

Large-Order Behavior of Two-coupling Constant ϕ^4 -Theory with Cubic Anisotropy

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Abstract

For the anisotropic $[u(\sum_{i=1}^N \phi_i^2)^2 + v \sum_{i=1}^N \phi_i^4]$ -theory with $N = 2, 3$ we calculate the imaginary parts of the renormalization-group functions in the form of a series expansion in v , i. e., around the isotropic case. Dimensional regularization is used to evaluate the fluctuation determinants for the isotropic instanton near the space dimension 4. The vertex functions in the presence of instantons are renormalized with the help of a nonperturbative procedure introduced for the simple $g\phi^4$ -theory by McKane et al.

1 Introduction

More than twenty years ago, Brezin, Le Guillou and Zinn-Justin (BGZ) studied the phase transition of a cubic anisotropic system by means of renormalization group equations [1]. Within a $(4 - \varepsilon)$ -expansion, they found that to lowest nontrivial order in ε , the only stable fixed point for $N < N_c = 4$ is the $O(N)$ -symmetric one, where N is the number of field components appearing in the cubic anisotropic model. They interpreted this as an indication that the anisotropy is irrelevant as long as N is smaller than four. For $N > 4$, the isotropic fixed point destabilized and the trajectories crossed over to the cubic fixed point.

Recently, our knowledge of perturbation coefficients of the renormalization group functions of the anisotropic system was extended up to the five-loop level by Kleinert and Schulte-Frohlinde [2]. Since the perturbation expansions are badly divergent, they do not directly yield improved estimates for the crossover value N_c where the isotropic fixed point destabilizes in favor of the cubic one. An estimate using Padé approximants [2] indicates N_c to lie below 3, thus permitting real crystals to exhibit critical exponents of the cubic universality class.

For a simple ϕ^4 -theory, the Padé approximation is known to be inaccurate. In fact, the most accurate renormalization group functions for that theory have been obtained by combining perturbation expansions with large-order estimates and using a resummation procedure based on Borel-transformations [3]–[9].

It is the purpose of this paper to derive the large-order behavior of the renormalization group functions for the anisotropic $g\phi^4$ -theory. In a forthcoming paper we will combine these results with the five-loop perturbation expansion of Kleinert and Schulte-Frohlinde to derive the precise value for the crossover value N_c .

For the simple $g\phi^4$ -theory, the large-order behavior of perturbation coefficients has been derived by Lipatov [10, 11], BGZ [12] and others [13]–[15] in a number of papers. The generalization to the $O(N)$ -symmetric case was given in [12]. An equivalent method for calculating the large-order behavior is based on the observation that for a negative coupling constant Green functions possess an exponentially small imaginary part due to the fact that the ground state is unstable [16, 17]. The imaginary part is associated with the tunneling decay rate of the ground state. It determines directly the large-order behavior of the perturbation coefficients via a dispersion relation in the complex coupling constant plane.

In the semiclassical limit, the imaginary part of all Green functions can be calculated with the help of classical solutions called *instantons*. For a massless $g\phi^4$ -theory in $d = 4$ space dimensions, these instantons can be found analytically. The imaginary part is a consequence of a negative frequency mode in the spectrum of the fluctuation operator, whose determinant enters the one-loop correction to the instanton contribution. McKane, Wallace and de Alcantara Bonfim [18] found a way to continue the results of the $g\phi^4$ -theory in $d = 4$ to a field theory in $d = 4 - \varepsilon$ dimensions. They proposed an extended dimensional regularization scheme for nonperturbative renormaliz-

ing the imaginary parts of vertex functions.

In the present work we have to extend this scheme to the case of a ϕ^4 -theory with cubic anisotropy, where the energy functional has the following form:

$$H(\vec{\phi}) = \int d^d x \left[\frac{1}{2} \sum_{i=1}^N \phi_i (-\nabla^2) \phi_i + \frac{u}{4} \left(\sum_{i=1}^N \phi_i^2 \right)^2 + \frac{v}{4} \sum_{i=1}^N \phi_i^4 \right]. \quad (1)$$

For $N = 2$, the corresponding model in quantum mechanics was first studied by Banks, Bender and Wu [19, 20] who used multidimensional WKB-techniques to derive the large-order behavior of the perturbation expansion for the ground state energy. In 1990 Janke [21] presented a more efficient calculation using a path integral approach. In the present work, this approach will be generalized to quantum field theory and extended by a careful discussion of the region near the isotropic limit $v \rightarrow 0$. This is important, since the infrared-stable cubic fixed point is expected to appear very close to the $O(N)$ -symmetric one. In fact, it will be sufficient to give the quantum-field theoretical generalization of [21] in terms of an expansion about the isotropic case in powers of v .

