varphi)} \mathrm{Det} \left(R^A_\alpha \frac{\partial \Psi^\beta}{\partial \varphi^A} \right) \prod_\alpha \delta(\Psi^\alpha) \prod_A d\varphi^A \tag{2.3.2}$$

Indeed even in the case where basis of symmetries forms Lie algebra, (2.3.2) gives correct answer for the partition function. The localizing of nonlocal functional $\mathrm{Det} \left(R^A_\alpha \frac{\partial \Psi^\beta}{\partial \varphi^A} \right)$ in the enlarged space of ghosts

$$\mathrm{Det} \left(R^A_\alpha \frac{\partial \Psi^\beta}{\partial \varphi^A} \right) = \int e^{c^\alpha R^A_\alpha \frac{\partial \Psi^\beta}{\partial \varphi^A} \nu_\beta} \prod_\alpha dc^\alpha d\nu_\alpha \tag{2.3.3}$$

gives us well–known Faddeev–Popov trick.

(The geometrical meaning of (2.3.2) and of (2.3.3) see in 3-th Section)

Before going in delivering the eq. (2.3.2) we will show that it coincides with BV partition function (2.2.10).

Indeed in the case (2.3.1) the solution of (2.2.7) is

$$\mathcal{S} = S(\varphi) + c^\alpha R^A_\alpha \varphi^*_A + \nu^{*\alpha} \lambda_\alpha \tag{2.3.4}$$

Indeed it is easy to see that in this case

$$\{\mathcal{S}, \mathcal{S}\} = 2 R^A_\alpha \frac{\partial S}{\partial \varphi^A} = 0. \tag{2.3.5}$$

(We consider the case where

$$\frac{\partial R^A}{\partial \varphi^A} = 0 \tag{2.3.6}$$

(the symmetries preserve volume form). See also remark after (2.3.15)).

In this case using (2.3.3) we can rewrite (2.3.2) in the form (2.2.10)

$$\int e^{S(\varphi)} \prod_\alpha \delta(\Psi^\alpha) Det(R^A_\alpha \frac{\partial \Psi^\beta}{\partial \varphi^A}) \prod_A d\varphi^A =$$

$$\int e^{S(\varphi) + \lambda_\alpha \Psi^\alpha + c^\alpha R^A_\alpha \frac{\partial \Psi^\beta}{\partial \varphi^A} \nu_\beta} \prod_{A,\alpha} d\lambda_\alpha d\nu_\alpha dc^\alpha d\varphi^A =$$

$$\int e^{S(\varphi) + c^\alpha R^A_\alpha \varphi^*_A + \nu^{*\alpha} \lambda_\alpha} \delta(\varphi^*_A - \frac{\partial \Psi}{\partial \varphi^A}) \delta(\nu^{\alpha *} - \frac{\partial \Psi}{\partial \nu_\alpha}) \delta(c^*_\alpha) \prod_{A,\alpha} d\lambda_\alpha dc^\alpha dc^*_\alpha d\nu_\alpha d\nu^{\alpha *} d\varphi^A d\varphi^*_A =$$

$$\int e^{\mathcal{S}(\Phi, \Phi^*)} \delta(\Phi^*_A - \frac{\partial \Psi}{\partial \Phi^A}) \prod_A d\Phi^A d\Phi^*_A \tag{2.3.7}$$

where $\Phi^A = (\varphi^a, c^\alpha, \nu_\alpha, \lambda_\alpha)$ is given by (2.2.3) and Ψ is given by (2.2.9).

In general case (2.1.21), (2.3.2) depends on the gauge conditions (2.2.1) because the integrand in (2.3.2) is not anymore closed density (see Section 4). For obtaining (2.2.10) we do following:

1) From the basis of symmetries $\{\mathbf{R}_\alpha\}$ we go to abelian basis of symmetries $\{\mathcal{R}_\alpha\}$ (temporary ignoring the problem of locality of symmetries)

2) We will show that in abelian basis we will come to (2.3.2) - so (2.2.10) is valid in this case (See eq.(2.3.7) above).

3) Then we will return from non-local abelian basis $\{\mathcal{R}_\alpha\}$ to local physical basis $\{\mathbf{R}_\alpha\}$. We will see that in the enlarged space $S\mathcal{E}^e$ of the fields-antifields the returning to initial symmetries corresponds to the canonical transformation preserving (2.2.5) and master-equation (2.2.7). Using uniqueness of the solution of (2.2.7) with boundary condition (2.2.8) we come to (2.2.10).

1) Let $\{\mathbf{R}_\alpha\}$ be basis of symmetries of theory $S(\varphi)$. Let ξ^a be the coordinates on some surface Ω_0 given by the equation

$$\Psi_0^\alpha = 0 \tag{2.3.8}$$

which is transversal to vector fields $\{\mathbf{R}_\alpha\}$. One can introduce in the space $\mathcal{E}$ the new coordinates (ξ^a, η^α), which correspond to symmetries $\{\mathbf{R}_\alpha\}$: for every set (ξ_0^a, η_0^α) we consider the integral curve (the exponent) of vector field $\mathbf{R}(\eta_0) = \eta_0^\alpha \mathbf{R}_\alpha$:

$$\begin{aligned} \gamma_{\eta_0^\alpha}(t) &= \exp(t\eta_0^\alpha \mathbf{R}_\alpha)|\varphi_0\rangle, \qquad (0 \le t \le 1) \\ \frac{d\gamma_{\eta_0^\alpha}(t)}{dt} &= \eta_0^\alpha R_\alpha(\gamma) \end{aligned} \tag{2.3.9}$$

beginning at the point φ_0 with coordinates ξ_0^a on the surface Ω_0. To the ending point of this curve corresponds the set (ξ_0^a, η_0^α).

Of course, these new coordinates in general are non-local. But we do not pay attention on this fact because at very end we return to initial local coordinates.

It is evident that the action S does not change along the integral curves $\gamma^A(t, \xi, \eta)$ so in the new coordinates, S does not depend on η^α

$$S = S(\xi^\alpha) \tag{2.3.10}$$

and $\mathcal{R}_\alpha = \{\frac{\partial}{\partial \eta^\alpha}\}$ is evidently the abelian basis of symmetries. In the initial coordinates φ^A this abelian basis is equal to

$$\mathcal{R}_\alpha = \frac{\partial}{\partial \eta^\alpha} = \frac{\partial \varphi^A(\xi, \eta)}{\partial \eta^\alpha} \frac{\partial}{\partial \varphi^A}, \tag{2.3.11}$$

$$\mathcal{R}_\alpha^{\mathcal{A}} = \frac{\partial \varphi^A}{\partial \eta^\alpha}. \tag{2.3.12}$$

In the coordinates (ξ^a, η^α) the problem of excluding the gauge degrees of freedom is trivial:

$$Z' = \int e^{S(\xi)} \prod_a d\xi^a = \int e^{S(\xi)} \prod_{a,\alpha} \delta(\Psi^\alpha) |\frac{\partial \Psi^\alpha}{\partial \eta^\beta}| d\xi^a d\eta^\alpha . \tag{2.3.13}$$

Using that $R^\beta_\alpha = \delta^\beta_\alpha$, $R^a_\alpha = 0$ in these coordinates we come to (2.3.2). Master–action in these coordinates is

$$\mathcal{S} = S(\xi) + c^\alpha \eta^*_\alpha + \nu^{*\alpha} \lambda_\alpha \, .$$

In the initial coordinates (φ^A)

$$\begin{aligned} Z' = \int e^{S(\xi)} \prod_\alpha \delta(\Psi^\alpha) |\frac{\partial \Psi^\alpha}{\partial \eta^\beta}| \prod_{a,\alpha} d\xi^a d\eta^\alpha = \\ \int e^{S(\xi)} \prod_\alpha \delta(\Psi^\alpha) |\frac{\partial \Psi^\alpha}{\partial \eta^\beta}| \prod_A d\varphi^A \end{aligned} \tag{2.3.14}$$

Using (2.3.12) we come to (2.3.2):

$$Z' = \int e^{S(\varphi)} \mathrm{Det} \left(\mathcal{R}^A_\alpha \frac{\partial \Psi^\alpha}{\partial \varphi^A} \right) \prod_\alpha \delta(\Psi^\alpha) \prod_A d\varphi^A \tag{2.3.15}$$

We see (using (2.3.7)) that in the basis $\{\mathcal{R}_\alpha\}$ (2.2.10) is valid.

The basis is abelian, exponent of action evidently obeys to master-equation. But the price for receiving this simple formula is very high: the symmetries $\mathcal{R}^{\mathcal{A}}_\alpha$ are nonlocal.

Remark. Our considerations in this section are precise up to the changing of volume form. It corresponds to the classical limit ($\hbar \to 0$) of master equation (2.2.7a).

3) The returning to initial symmetries $\{\mathcal{R}_\alpha\}$: It is here where canonical structure plays crucial role: The relation between new abelian basis $\{\mathcal{R}_\alpha\}$ and initial one is given by equation

$$\mathbf{R}_\alpha = \lambda^\beta_\alpha \mathcal{R}_\beta + E^{[AB]}_\alpha \mathcal{F}_B \tag{2.3.16}$$

(See equation (2.1.17, 2.1.18)).

One can show that the transformation (2.3.16) can be realized *by canonical transformation in the space of fields, antifields.* We will show it infinitesimally. We note (see in details section 4) that to arbitrary odd function

$$Q(\Phi_{min}, \Phi^*_{min}) = Q(\varphi, c, \varphi^*, c^*)$$

corresponds canonical infinitesimal transformation:

$$\begin{aligned} \delta \Phi^A = \epsilon \{Q, \Phi^A\} \\ \delta \Phi^*_A = \epsilon \{Q, \Phi^*_A\} \end{aligned} \tag{2.3.17}$$

and:

$$\delta \mathcal{S} = \epsilon \{Q, \mathcal{S}\} \, . \tag{2.3.18}$$

If we consider

$$Q = c^{\alpha}\lambda^{\beta}_{\alpha}c^{*}_{\beta} + c^{\alpha}c^{\beta}E^{AB}_{\alpha\beta}\varphi^{*}_{A}\varphi^{*}_{B}$$

then putting (2.3.4) in (2.3.18) and using (2.3.17) we obtain that

$$\mathcal{S} \to \mathcal{S} + \delta\mathcal{S} = S(\varphi) + c^{\alpha}\left(\mathcal{R}^{A}_{\alpha}\varphi^{*}_{A} + \epsilon\lambda^{\beta}_{\alpha}\mathcal{R}^{A}_{\beta} + \epsilon E^{AB}_{\alpha}\mathcal{F}_{B}\right) + \dots \tag{2.3.19}$$

Using (2.2.8a) we see that (2.3.19) corresponds to infinitesimal transformation (2.3.16).

We note that if the generator Q of canonical transformation obeys to equation

$$\Delta Q = 0 \tag{2.3.20}$$

then one can see that the canonical transformation (2.3.17) preserves volume form $dv = \prod_{A} d\Phi^{A}d\Phi^{*}_{A}$. Indeed from (2.3.17) follows that

$$\delta dv = 0 \quad \text{if} \quad \Delta Q = 0. \tag{2.3.21}$$

The classical master–equation (2.2.7a) is invariant under canonical transformations (transformations preserving odd bracket $\{\ ,\ \}$), the quantum master–equation (2.2.7) is invariant under the canonical transformations preserving the volume form. So from the fact that to the changing of the basis of the symmetries corresponds canonical transformation (canonical transformation preserving the volume form) and from the fact that master–equation have unique solution follows (2.2.10).

