

QUANTIZATION OF THE SCHWARZSCHILD BLACK HOLE

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ABSTRACT

We quantize by the Dirac – Wheeler–DeWitt method the canonical formulation of the Schwarzschild black hole developed in a previous paper. We investigate the properties of the operators that generate rigid symmetries of the Hamiltonian, establish the form of the invariant measure under the rigid transformations, and determine the gauge fixed Hilbert space of states. We also prove that the reduced quantization method leads to the same Hilbert space for a suitable gauge fixing.

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1. Introduction.

Recently a good deal of work has been dedicated to the canonical formulation of the spherically symmetric gravity and to its quantization [1-3]. In Ref. [3] (hereafter referred as I) we developed a canonical approach to the study of the spherically symmetric metric by proposing a foliation in the radial parameter r and considering the Lagrangian coordinates as functions only of r . Thus this leads to a structure of minisuperspace. The theory is of course endowed with a gauge invariance (reparametrization in r) and a constraint. In I we have developed the theory and shown a number of points that we recall briefly.

We expressed the Einstein equations as a canonical system in a finite, 2×2 dimensional phase space. The gauge transformation is integrable; in particular the solution of the equations of motion is the Schwarzschild solution. This property allows the identification of the canonical quantity that corresponds to the Schwarzschild mass. There is an interesting algebraic structure of three gauge invariant canonical quantities, whose physical meaning was clarified, that form an affine algebra.

We also started to investigate the Dirac – Wheeler–De Witt (WDW) quantization discussing the general form of the solutions and showed that they are oscillating in the classically allowed regions and exponentially decreasing in the forbidden regions. We briefly discussed the form of the eigenfunctions of the mass operator and of the generator of dilatations. We noted that a set of solutions coincides with that of Kantowski-Sachs (KS) wormholes [4,5]. This is hardly surprising. The geometry inside the horizon of a black hole coincides with the KS geometry, and further the foliation parameter r is timelike inside the horizon, so what we expose here is, for the part internal to the horizon, isomorphic to the theory of the KS spacetime. This property enforces the much discussed possibility that a black hole can be connected to a KS wormhole [6].

What remained to be done was the determination of the measure in the inner product and the gauge fixing with the consequent establishment of a positive definite Hilbert space. This is essentially the content of the present paper.

We start by introducing the classical Lagrangian and Hamiltonian and integrate the gauge transformations and rigid symmetries. Then we carry on the construction of the quantum theory. We start with the Dirac method and establish the WDW equation.

The request of preserving at the quantum level both the gauge invariance and the classical rigid symmetries, together with the support properties of the variables used as quantum coordinates, determines completely the quantum measure and fixes the representation of the quantum operators. We identify the solutions of the WDW equation that are eigenfunctions of the operators corresponding to the most important invariants of the classical theory. A Fourier transform gives the solutions in the configuration space already found in I using the covariant measure.

Up to this point there has been no gauge fixing nor definition of a norm. We then fix the gauge by defining the inner product by the Faddeev–Popov (FP) procedure [7] and prove the existence of a class of gauges. This leads to a positive definite Hilbert space.

We may also start by first fixing the gauge in the classical frame by a suitable canonical gauge fixing identity that contains the coordinate r (for the method, see e.g. [8]); the results coincide with those obtained by the Dirac method.

Having a positive definite Hilbert space, we are able to prove that, due to the support properties of its conjugate variable, the hermitian operator corresponding to the Schwarzschild mass in the gauge fixed, positive norm, Hilbert space is not self-adjoint, while its square is a self-adjoint operator with positive eigenvalues, analogously to what happens for the radial momentum in ordinary quantum mechanics.

Of course in the classical theory the mass is perfectly defined. Again take the example of the radial momentum: although it is not a self-adjoint operator in the Hilbert space, a classical radial momentum is defined, namely $p_r = m\dot{r}$, and its square is self-adjoint. This difference between classical and quantum behavior is due to the fact that a classical canonical quantity is a purely local entity while the definition of a self-adjoint operator conveys general informations about the Hilbert space. Alternatively, one may think that the operator corresponding to the mass of the black hole should be defined in a different way.

No quantization of the mass is required by the quantum theory as it stands. Quantization of the mass can be surely achieved by a modification of the boundary conditions and/or of the original Hamiltonian, and we point in the conclusions how it could be carried on in a gauge invariant way. It would be worth exploring this weird and fascinating possibility. It would also be interesting to introduce matter degrees of freedom with the aim of summing over degrees of freedom in order to compute entropy (hopefully, this may allow us to define a more satisfactory operator for the mass of the black hole).

2. Classical Theory.

In this section we summarize the main classical results obtained in I with some minor changes in notation and some added considerations about the symmetries of the action.

The spherically symmetric line element can be written in the following convenient form

$$ds^2 = -4a(r)dt^2 + 4n(r)dr^2 + b^2(r)d\Omega_2^2, \quad (2.1)$$

where a , n , and b only depend on the radial coordinate r and $d\Omega_2^2$ is the line element of the two-sphere.

We allow in principle for changes of signs in the metric tensor. Depending on the sign of a and n , the coordinates t and r may be timelike and spacelike

or vice versa. $n(r)$ plays essentially the role of the r -lapse function and it is just a Lagrange multiplier in the action enforcing the constraint that generates reparametrizations of r .

We start from the action

$$S = \frac{1}{16\pi G} \int_{V_4} d^4x \sqrt{-g} (R + 2\Lambda) - \frac{1}{8\pi G} \int_{\partial V_4} d^3x \sqrt{h} \mathbf{K}. \quad (2.2)$$

Introducing the Ansatz (2.1) one obtains

$$S = \int_{t_1}^{t_2} dt \int_{r_1}^{r_2} dr \mathcal{L}(a, b, l), \quad (2.3)$$

where the Lagrangian \mathcal{L} is

$$\mathcal{L} = 2l \left(\frac{\dot{a}b\dot{b}}{l^2} + \frac{a\dot{b}^2}{l^2} + \frac{1 + \Lambda b^2}{4} \right). \quad (2.4)$$

In Eq. (2.4), dots denote differentiation with respect to r and we have set $G = 1$. The Lagrangian multiplier l is given by

$$l(r) = 4\sqrt{an}. \quad (2.5)$$

Note that in I the Lagrangian multiplier was chosen as $l(r) = \sqrt{an}/2b^2$. Also (2.1) and (2.4) are different. We have preferred the present definitions as they lead to some simplification in the Hamiltonian treatment. All the results of the present paper, both classical and quantum, are of course identical.

As already pointed in I, the Lagrangian must be real, so a and n have the same sign and thus the line element (2.1) has lorentzian signature everywhere: as we will see in a moment, on the classical solutions positive values of a represent the exterior of the black hole and negative values of a represent the region inside the horizon.