The paper is organized as follows. The method is developed by treating first the case $N = 2$. In Section 2 we derive the Feynman rules for the power series expansion of all Green functions around the isotropic limit. In Section 3 we calculate the small-oscillation determinants for the transversal and longitudinal fluctuations. In Section 4 we use the extended renormalization scheme of [18] to find the full (real and imaginary) vertex functions, and derive the renormalization constants to one loop. In Section 5 we calculate the imaginary parts of the renormalization-group functions and thus the large-order behavior of the perturbation coefficients. In Section 6, finally, we extend the results to the physically relevant case $N = 3$.

2 Fluctuations around the isotropic instanton

For positive coupling constants u and v , the system defined by the energy functional (1) is stable and the Green functions are real. On the other hand, if both coupling constants are negative the system is unstable and the Green

functions acquire an imaginary part. The corresponding functional integrals can be calculated by an analytical continuation from positive to negative coupling constants, keeping the factor $\exp[\int(-\frac{u}{4}(\sum_{i=1}^N \phi_i^2)^2 - \frac{v}{4} \sum_{i=1}^N \phi_i^4) d^4x]$ real. We perform this analytical continuation by means of a joint rotation in the two complex planes, substituting $u \rightarrow u \exp(i\theta)$ and $v \rightarrow v \exp(i\theta)$, and rotating the azimuthal angle θ from 0 to π . At the same time, we rotate the contour of integration in the field space. The convergence of the functional integrals is maintained by the field rotation $\phi \rightarrow \phi \exp(-i\theta/4)$.

A natural parameter for the anisotropy of the system is the ratio $\delta = v/(u+v)$. The isotropic limit corresponds to $\delta = 0$. During the joint rotation of u and v , the parameter δ remains constant. Thus, δ is a good parameter for the anisotropy at both positive and negative couplings u and v . We shall use the coupling constants $g = u + v$ and δ for a calculation of the Green functions. These are given by the functional integrals

$$G^{(2M)}(x_1, x_2, \dots, x_{2M})_{i_1, i_2, \dots, i_{2M}} = \frac{\int D\phi \phi_{i_1}(x_1) \phi_{i_2}(x_2) \cdot \dots \cdot \phi_{i_{2M}}(x_{2M}) \exp\{-H[\phi]\}}{\int D\phi \exp\{-H[\phi]\}}, \quad (2)$$

where the subscripts i_k run through the N components of the field ϕ . As explained in the introduction, we shall first study the case $N = 2$ with the free energy functional

$$H[\phi_1, \phi_2] = \int d^d x \left\{ \frac{1}{2} [\phi_1(-\nabla^2)\phi_1 + \phi_2(-\nabla^2)\phi_2] + \frac{g}{4} [\phi_1^4 + 2(1-\delta)\phi_1^2\phi_2^2 + \phi_2^4] \right\}. \quad (3)$$

When expressed in terms of the coupling constants g and δ , the Green functions possess an imaginary part for $g < 0$. For the reasons explained above, we shall derive an expansion of the imaginary parts of the Green functions (2) around the isotropic case, i. e., in powers of δ :

$$\text{Im } G = \sum_{n=0}^{\infty} a_n \delta^n \left(\frac{A}{-g} \right)^{p(n)} \exp\left(\frac{A}{g} \right) [1 + \mathcal{O}(g)]. \quad (4)$$

Each power δ^n has its own n -dependent imaginary part. Given such an expansion, the large-order estimates for the coefficients of the powers g^k follows from a dispersion relation in g :

$$G = \frac{1}{\pi} \int_{-\infty}^0 d\bar{g} \frac{\text{Im} G(\bar{g} + i0)}{\bar{g} - g}. \quad (5)$$

For a general discussion of the relationship between imaginary parts and large-order behavior see, for example, ch. 17 of [22]. If the power-series expansion of G is

$$G = \sum_{k,n=0}^{\infty} G_{kn} g^k \delta^n, \quad (6)$$

the coefficients G_{kn} have the asymptotic behavior

$$G_{kn} \xrightarrow{k \rightarrow \infty} -\frac{a_n}{\pi} (-1)^k \left(\frac{1}{A}\right)^k k! k^{p(n)-1} [1 + \mathcal{O}(1/k)]. \quad (7)$$

These estimates apply to perturbation coefficients, in which the maximal power of g is much larger than the power of δ . Being interested in the region close to the isotropic limit, this restricted large-order estimation will be sufficient.