3.Integration Theory over Surfaces in the Superspace

3.1.Densities in the superspace and Pseudodifferential Forms

In this section we present some results of geometric integration theory on the surfaces in the superspace (see [28], [64], [40]).

Let $\Omega^{m.n}$ be an $(m.n)$–dimensional supersurface in the superspace $E^{M.N}$ given by parametrization $z^{A} = z^{A}(\zeta^{B})$ the mapping of superspace $E^{m.n}$ in superspace $E^{M.N}$ where $z^{A} = (x^{1}, \dots, x^{M}, \theta^{1}, \dots, \theta^{N})$ are coordinates of superspace $E^{M.N}$ and $\zeta^{B} = (\xi^{1}, \dots, \xi^{m}, \nu^{1}, \dots, \nu^{n})$ are the coordinates of $E^{m.n}$

One can consider the functional $\Phi_{A}(\Omega)$ given on (m.n)–supersurfaces by the following expression:

$$\Phi_{A}(\Omega) = \int A\left(z^{A}(\zeta), \frac{\partial z^{A}}{\partial\zeta^{B}}, \dots, \frac{\partial^{k}z^{A}}{\partial\zeta^{B_1}\dots\partial\zeta^{B_k}}\right)d^{m+n}\zeta \tag{3.1.1}$$

where the function A is obeyed to the following condition

$$A\left(z^{A}, \frac{\partial z^{A}}{\partial\tilde{\zeta}^{B}}, \cdots, \frac{\partial^{k}z^{A}}{\partial\tilde{\zeta}^{B_1}\dots\partial\tilde{\zeta}^{B_k}}\right) = \mathrm{Ber}\left(\frac{\partial\zeta}{\partial\tilde{\zeta}}\right)\cdot A\left(z^{A}, \frac{\partial z^{A}}{\partial\zeta^{B}}, \cdots, \frac{\partial^{k}z^{A}}{\partial\zeta^{B_1}\dots\partial\zeta^{B_k}}\right). \tag{3.1.2}$$

In the case if the condition (3.1.2) holds the functional (3.1.1) does not depend on the choice of parametrization $z(\zeta)$ of the supersurface Ω.

The function A obeying the condition (3.1.2) is called (m.n) density of rank k.

The (m.n) density A defines the functional $\Phi_A(\Omega)$ on (m.n) surfaces obeying to additivity condition

$$\Phi_A(\Omega_1 + \Omega_2) = \Phi_A(\Omega_1) + \Phi_A(\Omega_2)\,. \tag{3.1.3}$$

The densities are the most general object of integration over surfaces [28].

Let us consider in a more details the case where the rank of density is equal one:

$$A = A\left(z^A, \frac{\partial z^A}{\partial \zeta^B}\right)\,. \tag{3.1.4}$$

The condition (3.1.2) can be rewritten in a following way

$$A\left(z^A, K^B_{B'}\frac{\partial z^A}{\partial \zeta^B}\right) = \mathrm{Ber}\, K \cdot A\left(z^A, \frac{\partial z^A}{\partial \zeta^B}\right)\,. \tag{3.1.5}$$

$$\mathrm{Ber}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\mathrm{Det}(A - BD^{-1}C)}{\mathrm{Det}\, D} \tag{3.1.6}$$

is superdeterminant of the matrix.

In the bosonic case (if there are no odd variables) it is easy to see that the densities which are linear functions on the $\frac{\partial z^A}{\partial \zeta^B}$ are in one–one correspondence with differential forms: to k–form $\omega = \omega_{i_1\ldots i_k} dz^{i_1} \wedge \ldots \wedge dz^{i_k}$ corresponds density

$$A_\omega = < \frac{\partial z^{i_1}}{\partial \zeta^1}, \ldots, \frac{\partial z^{i_k}}{\partial \zeta^k}\,,\, \omega > = k!\omega_{i_1\ldots i_k}\frac{\partial z^{i_1}}{\partial \zeta^1}\ldots\frac{\partial z^{i_k}}{\partial \zeta^k}\,,$$

$$\Phi_A(\Omega^k) = \int_{\Omega^k} \omega\,. \tag{3.1.7}$$

The equation (3.1.5) holds because Det (Ber $\to$ Det in bosonic case) is polylinear antisymmetric function on tangent vectors $\frac{\partial x^a}{\partial \zeta^b}$.

In the case if the density A corresponds to differential form by (3.1.7) then Stokes theorem is obeyed

$$\Phi_{A\omega}(\partial\Omega) = \Phi_{A_{d\omega}}(\Omega) \tag{3.1.8}$$

One can show that *in bosonic case the densities obeying to Stokes theorem correspond to differential forms.*

What happens in supercase?

In the bosonic case differential forms are simultaneously the linear functions on tangent vectors on which exterior differentiation operator can be defined and on other hand they are integration object (3.1.5)

In the supercase the differential form ω can be defined as the function linear on tangent vectors which is superantisymmetric:

$$\omega(\ldots, \mathbf{u}, \mathbf{v}, \ldots) = -\omega(\ldots, \mathbf{v}, \mathbf{u}, \ldots)(-1)^{p(u)p(v)}\,. \tag{3.1.9}$$

In supercase (3.1.9) is not in accordance with (3.1.6)—to differential form (3.1.7) does not correspond density.

One have to construct the right generalization of differential form (considering as integration object), so called psendodifferential forms as a density obeying to Stokes theorem. It is the way which was at beginning developed in [19,20] and was studied in general case in [2,3,62–68].

For defining pseudodifferential forms we have to check the conditions which one have to put on the density (3.1.4) for having the Stokes theorem (3.1.8) (see for details [64]).

Let two (m.n) surfaces Ω_0 and Ω_1 are given by parametrization $z_0^A = z_0^A(\zeta^B)$ and $z_1^A = z_1^A(\zeta^B)$ correspondingly and

$$z^A = z^A(t,\zeta^B)\,,\,(0 \le t \le 1): \quad z(0,\zeta^B) = z_0(\zeta^B),\; z(1,\zeta^B) = z_1(\zeta^B) \tag{3.1.10}$$

is a parametrization of $(m+1.n)$ surface $\mathcal{V}$

$$\partial\mathcal{V} = \Omega_1 - \Omega_0 \tag{3.1.11}$$

(up to a boundary terms) Then if A is a density of rank 1 we have

$$\Phi_A(\partial\mathcal{V}) = \Phi_A(\Omega_1) - \Phi_A(\Omega_0) = \int d\zeta^{m+n} \int_0^1 dt \frac{d}{dt} A\left(z^A(t,\zeta^B), \frac{\partial z^A(t,\zeta^B)}{\partial\zeta^B}\right) =$$

$$\int d\zeta^{m+n} \int dt \left[\left(\frac{dz^A}{dt}\,\frac{\partial A}{\partial z^A} + \frac{dz_B^A}{dt}\,\frac{\partial A}{\partial z_B^A}\right)\right] =$$

$$\int d\zeta^{m+n} \int dt \left[\frac{dz^A}{dt}\,\frac{\partial A}{\partial z^A} + \frac{d}{d\zeta^B}\left(\frac{dz^A}{dt}\,\frac{\partial A}{\partial z_B^A}\right) - \frac{dz^A}{dt}\frac{d}{\partial\zeta^B}\frac{\partial A}{\partial z_B^A}(-1)^{p(B)p(A)}\right] =$$

$$\int d\zeta^{m+n} dt\Big(\frac{dz^A}{dt}\frac{\partial A}{\partial z^A} - \Big(\frac{dz^A}{dt}\frac{\partial z^{A'}}{\partial\zeta^B}\frac{\partial^2 A}{\partial z^{A'}\partial z_B^A} + \frac{dz^A}{dt}\frac{\partial z_{B'}^{A'}}{\partial\zeta^B}\frac{\partial^2 A}{\partial z_{B'}^{A'}\partial z_B^A}\Big)(-1)^{p(A)p(B)}\Big) \tag{3.1.12}$$
$$+\text{boundary terms}\,.$$

(We use notation $z_B^A = \frac{\partial z^A}{\partial\zeta^B}$).

From (3.1.11), (3.1.12) one can see that if the last term in integral (3.1.12) vanishes:

$$\frac{dz^A}{dt}\frac{\partial^2 z^{A'}}{\partial\zeta^B\partial\zeta^{B'}}\frac{\partial^2 A}{\partial z_{B'}^{A'}\partial z_B^A}(-1)^{p(A)p(B)} = 0 \quad \text{i.e}$$

$$\frac{\partial^2 A}{\partial z_{B'}^{A'}\partial z_B^A} = -(-1)^{p(B)p(B')+(p(B)+p(B'))p(A)}\frac{\partial^2 A}{\partial z_B^{A'}\partial z_{B'}^A} \tag{3.1.13}$$

then this integral can be considered as $(m+1.n)$ density dA of rank 1. The differential is defined by the relation

$$dA\left(z^A, \frac{\partial z^A}{\partial\zeta^B}, \frac{dz^A}{dt}\right) = \frac{dz^A}{dt}\,\frac{\partial A}{\partial z^A} - \frac{dz^A}{dt}\,\frac{\partial z^{A'}}{\partial\zeta^B}\,\frac{\partial^2 A}{\partial z^{A'}\partial z_B^A}(-1)^{p(A)p(B)}\,. \tag{3.1.14}$$

We come to correct definition (3.1.14) of the exterior differential d of the density of rank 1 in supercase if the condition (3.1.13) holds (see for details [Vor]).(Of course in usual case from (3.1.13)) immediately follows the statement after (3.1.8)).

The density is called pseudodifferential form if condition (3.1.13) holds.

If A is pseudodifferential form then dA is pseudodifferential form too.

Example 3.1.1.

In the superspace $E^{M.N}$ with coordinates $z^A = (x^1, \dots, x^M, \theta^1, \dots, \theta^N)$ we consider $(m.n)$ density of rank 1.

$$A = \mathrm{Ber}\left(\frac{\partial z^A}{\partial \zeta^B} L_A^{B'}\right) \tag{3.1.15}$$

Where ζ^B are coordinates of $E^{m.n}$, L_A^B is $(m.n) \times (M.N)$ arbitrary matrix.

A is density because condition (3.1.2) is evidently satisfied.

Indeed (3.1.15) is pseudodifferential form. The condition (3.1.13) can be checked by straightforward but long computations. (Alternatively (3.1.13) for (3.1.15) follows from the fact that (3.1.15) is proportional to volume form on $E^{m.n}$. The volume form evidently obeys to (3.1.13) because the conditions (3.1.13) are reparametrization invariant).

It is useful to consider two particular cases of (3.1.15).

a) $n = 0$ ($L_A^{B'} = 0$ if $p(B') = 1$). In this case Ber $\to$ Det and to A corresponds differential form

$$\omega_A = L^1_{A_1} \dots L^m_{A_m} dz^{A_1} \wedge \dots \wedge dz^{A_m} \tag{3.1.16}$$

b) $m = n = 1$ and L_A^B is such that

$$A = \mathrm{Ber} \frac{\partial(x^1, \theta^1)}{\partial(\xi, \eta)} = \frac{x^1_\xi}{\theta^1_\eta} - \frac{x^1_\eta \theta^1_\xi}{(\theta^1_\eta)^2} \tag{3.1.17}$$

$z^A = (x^1, \dots, x^M, \theta^1, \dots, \theta^N)$, $\zeta^B = (\xi, \eta)$ (ζ is even and η is odd.)

