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# Top-bottom doublet in the sphaleron background *

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**Abstract**

We consider the top-bottom doublet in the background of the sphaleron for the realistic case of large non-degeneracy of fermion masses, in particular $m_b = 5$ GeV and $m_t = 175$ GeV. We propose an axially symmetric $(r, \theta)$-dependent ansatz for fermion fields and investigate the effects of the non-degeneracy on them. The exact solution is described, with an error less than 0.01%, by a set of ten radial functions. We also propose an approximate solution, in the $m_b/m_t \to 0$ limit, with an error $\mathcal{O}(m_b/m_t)$. We have found that the effects of non-degeneracy provide a $\theta$-dependence typically $\sim 10\%$.

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After the discovery by 't Hooft [1] that baryon and lepton numbers are anomalous global symmetries in the Standard Model, violated by non-perturbative effects, and the observation [2] that the three Sakharov's conditions for baryogenesis can be fulfilled at the electroweak phase transition, there has been renewed interest in understanding the dynamics of baryon asymmetry generation near the electroweak critical temperature. In particular the sphaleron, a static unstable solution of the classical equations of motion [3], has played a key rôle because the baryon violation rate for fluctuations between neighboring minima contains a Boltzmann suppression factor proportional to the energy of the sphaleron $E_{\rm sph}$. In particular the rate per unit time and unit volume is given by [4]

$$\Gamma \sim \omega T^4 \exp\left(-\frac{E_{\rm sph}}{v}\frac{\phi(T)}{T}\right) \tag{1}$$

where the sphaleron solution is usually computed in the limit of zero Weinberg angle (corrections of $\mathcal{O}(g')$ have been proved to be negligible [5] in the absence of fermions) and use has been made that the energy of the sphaleron at finite temperature follows an approximate scaling law with $\phi(T)/v$ [6]. The prefactor $\omega$ in (1) which contains, in particular, the product of bosonic and fermionic fluctuation determinants ($\kappa_{\rm bos}\kappa_{\rm fer}$) around the sphaleron, was evaluated [7] in the high $T$ limit (three-dimensional effective theory with all fermionic modes decoupled, i.e. $\kappa_{\rm fer}=1$).

In view of the recent experimental detection of the top-quark and its mass measurement [8] it seems of utmost importance to consider the modification, due to the presence of the top-quark, of the classical sphaleron energy and its contribution to $\kappa_{\rm fer}$. Progress in this direction has been done by considering both effects, in Refs. [9] and [10], respectively, as triggered by the presence of mass degenerate fermion doublets. The reason for studying degenerate fermions is simplicity since the spherical symmetry of the sphaleron solution is not spoiled by the presence of degenerate fermion doublets.

However, for the top-bottom doublet, with hierarchically different masses, $m_b \sim 5$ GeV and $m_t \sim 175$ GeV, the breaking of degeneracy (custodial symmetry), controlled by $(m_t - m_b)/m_t$, cannot be considered *a priori* as a small perturbation and the previous results might be spoiled by the large non-degeneracy effects. It is the purpose of this note to study the effects of the non-degeneracy of the top-bottom doublet in the background of the sphaleron. Our results have to be considered as a first step towards quantifying the effects of the non-degeneracy of fermion doublets on the sphaleron energy. They contain interesting (and somewhat unexpected) results which could be used as hints towards computing the effects of the top-bottom doublet with physical masses on the classical sphaleron barrier and fermion determinant.

After completion of this work we have received Ref. [11] where the same problem is considered using a different expansion for the fermion fields. We will compare both approaches and point out the differences in both expansions.

We will now study the fermionic distribution of a realistic quark doublet, non-degenerate in mass, in the sphaleron background. To simplify, we will work in the $g' = 0$ approximation. In this limit the hypercharge field $B_\mu$ decouples and the sphaleron energy density is spherically symmetric. The sphaleron configuration can be parametrized as:

$$\begin{aligned} \Phi(\hat{\mathbf{r}}) &= i\hat{r}.\vec{\tau}h(r)\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ W_i^a &= \tfrac{2}{gr}\epsilon_{aij}\hat{r}_j f(r) \end{aligned} \tag{2}$$

The relevant lagrangian density for fermions is:

$$\begin{aligned} \mathcal{L} = & \quad i\overline{\mathbf{Q}}\not{D}\mathbf{Q} + i\overline{\mathbf{u}}_R\not{\partial}\mathbf{u}_R + i\overline{\mathbf{d}}_R\not{\partial}\mathbf{d}_R \\ & -h_t\left(\overline{\mathbf{Q}}\tilde{\Phi}\mathbf{u}_R + h.c.\right) - h_b\left(\overline{\mathbf{Q}}\Phi\mathbf{d}_R + h.c.\right) \end{aligned} \tag{3}$$

where

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u}_L \\ \mathbf{d}_L \end{pmatrix}$$

is the left-handed quark doublet, $W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\varepsilon^{abc}W^b_\mu W^c_\nu$, $D_\mu = \partial_\mu - i\frac{g}{2}\tau^a W^a_\mu$, $\tilde{\Phi} \equiv i\tau^2\Phi^*$.