An expansion of the Green function (2) around the instanton yields exponentially small imaginary parts in both numerator and denominator. In order to isolate the imaginary part of the numerator, we simplify the denominator in (2) and calculate first the imaginary part of the approximate Green functions

$$\begin{aligned} \tilde{G}^{(2M)}(x_1, x_2, \dots, x_{2M})_{i_1, i_2, \dots, i_{2M}} = \\ \frac{\int D\phi \phi_{i_1}(x_1) \phi_{i_2}(x_2) \cdot \dots \cdot \phi_{i_{2M}}(x_{2M}) \exp\{-H[\phi]\}}{\int D\phi \exp\{-H_0[\phi]\}}, \end{aligned} \quad (8)$$

where the denominator contains only the free energy functional H_0 .

With the aim of calculating (4), we expand the fields around the classical solution of the isotropic limit $\delta = 0$ for the space dimension $d = 4$. Thus we write:

$$\vec{\phi} = \vec{u}_L \phi_c + \vec{u}_L \xi + \vec{u}_T \eta = \begin{pmatrix} \cos \vartheta \\ \sin \vartheta \end{pmatrix} \phi_c + \begin{pmatrix} \cos \vartheta \\ \sin \vartheta \end{pmatrix} \xi + \begin{pmatrix} -\sin \vartheta \\ \cos \vartheta \end{pmatrix} \eta, \quad (9)$$

where ϕ_c is the well-known $g\phi^4$ -instanton in four dimensions

$$\phi_c = \left(\frac{8}{-g} \right)^{1/2} \frac{\lambda}{1 + \lambda^2(x - x_0)^2}, \quad (10)$$

and ϑ is the rotation angle of the isotropic instanton in the (ϕ_1, ϕ_2) -plane. The fields ξ and η correspond to the degrees of freedom orthogonal to the rotation of that instanton. The parameters x_0 and λ are position and scale size of the instanton, respectively. Inserting the expansion (9) in $H(\phi_1, \phi_2)$, we obtain the expression:

$$H(\phi_1, \phi_2) = H(\phi_{1c}, \phi_{2c}) + H_1 + H_2 + H_3 + H_4, \quad (11)$$

where H_i collects all terms in ξ and η of order i . They are given by:

(1) Linear terms:

$$\begin{aligned} H_1(\xi, \eta) &= -\frac{\varepsilon 4(2)^{1/2}}{(-g)^{1/2}} \int d^d x \frac{\lambda^3}{[1 + \lambda^2(x - x_0)^2]^2} \xi \\ &\quad + \frac{\delta}{(-g)^{1/2}} \sin^2(2\vartheta) 8(2)^{1/2} \int d^d x \frac{\lambda^3}{[1 + \lambda^2(x - x_0)^2]^3} \xi \\ &\quad + \frac{\delta}{(-g)^{1/2}} \sin(4\vartheta) 4(2)^{1/2} \int d^d x \frac{\lambda^3}{[1 + \lambda^2(x - x_0)^2]^3} \eta \end{aligned} \quad (12)$$

(2) Quadratic terms:

$$\begin{aligned} H_2(\xi, \eta) &= \frac{1}{2} \int d^d x \xi M_L \xi + \frac{1}{2} \int d^d x \eta M_T \eta \\ &\quad + \delta \sin^2(2\vartheta) 6 \int d^d x \frac{\lambda^2}{[1 + \lambda^2(x - x_0)^2]^2} (\xi^2 - \eta^2) \\ &\quad + \delta \sin(4\vartheta) 6 \int d^d x \frac{\lambda^2}{[1 + \lambda^2(x - x_0)^2]^2} \xi \eta, \end{aligned} \quad (13)$$

where M_L and M_T are the operators

$$M_L = -\nabla^2 - \frac{24\lambda^2}{[1 + \lambda^2(x - x_0)^2]^2}, \quad M_T = -\nabla^2 - \frac{8(1 - \delta)\lambda^2}{[1 + \lambda^2(x - x_0)^2]^2} \quad (14)$$