The Hamiltonian \mathcal{H} can be calculated in the usual way by defining canonical r -momenta as

$$p_a = \frac{2b\dot{b}}{l}, \quad (2.6a)$$

$$p_b = \frac{2}{l}(\dot{a}b + 2a\dot{b}). \quad (2.6b)$$

We have:

$$\mathcal{H} = lH, \quad (2.7a)$$

$$H = \frac{1}{2b^2}[p_a(bp_b - ap_a)] - \frac{1}{2}(1 + \Lambda b^2), \quad (2.7b)$$

where H is the generator of r -reparametrizations (gauge transformations) that we will simply call the ‘‘Hamiltonian’’ of the system. As a consequence of the form of \mathcal{H} we have the constraint

$$H = 0, \quad (2.8)$$

which expresses the invariance under r -reparametrization.

Let us set from now on $\Lambda = 0$; the case of non zero cosmological constant will be examined in Appendix B. The Hamiltonian (2.7b) has very interesting invariance transformations; first, the gauge transformations generated by H (denoted as \mathcal{H}_h)

$$\delta q_i = \epsilon \frac{\partial H}{\partial p_i} = \epsilon [q_i, H]_P, \quad (2.9a)$$

$$\delta p_i = -\epsilon \frac{\partial H}{\partial q_i} = \epsilon [p_i, H]_P, \quad (2.9b)$$

$$\delta l = \frac{d\epsilon}{dr}, \quad (2.9c)$$

can be integrated explicitly. For $H = 0$ the result is:

$$b \rightarrow \bar{b} = b + h(r) \frac{p_a}{2b}, \quad (2.10a)$$

$$p_a \rightarrow \bar{p}_a = p_a + h(r) \frac{p_a^2}{2b^2}, \quad (2.10b)$$

$$a \rightarrow \bar{a} = a + \frac{N}{b^2} \frac{h(r)/2}{1 + h(r)p_a/2b^2}, \quad (2.10c)$$

$$p_b \rightarrow \bar{p}_b = p_b + \frac{J}{b^2} \frac{h(r)/2}{1 + h(r)p_a/2b^2}, \quad (2.10d)$$

$$l(r) \rightarrow \bar{l}(r) = l(r) + \frac{dh}{dr}, \quad (2.10e)$$

where J and N are gauge invariant quantities defined below. Note the simplicity of the gauge transformations of b and p_a . This fact will be exploited later.

We have three gauge invariant canonical quantities, namely

$$I = b/p_a, \quad (2.11a)$$

$$J = 2b - p_a p_b + 4bH, \quad (2.11b)$$

$$N = IJ = b p_b - 2a p_a. \quad (2.11c)$$

I, J, N play a fundamental role in the theory. The algebra of H, I, J, N is:

$$\begin{aligned} [I, H]_P &= 0, & [J, H]_P &= 0, & [J, I]_P &= 1, \\ [N, H]_P &= 0, & [N, I]_P &= I, & [N, J]_P &= -J. \end{aligned} \tag{2.12}$$

We write also the unconstrained solution of the equations of motion; for $H = 0$ they have of course the same content as the gauge equations (2.10). We have

$$b = \frac{\tau}{2I}, \tag{2.13a}$$

$$p_a = \frac{b}{I}, \tag{2.13b}$$

$$a = I^2 \left(2H + 1 - \frac{J}{b} \right), \tag{2.13c}$$

$$p_b = I \left(4H + 2 - \frac{J}{b} \right), \tag{2.13d}$$

$$\tau = \int_{r_0}^r l(r) dr, \quad l(r) > 0. \tag{2.13e}$$

τ will be chosen positive without loss of generality.

Eq. (2.13c) corresponds to the Schwarzschild solution if we set $H = 0$. Then a vanishes for $b = J$ and so $J/2 \equiv M$ is the classical canonical expression of the Schwarzschild mass M . Again from (2.13c), remembering (2.1), we see that $T \equiv 2I$ is the ratio between proper and coordinate time in the asymptotic region $b \rightarrow \infty$.

There is a very important point concerning the support of the variables b and p_a . b is positive definite as it is natural since it is classically a radial variable. Then from the positivity of τ it follows from (2.13a) that $I > 0$. Also, p_a is positive. These properties will be essential in the following.

Let us call rigid those symmetries generated by I, J and N . Any gauge invariant function of the canonical variables can be written as $F(H, I, J)$. The requests that it be N and I invariant, or N and J , or I and J , are equivalent as they leave us with $F(H)$, so we need only consider two of the three rigid transformations.

Invariance under rigid transformations will be used in the next chapter to investigate the quantum measure. For the moment let us write down these transformations.

The finite transformations generated by I (denoted as \mathcal{I}_f) are:

$$\begin{aligned}
b &\rightarrow \bar{b} = b, \\
p_a &\rightarrow \bar{p}_a = p_a, \\
a &\rightarrow \bar{a} = a - fb p_a^{-2}, \\
p_b &\rightarrow \bar{p}_b = p_b - f p_a^{-1}.
\end{aligned} \tag{2.14}$$

The finite transformations generated by J (denoted as \mathcal{J}_q) are:

$$\begin{aligned}
b &\rightarrow \bar{b} = \frac{b}{1 - q/I}, \\
p_a &\rightarrow \bar{p}_a = \frac{p_a}{(1 - q/I)^2}, \\
a &\rightarrow \bar{a} = a(1 - q/I)^2 + \frac{N}{b}q(1 - q/I)^2, \\
p_b &\rightarrow \bar{p}_b = p_b(1 - q/I) + \frac{J}{b}q(1 - q/I).
\end{aligned} \tag{2.15}$$

The finite transformations generated by N (denoted as \mathcal{N}_g) on the canonical variables are dilatations, due to the form of N :

$$\begin{aligned}
b &\rightarrow \bar{b} = e^g b, \\
p_a &\rightarrow \bar{p}_a = e^{2g} p_a, \\
a &\rightarrow \bar{a} = e^{-2g} a, \\
p_b &\rightarrow \bar{p}_b = e^{-g} p_b.
\end{aligned} \tag{2.16}$$

Now looking at these three sets of transformations and at the gauge transformations (2.10) we see that the canonical variables b, p_a transform separately under all transformations. These variables will be most appropriate as coordinates in the quantum case. We also note that the J transformations may change the sign of b , in contrast with our assumption that $b > 0$. Thus we consider as fundamental symmetries \mathcal{N}_g and \mathcal{I}_f . The analysis of the consequences of relaxing the condition $b > 0$ (unfolding of b) will be carried on elsewhere.