The equations of motion for the fermionic fields can be written as:

$$\begin{aligned} i\not{D}\mathbf{Q} &= h_t\tilde{\Phi}\mathbf{u}_R + h_b\Phi\mathbf{d}_R \\ i\not{\partial}\mathbf{u}_R &= h_t\tilde{\Phi}^\dagger\mathbf{Q} \\ i\not{\partial}\mathbf{d}_R &= h_b\Phi^\dagger\mathbf{Q} \end{aligned} \tag{4}$$

The non degeneracy of fermions explicitly breaks the global custodial symmetry $SU(2)_L \times SU(2)_R$. Let us remember that in the degenerate case the Dirac hamiltonian deduced from (4) commutes with the grand spin operator $\vec{K} = \vec{L} + \vec{S} + \vec{T}$, where $\vec{L}$ stands for the angular momentum, $\vec{S}$ for the spin and $\vec{T}$ for the isospin. In this case the eigenstates $\Psi$ of the Dirac operator can be labelled by $k, k_3$, with $K^2\Psi = k(k+1)\Psi$ and $K_3\Psi = k_3\Psi$. In particular the zero mode is included in the $k = 0$ subspace. For non-degenerate fermions, the different Yukawa couplings to the Higgs field background single out a direction in the isospin space. As a consequence $K^2$ does not commute any longer with the hamiltonian and the energy eigenstates will have non-vanishing projections over different values of $k$. However, $K_3$ still commutes with the hamiltonian, and hence $k_3$ is a good quantum number to label the energy eigenstates. Since $k_3$ is a discrete label and we want to continuously deform the zero mode from the degenerate case to the non-degenerate one, we will restrict ourselves to the subspace $k_3 = 0$.

To describe the fermions, we deal with the Dirac representation:

$$\gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \tag{5}$$

with $\sigma^i$ the Pauli matrices and $\sigma^0 = \mathbf{1}$. Using two component spinors, $q_L$, $q_R$, defined as:

$$\mathbf{q}_L = \frac{1}{\sqrt{2}}\begin{bmatrix} q_L \\ -q_L \end{bmatrix}, \quad \mathbf{q}_R = \frac{1}{\sqrt{2}}\begin{bmatrix} q_R \\ q_R \end{bmatrix}, \tag{6}$$

we can write our ansatz, a general parametrization of the $K_3 = 0$ subspace, as:

$$
\begin{aligned}
u_L(\vec{r}) &= \left(u_{L2}(\rho,z) + u_{L1}(\rho,z)\,\hat{\rho}.\vec{\sigma}\right)\chi_- \\
d_L(\vec{r}) &= \left(d_{L1}(\rho,z) + d_{L2}(\rho,z)\,\hat{\rho}.\vec{\sigma}\right)\chi_+ \\
u_R(\vec{r}) &= \left(u_{R2}(\rho,z) + u_{R1}(\rho,z)\,\hat{\rho}.\vec{\sigma}\right)\chi_- \\
d_R(\vec{r}) &= \left(d_{R1}(\rho,z) + d_{R2}(\rho,z)\,\hat{\rho}.\vec{\sigma}\right)\chi_+
\end{aligned}
\tag{7}
$$

where we have decomposed $\vec{r}$ into the longitudinal and transverse components, $\vec{r} = \vec{\rho} + z\vec{k}$, $\hat{\rho}$ is the normalized vector in the XY plane $\vec{\rho}/\rho$, $\hat{\rho}.\vec{\sigma} = \rho_x\sigma_x + \rho_y\sigma_y$ and

$$
\chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \chi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\tag{8}
$$

