

# LINEAR FORM OF CANONICAL GRAVITY

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**Summary.** - Recent work in the literature has shown that general relativity can be formulated in terms of a jet bundle which, in local coordinates, has five entries: local coordinates on Lorentzian space-time, tetrads, connection one-forms, multivelocities corresponding to the tetrads and multivelocities corresponding to the connection one-forms. The derivatives of the Lagrangian with respect to the latter class of multivelocities give rise to a set of multimomenta which naturally occur in the constraint equations. Interestingly, all the constraint equations of general relativity are linear in terms of this class of multimomenta. This construction has been then extended to complex general relativity, where Lorentzian space-time is replaced by a four-complex-dimensional complex-Riemannian manifold. One then finds a holomorphic theory where the familiar constraint equations are replaced by a set of equations linear in the holomorphic multimomenta, providing such multimomenta vanish on a family of two-complex-dimensional surfaces. In quantum gravity, the problem arises to quantize a real or a holomorphic theory on the extended space where the multimomenta can be defined.

Recent work by the authors [1] has shown that a constraint analysis of general relativity can be performed starting from a one-jet bundle which, in local coordinates, is represented by  $(x^a, e^a_{\hat{b}}, \omega_a^{\hat{b}\hat{c}}, V^a_{b\hat{c}}, W_{ab}^{\hat{c}\hat{d}})$ . With our notation,  $x^a$  are local coordinates on space-time,  $e^a_{\hat{b}}$  are the tetrad vectors,  $\omega_a^{\hat{b}\hat{c}}$  the connection one-forms,  $V^a_{b\hat{c}}$  the multivelocities corresponding to  $e^a_{\hat{b}}$  and  $W_{ab}^{\hat{c}\hat{d}}$  the multivelocities corresponding to  $\omega_a^{\hat{b}\hat{c}}$ . In analogy with classical mechanics, where the momenta are the derivatives of the Lagrangian with respect to the velocities, one can here define the *multimomenta* (cf. ref. [1])

$$\tilde{\pi}_a^{\hat{b}\hat{c}} \equiv \frac{\partial L}{\partial V^a_{b\hat{c}}}, \quad (1)$$

and

$$\tilde{p}^{ab}_{\hat{c}\hat{d}} \equiv \frac{\partial L}{\partial W_{ab}^{\hat{c}\hat{d}}}. \quad (2)$$

In general relativity, the former vanish, while the latter take the form

$$\tilde{p}^{ab}_{\hat{c}\hat{d}} = \frac{\epsilon}{2} \left( e^a_{\hat{c}} e^b_{\hat{d}} - e^a_{\hat{d}} e^b_{\hat{c}} \right), \quad (3)$$

where  $\epsilon$  is the determinant of the tetrad. Note that the analogy with classical mechanics is more strict, if one thinks that in both cases one is working with a *finite* number of degrees of freedom [2]. However, since in field theory the base space is the whole space-time manifold, and jet bundles have as base space a fibre bundle over space-time, one has to introduce the multimomenta instead of the momenta. This leads to a fully covariant formalism [3].

The main result of ref. [1] was that, on any spacelike hypersurface  $\Sigma$ , the constraint equations are all linear in the multimomenta (3), and take the form (see ref. [1] for the notation)

$$\int_{\Sigma} \lambda^{\hat{c}\hat{d}} \left( D_a \tilde{p}^{ab} \right)_{\hat{c}\hat{d}} d^3 x_b = 0, \quad (4)$$

$$\int_{\Sigma} \text{Tr} \left[ \tilde{p}^{af} \Omega_{ad} - \frac{1}{2} \tilde{p}^{ab} \Omega_{ab} \delta^f_d \right] u^d d^3 x_f = 0, \quad (5)$$

providing the multimomenta (3) vanish on the boundary  $\partial\Sigma$  of  $\Sigma$ . The alternative possibility is to set to zero at the boundary the gauge parameters  $\lambda^{\hat{c}\hat{d}}$ , or the connection one-forms  $\omega_a^{\hat{b}\hat{c}}$ , jointly with the vector field  $u^a$  describing diffeomorphisms on space-time [1].