Different sets of canonical pairs will be used in what follows. We may perform a canonical transformation to the new canonical variables $\{J, I, Y, P_Y\}$, where

$$Y = \frac{2b^2}{p_a}, \quad P_Y = H. \tag{2.17}$$

This choice is motivated by their invariance properties: I, J, H are gauge invariant and Y behaves in the simplest way under gauge transformations \mathcal{H}_h . For completeness we give the generating function of the canonical transformation:

$$F = \frac{2ab^2}{Y} - \frac{YJ}{2b} + \frac{1}{2}Y. \tag{2.18}$$

We can use alternatively $N = IJ$ and $p_N = \ln I$ instead of J and I . Using the canonical variables $\{J, I, Y, P_Y\}$, the Hamiltonian reads simply

$$\mathcal{H} = IP_Y. \quad (2.19)$$

We list their transformation properties under gauge and rigid transformations:

• \mathcal{I}_f :

$$\begin{aligned} I &\rightarrow \bar{I} = I, \\ J &\rightarrow \bar{J} = J + f, \\ Y &\rightarrow \bar{Y} = Y, \\ P_Y &\rightarrow \bar{P}_Y = P_Y; \end{aligned} \quad (2.20)$$

• \mathcal{J}_q :

$$\begin{aligned} I &\rightarrow \bar{I} = I - q, \\ J &\rightarrow \bar{J} = J, \\ Y &\rightarrow \bar{Y} = Y, \\ P_Y &\rightarrow \bar{P}_Y = P_Y; \end{aligned} \quad (2.21)$$

• \mathcal{N}_g :

$$\begin{aligned} I &\rightarrow \bar{I} = e^{-g}I, \\ J &\rightarrow \bar{J} = e^g J, \\ Y &\rightarrow \bar{Y} = Y, \\ P_Y &\rightarrow \bar{P}_Y = P_Y; \end{aligned} \quad (2.22)$$

• \mathcal{H}_h :

$$\begin{aligned} I &\rightarrow \bar{I} = I, \\ J &\rightarrow \bar{J} = J, \\ Y &\rightarrow \bar{Y} = Y + h(r), \\ P_Y &\rightarrow \bar{P}_Y = P_Y. \end{aligned} \quad (2.23)$$

These formulas will be important for the discussion in the next section.

Finally, let us write another set of canonical variables that will be used in section 4:

$$\begin{aligned} \alpha &= \ln |a|, \\ c &= 2\sqrt{|a|}b, \\ p_\alpha &= -\frac{N}{2}, \\ p_c &= \frac{p_b}{2\sqrt{|a|}}. \end{aligned} \quad (2.24)$$

In this case there are two different canonical transformations, for positive and negative a ; the Hamiltonian (2.7b) becomes

$$H = \frac{1}{2} \left[\sigma \left(p_c^2 - 4\frac{p_\alpha^2}{c^2} \right) - 1 \right], \quad (2.25)$$

where $\sigma = a/|a|$. The gauge transformation laws of these canonical coordinates are not simple:

$$\begin{aligned}\delta p_\alpha &= 0, \\ \delta \alpha &= -4\epsilon\sigma \frac{p_\alpha}{c^2}, \\ \delta p_c &= -4\epsilon\sigma \frac{p_\alpha^2}{c^3}, \\ \delta c &= \epsilon\sigma p_c.\end{aligned}\tag{2.26}$$

Let us remark that these canonical variables become useless in the case of non vanishing cosmological constant.

3. Quantization.

There are two approaches to the quantization of gauge systems [7]. The first is the Dirac method that leads in our case to the WDW equation and needs gauge fixing before being interpreted. This method has the problem of the choice of the measure and the related problem of the representation of the operators. This difficulty is usually overcome by the definition of an invariant measure in superspace.

The second approach is the canonical gauge fixing method leading to a classical reduced phase space where quantization can be carried on as usual and wave functions have the customary interpretation.

In our treatment of the quantization of the black hole one may carry on both methods and we will be able to show that they lead to the same results for correct gauge fixing conditions, thus proving the equivalence of the two approaches. Most of this section is dedicated to the Dirac method, devoting the final part to the discussion of the canonical gauge fixing.

In order to implement the Dirac procedure, the first main problem we meet is the choice of the measure in superspace and, as consequence, the choice of the variables to be used for the wave functions. We start with the formal commutation relations

$$[a, p_a] = i, \tag{3.1a}$$

$$[b, p_b] = i. \tag{3.1b}$$

In order to represent them as differential operators we must first choose a pair of commuting variables as coordinates and establish the form of the (non gauge fixed) measure $d\mu$. The measure $d\mu$ can be determined by the requirement that it be invariant under the symmetry transformations of H , namely rigid and gauge transformations.

We shall see in section 4 that the wave functions obtained with this measure are connected by a Fourier transform to the solutions of the WDW equation that uses the covariant measure in the a, b space.

Let us come back to the algebra of I , J , N and H . This is a powerful inspiration for physical consequences to be found in the structure of the gauge fixed positive definite Hilbert space. The I , J , N algebra is a dilatation algebra, so it is useful to recall some important points about the self-adjointness of the dilatation operator [9].

Let us consider a realization of the dilatation algebra on differentiable functions of a single variable ξ . If the support of the eigenvalues of both $\hat{\xi}$ and \hat{p}_ξ extends from $-\infty$ to ∞ , then $\hat{\xi}$ and \hat{p}_ξ are self-adjoint while the dilatation operator $\hat{D} = (\xi p_\xi + p_\xi \xi)/2$ is not self-adjoint. If instead for instance $\xi \geq 0$, then (as typical for radial variables) the dilatation generator is self-adjoint and the conjugate momentum \hat{p}_ξ is not. So we expect that the support of the variables in the present problem will be the key to the properties of the Hilbert space.

In order to determine the quantum measure, we require that the measure be invariant under rigid and gauge transformations. We choose the \mathcal{N}_g and \mathcal{I}_f form of the rigid symmetries, (2.14,16), because they preserve the sign of b . Then the measure is (we denote by x , j , y the continuous eigenvalues of \hat{I} , \hat{J} , \hat{Y}):

$$d\mu(x, y) = \frac{dx}{x} dy. \quad (3.2)$$

This measure makes sense as we have seen that classically $I > 0$ since both $p_a, b > 0$. We cannot use the \mathcal{N}_g and \mathcal{J}_q form of the rigid symmetries, as they change the sign of b (and of I). The choice of implementing the rigid symmetries \mathcal{N}_g , \mathcal{J}_q implies that b becomes negative, for which there is no basis. In that case the invariant measure would be

$$d\mu(j, y) = \frac{dj}{j} dy, \quad (3.3)$$

that requires $j > 0$. Of course one could argue that $j > 0$ because we have to exclude negative masses, but this choice would introduce an external criterion into the discussion. As we will see in a moment, the measure (3.2) implies that the operator \hat{J} is not self-adjoint.