The equations for fermions can be written as functions of $r, \theta$, with $\rho = r\sin\theta$, as:

$$
\begin{aligned}
&\partial_r u_{L1}(r,\theta) + \tfrac{1}{r}u_{L1}(r,\theta) + \tfrac{1}{r}\partial_\theta u_{L2}(r,\theta) \\
&+\tfrac{1}{r}f(r)\left(\tfrac{1}{2}\sin 2\theta(d_{L1}(r,\theta) + u_{L2}(r,\theta)) + \sin^2\theta(d_{L2}(r,\theta) - u_{L1}(r,\theta))\right) \\
&-h(r)\left(\tfrac{1}{2}\sin 2\theta(h_b d_{R1}(r,\theta) + h_t u_{R2}(r,\theta)) + \cos^2\theta h_t u_{R1}(r,\theta) + \sin^2\theta h_b d_{R2}(r,\theta)\right) = 0 \\
\\
&\partial_r u_{L2}(r,\theta) - \cot\theta\tfrac{1}{r}u_{L1}(r,\theta) - \tfrac{1}{r}\partial_\theta u_{L1}(r,\theta) \\
&-\tfrac{1}{r}f(r)\big((d_{L1}(r,\theta) - u_{L2}(r,\theta)) + \cos^2\theta(d_{L1}(r,\theta) + u_{L2}(r,\theta)) \\
&\qquad\qquad + \tfrac{1}{2}\sin 2\theta(d_{L2}(r,\theta) - u_{L1}(r,\theta))\big) \\
&-h(r)\left(\tfrac{1}{2}\sin 2\theta(h_t u_{R1}(r,\theta) - h_b d_{R2}(r,\theta)) + \sin^2\theta h_b d_{R1}(r,\theta) - \cos^2\theta h_t u_{R2}(r,\theta)\right) = 0 \\
\\
&\partial_r d_{L1}(r,\theta) + \cot\theta\tfrac{1}{r}d_{L2}(r,\theta) + \tfrac{1}{r}\partial_\theta d_{L2}(r,\theta) \\
&-\tfrac{1}{r}f(r)\big((u_{L2}(r,\theta) - d_{L1}(r,\theta)) + \cos^2\theta(d_{L1}(r,\theta) + u_{L2}(r,\theta)) \\
&\qquad\qquad + \tfrac{1}{2}\sin 2\theta(d_{L2}(r,\theta) - u_{L1}(r,\theta))\big) \\
&-h(r)\left(\tfrac{1}{2}\sin 2\theta(h_t u_{R1}(r,\theta) - h_b d_{R2}(r,\theta)) + \sin^2\theta h_t u_{R2}(r,\theta) - \cos^2\theta h_b d_{R1}(r,\theta)\right) = 0 \\
\\
&\partial_r d_{L2}(r,\theta) + \tfrac{1}{r}d_{L2}(r,\theta) - \tfrac{1}{r}\partial_\theta d_{L1}(r,\theta) \\
&-\tfrac{1}{r}f(r)\left(\tfrac{1}{2}\sin 2\theta(d_{L1}(r,\theta) + u_{L2}(r,\theta)) + \sin^2\theta(d_{L2}(r,\theta) - u_{L1}(r,\theta))\right) \\
&+h(r)\left(\tfrac{1}{2}\sin 2\theta(h_t u_{R2}(r,\theta) + h_b d_{R1}(r,\theta)) - \sin^2\theta h_t u_{R1}(r,\theta) - \cos^2\theta h_b d_{R2}(r,\theta)\right) = 0 \\
\\
&\partial_r u_{R1}(r,\theta) + \tfrac{1}{r}u_{R1}(r,\theta) + \tfrac{1}{r}\partial_\theta u_{R2}(r,\theta) \\
&-h(r)h_t\left(\tfrac{1}{2}\sin 2\theta(d_{L1}(r,\theta) + u_{L2}(r,\theta)) + \cos^2\theta u_{L1}(r,\theta) + \sin^2\theta d_{L2}(r,\theta)\right) = 0 \\
\\
&\partial_r u_{R2}(r,\theta) - \cot\theta\tfrac{1}{r}u_{R1}(r,\theta) - \tfrac{1}{r}\partial_\theta u_{R1}(r,\theta) \\
&-h(r)h_t\left(\tfrac{1}{2}\sin 2\theta(u_{L1}(r,\theta) - d_{L2}(r,\theta)) + \sin^2\theta d_{L1}(r,\theta) - \cos^2\theta u_{L2}(r,\theta)\right) = 0
\end{aligned}
$$

$$-h(r)h_b\left(\tfrac{1}{2}\sin 2\theta(u_{L1}(r,\theta)-d_{L2}(r,\theta))+\sin^2\theta u_{L2}(r,\theta)-\cos^2\theta d_{L1}(r,\theta)\right)=0$$

$$\begin{aligned}&\partial_r d_{R2}(r,\theta)+\tfrac{1}{r}d_{R2}(r,\theta)-\tfrac{1}{r}\partial_\theta d_{R1}(r,\theta)\\&+h(r)h_b\left(\tfrac{1}{2}\sin 2\theta(u_{L2}(r,\theta)+d_{L1}(r,\theta))-\sin^2\theta u_{L1}(r,\theta)-\cos^2\theta d_{L2}(r,\theta)\right)=0\end{aligned} \tag{9}$$