(3) Cubic terms:

$$\begin{aligned}
H_3(\xi, \eta) &= -(-8g)^{1/2} \int d^d x \frac{\lambda}{1 + \lambda^2(x - x_0)^2} [\xi^3 + (1 - \delta)\xi\eta^2] \\
&\quad - \delta \frac{\sin^2(2\vartheta)}{2} (-8g)^{1/2} \int d^d x \frac{\lambda}{1 + \lambda^2(x - x_0)^2} (3\eta^2\xi - \xi^3) \\
&\quad - \delta \frac{\sin(4\vartheta)}{4} (-8g)^{1/2} \int d^d x \frac{\lambda}{1 + \lambda^2(x - x_0)^2} (\eta^3 - 3\eta\xi^2) \quad (15)
\end{aligned}$$

(4) Quartic terms:

$$\begin{aligned}
H_4(\xi, \eta) &= +\frac{g}{4} \int d^d x [\xi^4 + \eta^4 + 2(1 - \delta)\xi^2\eta^2] \\
&\quad - \delta g \frac{\sin^2(2\vartheta)}{8} \int d^d x (\xi^4 + \eta^4 - 6\xi^2\eta^2) \\
&\quad + \delta g \frac{\sin(4\vartheta)}{4} \int d^d x (\xi\eta^3 - \eta\xi^3) \quad (16)
\end{aligned}$$

The linear terms (12) contain factors ε or δ , since the expansion of the fields around the isotropic instanton is extremal only in the four-dimensional isotropic limit.

The classical contribution of the instanton is

$$\begin{aligned}
H(\phi_{1c}, \phi_{2c}) &= \quad (17) \\
&\quad -\frac{\lambda^\varepsilon 8\pi^2}{g} \frac{1}{3} \left[1 - \frac{\varepsilon}{2} (2 + \ln \pi + \gamma) \right] - \frac{\lambda^\varepsilon \delta}{g} \frac{8\pi^2}{3} \frac{\sin^2(2\vartheta)}{2} + \mathcal{O}\left(\frac{\delta}{g}\varepsilon\right).
\end{aligned}$$

The first term in $H(\phi_{1c}, \phi_{2c})$ is evaluated up to the first order in ε , because the one-loop renormalization will require replacing $1/g \rightarrow 1/g_r + \mathcal{O}[f(\delta_r)/\varepsilon]$, where g_r is the renormalized version of the coupling constant g . A contribution of order ε is not needed in the second, δ -dependent term, where it would produce a further factor δ , and thus be part of the neglected terms $\mathcal{O}(g)$ in (4).

Due to the anisotropy of the action, the fluctuation expansion (12)–(17) contains ϑ -dependent parts. However, all these terms are of order δ and can therefore be handled by straightforward perturbation theory.

Note that the angle ϑ disappears when the isotropic instanton is directed along the coordinate axis. Then the isotropic instanton coincides with the

exact solution in $d = 4$ for $\delta > 0$. This is also seen by inspecting the potential in (3). For $\delta > 0$, the term $\phi_1^4 + \phi_2^4$ is larger than $2(1 - \delta)\phi_1^2\phi_2^2$, so that the “tunneling-paths” of extremal action are obviously straight lines along the coordinate axis.

For the region $\delta < 0$, the exact instanton follows from the known fact that the action (3) is invariant under the orthogonal transformation:

$$\phi_1 = (\tilde{\phi}_1 + \tilde{\phi}_2)/\sqrt{2}, \quad \phi_2 = (\tilde{\phi}_1 - \tilde{\phi}_2)/\sqrt{2},$$

with the new coupling constants:

$$g = (2 - \tilde{\delta})\tilde{g}/2, \quad \delta = -\frac{2\tilde{\delta}}{2 - \tilde{\delta}},$$

satisfying $\delta < 0$ for $\tilde{\delta} > 0$.