(3.1.17) is the simplest example of non–linear pseudodifferential form.

In the [2,3] Baranov and Schwarz suggested the following construction producing the pseudodifferential form which seems natural in spirit of ghost technique.

For $(M.N)$ dimensional superspace $E^{M.N}$ let $STE^{M.N}$ be a superspace associated with tangent bundle $TE^{M.N}$ of the superspace $E^{M.N}$. (If z^A are coordinates on $E^{M.N}$ then (z^A, z^{*A}) are coordinates on $STE^{M.N}$ where z^{*A} transform as dz^A and have reversed parity

$$p(z^{*A}) = p(z^A) + 1$$

The superspace $STE^{M.N}$ have dimension $(M + N.M + N)$.

Then arbitrary function * $W(z, z^*)$ on $STE^{M.N}$ defines $(m.n)$ density of rank 1

$$A_W\left(z^A, \frac{\partial z^A}{\partial \zeta^B}\right) = \int W\left(z^A, z^{*A} = \nu^B \frac{\partial z^A}{\partial \zeta^B}\right) d\nu^{n+m} \tag{3.1.18}$$

---

* The function W have to obey the conditions on infinity by even variables for (3.1.18) being correct

where ν^B have reversed parity to ζ^B

$$p(\nu^B) = p(\zeta^B) + 1. \tag{3.1.19}$$

Indeed it is easy to check using (3.1.19) that (3.1.18) obeys to condition (3.1.2)

$$A_W\left(z^A, K^B_{B^1}\frac{\partial z^A}{\partial \zeta^B}\right) = \int W\left(z^A, \nu^{B'}K^B_{B'}\frac{\partial z^A}{\partial \zeta^B}\right) d\nu^{n+m} =$$

$$\int W\left(z^A, \tilde{\nu}^B\frac{\partial z^A}{\partial \zeta^B}\right) d\nu^{n+m} = \mathrm{Ber}(K^B_{B'}) \int W\left(z^A, \tilde{\nu}^B\frac{\partial z^A}{\partial \zeta^B}\right) d\tilde{\nu}^{n+m} \tag{3.1.20}$$

One can easy check by direct computation that the density A_W in (3.1.18) obeys to condition (3.1.13).

We say that the function W is BS representation of the pseudodifferential form A.

One can see by straightforward calculations that to the exterior differential d of the pseudodifferential form A (See (3.1.14)) corresponds the operator

$$\hat{d} = z^{*A}\frac{\partial}{\partial z^A} \tag{3.1.23}$$

in BS representation. Namely it can be checked using (3.1.14) and (3.1.18) that

$$A_{\hat{d}W} = \pm d(A_W) \tag{3.1.24}$$

if A_W is $(m.n)$ density and $A_{\hat{d}W}$ is $(m+1.n)$ density.

Example 3.1.2

Let us consider the function

$$W = \frac{1}{\sqrt{\pi}}e^{-(x^{1*})^2}\theta^{1*} \tag{3.1.25}$$

on the $STE^{M.N}$

$(z^A = (x^1, \ldots, x^M, \theta^1, \ldots, \theta^N)$, $z^{*A} = (x^{*1}, \ldots, x^{*M}, \theta^{*1}, \ldots, \theta^{*N})$.)

The (1.1) density corresponding to (3.1.25) by (3.1.18)

$$A\left(z, \frac{\partial z}{\partial \zeta}, \frac{\partial z}{\partial \theta}\right) = \frac{1}{\sqrt{\pi}}\int e^{-(\nu x^1_\zeta + t x^1_\eta)^2}\left(\nu\theta^1_\zeta + t\theta^1_\eta\right) d\nu dt = \frac{x^1_\zeta}{\theta^1_\eta} - \frac{x^1_\eta\theta^1_\xi}{(\theta^1_\eta)^2} \tag{3.1.26}$$

coincides with the density (3.1.17). To generate the density (3.1.15) from Example 3.1.1 by the construction (3.1.18)one can consider instead (3.1.25) the following formal expression

$$W = \int e^{z^{*A}L^B_A c_B}\, dc \tag{3.1.27}$$

where $p(c_B) = p(\zeta^B) + 1$. (Compare with (2.3.3))

It is easy to see that formally (3.1.27) gives (3.1.15). But it have sense only in the case where $L^B_A = 0$ if $p(B) = 1$.(See footnote before eq. (3.1.18)).

For us it is most interesting the case where pseudodifferential form is closed—i.e. the density obeys to condition (3.1.13) and

$$dA = 0 \quad \text{i.e.} \quad \frac{\partial A}{\partial z^A} - \frac{\partial z^{A\prime}}{\partial \zeta^B}\frac{\partial^2 A}{\partial z^{A\prime}\partial z^A_B}(-1)^{p(A)p(B)} = 0 \tag{3.1.28a}$$

or in BS representation

$$\hat{d}W_A = z^{*A}\frac{\partial W}{\partial z^A} = 0 \tag{3.1.28b}$$

In other words condition of closeness means that Euler–Lagrange equations of the functional (3.1.1) are trivial [64].

It is these densities which arise when we reduce the partition function integral (2.2.2) to the integral over the surfaces in the space of field configurations defined by gauge conditions. The gauge independence of this integral means that the corresponding density is closed. But in field theory this surface is defined not by parametrization but by equations ("gauge conditions") We need to consider corresponding integration objects.

3.2 Dual densities and closed pseudointegral forms

The surfaces in the superspace can be defined not by parametrization, but by dual construction—by equations.

Let $\Omega^{(m.n)}$ be a $(m.n)$ dimensional supersurface in the superspace $E^{M.N.}$ defined by equations

$$F^a(z^A) = 0 \tag{3.2.1}$$

Where $F^a = (f^1, \ldots, f^{M-m}, \varphi^1, \ldots, \varphi^{N-n})$ are coordinates of the superspace $E^{M-m.N-n}$ (f are even, φ are odd).

Let

$$dv = \rho(z)dz^1 \ldots dz^n \tag{3.2.2}$$

be a volume form on $E^{M.N}$ Then (3.1.1) can be replaced by the functional:

$$\Phi_{\tilde{A}}(\Omega) = \int \tilde{A}\left(z^A, \frac{\partial F^a(z)}{\partial z^A}, \ldots, \frac{\partial^k F(z)}{\partial z^{A_1}\ldots\partial z^{A_k}}\right)\prod_a \delta(F^a)dv. \tag{3.2.3}$$

where $\tilde{A}$ is obeyed to the condition

$$\tilde{A}\left(z^A, \frac{\partial F'^c(z)}{\partial z^A}, \ldots, \frac{\partial^k F'^c(z)}{\partial z^{A_1}\ldots\partial z^{A_k}}\right) = \mathrm{Ber}\,(\eta^c_d)\,\tilde{A}\left(z^A, \frac{\partial F^c(z)}{\partial z^A}, \ldots, \frac{\partial^k F^c(z)}{\partial z^{A_1}\ldots\partial z^{A_k}}\right) \tag{3.2.4}$$

($F'^c(z) = \eta^c_d F^d(z)$ determine the same surface $\Omega^{m.n}$).

$\tilde{A}$ is called $(m.n)$ D–density (dual density) of rank k [28]. It is easy to see that in the same way like for usual densities, if conditions (3.2.4) hold then (3.2.3) does not depend on the choice of the functions $\{F^a(z)\}$ defining the surface Ω by the equation (3.2.1).

D–density $\tilde{A}$ corresponds to density A if for arbitrary surface Ω

$$\Phi_{\tilde{A}}(\Omega) = \Phi_A(\Omega) \tag{3.2.5}$$

More precisely if the surface $\Omega^{m.n}$ is given by equation

$$F_0^a(z^A) = y^a - r^a(s^B)$$

and its parametrization by

$$z_0^A(\zeta): \quad y^a(\zeta^B) = r^a(\zeta^B),\ s^B(\zeta) = \zeta^B \qquad (z^A = (y^a, s^B)) \tag{3.2.6}$$

then the $\tilde{A}$ corresponds to A if

$$\tilde{A}\left(z^A, \frac{\partial F_0^a}{\partial z^A}, \ldots, \frac{\partial^k F_0^a}{\partial z^{A_1} \ldots \partial z^{A_k}}\right) = A\left(z_0^A(\zeta), \frac{\partial z_0^A}{\partial \zeta^B}, \ldots, \frac{\partial^k z_0^A}{\partial z^{B_1} \ldots \partial \zeta^{B_k}}\right) \tag{3.2.7}$$

in the case $\rho = 1$ in the (3.2.2).

In the next section we consider the examples of D–densities arising in odd symplectic geometry.

If the density A corresponds to differential form $w_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ in the space E^n (in bosonic case) then it is easy to see that dual density $\tilde{A}$ corresponds to integral form

$$W^{i_1 \ldots i_{n-k}} \frac{\partial}{\partial x^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_{n-k}}}$$

such that $W^{i_1 \ldots i_{n-k}} = \frac{1}{\rho} \epsilon^{i_1 \ldots i_{n-k}\, j_1 \ldots j_k} \omega_{j_1 \ldots j_k}$ and

$$\tilde{A} = W^{i_1 \ldots i_{n-k}} \frac{\partial f^1}{\partial x^{i_1}}, \ldots, \frac{\partial f^{n-k}}{\partial x^{i_{n-k}}} \tag{3.2.8}$$

where ρ is given by volume form (3.2.2).

To construct D–densities which are dual to pseudodifferential forms (so called pseudointegral forms) one have to check the eq. (3.1.8) on the language of D –densities.

We represent only the final results of these calculations:

Let $A(z^A, \frac{\partial F^a}{\partial z^A})$ be a D–density of rank 1.

$$A\left(z^A, K_b^a \frac{\partial F^b}{\partial z^A}\right) = \mathrm{Ber} K \cdot A\left(z^A, \frac{\partial F^a}{\partial z^A}\right) \tag{3.2.9}$$

This density is closed if

$$\frac{\partial^2 A}{\partial F_A^a \partial F_B^b} = -(-1)^{p(A)p(B)+(p(A)+p(B))p(a)} \frac{\partial^2 A}{\partial F_B^a \partial F_A^b} \tag{3.2.10}$$

(Compare with (3.1.13)) and

$$(-1)^{p(a)p(A)}\frac{1}{\rho}\frac{\partial}{\partial z^A}\left(\rho\frac{\partial A}{\partial F^a_A}\right)=0. \tag{3.2.11}$$

One can come to (3.2.10), (3.2.11) considering the variation of functional (3.2.3) under the infinitesimal variation of surface Ω (Compare with (3.1.12)).

Analogously to (3.1.18) one can develop Baranov–Schwarz procedure for pseudointegral forms [40].