We have, then, a set of eight partial differential equations. Instead of solving it in general, we will work out a systematic expansion of these functions in terms of a Fourier series. Notice that the commutator of the hamiltonian [defined on the $(q_L, q_R)^T$-space] with parity vanishes on the zero modes, which allows to label them with a definite parity. As parity is a discrete symmetry we will choose for our ansatz the same parity (i.e. even) as zero modes in the degenerate case. Therefore, our expansion can be written as [1]

$$\begin{aligned}
u_{L1}(r,\theta) &= \textstyle\sum_{n=1}^{\infty} u_{L1}^{(n)}(r)\sin(2n\theta)\\
u_{L2}(r,\theta) &= u_{L2}^{(0)}(r) + \textstyle\sum_{n=1}^{\infty} u_{L2}^{(n)}(r)\cos(2n\theta)\\
d_{L1}(r,\theta) &= d_{L1}^{(0)}(r) + \textstyle\sum_{n=1}^{\infty} d_{L1}^{(n)}(r)\cos(2n\theta)\\
d_{L2}(r,\theta) &= \textstyle\sum_{n=1}^{\infty} d_{L2}^{(n)}(r)\sin(2n\theta)\\
u_{R1}(r,\theta) &= \textstyle\sum_{n=1}^{\infty} u_{R1}^{(n)}(r)\sin(2n\theta)\\
u_{R2}(r,\theta) &= u_{R2}^{(0)}(r) + \textstyle\sum_{n=1}^{\infty} u_{R2}^{(n)}(r)\cos(2n\theta)\\
d_{R1}(r,\theta) &= d_{R1}^{(0)}(r) + \textstyle\sum_{n=1}^{\infty} d_{R1}^{(n)}(r)\cos(2n\theta)\\
d_{R2}(r,\theta) &= \textstyle\sum_{n=1}^{\infty} d_{R2}^{(n)}(r)\sin(2n\theta)
\end{aligned} \tag{10}$$

We expect the convergence of the series (10) to be good enough to provide a reliable solution to the system of equations (9). In fact, cutting off the series at $n = 0$ would include the appropriate ansatz to describe a situation in which the fermion doublet is degenerate. However the case we want to address, i.e. that of the top-bottom doublet, is very massive and highly non-degenerate. This means, on the one hand, that we need the modes $n \geq 1$ in (10) and, on the other hand, that the convergence of the series (10) might be slow enough to require many terms for an accurate description of the solution. Fortunately we will see this is not the case, and, as we will prove, taking only the terms with $n = 0, 1$ provides a reliable solution with an accuracy better than 0.01%.

[1] An alternative expansion is given by developing the fermionic fields in terms of the even spherical harmonic subsets $Y_{2n}^0(\theta)$, $Y_{2n}^{\pm 1}(\theta,\phi)$ instead of $\cos(2n\theta)$, $\sin(2n\theta)e^{\pm i\phi}$ respectively. In this basis it is straightforward to check that $k_3 = 0$.

order $n$ (i.e. neglecting all modes with higher values of $n$) and estimate the accuracy of the different approximations. A first step along this direction is solving (9) to order $n = 0$, i.e. in the approximation of a spherically symmetric ansatz for fermions (as in the case $m_u = m_d$). Only four spinor components do not vanish:

$$u_{L2}^{(0)},\ d_{L1}^{(0)},\ u_{R2}^{(0)},\ d_{R1}^{(0)}. \tag{11}$$

Replacing them into (9) one obtains a system of four differential equations whose solution is shown in Fig. 1 (dashed lines) [2]. It is interesting to notice that the relation

$$u_{L2}^{(0)}(r) + d_{L1}^{(0)}(r) = 0, \tag{12}$$

valid in the degenerate case, also holds in this approximation. On the other hand, the similar relation valid in the degenerate case, $u_{R2}^{(0)}(r) + d_{R1}^{(0)}(r) = 0$ is spoiled by non-degeneration effects, and replaced by

$$h_t d_{R1}^{(0)}(r) + h_b u_{R2}^{(0)}(r) = 0, \tag{13}$$

In fact, the functions defined by the left-hand sides of Eqs. (12) and (13) only couple to themselves and to higher modes, and hence can be consistently fixed to zero to this order.