The measure (3.2) can be obtained through different considerations, i.e. using as variables the pair $\{b, p_a\}$ whose behaviour is simple under both rigid and gauge transformations. This pair of non conjugate variables is a basis for a representation of the gauge group and therefore b and p_a are good candidates as coordinates in the wave functions. It is straightforward to determine the form of the invariant measure in this representation. Let

$$d\mu(b, p_a) = F(b, p_a) db dp_a; \quad d\bar{\mu}(\bar{b}, \bar{p}_a) = F(\bar{b}, \bar{p}_a) d\bar{b} d\bar{p}_a. \quad (3.4)$$

We have:

$$d\bar{\mu} \approx d\mu (1 + \Delta + f_b \delta b + f_{p_a} \delta p_a) db dp_a, \quad (3.5)$$

where

$$f = \ln F, \quad f_k = \partial_k f, \quad \frac{\partial(\bar{b}, \bar{p}_a)}{\partial(b, p_a)} - 1 \approx \Delta, \quad (3.6)$$

and

$$\Delta = 3g + \frac{hp_a}{2b^2}. \quad (3.7)$$

The condition of invariance determines completely F :

$$F = \frac{b}{p_a^2}. \quad (3.8)$$

The measure invariant under the continuous transformations $\mathcal{N}_g, \mathcal{I}_f$ that leave H invariant is thus

$$d\mu(b, p_a) = \frac{bdb dp_a}{p_a^2}. \quad (3.9)$$

It is immediate to see that it coincides with (3.2).

Let us consider for a moment the set of rigid transformations \mathcal{N}_g and \mathcal{J}_q . In spite of the simple transformation properties of b, p_a under them, it is easy to see by the above method that an invariant measure of the form (3.4) cannot be determined. Furthermore, the measure (3.3) is invariant under $\mathcal{J}_q, \mathcal{N}_g$ and \mathcal{H}_h but cannot be transformed back to the canonical variables $\{b, p_a\}$.

So let us go back to the measure (3.9) or (3.2). We can define the operators $\hat{H}, \hat{N}, \hat{J}$ both in the $\{b, p_a\}$ and in the $\{x, y\}$ representation. Using the first pair of coordinates we have the hermitian operators

$$\hat{a} = i p_a \partial_{p_a} p_a^{-1}, \quad (3.10a)$$

$$\hat{p}_b = -i b^{-1/2} \partial_b b^{1/2}. \quad (3.10b)$$

Note that \hat{H} is first order in derivatives, as well as \hat{N} and \hat{J} . Using the Weyl ordering we obtain:

$$\hat{H} = -i \frac{p_a}{2b^2} (b\partial_b + p_a\partial_{p_a}) - \frac{1}{2}, \quad (3.11a)$$

$$\hat{N} = -i (b\partial_b + 2p_a\partial_{p_a}), \quad (3.11b)$$

$$\hat{J} = -i \frac{p_a}{b} \left(b\partial_b + 2p_a\partial_{p_a} + \frac{1}{2} \right). \quad (3.11c)$$

Let us first discuss the eigenfunctions of \hat{N} . The solution of

$$\hat{H}\Psi = 0, \quad \hat{N}\Psi = \nu\Psi, \quad (3.12)$$

is

$$\Psi_\nu(b, p_a) = c(\nu) b^{-i\nu} p_a^{i\nu} e^{ib^2/p_a}, \quad (3.13a)$$

or, in terms of x, y :

$$\Psi_\nu(x, y) = c(\nu) x^{-i\nu} e^{iy/2}. \quad (3.13b)$$

The eigenfunctions of the mass operator \hat{J} are the solution of the equations:

$$\hat{H}\Psi = 0, \quad \hat{J}\Psi = j\Psi, \quad (3.14)$$

namely,

$$\Psi_j(b, p_a) = c(j) \sqrt{\frac{b}{p_a}} e^{ib(b-j)/p_a}, \quad (3.15a)$$

or, in the $\{x, y\}$ representation:

$$\Psi_j(x, y) = c(j) \sqrt{x} e^{i(y/2 - jx)}. \quad (3.15b)$$

For sake of completeness, let us obtain from the differential representation (3.11) the form of the operators \hat{H} , \hat{J} , \hat{N} in the $\{x, y\}$ representation:

$$\hat{H} = P_y = -i\partial_y - \frac{1}{2}, \quad (3.16a)$$

$$\hat{J} = i\sqrt{x}\partial_x \frac{1}{\sqrt{x}}, \quad (3.16b)$$

$$\hat{N} = i\frac{\partial}{\partial \ln x}, \quad (3.16c)$$

Now in order to progress we have to introduce the gauge fixing via the FP method [7]. We will prove that there is a class of viable gauges for which there are no Gribov copies and the FP determinant Δ_{FP} is invariant under gauge transformations. Indeed, let us suppose that the gauge be enforced by

$$\Phi(x, y) = 0, \quad (3.17)$$

and let Φ have the form

$$\Phi(x, y) = \psi(x, y) \prod_i (y - \phi_i(x)), \quad (3.18)$$

where $\psi(x, \phi_i(x)) \neq 0$ and $\phi_i(x) \neq \phi_j(x)$ for any x . Then

$$\delta(\Phi) = \sum_i \delta(y - \phi_i(x)) \psi_i(x), \quad (3.19)$$

where

$$\psi_i(x) = \psi(x, \phi_i(x)) \prod_{j \neq i} (\phi_i(x) - \phi_j(x)). \quad (3.20)$$

So, finally,

$$\Delta_{FP}^{-1} = \int dh \delta(\Phi(h)) = \sum_i (\psi_i(x))^{-1}. \quad (3.21)$$

Note that since x is gauge invariant, so is Δ_{FP} . The gauge fixed invariant measure is then

$$\int d\mu(x, y) \delta(\Phi(x, y)) \Delta_{FP} = \int \frac{dx}{x} dy \delta(\Phi(x, y)) \Delta_{FP}. \quad (3.22)$$

In our case the most convenient gauge (3.17) is:

$$\Phi(x, y) = y - 1 = \frac{2b^2}{p_a} - 1. \quad (3.23)$$

This gauge fixing implies obviously $\Delta_{FP} = 1$ and it determines uniquely the gauge. Indeed,

$$\frac{2\bar{b}^2}{\bar{p}_a} = 1 \quad (3.24)$$

defines uniquely $h = 1 - 2b^2/p_a$.

Now we may discuss the form of the wave functions in the gauge (3.23). Denote by lower case greek letters the wave functions in the gauge fixed representation and start from the eigenfunctions of \hat{N} . Choosing $c(\nu) = (2\pi)^{-1/2}$, the gauge fixed eigenfunctions of \hat{N} are

$$\psi_\nu(x) = \frac{1}{\sqrt{2\pi}} x^{-i\nu}. \quad (3.25)$$

They are of course orthonormal in the gauge fixed measure:

$$(\psi_{\nu_2}, \psi_{\nu_1}) = \int_0^\infty \frac{dx}{x} \psi_{\nu_2}^*(x) \psi_{\nu_1}(x) = \delta(\nu_1 - \nu_2). \quad (3.26)$$

Now consider the gauge fixed eigenfunctions of \hat{J} :

$$\psi_j(x) = c'(j) \sqrt{x} e^{-ixj}. \quad (3.27)$$

This makes clear the important point already stressed. It is indeed immediate to verify that \hat{J} is not self-adjoint in that space. As already remarked, the situation is similar to the familiar case of the radial coordinate r in flat space: its conjugate p_r is not a self-adjoint operator on the Hilbert space of the Laplace operator, although it is of course a well defined classical quantity.