In complex general relativity, the five entries of the corresponding jet bundle (see above) are all of holomorphic nature [4], and it is incorrect to talk about a Cauchy problem, since the concepts of spacelike hypersurface and time evolution are meaningless. The basic *postulate* of our multisymplectic approach is instead that, on evaluating the holomorphic multimomentum map [4] on an arbitrary three-complex-dimensional surface  $\Sigma_c$ , and setting the resulting geometric object to zero, one gets all the basic equations of the theory, which correspond to the constraint equations of the Lorentzian theory. They are the holomorphic counterpart of these constraints, but cannot quite be called constraints themselves. The explicit calculation [4] shows indeed that, providing the holomorphic multimomenta vanish on the two-complex-dimensional boundary  $\partial\Sigma_c$  of  $\Sigma_c$ , the holomorphic equations still take the forms (4) and (5), where  $\Sigma$  is replaced by  $\Sigma_c$ , and all geometric objects have now a holomorphic nature [4].

The implications for twistor theory are discussed in ref. [4]. We are instead interested in the relevance of the multisymplectic scheme for quantum gravity. The basic problems and properties can be described as follows.

- (i) How to define suitable brackets for a space-time covariant analysis.
- (ii) How to build the counterpart of Dirac's map and Dirac's quantization scheme.
- (iii) What are the arguments of the state vectors, and how to define a Feynman path integral with our Lagrangian [1].
- (iv) Our holomorphic framework yields a theory *without time*, and the Lorentzian theory cannot be recovered by imposing suitable reality conditions. In our complex base space the complex conjugation of spinors is not invariant under holomorphic coordinate transformations [5], and hence cannot be defined. The unprimed and primed spin-spaces become then independent of each other, not related by any conjugation [5]. By contrast, in the Ashtekar

programme [6], one studies complex tetrads on a four-real-dimensional Lorentzian manifold. This hybrid scheme makes it possible, in principle, to recover real general relativity. However, it does not make full use of the holomorphic formalism (the base space remaining real Lorentzian), and it can be quite hard to find a suitable set of reality conditions.

(v) If a complex Ricci-flat space-time, not necessarily anti-self-dual, could be reconstructed out of a twistor space consisting of charges for massless spin-3/2 fields as suggested by Penrose (see chapter 5 of ref. [5] and references therein), one might try to relate the quantization of the classical multisymplectic formalism to the quantization of the twistor scheme.

These exciting problems are being investigated for the first time within the multimomenta formulation. The elegance of the mathematical formalism seems to suggest that one has to learn how to formulate physical laws and quantization schemes on the extended spaces where multimomenta can be defined, instead of the original spaces, where the constraint equations are quadratic in the momenta [6]. The results obtained in refs. [1,4], and the quantization programme discussed in this note, seem to point out that yet a new formalism is available for canonical gravity and the non-perturbative quantization of the gravitational field. Whether or not this can improve the current theories of classical and quantum gravity, will depend on a deeper understanding of the mathematical structures underlying Lagrangian field theory and spinor geometry.

## REFERENCES

- [1] ESPOSITO G., GIONTI G. and STORNAIOLO C., *Nuovo Cimento B*, **110** (1995).  
[GR-QC 9506008]
- [2] KIJOWSKI J., *Commun. Math. Phys.*, **30** (1973) 99.
- [3] GOTAY M. J., A Multisymplectic Framework for Classical Field Theory and the Calculus of Variations I: Covariant Hamiltonian Formalism, in: M. Francaviglia, ed.,

*Mechanics, Analysis and Geometry: 200 Years After Lagrange* (North Holland, Amsterdam, 1991) p. 203.

[4] ESPOSITO G. and STORNAIOLO C., *Class. Quantum Grav.*, **12** (1995) 1733.

[5] ESPOSITO G., *Complex General Relativity, Fundamental Theories of Physics*, Vol. 69 (Kluwer, Dordrecht) 1995.

[6] ASHTEKAR A., *Lectures on Non-Perturbative Canonical Gravity* (World Scientific, Singapore) 1991.