In general, solving the system to a given order $n$, amounts to keeping the system of fields

$$u_{L1}^{(p)},\ u_{L2}^{(p)},\ d_{L1}^{(p)},\ d_{L2}^{(p)},$$

$$u_{R1}^{(p)},\ u_{R2}^{(p)},\ d_{R1}^{(p)},\ d_{R2}^{(p)} \tag{14}$$

for $p = 1, ..., n$, and the spherically symmetric components (11): a system of $8n + 4$ differential equations with $8n + 4$ unknown functions. We will solve the system of differential equations using the following boundary conditions. There are two zeroes in the $(8n+4) \times (8n+4)$ asymptotic mass matrix which correspond to two directions, in the space of highest, $n$-th, components of the spinorial fields. They must be fixed to zero at infinity, and thus elsewhere since these directions have (up to higher order contributions) fixed points at zero. Fixing to zero these two directions amounts to the conditions:

$$u_{L1}^{(n)}(r) + u_{L2}^{(n)}(r) + d_{L1}^{(n)}(r) - d_{L2}^{(n)}(r) = 0 \tag{15}$$

and

$$h_b u_{R1}^{(n)}(r) + h_b u_{R2}^{(n)}(r) + h_t d_{R1}^{(n)}(r) - h_t d_{R2}^{(n)}(r) = 0 \tag{16}$$

Out of the remaining $8n + 2$ fields, $4n + 1$ correspond to positive mass eigenvalues, and therefore are fixed to zero at infinity imposing normalizability of the fermionic configuration, and $4n + 1$ have negative mass eigenvalues. $4n$ of them are fixed to zero

[2] All dimensional quantities are expressed in units of the corresponding power of $v = \langle\phi_0\rangle$, and we have arbitrarily normalized the total fermionic density to $4\pi$.

and the last one is used to fix the global normalization. In particular, for $n = 1$ we obtain a system of twelve differential equations with twelve unknown functions: the set (11) and (14) with $p = 1$. The solution is presented in Figs. 1 and 2 (solid lines). We can see from Fig. 1 that the relations (12) and (13) are spoiled by the presence in the equations of the angular dependent components. The spoiling is $\sim$10%, which is the expected order of magnitude of the angular dependent part of the solution. The latter, i.e. the radial functions of (14) with $p = 1$ are plotted in Fig. 2 (solid lines). We can see that the typical size of these functions is $\sim$ 10% those contributing to the spherically symmetric part of Fig. 1. This is the amount by which the relations between the components with $n = 0$, Eqs. (12) and (13), are spoiled. The latter are replaced by the similar relations (15) and (16) between the $n = 1$ components. One can easily check from the curves in Fig. 2 that relations (15) and (16) are exactly satisfied. Moreover, we have numerically found that Eq. (15) splits with a good accuracy, for $n \geq 1$, into the couple of constraints:

$$u_{L1}^{(n)}(r) + d_{L1}^{(n)}(r) = 0 \tag{17}$$

and

$$u_{L2}^{(n)}(r) - d_{L2}^{(n)}(r) = 0. \tag{18}$$

We believe that Eqs. (17) and (18) are not a numerical coincidence. The fermionic fields expanded up to a given order $n$ can be decomposed into eigenstates of angular momentum $\ell = 0,\ 2,\ ,\ldots,\ \leq 2n$, isospin $T_3 = \pm 1/2$ and spin $S_3 = \pm 1/2$, given by

$$|\ell, -(T_3 + S_3)\rangle \otimes \left|\frac{1}{2}, T_3\right\rangle \otimes \left|\frac{1}{2}, S_3\right\rangle .$$

Defining $\vec{J} = \vec{L} + \vec{T}$ and using the basis of $J^2, J_3$ eigenstates, condition (17) is equivalent to projecting out the left-handed states on $|j, j_3\rangle = \left|2n + \frac{1}{2}, -\frac{1}{2}\right\rangle$. Similarly, condition (18) amounts to projecting out on $|j, j_3\rangle = \left|2n - \frac{1}{2}, \frac{1}{2}\right\rangle$. Therefore a consistent ansatz using the latter basis should take advantage of this approximate symmetry.