If, as it is suggested by the classical correspondence, we identify \hat{J} with the mass operator, we must conclude that there is no self-adjoint mass operator in

this reduced theory. In other words, with this definition the mass operator is not an observable.

To conclude this section, let us now investigate the operator \hat{J}^2 . In order to be a self-adjoint operator, the eigenfunctions of \hat{J}^2 with eigenvalue j^2 must meet one of the two conditions:

$$\lim_{x \rightarrow 0} \frac{\psi_{j^2}^{(1)}(x)}{\sqrt{x}} = 0, \quad (3.28a)$$

or

$$\lim_{x \rightarrow 0} \left[\frac{\psi_{j^2}^{(2)}(x)}{\sqrt{x}} \right]' = 0. \quad (3.28b)$$

The two separate sets are given of course by ($j > 0$)

$$\psi_{j^2}^{(1)}(x) = \frac{1}{\sqrt{\pi j}} \sqrt{x} \sin jx, \quad (3.29a)$$

$$\psi_{j^2}^{(2)}(x) = \frac{1}{\sqrt{\pi j}} \sqrt{x} \cos jx. \quad (3.29b)$$

Either the set (3.29a) or the set (3.29b) must be chosen. The eigenfunctions of each set are orthonormal

$$\left(\psi_{j_2^2}^{(k)}, \psi_{j_1^2}^{(k)} \right) = \int_0^\infty \frac{dx}{x} \psi_{j_2^2}^{(k)*}(x) \psi_{j_1^2}^{(k)}(x) = \delta(j_2^2 - j_1^2), \quad k = 1, 2. \quad (3.30)$$

Thus the operator \hat{J}^2 is self-adjoint. The effect of the non self-adjoint operator \hat{J} is to transform the set (1) into the set (2) and viceversa.

The same results can be obtained by the canonical gauge fixing method (see [8]) using the gauge fixing condition

$$Y = r. \quad (3.31)$$

This gauge fixing (3.31) corresponds to the “area gauge” $b = \text{const} \cdot r$ since $Y = 2bI$. Indeed, we have $l = 1$. The effective Hamiltonian on the physical shell is

$$H_{\text{eff}} = -P_Y \equiv -H = 0. \quad (3.32)$$

So the functions do not depend on r . Diagonalizing \hat{N} or \hat{J} using (3.16b,c) one obtains the gauge fixed wave functions (3.25) and (3.27). This proves the equivalence of the Dirac-WDW and reduced canonical quantization methods for the gauge fixings that we have implemented.

4. WDW solutions in the $\{\mathbf{a}, \mathbf{b}\}$ representation.

We now follow the traditional path of determining the measure by defining the kinetic part of the Hamiltonian as a Laplace-Beltrami operator. We use the couple of variables a, b . From (2.4) we read the covariant measure in superspace

$$d\mu(a, b) = b da db. \quad (4.1)$$

The representation for \hat{p}_a and \hat{p}_b is thus:

$$\hat{p}_a = -i\partial_a, \quad \hat{p}_b = -i(\partial_b + 1/2b). \quad (4.2)$$

In the $\{\alpha, c\}$ representation the covariant measure is $c \, d\alpha dc$ and we have:

$$\hat{p}_\alpha = -i\partial_\alpha, \quad \hat{p}_c = -i(\partial_c + 1/2c). \quad (4.3)$$

Using the covariant Laplace–Beltrami ordering for the Hamiltonian (note that it coincides with the Weyl ordering) the WDW equation becomes:

$$[ab\partial_a\partial_b - (a\partial_a)^2 + ab^2]\Psi = 0, \quad (4.4)$$

or, in terms of α and c :

$$[-(c\partial_c)^2 + 4\partial_\alpha^2 - \sigma c^2]\Psi = 0, \quad (4.5)$$

where $\sigma = a/|a|$. The representations for the operators \hat{J} and \hat{N} are

$$\hat{J} = \partial_a\partial_b + 2b - \frac{1}{2b}\partial_a, \quad (4.6)$$

$$\hat{N} = -i(b\partial_b - 2a\partial_a). \quad (4.7)$$

It is easy to check that, using a different definition of the Lagrange multiplier, the WDW differential equation (4.4) and the differential expressions for \hat{J} and \hat{N} (4.6,7) remain unchanged.

Now let us discuss the diagonalization of \hat{N} . We have to discuss separately the cases $a > 0$ and $a < 0$. The solutions are:

$$\Psi_\nu(a, b) = c(\nu)(-a)^{-i\nu/2} K_{i\nu}(2b\sqrt{-a}), \quad (4.8a)$$

for $a < 0$, where $K_{i\nu}$ is the modified Bessel function of order $i\nu$ [10] (we have chosen this solution because of its asymptotic properties for large argument). For $a > 0$, we have

$$\Psi_\nu(a, b) = c'(\nu)a^{-i\nu/2} C_{i\nu}(2b\sqrt{a}), \quad (4.8b)$$

where the function $C_{i\nu}$ is any combination of Hankel functions. For the \hat{J} operator, the solutions with eigenvalue j are

$$\Psi_j(a, b) = \frac{K(j)}{\sqrt{|b-j|}} e^{\pm 2i\sqrt{ab(b-j)}}, \quad (4.9a)$$

in the classically allowed region ($a(b-j) > 0$, oscillating behavior), and

$$\Psi_j(a, b) = \frac{K(j)}{\sqrt{|b-j|}} e^{-2\sqrt{ab(j-b)}}, \quad (4.9b)$$

in the classically forbidden region $a(b - j) < 0$, where we have chosen the decreasing exponential behavior, analogously to (4.8a).

Now we may see that these solutions are the Fourier transforms of the solutions in the $\{b, p_a\}$ space obtained in the previous section. The Fourier transform is defined as:

$$\Psi(a, b) = \int_0^\infty \frac{dp_a}{p_a^2} \Psi(p_a, b) p_a e^{iap_a} . \quad (4.10)$$

Introducing in (4.10) $\Psi_\nu(p_a, b)$ given in (3.13a) and using Ref. [11] (Vol. I, p. 313, formula (17)), one obtains (4.8a); (4.8b) is obtained by elementary analytic continuations. Analogously, introducing (3.15a) one obtains (4.9a) or (4.9b). This proves the equivalence of the invariant measure (3.9) and of the representations (3.10) with the covariant measure (4.1) and representation (4.2).

Possibly there is no gauge fixing in the Dirac method leading to a positive definite Hilbert space of states $\Psi(a, b)$, analogously to what happens in the very similar Klein–Gordon (KG) case. Note that if we use the KG light cone variables p_+ and x_+/p_+ ($p_+ = p_0 + p$, $x_+ = x_0 + x$, see below Eqs. (4.11,12)), we may apply the procedures of the previous section to the KG case. This fact follows from the canonical equivalence of the classical black hole to the classical KG theory. This equivalence will be discussed in a forthcoming publication.