For $(M.N)$ dimensional superspace $E^{M.N}$ let $ST^*E^{M.N}$ be a superspace associated with cotangent bundle $T^*E^{M.N}$ of the superspace $E^{M.N}$. (z^A, z^*_A) are the coordinates of $T^*E^{M.N}$, z^*_A transform as $\frac{\partial}{\partial z^A}$ and have reversed parity

$$p(z^*_A)=p(z^A)+1.$$

Then the arbitrary function $W(z,z^*)$ on $T^*E^{M.N}$ (see the footnote before (3.1.18)) defines D–density of rank 1:

$$A\left(z^A,\frac{\partial F^a}{\partial z^A}\right)=\int W\left(z^A,z^*_A=\nu_a\frac{\partial F^a}{\partial z^A}\right)\prod_a d\nu_a, \tag{3.2.12}$$

$$p(\nu_a)=p(F^a)+1.$$

(Compare with (3.1.18)).

A indeed is density (The condition (3.2.9) is evidently satisfied for (3.2.12) as in (3.1.20)). The conditions (3.2.10) can checked by direct computation.

Comparing (3.2.11) with (3.2.12) one can see that the density (3.2.12) is closed if

$$\frac{1}{\rho}\frac{\partial\rho}{\partial z^A}\frac{\partial W}{\partial z^*_A}+\frac{\partial^2 W}{\partial z^A\partial z^*_A}=0. \tag{3.2.13}$$

In the 4–th Section we give the interpretation of (3.2.13) in the terms of odd symplectic geometry.—Indeed this formula is strictly connected with BV master—equation.

Now let us consider example which we will use later:

Example 3.2.1

Let $\{R^A_a\}$ be a set of vector fields on $E^{M.N}$. One can consider D–density— pseudointegral form:

$$\tilde{A}\left(\frac{\partial F^a}{\partial z^A}\right)=\mathrm{Ber}\left(\frac{\partial F^a}{\partial z^A}R^A_b\right). \tag{3.2.14}$$

(Compare with (3.1.15)).

It is the density which arise in (2.3.2).

One have to note that $(m.N)$ D–density (of maximal odd dimension) are polynomial by $\frac{\partial F^a}{\partial z^A}$. It is just Bernstein—Leites integral forms [19,20] (see also [28]).

The D–density (3.2.14) formally generates by the function

$$W\left(z^A,z^A_*\right)=\int e^{c^a R^A_a z^*_A}dc \tag{3.2.15}$$

$$(p(c^a) = p(a) + 1)$$

The equation (3.2.15) is correct in the case if all vector fields $R_a^A \frac{\partial}{\partial z^A}$ are even.

c^a in (3.2.15) are nothing that usual ghosts in Faddeev—Popov trick (Compare with (2.3.3))

4 Odd Symplectic Geometry

4.1 Basic Definitions

Let M^{2n} be an $2n$–dimensional manifold provided with closed non–degenerated two form w:

$$dw = 0\,, \quad \mathrm{Det} w_{ij} \neq 0\,, \tag{4.1.1}$$

where $w = w_{ij} dx^i \wedge dx^j$ in the local coordinates $(x^1, \ldots, x^{2n})$.

The pair (M^{2n}, w) is called symplectic manifold.

The non–degenerated two–form (4.1.1) establishes one–one correspondence between TM and T^*M:

$$\forall \mathbf{u} \in T_m M \ (m \in M) \rightarrow \ \mathrm{one form}\, w_m \in T_m^* M : \forall \xi \in T_m M \, w_m(\xi) = w(\xi, \mathbf{u})\,. \tag{4.1.2}$$

According to (4.1.2) to every function f on M corresponds a vector field $\mathbf{D}_f$ which in coordinates is

$$D_f^i \partial_i = (w^{-1})^{ij} \partial_j f \partial_i\,. \tag{4.1.3}$$

(To vector field $\mathbf{D}_f$ corresponds one form df by (4.1.2)).

The Poisson bracket of two functions f and g is equal

$$\{f, g\} = w(\mathbf{D}_f, \mathbf{D}_g) = \frac{\partial f}{\partial x^i} (w^{-1})^{ij} \frac{\partial g}{\partial x^j}. \tag{4.1.4}$$

It obeys to Jacoby identity

$$\{\{f, g\}, h\} + \mathrm{cyclic\,permutation} = 0 \tag{4.1.5}$$

which follows from (4.1.1).

The group G^{can} of the symplectomorphisms (canonical transformations) of (M^{2n}, w) i.e. the diffeomorphisms preserving the two form w is infinite–dimensional.— To every function (Hamiltonian) H corresponds infinitesimal transformation $\mathbf{D}_H$

$$\frac{dx^i}{dt} = D_H^i = \{H, x^i\}\,, \quad \mathcal{L}_{\mathbf{D}_H} w = 0\,. \tag{4.1.6}$$

There exists unique (up to multiplication on constant) $2k$ density which is G^{can}–invariant. (We say that a density A on E is invariant under the action of a group G of transformations of E if in (3.1.1)

$$\forall g \in G\,, \quad \Phi_A(\Omega^g) = \Phi_A(\Omega)\,.) \tag{4.1.7}$$

It is closed density which corresponds to k–times wedged product of the form w :

$$\Phi_A(\Omega) = \int_\Omega w \wedge \ldots \wedge w \tag{4.1.8}$$

It is a well–known Poincare–Cartan integral invariant of canonical transformations [1].

The integrand in (4.1.8) is G^{can} invariant closed $2k$–density of rank 1

$$A(x^i, \frac{\partial x^i}{\partial \xi^\alpha}) = \sqrt{\mathrm{Det}\left(\frac{\partial x^i}{\partial \xi^\alpha} w_{ij} \frac{\partial x^i}{\partial \xi^\beta}\right)} \tag{4.1.9}$$

where $x^i(\xi^\alpha)$ is the parametrization of surface Ω.

The dual D–density $\tilde{A}$ corresponding to A is

$$\tilde{A}(x^i, \frac{\partial f^a}{\partial x^i}) = \sqrt{Det\,(\{f^a, f^b\})} \tag{4.1.1.9a}$$

where the equations $f^a = 0$ define the surface Ω .

In the case $k = n$ the density (4.1.9) is G^{can}–invariant volume form corresponding to the symplectic structure.

Locally there exist coordinates in which the form w (4.1.1) defining symplectic structure have canonical form (Darboux Theorem):

$$w = \sum_{i=1}^{n} dx^i \wedge dx^{i+n} \,. \tag{4.1.10}$$

Remark. Indeed one can prove more : By suitable canonical transformation one can make "flat" a surface in a vicinity of arbitrary point (for any ξ_0 the derivatives $\frac{\partial^k x^i}{\partial \xi^{[k]}}|_{\xi_0}$ for $k \geq 2$ can be cancelled by suitable canonical transformation). From this fact in particularly follows that the density (4.1.9) is a unique G^{can}–invariant density in the class of densities of arbitrary rank k [41].

The symplectic geometry in pure bosonic case in contrary to Riemannian one is "poor" because the group of transformations is "rich" . In Riemannian geometry there are the invariant densities of higher degrees constructed via the curvature and its covariant derivatives.— The analogue of (4.1.9) which is a volume form of the surface ($w_{ij} \to g_{ij}$ in (4.1.9)) is not unique invariant density.

Now we represent the superizations of the constructions above.

Let

$$w = w_{AB} dz^A dz^B \tag{4.1.11}$$

be closed non–degenerated two form on the superspace $E^{M.N}$ with coordinates $z^A = (x^1, \ldots, x^M, \theta^1, \cdots, \theta^N)$.

w is a linear superantisymmetric function on tangent vectors:

$$\begin{aligned} w(\mathbf{u}, \mathbf{v}\lambda) &= w(\mathbf{u}, \mathbf{v})\lambda \,, \text{if } \lambda \in \mathbf{R} \\ w(\mathbf{u}, \mathbf{v}) &= -w(\mathbf{v}, \mathbf{u})(-1)^{p(\mathbf{u})p(\mathbf{v})}. \end{aligned} \tag{4.1.12}$$

In coordinates

$$\begin{aligned} w_{AB} &= (-1)^{1+p(A)p(B)} w_{BA}\,, \\ w(u^A\partial_A, v^B\partial_B) &= u^A w_{AB} v^B (-1)^{(p(u)+p(A))p(w)+(p(v)+p(B))p(B)}. \end{aligned} \tag{4.1.13}$$

The closeness condition $dw = 0$ is

$$(-1)^{p(A)p(C)}\partial_A w_{BC} + \text{cyclic permutation } = 0\,. \tag{4.1.14}$$

The non–degeneracy of w (i.e. the matrix w_{AB} is invertible) on $E^{M.N}$ means that M is even if w is even and $M = N$ if w is odd.

The analogue of Darboux Theorem [57] states that there exist coordinates (Darboux coordinates) in which the two form w_0 defining an even symplectic structure on $E^{2M.N}$ have the following canonical form:

$$w_0 = \sum_{i=1}^{M} dx^i \wedge dx^{i+M} + \sum_{\alpha=1}^{N} \epsilon_\alpha (d\theta^\alpha)^2\,, \quad (\epsilon_\alpha = \pm 1) \tag{4.1.15}$$

and the two form w_1 defining an odd symplectic structure on $E^{M.M}$ have the following canonical form:

$$w_1 = \sum_{i=1}^{M} dx^i \wedge d\theta^i\,. \tag{4.1.16}$$

(On $E^{2M.2M}$ one can consider two simplectic structures of the different grading simultaneously (see [34,37,38,47])).

Using (4.1.13) and (4.1.14) one can establish a superversion of the equations (4.1.3) and (4.1.4):

$$D_f^A \partial_A = (w^{-1})^{AB}\partial_B f (-1)^{(p(f)+p(w)+p(B))p(B)}\partial_A \tag{4.1.17}$$

and formulae for Poisson bracket:

$$\{f,g\} = \frac{\partial f}{\partial z_A}(w^{-1})^{AB}\frac{\partial g}{\partial z_B}(-1)^{(p(f)+p(w))p(A)}\,. \tag{4.1.17a}$$

For computing (4.1.17a) we have to note that the inverse matrix in the superspace has the inverse parity:

$$(w^{-1})^{AB} = (-1)^{(p(A)+1)(p(B)+1)}(w^{-1})^{BA}.$$

(Compare with (4.1.13)). In Darboux coordinates on $E^{2M.N}$ the even Poisson bracket corresponding to (4.1.15) have the form:

$$\{f,g\} = \sum_{i=1}^{M}\left(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^{i+M}} - \frac{\partial f}{\partial x^{i+M}}\frac{\partial g}{\partial x^i}\right) + \sum_{\alpha=1}^{N}\epsilon_\alpha (-1)^{p(f)}\frac{\partial f}{\partial \theta^\alpha}\frac{\partial g}{\partial \theta^\alpha} \tag{4.1.18}$$

and in Darboux coordinates on $E^{M.M}$ the odd Poisson bracket corresponding to (4.1.16) have the form:

$$\{f,g\}_1 = \sum_{i=1}^{M} \left(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial \theta^i} + (-1)^{p(f)}\frac{\partial f}{\partial \theta^i}\frac{\partial g}{\partial x^i} \right) . \tag{4.1.19}$$

The Hamiltonian mechanics can be formulated in the terms of even as well as odd symplectic structures [43,60,36].

The formulae above are similar for even and odd structures. But there is essential difference between these structures.