We have solved the system (9) to order $n = 2$. In this case we have a system of twenty differential equations with twenty unknown functions, those in (14) with $p = 1, 2$ and in (11). The new components, with $p = 2$ in (14), are less than $10^{-4}$ the large components in Fig. 1 while the $p = 1$ components in (14) and the components in (11), computed to this order of approximation, are indistinguishable from those previously computed. In fact, we have plotted in Figs. 1 and 2 the latter in solid lines and they are superimposed on those computed to order $n = 1$. In the same way the relationships (17) and (18) remain very accurately satisfied.

To summarize the previous results, we can keep, out of the general expansion (10), the particular reduction

$$
u_L(\vec{r}) = \begin{bmatrix} & & u_{L1}^{(1)}(r)\sin 2\theta e^{-i\phi} \\ u_{L2}^{(0)}(r) & + & u_{L2}^{(1)}(r)\cos 2\theta \end{bmatrix} \quad u_R(\vec{r}) = \begin{bmatrix} & & u_{R1}^{(1)}(r)\sin 2\theta e^{-i\phi} \\ u_{R2}^{(0)}(r) & + & u_{R2}^{(1)}(r)\cos 2\theta \end{bmatrix}
$$

$$
d_L(\vec{r}) = \begin{bmatrix} d_{L1}^{(0)}(r) & + & d_{L1}^{(1)}(r)\cos 2\theta \\ & & d_{L2}^{(1)}(r)\sin 2\theta e^{i\phi} \end{bmatrix} \quad d_R(\vec{r}) = \begin{bmatrix} d_{R1}^{(0)}(r) & + & d_{R1}^{(1)}(r)\cos 2\theta \\ & & d_{R2}^{(1)}(r)\sin 2\theta e^{i\phi} \end{bmatrix} \tag{19}
$$

where two radial functions can be eliminated using the relations (15) and (16).

Given the smallness of the ratio $m_b/m_t$ one can try to implement the $m_b/m_t \to 0$ limit in our solution. In that limit $d_R$ decouples from the equations of motion and can be consistently fixed to zero. In particular

$$
d_{R1}^{(n)}(r) = d_{R2}^{(n)}(r) \equiv 0. \tag{20}
$$

Eq. (20) is very accurately satisfied, as can be checked from Figs. 1b and 2b. We have solved the problem for values of $m_b$ in the range $0 \leq m_b \leq 5$ GeV. We have verified in this range that $d_R$ is proportional to $h_b$. The rest of fields remain essentially unchanged since their dependence on $m_b$ is through $\mathcal{O}(h_b^2)$ terms. Numerically these non-vanishing fields, computed in the strict limit $m_b = 0$, are superimposed with the corresponding functions plotted in Figs. 1 and 2 for the physical value of $m_b$. We can conclude that neglecting all right-handed components of spinor fields, in the limit $m_b/m_t \to 0$, is accurate to $\mathcal{O}(m_b/m_t)$.

We will compare now the previous solution with that recently proposed in Ref. [11]. The solution in Ref. [11] corresponds to the parametrization in terms of eight radial functions, as:

$$
\begin{aligned}
u_L(\vec{r}) &= \begin{bmatrix} & - & \frac{1}{2}F_L^+ \sin 2\theta e^{-i\phi} \\ G_L^+ + \frac{1}{2}F_L^+ & + & \frac{1}{2}F_L^+ \cos 2\theta \end{bmatrix} \\
u_R(\vec{r}) &= \begin{bmatrix} & - & \frac{1}{2}F_R^+ \sin 2\theta e^{-i\phi} \\ G_R^+ + \frac{1}{2}F_R^+ & + & \frac{1}{2}F_R^+ \cos 2\theta \end{bmatrix} \\
d_L(\vec{r}) &= \begin{bmatrix} -G_L^- + \frac{1}{2}F_L^- & + & \frac{1}{2}F_L^- \cos 2\theta \\ & & \frac{1}{2}F_L^- \sin 2\theta e^{i\phi} \end{bmatrix} \\
d_R(\vec{r}) &= \begin{bmatrix} -G_R^- + \frac{1}{2}F_R^- & + & \frac{1}{2}F_R^- \cos 2\theta \\ & & \frac{1}{2}F_R^- \sin 2\theta e^{i\phi} \end{bmatrix}
\end{aligned} \tag{21}
$$

$$F_L^{\pm} = F_L(r) \pm \Delta F_L(r), \quad G_L^{\pm} = G_L(r) \pm \Delta F_L(r),$$
$$F_R^{\pm} = F_R(r) \pm \Delta F_R(r), \quad G_R^{\pm} = G_R(r) \pm \Delta F_R(r) \tag{22}$$