Let us now discuss the gauge fixing by the canonical method, i.e. by a canonical identity and quantizing in the reduced phase space. In connection with this it will be interesting to recall a few interesting facts about the KG theory. Let us consider the relativistic particle in two dimensions. The Hamiltonian is

$$\mathcal{H} = l(t)H , \quad H = \frac{1}{2}(p^2 + m^2 - p_0^2) . \quad (4.11)$$

The equations of motion are:

$$\dot{x} = lp , \quad \dot{p} = 0 , \quad \dot{x}_0 = -lp_0 , \quad \dot{p}_0 = 0 ; \quad (4.12a)$$

$$H = 0 . \quad (4.12b)$$

The gauge can be fixed via the canonical method imposing the identity

$$x_0 + t = 0 . \quad (4.13)$$

As a consequence, (4.12b) and (4.13) become second class and the system can be reduced. Eq. (4.13) gives the Lagrangian multiplier $l = 1/p_0$. Using (4.12b) and (4.13) we find

$$H_{\text{eff}} = p_0 = \pm \sqrt{p^2 + m^2} . \quad (4.14)$$

In order to have a positive Lagrange multiplier and a sensible quantum mechanics of a single particle, we have to choose the positive sign in (4.14). So we end with the reduced space Hamiltonian

$$H_{\text{eff}} = \sqrt{p^2 + m^2} . \quad (4.15)$$

This is the gauge fixed Hamiltonian of the relativistic particle. The choice of the $-$ sign in (4.14) would be wrong in quantum mechanics of a single particle (see e.g. [12]). The Schrödinger equation is ($+$ stands for $l > 0$)

$$i \frac{\partial}{\partial t} \psi_+(x, t) = \hat{H}_{\text{eff}} \psi_+, \quad (4.16)$$

and the eigenfunctions of \hat{p} are

$$\psi_+(k; x, t) = (2\pi)^{-1/2} \exp[-it\omega + ikx], \quad (4.17)$$

where of course $\omega = \sqrt{k^2 + m^2}$. This is obviously a positive definite Hilbert space since the Hamiltonian is positive definite and hermitian. Usual quantum mechanics applies.

Now go back to our problem and discuss the canonical gauge fixing for the black hole. The discussion parallels that of the KG, as (4.5) is essentially a KG system in the $\{\alpha, c\}$ representation.

Using the variables α and c and the Hamiltonian (2.25), the equations of motion are

$$\dot{c} = l\sigma p_c, \quad \dot{\alpha} = -4l\sigma p_\alpha/c^2 \quad \dot{p}_\alpha = 0, \quad \dot{p}_c = -4l\sigma p_\alpha^2/c^3; \quad (4.18a)$$

$$H = 0. \quad (4.18b)$$

It is convenient to choose the gauge fixing canonical identity (analogous to (4.13)):

$$\alpha = r, \quad (4.19)$$

(of course with the gauge above r is not the area coordinate). The effective Hamiltonian is thus:

$$H_{\text{eff}} = -p_\alpha = N/2, \quad (4.20)$$

where p_α can be obtained from $H = 0$. Hence

$$H_{\text{eff}} = \pm \frac{1}{2} \sqrt{c^2(p_c^2 - \sigma)}. \quad (4.21)$$

Note that in the classical motion the argument in the square root never becomes negative. This is obvious for $a < 0$. For $a > 0$ it can be seen as follows: from (2.13c,d) we have the relation $p_b = a/I + I$ and using the definition of p_c in Eqs. (2.24) it follows that $p_c^2 = p_b^2/4a = (a/I + I)^2/4a \geq 1$.

Let us look at the value of the Lagrange multiplier. From (4.19) and from the equations of motion (4.18) we have

$$l = -\frac{c^2 \sigma}{4p_\alpha}. \quad (4.22)$$

Now, as in the KG case, we impose that $l > 0$, that is $\sigma p_\alpha < 0$. This means that for $a > 0$ we must choose the $+$ sign in (4.21), while for $a < 0$ we have to choose the $-$ sign. Let us use (4.3) and the covariant ordering. First discuss $a < 0$. The eigenstates of H_{eff} with eigenvalue $E = -\nu/2$, $\nu > 0$ are obtained by solving the equation

$$[-(c\partial_c)^2 + c^2] \psi_\nu(c) = \nu^2 \psi_\nu(c). \quad (4.23)$$

The solution is

$$\psi_\nu(c) = \sqrt{\frac{2\nu \sinh \pi\nu}{\pi^2}} K_{i\nu}(c). \quad (4.24)$$

For the case $a > 0$ we look for eigenstates of H_{eff}

$$[-(c\partial_c)^2 - c^2] \chi_\nu(c) = \nu^2 \chi_\nu(c), \quad (4.25)$$

with solution ($\nu > 0$)

$$\chi_\nu(c) = i \sqrt{\frac{\nu \sinh(\pi\nu/2)}{4 \cosh(\pi\nu/2)}} \left[e^{-\pi\nu/2} H_{i\nu}^{(1)}(c) - e^{\pi\nu/2} H_{i\nu}^{(2)}(c) \right]. \quad (4.26)$$

The above solutions of the Hamiltonian are orthonormal (see Appendix A):

$$(\psi_{\nu_1}, \psi_{\nu_2}) = \int_0^\infty \frac{dc}{c} \psi_{\nu_1}^*(c) \psi_{\nu_2}(c) = \delta(\nu_1 - \nu_2), \quad (4.27a)$$

$$(\chi_{\nu_1}, \chi_{\nu_2}) = \int_0^\infty \frac{dc}{c} \chi_{\nu_1}^*(c) \chi_{\nu_2}(c) = \delta(\nu_1 - \nu_2). \quad (4.27b)$$

These eigenfunctions span positive norm Hilbert spaces.

Now let us solve the Schrödinger equation

$$i \frac{\partial}{\partial \alpha} \Psi_+(\nu; c, \alpha) = H_{\text{eff}} \Psi_+(\nu; c, \alpha) \quad (4.28)$$

for the stationary states. We have (remember $\nu > 0$)

$$\Psi_+(\nu; c, \alpha) = e^{i\alpha\nu/2} \psi_\nu(c) \quad (4.29a)$$

for $E < 0$, $a < 0$, and

$$\Psi_+(\nu; c, \alpha) = e^{-i\alpha\nu/2} \chi_\nu(c) \quad (4.29b)$$

for $E > 0$, $a > 0$. On the other hand the solutions corresponding to $l < 0$ are

$$\Psi_-(\nu; c, \alpha) = e^{-i\alpha\nu/2} \psi_\nu(c) \quad (4.30a)$$

for $E > 0$, $a < 0$, and

$$\Psi_{-}(\nu; c, \alpha) = e^{i\alpha\nu/2} \chi_{\nu}(c) \quad (4.30b)$$

for $E < 0$, $a > 0$. Solutions (4.29-30) are the gauge-fixed wave functions correspondent of (4.8). Analogously to the KG case the use of both positive and negative l is appropriate if one reinterprets the wave function as a quantum operator (second quantization of BH). For instance,

$$\Psi_{\text{BH}}(\alpha, c) = \int_0^{\infty} d\nu \sqrt{\frac{2\nu \sinh \pi\nu}{\pi^2}} K_{i\nu}(c) [A^{\dagger}(\nu)e^{-i\nu\alpha/2} + B(\nu)e^{i\nu\alpha/2}] \quad (4.31)$$

is the representation of the BH quantum field for $a < 0$.