An even symplectic structure is nothing but natural lifting on $E^{2M.N}$ of the symplectic structure of the underlying space E^{2M}. And it is natural that it is very similar to symplectic structure in pure bosonic case.

For example the changing Det$\rightarrow Ber$ in (4.1.9,4.1.9a) leads to straightforward gene ralization of Poincare–Cartan invariant on the supercase (if the structure is even) [41]:

$$A(z^A, \frac{\partial z^A}{\partial \zeta^\alpha}) = \sqrt{\mathrm{Ber}\left(\frac{\partial z^A}{\partial \zeta^\alpha} w_{AB} \frac{\partial z^A}{\partial \zeta^\beta}\right)} \tag{4.1.20}$$

and corresponding D–density:

$$\tilde{A}(z^A, \frac{\partial F^a}{\partial z^A}) = \sqrt{\mathrm{Ber}\,(\{F^a, F^b\})}. \tag{4.1.21}$$

Of course in the supercase the invariant density cannot be anymore represented as integrand in (4.1.8) because form is not anymore integration object.(See section 3.1). But one can show that as well as in bosonic case (4.1.8) the density (4.1.20) is closed and there is no invariants in higher derivatives [41,2,3]. (The Remark above is valid in this case too.)

It is not the case for odd symplectic geometry. At first its ancestor in pure bosonic case is Lie derivative construction, not the symplectic geometry:

Example 4.1.1

Let $E^{M.M} = ST^*E^M$ be a superspace associated with cotangent bundle of the space E^M. (See subsection 3.2).

ST^*E^M is naturally provided with odd symplectic structure

$$w_1 = dx^i d\theta_i, \quad (\theta_i = x_i^*). \tag{4.1.22}$$

Then to vector field $\mathbf{R} = R^i(x)\partial_i$ on E^M corresponds the function $R = R^i(x)\theta_i$ and

$$\mathcal{L}_{\mathbf{R}} f = R^i(x)\frac{\partial f}{\partial x^i} = \{f, R\}. \tag{4.1.23}$$

More generally to every polyvectorial field $\mathbf{T} = T^{A_1 \ldots A_n}\partial_{A_1} \wedge \ldots \wedge \partial_{A_n}$ corresponds the function

$$\sigma(\mathbf{T}) = W_{\mathbf{T}}(x,\theta) == T^{A_1 \ldots A_n}\theta_1 \ldots \theta_n \tag{4.1.24}$$

and

$$\sigma([\mathbf{T}_1, \mathbf{T}_2]) = \{\sigma(\mathbf{T}_1), \sigma(\mathbf{T}_2)\} \tag{4.1.25}$$

where $[\quad , \quad]$ is Schouten bracket.

We see from this example that odd symplectic geometry is strictly connected with classical geometrical objects. And it is not surprising that in the terms of odd bracket some classical geometrical constructions can be formulated in a elegant way ([38,47–49]).

In the next subsections we will consider the geometrical constructions in odd symplectic geometry which have no analogues for even one and which play a crucial role in the formulating BV formalism. The essential difference of odd symplectic geometry from even one is that the transformations preserving odd bracket do not preserve any volume form. (The formulae (4.1.20,4.1.21) have not sense in the case if w is odd.) To consider the integration theory we provide an odd symplectic space with additional structure—volume form.

4.2 Δ–operator in odd symplectic geometry.

Let $E^{M.M}$ be provided with odd symplectic structure and with volume form dv

$$dv = \rho(x, \theta) dx^1 \dots dx^M d\theta_1 \dots \theta_M \tag{4.2.1}$$

We suppose $z^A = (x^i, \theta_j)$ be Darboux coordinates (4.1.16) on $E^{M.M}$. We consider $E^{M.M}$ provided with a structure defined by a pair $(dv, \{\quad , \quad\})$ where $\{\quad , \quad\}$ is the odd Poisson bracket (4.1.19). $G^{can}_{dv} \leq G^{can}$ is the group of the transformations preserving the pair $(dv, \{\quad , \quad\})$. From here and later we call the structure defined by the pair $(dv, \{\quad , \quad\})$ the odd symplectic structure.

We define the G^{can}_{dv}–invariant second order differential operator Δ–operator [34] corresponding to the structure $(dv, \{\quad , \quad\})$ in the following way

$$\Delta f = \frac{1}{2} \frac{\mathcal{L}_{\mathbf{D}_f} dv}{dv} = div_{dv} \mathbf{D}_f. \tag{4.2.2}$$

One can see by direct computation that

$$\Delta f = \frac{1}{2\rho} \frac{\partial}{\partial z^A} \left((-1)^{p(A)} \rho \{z^A, f\} \right) \tag{4.2.3}$$

where ρ defines volume form dv by (4.2.1).

$$\Delta f = \frac{1}{2} \{log\, \rho, f\} + \frac{\partial^2 f}{\partial x^i \partial \theta_i} \tag{4.2.3a}$$

in Darboux coordinates.

(In the case where $\{\,,\,\}$ is even Poisson bracket it is easy to see that the corresponding operator (4.2.2) is trivial: it is a first order differential operator which vanishes if a volume form corresponds to even symplectic structure by (4.1.20)).

Example 4.2.1 $\rho = 1$ in (4.2.1) then

$$\Delta = \Delta_0 = \frac{\partial^2}{\partial x^i \partial \theta_i} \tag{4.2.4}$$

In this form this operator was introduced by Batalin and Vilkovisky for formulating master–equation [11,12] (see (2.2.6)).

Example 4.2.2 Let $E^{M.M} = ST^*E^M$ be provided with natural symplectic structure (See Example 4.1.1). Let

$$dv = \rho(x^1, \dots, x^M) dx^1 \dots dx^M \tag{4.2.5}$$

be volume form on E^M. We consider the pair $(d\hat{v}, \{\ ,\ \})$ on ST^*E^M where

$$d\hat{v} = \rho^2(x^1, \dots, x^M) dx^1 \dots dx^M d\theta_1 \dots d\theta_M \tag{4.2.6}$$

is the volume form on ST^*E^M and $\{\ ,\ \}$ is the Poisson bracket (4.1.19) which corresponds to natural symplectic structure (4.1.22.). Then using (4.2.3a), (4.2.6) and (4.1.24) we see that Δ–operator on ST^*E^M corresponds to divergence on E^M:

$$\Delta_{d\hat{v}} \sigma(\mathbf{T}) = \frac{1}{\rho} \frac{\partial \rho}{\partial x^i} \frac{\partial W_{\mathbf{T}}}{\partial \theta_i} + \frac{\partial^2 W_{\mathbf{T}}}{\partial x^i \partial \theta_i} = \sigma(div_{dv} \mathbf{T}). \tag{4.2.7}$$

where $\sigma(\mathbf{T}) = W_{\mathbf{T}}$ is given by (4.1.24).

Moreover comparing (4.1.24) (3.2.8) and (3.2.12) one can see that $\sigma(\mathbf{T}) = W_{\mathbf{T}}$ is BS representation (3.2.12) of the D–density $\tilde{A}_{\mathbf{T}}$ corresponded to polyvectorial field $\mathbf{T}$ by (3.2.8). Then comparing (3.2.13) and (4.2.7) we see that closeness condition can be expressed in the terms of corresponding Δ–operator [56], [40].

$$\mathbf{T} \text{ is closed } \Leftrightarrow \Delta\sigma(\mathbf{T}) = 0. \tag{4.2.7a}$$

This example where Δ–operator corresponds to divergence describes an important but special case of the Δ–operator (4.2.2).(See Theorem below).

(In the examples 4.2.2 as well as in the examples 4.1.1 and 4.2.1 it was considered a case where $E^{M.M} = ST^*E^M$ ($z^A \to x^i$ and $z^*_A \to \theta_i$). By the slight modification of the considerations above one can consider in these examples $E^{M.M} = ST^*E^{M-k.k}$ where $k \neq 0$.)

Using (4.2.3) one can see that Δ–operator in general case obeys to conditions

$$\Delta_{dv'} f = \Delta_{dv} f + \frac{1}{2}\{log\lambda, f\} \tag{4.2.8}$$

and

$$\Delta^2_{dv'} f = \Delta^2_{dv} f + \{\lambda^{-\frac{1}{2}} \Delta_{dv} \lambda^{\frac{1}{2}}, f\} \tag{4.2.9}$$

where $dv' = \lambda dv$.

Following to A.S.Schwarz [56] we call the structure $(dv, \{\ ,\ \})$ SP structure if there exist Darboux coordinates in which

$$\Delta = \Delta_0 \tag{4.2.10}$$

i.e. $\rho = 1$ in (4.2.1) (see eq.(4.2.4) in the Example (4.2.1)).

Theorem The following statements are equivalent:
i) $(dv, \{\ ,\ \})$ structure is SP structure
ii) The Δ–operator corresponding to $(dv, \{\ ,\ \})$ structure is nilpotent:

$$\Delta_{dv}^2 = 0 \tag{4.2.11}$$

iii) the function ρ corresponding to volume form dv by (4.2.1) obeys to equation:

$$\Delta_0 \sqrt{\rho} = 0. \tag{4.2.11a}$$

(This Theorem is stated in [56], [39], [30]. The complete proof belongs to A.S.Schwarz [56])

For example for the structure $(d\hat{v}, \{\ ,\ \})$ from the Example (4.2.2) we come to i) if we choose coordinates on E^M in which volume form (4.2.5) is trivial on E^M ($dv = dx^1 \dots dx^M$). (The corresponding transformation of $\theta_i = x_i^*$ preserves symplectic structure.) The nilpotency condition ii) follows from the fact that $\Delta_{d\hat{v}}$ corresponds to divergence (4.2.7), The equation iii) is evidently obeyed.

In general case ii)$\Leftrightarrow$iii) follows from (4.2.8, 4.2.9) and i)$\Rightarrow$ii) is evident from invariant definition (4.2.2). The ii)$\Rightarrow$i) needs a more detailed analysis.

Remark In the paper [Kh] where was first introduced the structure $(dv, \{\ ,\ \})$ for arbitrary volume form dv, was done the false statement that every $(dv, \{\ ,\ \})$ structure is SP structure.

4.3 Invariant densities in odd symplectic geometry.

In contrary to even symplectic geometry where the invariant densities are exhausted by the density (4.1.20) depended on first derivatives, in odd symplectic geometry the situation is more non–trivial.

On one hand as it was mentioned above there are no G^{can} –invariant densities, because the group of transformations preserving odd symplectic structure does not preserve any volume form. In the class of densities which are invariant under canonical transformations preserving a fixed volume form dv the first non–trivial density (except the volume form itself) appears in a second derivatives [35].

In spite of this fact one can consider the density of rank 1 which is naturally defined on Langrangian surfaces and does not change under infinitesimal transformations in the class of Lagrangian surfaces in the case if $(dv, \{\ ,\ \})$–structure is SP structure [55,56,40]. We consider now this density.

Let a superspace $E^{N.N}$ be provided with a structure $(dv, \{\ ,\ \})$ defined in previous subsection.

Let Λ–be Lagrangian surface in it (i.e the form w defining simplectic structure vanishes on it)

$$w|_\Lambda = 0 \tag{4.3.1}$$

and Λ is $(N-k,k)$–dimensional.