Since we have proved that our expansion (19) describes the exact solution with an error less than 0.01%, we can compare it with the parametrization in Eq. (21). In the left-handed sector Eq. (21) would imply that $u_{L1}^{(1)} + u_{L2}^{(1)} = 0$ and $d_{L1}^{(1)} = d_{L2}^{(1)}$: these conditions are consistent with (15) though a quick glance at Fig. 2 shows that they are violated in the exact solution by $\sim$10% of the corresponding components, an error much greater than that induced by the next order. Moreover, these conditions conditions are not controlled by $m_b/m_t$ but, contrarily, become exact in the limit of large degeneracy. Similarly, in the right-handed sector, Eq. (21) would predict $u_{R1}^{(1)} + u_{R2}^{(1)} = 0$ and $d_{R1}^{(1)} = d_{R2}^{(1)}$, which are also consistent with our $n = 1$ constraint (16) but not satisfied by the exact solution. In summary, we have found that the parametrization (21) is accurate in the case of large degeneracy, but has corrections not controlled by the parameter $m_b/m_t$, which are much greater than those corresponding to the next order in our expansion.

In conclusion, we have performed a consistent expansion for the top-bottom fermionic fields in the background of the sphaleron. We have found that the first two terms in the expansion provide the fermionic fields with an accuracy better than 0.01%. We have quantified the effect of the top-bottom non-degeneracy in the $\theta$-dependent part of the fermionic components as $\sim$10%. We have proved that the approximation $m_b/m_t \to 0$ provides a solution accurate to $\mathcal{O}(m_b/m_t)$. Our work should be considered as a first step towards evaluating the effects of the non-degeneracy of the top-bottom doublet in the sphaleron energy.

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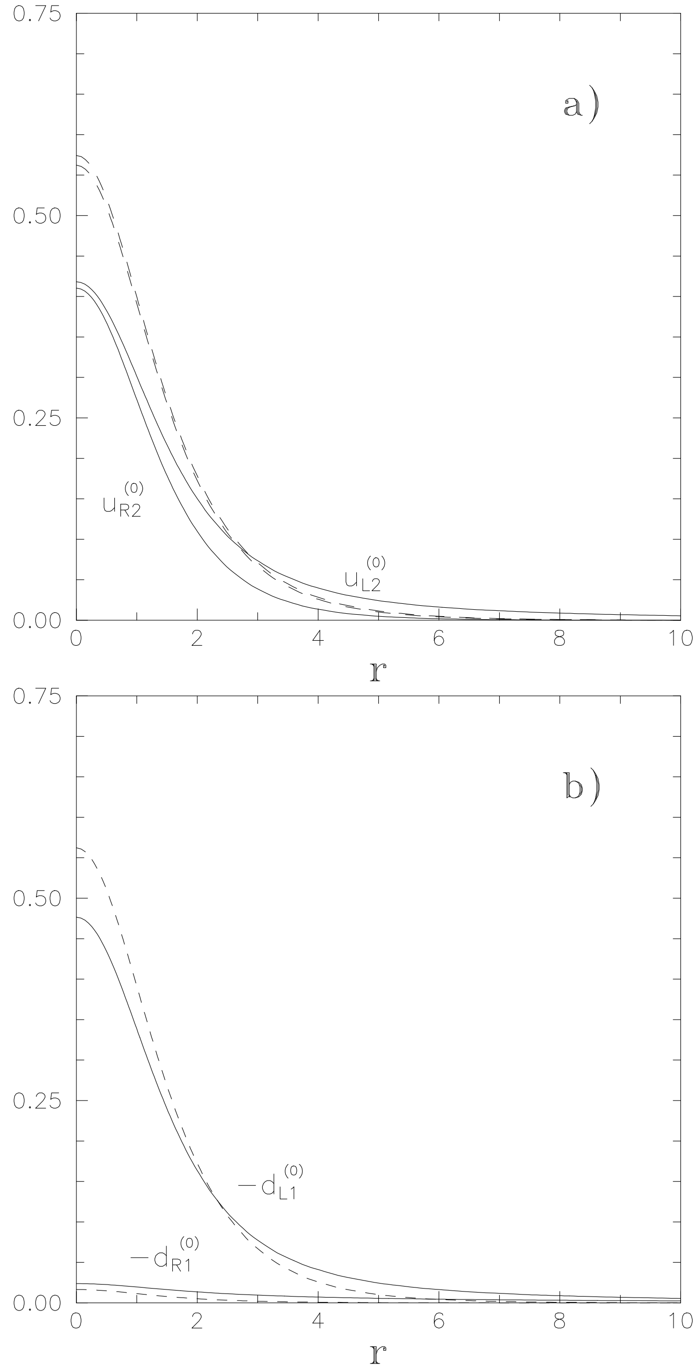