5. Conclusions.

The quantization of the canonical approach to the black hole proposed in I shows that, as a consequence of the positive definiteness of the canonical variable b , \hat{J} does not have a self-adjoint extension since its conjugate variable I has positive support. Instead, eigenfunctions of \hat{J}^2 can be defined in the Hilbert space.

This possibly signals that the identification of J with the mass carried at the classical level is not the correct one in the quantum formulation. Alternatively, this may have something to do with the fact that in classical physics only positive masses are present. To look into this question in the present frame one has to construct a procedure of classical limit that yields the Schwarzschild metric and investigate the role of eigenfunctions of \hat{J}^2 . Maybe some light could come.

Another subject that must be explored is the introduction of matter fields. This could be of importance in order to specify the physical degrees of freedom inaccessible for observation by an external observer, whose tracing out could explain the origin of the black hole entropy (see e.g. [13]). Hopefully, this may also shed light on the quantum definition of the mass of the black hole.

The set of solutions of wormholes for the KS metric coincides with the set of Schwarzschild wave functions inside the black hole, as the KS geometry coincides with the internal one of the black hole, and the parameter r in which we foliate is timelike there.

Let us also remark that no quantization of the mass appears from this theory. It is interesting to stress though that in the frame developed here quantization of the mass squared could be achieved in a gauge invariant way by a modification of the theory. For instance a very crude way is just to set the support condition $x < x_0$. This is a gauge invariant cut-off that leads to quantization of the eigenvalues of \hat{J}^2 . Now, this cut-off is performed in the gauge $y = 1$, that is $2bx = 1$. Thus a modification of the theory for large x corresponds to a gauge invariant modification for small b . It will be interesting to explore the consequences of less crude models leading to quantization of the mass; this requires a reliable definition of the quantum mass operator of course.

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Appendix A.

In this appendix we discuss the orthonormality of the eigenfunctions of \hat{N} that we used in section 4 (Eqs. (4.27)). Let us start considering $a < 0$. Using

$$\int_0^\infty \frac{dx}{x} K_{i\mu}(x) K_{i\nu}(x) = \frac{\pi^2}{2\mu \sinh \pi\mu} [\delta(\mu - \nu) + \delta(\mu + \nu)] , \quad (\text{A.1})$$

(see [14]) and recalling that ν is positive, we obtain (4.27a).

Let us discuss now in detail the case $a > 0$. The most general solution of the Eq. (4.25) has the form

$$\chi_\nu(\alpha, c) = \lambda_1 H_{i\nu}^{(1)}(c) + \lambda_2 H_{i\nu}^{(2)}(c) , \quad (\text{A.2})$$

where $\lambda_{1,2}$ have to be determined by orthonormality. We have to compute the integrals:

$$I^{(k,l)}(\mu, \nu) = \int_0^\infty \frac{dx}{x} H_{i\mu}^{(k)}(x) H_{i\nu}^{(l)}(x) , \quad (\text{A.3})$$

where $k, l = 1, 2$. From Bateman (see Ref. [11], Vol. I, p. 333, formulae (40) and (48)) we have the relation:

$$\int_0^\infty \frac{dx}{x} H_\mu^{(k)}(x) H_\nu^{(k)}(x) = -\frac{4}{\pi^2} e^{i\pi(2k-3)(\mu+\nu)/2} \int_0^\infty \frac{dx}{x} K_\mu(x) K_\nu(x) , \quad (\text{A.4})$$

where $k = 1, 2$. Setting $\mu \rightarrow i\mu$, $\nu \rightarrow i\nu$ in (A.4) and using (A.1) we obtain

$$I^{(1,1)}(\mu, \nu) = -\frac{2}{\mu \sinh \pi\mu} e^{\pi(\mu+\nu)/2} [\delta(\mu - \nu) + \delta(\mu + \nu)] , \quad (\text{A.5a})$$

$$I^{(2,2)}(\mu, \nu) = -\frac{2}{\mu \sinh \pi\mu} e^{-\pi(\mu+\nu)/2} [\delta(\mu - \nu) + \delta(\mu + \nu)] . \quad (\text{A.5b})$$

Now, let us calculate $I^{(1,2)}$. In order to do this we have to compute $\int_0^\infty J_{i\mu} J_{i\nu} dx/x$ and $\int_0^\infty J_{i\mu} Y_{i\nu} dx/x$. These integrals can be easily calculated using Bateman (see Ref. [11], Vol. I, p. 331-332, formulae (33) and (36)) and suitable analytic continuations. We have:

$$\int_0^\infty \frac{dx}{x} J_{i\mu} J_{i\nu} = -\frac{2i}{\pi} P \frac{\sinh [\pi(\mu - \nu)/2]}{\mu^2 - \nu^2} + \frac{1}{\mu} \sinh \pi\mu \delta(\mu + \nu) , \quad (\text{A.6a})$$

$$\begin{aligned} \int_0^\infty \frac{dx}{x} J_{i\mu} Y_{i\nu} &= \frac{2}{\pi} P \frac{\cosh [\pi(\mu - \nu)/2]}{\mu^2 - \nu^2} + \frac{i}{\mu} \delta(\mu - \nu) + \\ &+ \frac{i}{\mu} \cosh \pi\mu \delta(\mu + \nu) . \end{aligned} \quad (\text{A.6b})$$

Hence, using (A.5-6) we find:

$$I^{(1,2)}(\mu, \nu) = -\frac{4i}{\pi} P \frac{e^{\pi(\mu-\nu)/2}}{\mu^2 - \nu^2} + \frac{2}{\mu} \coth \pi \mu [\delta(\mu - \nu) + e^{\pi \mu} \delta(\mu + \nu)] . \quad (\text{A.7})$$

Now, we can calculate $\lambda_{1,2}$ imposing the inner product (4.27b). We have two sets of real orthonormal functions:

$$\chi_\nu^{(1)} = \sqrt{\frac{\nu \cosh(\pi\nu/2)}{4 \sinh(\pi\nu/2)}} \left[e^{-\pi\nu/2} H_{i\nu}^{(1)}(c) + e^{\pi\nu/2} H_{i\nu}^{(2)}(c) \right] , \quad (\text{A.8a})$$

$$\chi_\nu^{(2)} = i \sqrt{\frac{\nu \sinh(\pi\nu/2)}{4 \cosh(\pi\nu/2)}} \left[e^{-\pi\nu/2} H_{i\nu}^{(1)}(c) - e^{\pi\nu/2} H_{i\nu}^{(2)}(c) \right] . \quad (\text{A.8b})$$

In (4.26) we have chosen the set (A.8b) because it has the same properties as (4.24), i.e. the wave functions vanish for $\nu \rightarrow 0$. Also, the asymptotic behaviors for $c \rightarrow 0$ of (4.24) and (A.8b) are identical.