For example if $E^{N.N} = ST^*E^N$ then to every odd function $\Psi(x)$ on E^N corresponds $(N.0)$–Lagrangian surface in ST^*E^N defined by the equation

$$\theta_i = \frac{\partial \Psi(x)}{\partial x^i} \tag{4.3.2}$$

We consider later only the case $k = 0$. (The case $0 < k \leq n$ can be received by slight modifications of corresponding formulae. For example in (4.3.2) we come to $(N-k.k)$ dimensional Lagrangian surface if we consider instead $E^{N.N}$ $ST^*E^{N-k.k}$, ($x^i \to z^i$,$\theta_i \to z_i^*$).

If $\{\mathbf{t}_1, \ldots, \mathbf{t}_n\}$ are the vectors tangent to Lagrangian manifold Λ in the point $\lambda_0 \in \Lambda$ then we consider arbitrary vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ such that

$$w(\mathbf{t}_i, \mathbf{u}_k) = \delta_{ik} \tag{4.3.3}$$

and define a density A by equation [56]:

$$A\,(\lambda_0\,, \mathbf{t}_1, \ldots \mathbf{t}_n) = \sqrt{dv(\lambda_0\,, \mathbf{t}_1, \ldots, \mathbf{t}_n)} \tag{4.3.4}$$

(The volume form dv is $(N.N)$ density of rank 1 on $E^{N.N}$. (see Section 3.)

It can be proved that (4.3.4) is a density which does not depend on the choice of the vectors $\{\mathbf{u}_i\}$ obeying to (4.3.3) and this density (more exactly the functional (3.1.1)) is invariant under infinitesimal variation of the Lagrangian surface Λ if the $(dv, \{\,,\,\})$–structure is SP structure [56]. We prove it later.

Instead (4.3.4) we consider D–density which is defined on all $(N.0)$–dimensional surfaces and corresponds to the density (4.3.4) in the case if the surface is Lagrangian [40]. Let a $(N.0)$-dimensional surface Ω be defined in $E^{N.N}$ by equations

$$F^a = 0, \quad (a = 1, \ldots, N), \qquad (F^a \,\text{are odd}). \tag{4.3.5}$$

One can consider [40]

$$\tilde{A}\left(z^A, \frac{\partial F^a}{\partial z^A}\right) = \frac{1}{\sqrt{\rho}} \frac{\sqrt{\mathrm{Ber}\frac{\partial(G,F)}{\partial(x,\theta)}}}{\sqrt{\mathrm{Det}\,\{G^a, F^b\}}} \tag{4.3.6}$$

where $z^A = (x^1, \ldots x^N, \theta_1, \ldots \theta_N)$, are the coordinates in $E^{N.N} = ST^*E^N$, ρ defines the volume form

$$dv = \rho(x^1, \ldots x^N, \theta^1, \ldots, \theta^N) dx^1 \ldots dx^N d\theta_1 \ldots d\theta_N$$

and $\{G^a\}(a = 1, \ldots, N)$ are arbitrary even functions.

One can see that (4.3.6) is indeed $(N.0)$ D–density. (F^a are odd so $\mathrm{Det}^{-1} \sim \mathrm{Ber}$).

Moreover the D–density (4.3.6) on the surface (4.3.5) and corresponding to it functional (3.2.3) $\Phi_A(\Omega)$ does not depend on the choice of the functions $\{G^a\}$ if Ω is Lagrangian surface. Indeed in this case the functions F^a defining Ω by (4.3.5) obey to equation

$$\{F^a, F^b\}|_{F^a=0} = 0 \tag{4.3.7}$$

(Compare with (4.3.1)).

Let

$$\tilde{G}^a = \tilde{G}^a(G^1, \dots, G^N, F^1, \dots F^N) \tag{4.3.8}$$

be another set of even functions $\{\tilde{G}^a\}$.

Then it is easy to see that under the transformation $G^a \to \tilde{G}^a$ the numerator and denumerator in (4.3.6) are multiplied by the $\mathrm{Det}\frac{\partial \tilde{G}}{\partial G}$ in the case if (4.3.5) and (4.3.7) hold. It is easy to see (see for details Section 3, eq.(3.2.5)–(3.2.7)) that (4.3.6) corresponds to (4.3.4) on Lagrangian surfaces if we put

$$F_i = \theta_i - \frac{\partial \Psi(x)}{\partial x^i} \tag{4.3.10}$$

(Compare with (4.3.2))

In this case the functional (3.2.3) on Langrangian surface (4.3.10) is equal to

$$\Phi_{\tilde{A}}(\Lambda) = \int \sqrt{\rho} \prod_a \delta(F^a) dx^1 \dots dx^N d\theta_1 \dots d\theta_N \tag{4.3.11}$$

(We come immediately from (4.3.6) to (4.3.11) choosing $G^i = x^i$ in (4.3.6).)

To prove that this functional is invariant under infinitesimal variation of Lagrangian surface $\Lambda \to \Lambda + \delta\Lambda$ in the case if $(dv, \{\,,\,\})$ is SP structure we note that under the infinitesimal transformation $\Psi(x) \to \Psi(x) + \delta\Psi(x)$ in (4.3.10)

$$\delta\Phi_{\tilde{A}} = \int \frac{\partial \sqrt{\rho}}{\partial \theta^i} \frac{\partial \delta\Psi}{\partial x^i} \prod_a \delta(F_a) dx^1 \dots dx^N d\theta^1 \dots d\theta^N. \tag{4.3.12}$$

If $(dv, \{\,,\,\})$ is SP structure then from Theorem follows that

$$\Delta_0 \sqrt{\rho} = 0 \quad \text{so} \quad \delta\Phi_{\tilde{A}} = 0\,. \tag{4.3.13}$$

Remark. In the case if Langrangian surface is $(n-k.k)$ dimensional surface in the superspace $ST^*E^{N-k.k}$ (for arbitrary $1 \le k \le n$) one have consider instead (4.3.6) a density.

$$A\left(z^A, \frac{\partial F^a}{\partial z^A}, \frac{\partial F^a}{\partial z^*_A}\right) = \frac{1}{\sqrt{\rho}} \sqrt{\mathrm{Ber}\frac{\partial(G,F)}{\partial(z,z^*)}} \sqrt{\mathrm{Ber}\{G^{\tilde{a}}, F^b\}} \tag{4.3.14}$$

where index $\tilde{a}$ have a reversed parity to index a.

The density studied above is very essential for our considerations but even in the case of SP structure it is not G_{dv}^{can}–invariant density on all surfaces. We present here the example of non–trivial G_{dv}^{can}–invariant density of a second rank.

Let a $(N-1.N-1)$–dimensional surface in the superspace $E^{N.N}$ is defined by the equations

$$f=0, \quad \varphi=0 \quad (f \text{ is even } \varphi \text{ is odd}). \tag{4.3.15}$$

$E^{N.N}$is provided with $(dv,\{\ ,\ \})$ structure.

One can consider [35]:

$$\tilde{A}=\frac{1}{\sqrt{\{f,\varphi\}}}\left(\Delta f-\frac{\{f,f\}}{2\{f,\varphi\}}\Delta\varphi-\frac{\{f,\{f,\varphi\}\}}{\{f,\varphi\}}-\frac{\{f,f\}}{2\{f,\varphi\}^2}\{\varphi,\{f,\varphi\}\}\right). \tag{4.3.16}$$

(4.3.16) is G_{dv}^{can}–invariant semidensity—density of weight $\sigma=\frac{1}{2}$ (A density have weight σ if it multiplies by the σ–th power of Ber in (3.2.4)). For example if in the point z_0 the functions f and φ defining surface by the equations (4.3.15) obey to normalization conditions:

$$\{f,f\}|_{z_0}=\{f,\{f,\varphi\}\}|_{z_0}=0 \tag{4.3.17}$$

then

$$A|_{z_0}=\frac{\Delta f}{\sqrt{\{f,\varphi\}}} \tag{4.3.18}$$

and under the transformation $f\to\lambda f$, $\varphi\to\mu\varphi$ which does not change (4.3.17), (4.3.18) multiplies by the $(\frac{\lambda}{\mu})^{-\frac{1}{2}}$—the square root of the *Berezinian* of this transformation. ((4.3.16) can be directly computed from (4.3.18) and (4.3.17)).

One can show that the density (4.3.16) is unique (up to multiplication on a constant) in the class of the densities of the rank $k\le 2$ defined on the surfaces of the dimension $(p.p)$ which are invariant under the transformations preserving $(dv,\{\ ,\ \})$ structure (except the volume form itself) [35]. The semidensity (4.3.16) takes odd values. It is exotic analogue of Poincare–Cartan invariant.—$\tilde{A}^2=0$ so it cannot be integrated over surfaces.

4.4. SP–structure and Batalin–Vilkovisky Formalism

In this subsection we again return to considerations of the section 2 on the basis of odd symplectic geometry.

The space of fields and antifields described in a 2-nd section can be naturally provided with odd symplectic structures $(dv,\{\ ,\ \})$ which in fact are SP structures.

We recall that $\mathcal{E}$ is the space of initial fields $\{\varphi^A\}$, $\mathcal{E}^e_{min}$ is a space of fields $\{\varphi^A_{min}\}=\{\varphi^A,c^\alpha\}$ (the "ghosts" c^α have the parity opposite to $\mathbf{R}_\alpha$) and $\mathcal{E}^e$ is a space of fields

$\{\Phi^A\}=\{\varphi^A,c^\alpha,\lambda_\alpha,\nu_\alpha\}$, $\quad (p(c^\alpha)=p(\nu_\alpha)=p(\mathbf{R}_\alpha)+1=p(\lambda_\alpha)+1)$. The space of fields-antifields is nothing but a superspace associated to cotangent bundle of a corresponding space of fields (see Section 3). The superspace $ST^*\mathcal{E}$ have coordinates φ^A,φ^*_A. Analogously

$$ST^*\mathcal{E}^e_{min} \ - \ \{\Phi^A_{min},\Phi^*_{Amin}=\varphi^A,c^\alpha,\varphi^*_A,c^*_\alpha\} \ \text{ and}$$
$$ST^*\mathcal{E}^e \ - \ \{\Phi^A,\Phi^*_A=\varphi^A,c^\alpha,\lambda_\alpha,\nu_\alpha,\varphi^*_A,c^*_\alpha,\lambda^{*\alpha},\nu^{*\alpha}\}.$$

On the space $\mathcal{E}$ of initial fields $\{\varphi^A\}$ one can consider two volume forms:

$$dV_0 = \prod_A d\varphi^A \tag{4.4.1}$$

(canonical one) and

$$dV = e^{S(\varphi)} dV_0 \tag{4.4.2}$$

related with the action $S(\varphi)$ of theory.