Figure 1: **a)** The radial components $u_{L2}^{(0)}$ and $u_{R2}^{(0)}$ computed to order $n = 0$ (dashed lines) and $n = 1$ and 2 (solid lines). **b)** The radial components $-d_{L1}^{(0)}$ and $-d_{R1}^{(0)}$ computed to order $n = 0$ (dashed lines) and $n = 1$ and 2 (solid lines).

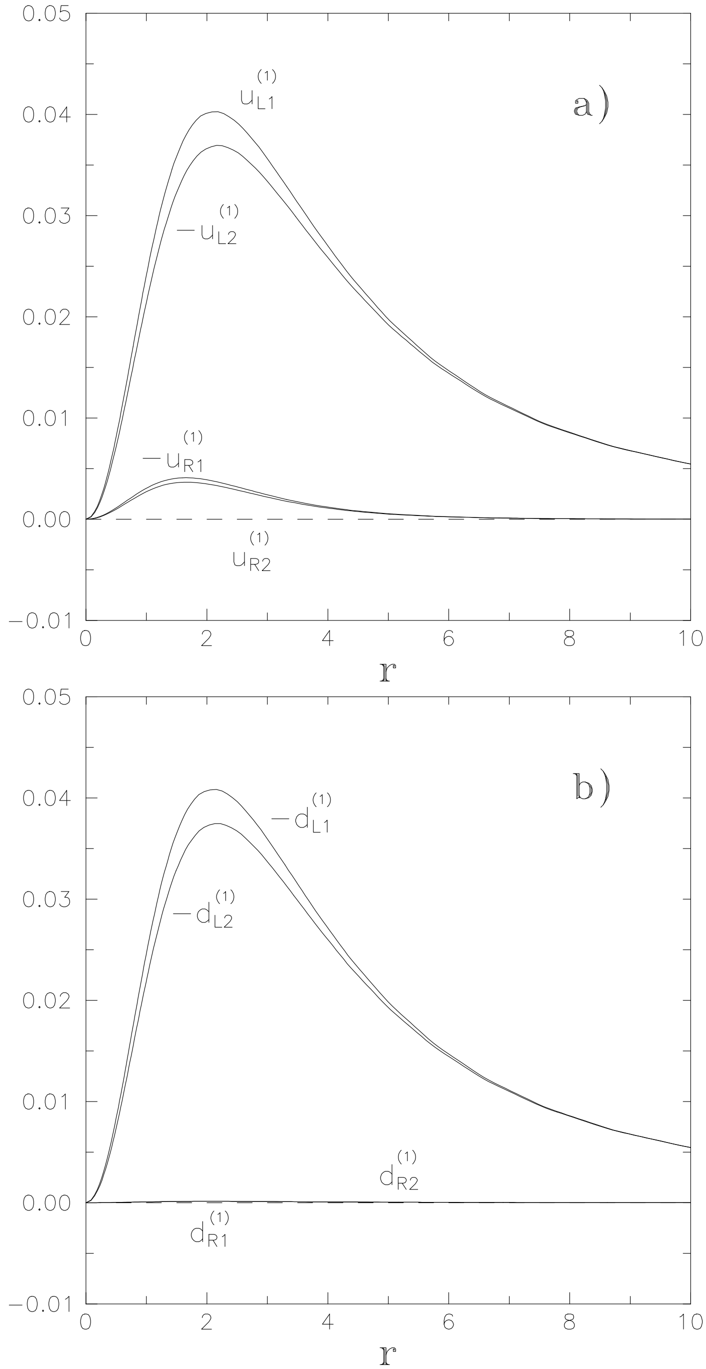


Figure 2: **a)** The radial components $u^{(1)}_{L1}$, $-u^{(1)}_{L2}$, $-u^{(1)}_{R1}$ and $u^{(1)}_{R2}$ computed to order $n = 1$ and 2. **a)** The radial components $-d^{(1)}_{L1}$, $-d^{(1)}_{L2}$, $d^{(1)}_{R1}$ and $d^{(1)}_{R2}$ computed to order $n = 1$ and 2.