Appendix B.

In this Appendix we collect the main formulae of sections 2–4 when the cosmological constant Λ is different from zero.

The system described by the Hamiltonian (2.7b) with $\Lambda \neq 0$ (hereafter denoted \tilde{H} to distinguish it from the Hamiltonian H of the previous sections) is again completely integrable. The finite gauge transformations are in this case:

$$\begin{aligned} b &\rightarrow \bar{b} = b + h(r) \frac{p_a}{2b} , \\ p_a &\rightarrow \bar{p}_a = p_a + h(r) \frac{p_a^2}{2b^2} , \\ a &\rightarrow \bar{a} = a + \frac{\tilde{N}}{b^2} \frac{h(r)/2}{1 + h(r)p_a/2b^2} + \frac{\Lambda b^2}{3 p_a} h(r) \left(1 + \frac{h(r)p_a}{4b^2} \right) , \\ p_b &\rightarrow \bar{p}_b = p_b + \frac{\tilde{J}}{b^2} \frac{h(r)/2}{1 + h(r)p_a/2b^2} + \frac{4\Lambda}{3} b h(r) \left(1 + \frac{h(r)p_a}{4b^2} \right) , \\ l(r) &\rightarrow \bar{l}(r) = l(r) + \frac{dh}{dr} , \end{aligned} \quad (\text{B.1})$$

where we have used the constraint $\tilde{H} = 0$. The gauge invariant quantities \tilde{J} and \tilde{N} are defined as:

$$\tilde{J} = J - \frac{2\Lambda}{3} b^3 = 2b - p_a p_b + 4b\tilde{H} + \frac{4}{3}\Lambda b^3 , \quad (\text{B.2a})$$

$$\tilde{N} = N - \frac{2\Lambda}{3} \frac{b^4}{p_a} = b p_b - 2a p_a - \frac{2\Lambda}{3} \frac{b^4}{p_a} . \quad (\text{B.2b})$$

The quantities (B.2a,b), together with

$$\tilde{I} = \tilde{N}\tilde{J}^{-1} = \frac{b}{p_a}, \quad (\text{B.2c})$$

satisfy the algebra (2.12). Note that the gauge transformations for b and p_a are unaffected by the presence of the cosmological constant.

The rigid transformations generated by \tilde{I} are identical to the ones generated by I when $\Lambda = 0$; the finite transformations generated by the dilatation operator \tilde{N} are instead:

$$\begin{aligned} b &\rightarrow \bar{b} = e^g b, \\ p_a &\rightarrow \bar{p}_a = e^{2g} p_a, \\ a &\rightarrow \bar{a} = e^{-2g} a + \frac{\Lambda}{3} \frac{b^4}{p_a^2} (1 - e^{-2g}), \\ p_b &\rightarrow \bar{p}_b = e^{-g} p_b + \frac{8\Lambda}{3} \frac{b^3}{p_a} (1 - e^{-g}). \end{aligned} \quad (\text{B.3})$$

Note that the presence of the cosmological constant does not affect the rigid transformations of b and p_a , again as it happens for the gauge transformations (B.1). As a consequence, the discussion about the gauge and the rigid invariant measure of the section 3 is applicable, as well as the FP gauge fixing method described there.

For sake of completeness, let us write the gauge invariant relation between a and b :

$$a = \tilde{I}^2 \left(2\tilde{H} + 1 - \frac{\tilde{J}}{b} + \frac{\Lambda}{3} b^2 \right). \quad (\text{B.4})$$

Of course, \tilde{I} and \tilde{J} have the same physical meaning of I and J in (2.13c).

In the $\{b, p_a\}$ representation the solutions of the WDW equation, eigenfunctions of \tilde{N} , are:

$$\Psi_\nu(b, p_a) = c(\nu) \left(\frac{p_a}{b} \right)^{i\nu} \exp \left[i \frac{b^2}{p_a} \left(1 + \frac{\Lambda}{3} b^2 \right) \right]. \quad (\text{B.5})$$

The eigenfunctions of the mass operator \tilde{J} are instead:

$$\Psi_j(b, p_a) = c(j) \sqrt{\frac{b}{p_a}} \exp \left\{ i \frac{b}{p_a} \left[b \left(1 + \frac{\Lambda}{3} b^2 \right) - j \right] \right\}. \quad (\text{B.6})$$

In the $\{\tilde{J}, \tilde{I}, \tilde{Y} \equiv Y, \tilde{P}_Y \equiv \tilde{H}\}$ representation (B.5) reads:

$$\Psi_\nu(\tilde{x}, \tilde{y}) = c(\nu) \tilde{x}^{-i\nu} \exp \left[i \frac{\tilde{y}}{2} \left(1 + \frac{\Lambda}{12} \frac{\tilde{y}^2}{\tilde{x}^2} \right) \right]. \quad (\text{B.7})$$

Fixing the gauge $\Phi(\tilde{y}) = \tilde{y} - 1 = 0$, and using the measure (3.2) (where of course $x \rightarrow \tilde{x}$, $y \rightarrow \tilde{y}$), the orthonormal gauge fixed wave functions are then:

$$\psi_\nu(\tilde{x}) = \frac{1}{\sqrt{2\pi}} \tilde{x}^{-i\nu} \exp\left(i \frac{\Lambda}{24\tilde{x}^2}\right). \quad (\text{B.8})$$

To conclude this Appendix, let us find the eigenfunctions of \tilde{N} and \tilde{J} in the $\{a, b\}$ representation, using the Fourier transform (4.10). For the eigenfunctions of \tilde{N} , from (B.5) one obtains:

$$\Psi_\nu(a, b) = c(\nu) \left[\frac{|a|}{1 + \Lambda b^2/3} \right]^{-i\nu/2} B_{i\nu} \left(2b\sqrt{|a|(1 + \Lambda b^2/3)} \right), \quad (\text{B.9})$$

where $B_{i\nu} = K_{i\nu}$ for $a < 0$ and $B_{i\nu} = C_{i\nu}$ for $a > 0$. Analogously, from (B.6) the eigenfunctions of \tilde{J} are

$$\Psi_j(a, b) = \frac{K(j)}{\sqrt{|b(1 + \Lambda b^2/3) - j|}} \exp \left[\pm 2i\sqrt{ab[b(1 + \Lambda b^2/3) - j]} \right] \quad (\text{B.10a})$$

in the classically allowed region, and

$$\Psi_j(a, b) = \frac{K(j)}{\sqrt{|b(1 + \Lambda b^2/3) - j|}} \exp \left[-2\sqrt{ab[j - b(1 + \Lambda b^2/3)]} \right] \quad (\text{B.10b})$$

in the classically forbidden region. It is straightforward to verify that Eqs. (B.9,10) satisfy the WDW equation and are respectively eigenfunctions of \tilde{N} and \tilde{J} in the $\{a, b\}$ representation.

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