The canonical form (4.4.1) is naturally prolonged on $\mathcal{E}^e_{min}$ and $\mathcal{E}^e$:

$$\begin{aligned} dV^e_{0\,min} &= \prod_{A,\alpha} d\varphi^A dc^\alpha \\ dV^e_0 &= \prod_{A,\alpha} d\varphi^A dc^\alpha d\lambda_\alpha d\nu_\alpha . \end{aligned} \tag{4.4.3}$$

Using the construction of example 4.2.2 one can consider the lifting of volume forms (4.4.1)-(4.4.3) on the corresponding spaces of fields-antifields

$$\begin{aligned} d\hat{V}_0|_{ST^*\mathcal{E}} &= \prod_A d\varphi^A d\varphi^*_A, \\ d\hat{V}^e_0|_{ST^*\mathcal{E}^e_{min}} &= \prod_A d\Phi^A_{min} d\Phi^*_{min\,A}, \\ d\hat{V}^e_0|_{ST^*\mathcal{E}^e} &= \prod_A d\Phi^A d\Phi^*_A \end{aligned} \tag{4.4.4}$$

and correspondingly:

$$\begin{aligned} d\hat{V}|_{ST^*\mathcal{E}} &= e^{2\,S(\varphi)} d\hat{V}_0|_{ST^*\mathcal{E}} \\ d\hat{V}^e|_{ST^*\mathcal{E}^e_{min}} &= e^{2\,S(\varphi)} d\hat{V}_0|_{ST^*\mathcal{E}^e_{min}} \\ d\hat{V}^e|_{ST^*\mathcal{E}^e} &= e^{2\,S(\varphi)} d\hat{V}_0|_{ST^*\mathcal{E}^e} \end{aligned} \tag{4.4.5}$$

On the space $ST^*\mathcal{E}^e$ ($ST^*\mathcal{E}^e_{min}$) of fields-antifields there is a third possibility to consider a volume form

$$d\hat{V}^m = e^{2\mathcal{S}(\Phi,\Phi^*)} d\hat{V}^e_0 \tag{4.4.6}$$

where $\mathcal{S}(\Phi,\Phi^*) = S(\varphi) + c^\alpha R^A_\alpha \varphi^*_A + \ldots$ is master-action obeying to equation (2.2.7).

The symplectic structure on $ST^*\mathcal{E}$, $ST^*\mathcal{E}^e$, ($ST^*\mathcal{E}^e_{min}$) can be naturally defined by the construction of example (4.1.1) ($x^i \to \Phi^A, \theta_i = x^*_i \to \Phi^*_A$). The (2.2.5) is the corresponding Poisson bracket.

Using a volume forms (4.4.4) – (4.4.6) and the odd symplectic structures we come to different structures ($d\hat{V}^e_0$, $\{\ ,\ \}$), ($d\hat{V}^e$, $\{\ ,\ \}$), ($d\hat{V}^m$, $\{\ ,\ \}$) on the space of fields-antifields.

The first two structures are SP structures. (See example 4.2.2 and the statements i), ii) of Theorem.)

From the master-equation (2.2.7) and Theorem (statement iii)) follows that the structure

$$(d\hat{V}^m\ ,\ \{\ ,\ \}) \tag{4.4.7}$$

constructed via master-action by (4.4.6) is SP–structure too. So using the Theorem one can interpret the master– equation in the following way:

To find a volume form $d\hat{V}^m$ in $ST^*\mathcal{E}^e$ such that it obeys to initial conditions

$$d\hat{V}^m = \left(e^{c^\alpha R^A_\alpha \varphi^*_A + \dots}\right) d\hat{V} \tag{4.4.8}$$

and there are Darboux coordinates (of course non-local) in which

$$d\hat{V}^m = 1 \cdot d\hat{V}_0 \tag{4.4.9}$$

(Action "dissolves".)

The basic formula (2.2.10) for reduced partition function is interpreted in a following way.

To SP structure $(d\hat{V}^m,\ \{\ ,\ \})$ on $ST^*\mathcal{E}$ corresponds D-density (4.3.6). To this D-density corresponds functional (3.2.3)— integral over Lagrangian surface Λ in $ST^*\mathcal{E}$ defined by gauge conditions (2.2.1), (2.2.9).

$$\Psi^\alpha = 0 \ \Rightarrow\ \Lambda: \quad \Phi^*_A - \frac{\partial(\Psi^\alpha \nu_\alpha)}{\partial \Phi^A} = 0. \tag{4.4.10}$$

The functional (3.2.3) with integrand (4.3.6) (with a volume form (4.4.6)) is covariant expression for (2.2.10). The gauge independence follows from the fact that $(d\hat{V}^m,\ \{\ ,\ \})$ structure is SP structure (see (4.3.11)–(4.3.13)).

What is the relation between symmetries of a theory and SP structure (4.4.7) ?

Let $\{R_\alpha\}$ be basis of symmetries of Theory. (See subsection 2.1):

$$R^A_\alpha \frac{\partial S}{\partial \varphi^A} = 0 \tag{4.4.11}$$

(symmetry condition)

$$(-1)^{p(A)} \frac{\partial R^A_\alpha}{\partial \varphi^A} = 0 \tag{4.4.12}$$

(preserving of canonical volume form (4.4.1)). One can consider D-density

$$\tilde{A} = Ber\left(\frac{\partial \Psi^\alpha}{\partial \varphi^A} R^A_\beta\right) \tag{4.4.13}$$

(See example 3.2.1) and corresponding functional

$$\Phi = \int Ber\left(\frac{\partial \Psi^\alpha}{\partial \varphi^A} R^A_\beta\right) \prod_\alpha \delta(\Psi^\alpha) dV. \tag{4.4.14}$$

In the case if volume form dV in $\mathcal{E}$ is given by (4.4.2) $dV = e^S dV_0$, the functional (4.4.14) is nothing but (2.3.2).

We study the problem of gauge independence of this functional – i.e. closeness of a density (4.4.13).

Let us consider first

Toy-example. The number of symmetries is one. (4.4.14) is reduced to

$$\Phi = \int \frac{\partial \Psi}{\partial \varphi^A} R^A \delta(\Psi) e^S \prod_A d\varphi^A$$

which is nothing but flux of vector field R through a surface $\Omega : \Psi = 0$.

"Gauge" independence means that

$$0 = div_{dV}\mathbf{R} == \frac{\partial R^A}{\partial \varphi^A} + \frac{1}{\rho} R^A \frac{\partial \rho}{\partial \varphi^A} = \frac{\partial R^A}{\partial \varphi^A} + R^A \frac{\partial S}{\partial \varphi^A}, \quad (\rho = e^S).$$

which follows from (4.4.11), (4.4.12).

In a general case to investigate a problem of closeness of (4.4.13) we go to BS representation of this density:

$$W_A = \int W_A^e \prod_\alpha dc^\alpha \tag{4.4.15}$$

where

$$W_A^e = e^{c^\alpha R_\alpha^A \varphi_A^*} \tag{4.4.16}$$

is a function on space $ST^*\mathcal{E}^e_{min}$ (see Example 3.2.1 and (2.3.3)).

and use the fact that the closeness condition (3.2.13) can be expressed in a terms of corresponding Δ operator. (See example 4.2.2).

For this purpose following to (4.2.7) we consider Δ–operator defined on $ST^*\mathcal{E}^e_{min}$ by the structure $(d\hat{V}, \{\,,\,\})$ where the volume form $d\hat{V}$ corresponded to dV in (4.4.14) is given by (4.4.5). Using that $\{\mathbf{R}_\alpha\}$ are the symmetries (4.4.11) we come to

$$\Delta W_A^e = c^\alpha c^\beta \left(t^\gamma_{\alpha\beta} R^A_\gamma + E^{AB}_{\alpha\beta} \frac{\partial S}{\partial \varphi^B} \right) \varphi^*_A \tag{4.4.17}$$

where

$$[\mathbf{R}_\alpha, \mathbf{R}_\beta] = t^\gamma_{\alpha\beta} \mathbf{R}_\gamma + E^{AB}_{\alpha\beta} \frac{\partial S}{\partial \varphi^B}\,. \tag{4.4.18}$$

(See (2.1.21)).

In the case if symmetries are abelian ($t^\gamma_{\alpha\beta} = E^{AB}_{\alpha\beta} = 0$) then

$$\Delta W_A^e = 0. \tag{4.4.19}$$

and evidently in the space $ST^*\mathcal{E}$

$$\Delta W_A = 0. \tag{4.4.20}$$

for BS representation (4.4.15) of the density (4.4.13). So this density is closed. It means that (4.4.14) is gauge independent. (Compare with (2.3.2),(2.3.3)).

The equation (4.4.19) means that to the function W^e_A on $ST^*\mathcal{E}^e_{min}$ corresponds the closed density in the space $\mathcal{E}^e_{min}$ of the fields φ^A and ghosts c^α (if a volume form $dV = e^S dV_0$ in $\mathcal{E}^e_{min}$).

W_A is odd function and W^e_A is even one. One can see that a volume form

$$d\hat{\tilde{V}} = (W^e)^2 d\hat{V}^e_0 = e^{2(S(\varphi)+c^\alpha R^A_\alpha \varphi^*_A)} dV^e_0 \tag{4.4.21}$$

provides $ST^*\mathcal{E}^e_{min}$ by SP structure because $(d\hat{V}, \{\ ,\ \})$ is SP structure and W^e given by (4.4.16) obeys to equation (4.4.19) (see the statement iii) of Theorem).

We come to SP structure (4.4.6) related with master–action in the case where symmetries are abelian (See eq. (2.3.4)).

To transformation (2.1.23) of the basis $\mathbf{R}_\alpha$ corresponds canonical transformation

$$\begin{aligned} W^e &\to W'^e = e^{c^\alpha R^A_\alpha \varphi^*_A + \dots} \\ d\hat{v}^e &\to e^{2(S + c^\alpha R^A_\alpha \varphi^*_A + \dots)} \end{aligned} \tag{4.4.22}$$

In the general case where initial basis of symmetries is not abelian one have to put $\mathcal{R}_\alpha$ instead $\mathbf{R}_\alpha$ in (4.4.16) where $\mathcal{R}_\alpha$ is abelian (in general non–local) basis of symmetries (2.3.11). To the transformation (4.4.22) corresponds the transformation from abelian basis to initial one performed in a subsection 2.3. The function W^e in (4.4.16) plays the role of initial conditions. (Compare with (4.4.8)).

At the end we note that in the case where initial symmetries constitute a group (even not–abelian) and $\sum_\alpha t^\alpha_{\alpha\beta} = 0$ one can see by direct computation using (4.4.17) that (4.4.20) is obeyed in spite of the fact that (4.4.19) is not obeyed. So the density (4.4.13) corresponded to (4.4.15) is closed in this case.

We come to Faddeev–Popov trick. (Compare with (2.3.2) and (2.3.3)).

5. Acknowledgments

I am deeply indebted to my collaborator A.P.Nersessian. The essential part of the results related to the interpretation of BV formalism in terms of Odd symplectic geometry is the product of our cooperative work.

I am especially grateful to T.Voronov. It would been impossible to write this review without his moral support and attention.

I am grateful to I.Batalin, O.Piguet, I. Tyutin for fruitful discussions.

I am grateful to T.S. Akobian, E.V.Tameyan and G.R. Khoudaverdian who helped me during the preparation of the manuscript of this paper.

And finally I want to use opportunity to express my indebtedness to my teacher A.S. Schwarz.

References

1.Arnold,I.: Mathematical methods of classical mechanics.– Moscow, Nauka (1974), pp.1–432.

2. Baranov,M.A.,, Schwarz,A.S.— *Funkts. Analiz i ego pril.*, **18** No.2,pp.53–54 , (1984).

3. Baranov,M.A, Schwarz,A.S.: Cohomologies of supermanifolds. *Funkts. analiz i ego pril.*. **18** No.3, pp.69–70,(1984) .

4. Batalin,I.A., Lavrov P., Tyutin I.V—*J.Math.Phys.*, **31**, 1487. (1990).

5. Batalin,I.A., Lavrov P., Tyutin I.V—*J.Math.Phys.*, **32**, 2513. (1991).

6. Batalin I.A.,Marnelius.R, Semikhatov,A.— *Preprint of Lebedev Institute* FIAN (1995), hep–th 9502031

7. Batalin,I.A.,Tyutin I.V.: On possible generalization of field– antifield formalism. *Int. J. of Mod. Phys.* **A8**, No.13, pp.2333–2350 (1993).

8. Batalin,I.A.,Tyutin I.V.: On the multilevel generalization of field– antifield formalism. *Mod. Phys.Lett.* **A8**, No.38, pp.3673–3681 (1993).

9. Batalin,I.A.,Tyutin I.V.: On local quantum deformation of antisymmetric differential. *Int. J. of Mod. Phys.* **A9**,No.4, pp.517–525,(1994).

10. Batalin,I.A., Vilkovisky,G.A.:Relativistic S–matrix of dynamical systems with boson and fermion constraints. *Phys.Lett.* **69B**, 309, (1977).

11. Batalin,I.A., Vilkovisky,G.A.: Gauge algebra and quantization. *Phys.Lett.* **102B** pp.27–31 (1981).

12. Batalin,I.A., Vilkovisky,G..A.: Closure of the gauge algebra, generalized Lie equations and Feynman rules. *Nucl.Phys.* **B234**, pp.106–124, (1984).

13. Batalin,I.A., Vilkovisky,G.A.: Quantization of Gauge theories with linearly dependent generators. *Phys.Rev.* **D28** ,pp.2567–2582, (1983)

14. Batalin,I.A, Vilkovisky,G.A.: Existence Theorem for gauge algebra. *J.Math.Phys.* **26** pp.172–194 (1985)

15. Becchi,C., Rouet,A., Stora,R.— *Commun. Math. Phys.* **42**,127 (1975)

16. Becchi,C., Rouet,A., Stora,R.: Renormalization of gauge theories. *Ann. Phys.* (N.Y) **98**, 287 (1976)

17. Berezin,F.A.: Introduction in analysis with anticommuting variables. Moscow Univ., (1983). (English translation is published by Reidel.)

18. Berezin,F.A., Marinov M.S.: Particle spin dynamics as the Grassmannian variant of classical mechanics. *Ann. Phys.* **104**, No.2, pp.336–362 ,(1977).

19. Bernstein,J.N, Leites,D.A.: Integral forms and the Stokes formula on supermanifolds. *Funkts. Analiz i ego pril.* **11** No.1 pp. 55–56 (1977)

20. Bernstein,J.N, Leites,D.A.: How to integrate differential forms on supermanifolds. *Funkts.. Analiz i ego pril.* **11** No.3 pp.70–71 (1977)

21. Browning,A.D., Mc–Millan,D.: The Batalin Fradkin amd Vilkovisky Formalism for higher order theories. *J.Math.Phys.* **28** , 438 (1987)

22. Buttin,C.—C.R.Acad. Sci. Paris, Ser.A–B, **269** A–87 (1969).

23. Dubois–Violette,M.: Systems dynamiques contraints: L'approche homologique *Ann. Inst. Fourier*, **37**,4, pp.45–57, (1987)

24. Fish,J.M.L., Henneaux M.: Homological perturbation theory and the algebraic structure of the antifield–antibracket formalism for gauge theories. *Comm.Math.Phys.* **128** pp. 627–639 (1990).

25. Fradkin,E.S, Vilkovisky,G.A.: Quantization of relativistic systems with constraints. *Phys.Lett.* **55B**, 224 ,(1975).

26. Fradkin,E., Vilkovisky,G.: Quantization of relativistic systems with constraints, equivalence of canonical and covariant formalism in quantum theory of gravitational field. *CERN preprint* TH-2332 (1977)

27.Fradkin,E.S., Fradkina,T.E.: Quantization of relativistic systems with boson and fermion first and second class constraints. *Phys. Lett.* **72B**,343, (1978).

28. Gayduk,A.V.,Khudaverdian,O.M, Schwarz,A.S,: Integration on surfaces in superspace. *Theor. Mat. Fiz.* **52**, pp.375–383, (1982).

29. Getzler,E.: Batalin–Vilkovisky algebras and two–dimensional topological field theory. *Comm.Math. Phys.* **159**,No.2, 265, (1994).

30. Hata.,H.,Zwiebach,B.— *Ann. of Phys.*, **229**, 177, (1994).
Zwiebach,B.— preprint IASSNS–HER–92/53.

31. Henneaux,M.: Hamiltonian form of path integral for a theories with a gauge freedom. *Phys.Rep.* **126**, (1985).

32. Henneaux.,M.: Space–time locality of the BRST formalism. *Comm. Math. Phys.* **140**, pp.1–13 (1991).

33. Kallosh,R.: Modified Feynman rules in supergravity. *Nucl. Phys.*, **B141**,pp. 141-152 (1978).

34. Khudaverdian,O.M.: Geometry of superspace provided by Poisson brackets of different gradings. *J. Math. Phys.* **32**, pp.1934–1937, (1991) (Preprint of Geneve Universite. UGVA–DPT 1989/05–613).

35. Khudaverdian,O.M., Mkrtchian,R.L.: Integral invariants of Buttin bracket. *Lett. Math. Phys.* **18**, pp. 229–234, (1989). (*Preprint of Yerevan Physical Institute* , Yerphi, 918(69)–86 (1986),).

36. Khudaverdian,O.M., Nersessian,A.P.: Formulation of Hamiltonian mechanics with even and odd Poisson brackets. *Preprint of Yerevan Physical Institute.* Yerphi 1031(81)–87 (1987).

37. Khudaverdian,O.M., Nersessian,A.P.: Poisson brackets of different grading and strange superalgebras. *J. Math. Phys.*, **32**, pp.1937–1941, (1991).

38. Khudaverdian,O., Nersessian,A. Even and Odd symplectic kahlerian structures on projective superspace. *J. Math. Phys.* **34**, pp.5533–5548, (1993).

39. Khudaverdian,O.M., Nersessian,A.P.: On the geometry of Batalin–Vilkovisky formalism. *Mod. Phys. Lett.* **A8**, pp. 2377–2385 (1993).

40. Khudaverdian,O.M, Nersessian,A.P.: Batalin–Vilkovisky formalism and integration theory on manifolds. *Preprint Geneve Universite* UGVA–DPT 1995/07–896. VGVA-DPT 1995/07-896

41. Khudaverdian,O.M, Schwartz, A.S, Tyupkin,Yu.S.: Integral invariants of supercanonical transformations. *Lett. Math. Phys.*, **5** pp.517–520, (1981).

42. Kostant,B., Sternberg,S.: Symplectic reduction, B.R.S. cohomology and infinite–dimensional Clifford algebras. *Annals of Physics* **176**, pp.49–113, (1987)

43. Leites,D.A.: The new Lie superalgebras and Mechanics. *Docl. Acad. Nauk SSSR* **236**, pp. 804–807 (1977).

44. Leites,D.A.: Introduction to the theory of supermanifolds. *Usp. Mat. Nauk* **35**, No.1, 3–57.

45. Leites, D.A.: The theory of Supermanifolds. Karelskij Filial AN SSSR (1983).

46. Mc–Millan,D.: Yang–Mills theory and the Batalin Fradkin and Vilkovisky formalism. *Journ.Math.Phys.* **28** , 428 (1987).

47. Nersessian, A.P: On geometry of supermanifolds with even and odd kahlerian structures. *Theor. Math.Phys.* **96**, pp.866–871, (1993).

48. Nersessian ,A.P.: Antibracket and localization of path integral. *JETP Lett.* **58** No.1 pp.66–70. (1993).

49. Nersessian A.P.: From Antibracket to equivariant characteristic classes. Preprint JINR, E2–94–377 (1994).

50. Nersessian A.P., Damgaard ,P.H.: Comments on the covariant $SP(2)$–symmetric lagrangian BRST formalism. *Phys.Lett.*, **B355**, 150 ,(1995).

51. Nielsen,N.— *Nucl. Phys.*, **B142**, 306, (1978).

52. Penkava,M. Schwarz,A.S.: On some algebraical structures arising in string theories. Preprint of Davis University (1992), *Comm.Math.Phys.*

53. Schwarz,A.S.: The partition function of degenerate functional. *Comm..Math.Phys.* **67**, 1 (1979).

54. Schwarz,A.S.: Supergravity, complex geometry and G–structures. *Comm. Math.Phys.*, **87**, pp.37–63, (1982)

55. Schwarz,A.S.: Semiclassical approximation in Batalin–Vilkovisky formalism. hep–th/9210115

56. Schwarz,A.S.; Geometry of Batalin–Vilkovisky Formalism. *Commun. Math. Phys.* **155**, pp.249–260 (1993).

57. Shander,V.N.: Darboux and Lioville theorems on the supermanifolds. *Dokl.Akad.Nauk, Bulgary*, **36**, 309,(1983).

58. Tyutin,I.V.: Gauge invariance in field theory and statistical mechanics in the operator formalism. *Lebedev Physical Institute preprint* FIAN **N39** (1975).

59. Vandoren, S.,Van Proeyen A.: Simplifications in lagrangian BV quantization exemplified by the anomalies of chiral W_3 gravity. *Nucl. Phys.* **B411** pp.257–306, (1994).

60. Volkov,D.V., Pashnev A.I.,Soroka,V.A., Tkatch V.I.: On a Hamiltonian systems with even and odd Poisson brackets and duality of energy conservations laws. *JETP Lett.*, **44**,55, (1986).

61. Volkov,D.V.,Soroka V.A.,Tkach,V.I.: On a representation of Witten supersymmetric quantum mechanics by quantum antibracket. *J.Nucl. Phys.*, **44**, 810, (1986).

62. Voronov,T.: Theory of integration on vector bundles and supermanifolds with application to integral geometry.— Baku, International Conference on Topology. Pt.2, p.76 (1987)

63. Voronov,T.: On a class of integral transformations induced by morphisms of vector bundles. *Mat. Zametki*, **44**,No.6, pp.735–749 (1988).

64. Voronov, T.: Geometric Integration Theory on supermanifolds. *Sov.Sci.Rev.C Math.* **9**, pp.1–138 (1992).

65. Voronov,T.T., Zorich, A.V.: Complexex of forms on supermanifolds. *Funkts. Analiz i ego pril.* **20**, No.2, pp.58–59., (1988)

66. Voronov, T., Zorich ,A.V.:Integral transformations of pseudodifferential forms. *Usp. Mat.Nauk* **41**, No.6, pp.167–168 (1986).

67. Voronov,T.T., Zorich, A.V.: Cohomologies of supermanifolds and integral geometry. *Dokl. Akad. Nauk SSSR.* **298**, No.3, pp.528–533., (1988).

68. Voronov,T.T., Zorich, A.V.: Integration on vector bundles. *Funkts. Analiz i ego pril.* **22**, No.2, pp.14–25., (1988)

69. de Witt,B.S.: Dynamical theory of groups and fields. Gordon and Breach. New–York (1965).

70. Witten,E. — *Mod. Phys. Lett.*, A5, 487,(1990).