In contrast to the method in [21], the treatment of the fluctuations in a power series in δ does not allow us to deal with all modes perturbatively. Near the O(2)-symmetric case, the Gaussian approximation for the rotation of the instanton becomes invalid, and the Gaussian integral must be replaced by an exact angle integration. The separation of the rotation mode must be done with the help of collective coordinates. The Jacobian of the relevant transformation can be deduced from the isotropic system. The field η of small oscillations must not contain modes of M_T with eigenvalues of the order δ , since these would vanish for $\delta = 0$. The discussion of the longitudinal part is given in [23].

In dimensional regularization, all eigenvalues of M_L which would be zero for $d = 4$ are of order ε in $4 - \varepsilon$ dimensions. To avoid eigenvalues of the order $1/\varepsilon$ in the propagator for the longitudinal fluctuations, all collective coordinates of the four-dimensional case must be retained in d dimensions. Therefore the field ξ in (9) contains no modes of M_L with eigenvalues proportional to ε .

In order to calculate the fluctuation factor and to separate the almost-zero modes from $\det M_L$ and $\det M_T$, it is convenient to do a conformal transformation onto a sphere in $d + 1$ dimensions leading to a discrete spectrum for the transformed differential operators M_L and M_T [24, 25]. Their products of eigenvalues can be given in terms of the Riemann ζ -function which we easily expanded near $\varepsilon = 0$. Simple $1/\varepsilon$ -poles, which are characteristic of dimensional regularization, appear directly from the known singularity of that

function. In place of the fields ξ and η , we define the fields $\Phi_1(\rho)$ and $\Phi_2(\rho)$ on the unit sphere in $d + 1$ dimensions:

$$\Phi_1 = \sigma^{1-d/2}\xi \quad \Phi_2 = \sigma^{1-d/2}\eta, \quad (18)$$

where $\sigma = 2/(1 + x^2)$. The instanton is supposed to be centered at the origin and rescaled by λ . The spherical operator corresponding to the differential operator ∇^2 is

$$V_0 = \frac{1}{2}L^2 - \frac{1}{4}d(d - 2), \quad (19)$$

where $L^2 = \sum_{a,b}(\rho_a\partial_b - \rho_b\partial_a)^2$ is the total angular momentum operator on the $(d + 1)$ -dimensional sphere.

The eigenfunctions of V_0 are the spherical harmonics $Y_m^l(\rho)$ in $d + 1$ dimensions [26]. They satisfy

$$V_0 Y_m^l(\rho) = -(l + \frac{1}{2}d - 1)(l + \frac{1}{2}d) Y_m^l(\rho), \quad (20)$$

and have the degeneracy

$$\nu_l(d + 1) = \frac{(2l + d - 1)\Gamma(l + d - 1)}{\Gamma(d)\Gamma(l + 1)}. \quad (21)$$

After the transformation, the angle-independent quadratic part of H_2 in (13) becomes

$$\frac{1}{2} \int d\Omega \Phi_1(-V_0 - 6)\Phi_1 + \frac{1}{2} \int d\Omega \Phi_2[-V_0 - 2(1 - \delta)]\Phi_2, \quad (22)$$

where $d\Omega$ is the surface element of the unit sphere in $d + 1$ dimensions.

After rewriting the entire energy functional(11) in terms of the new fields, we can summarize the Feynman rules for the diagrammatic evaluation of the functional integrals as follows:

(1) Propagators (from the ϑ -independent part of H_2):

longitudinal propagator:

$$\text{—————} = [(-V_0 - 6)']^{-1} \quad (23)$$

transversal propagator:

$$\text{-----} = \{[-V_0 - 2(1 - \delta)]'\}^{-1} \quad (24)$$

The prime indicates that the almost-zero modes are omitted when forming the inverse.

(2) Tadpoles (H_1):

$$\begin{aligned}
\text{X} \text{---} &= -\varepsilon \left(\frac{2}{-g} \right)^{1/2} \sigma^{-1+\varepsilon/2} \Phi_1 \\
\bullet \text{---} &= \left(\frac{2}{-g} \right)^{1/2} \delta \sin^2(2\vartheta) \sigma^{\varepsilon/2} \Phi_1 \\
\bullet \text{---} &= \left(\frac{1}{-2g} \right)^{1/2} \delta \sin(4\vartheta) \sigma^{\varepsilon/2} \Phi_2
\end{aligned} \tag{25}$$

The bold dot stands for the ϑ -dependence of the vertex.

(3) Two-point vertices (ϑ -dependent part of H_2):

$$\begin{aligned}
\text{---} \bullet \text{---} + \text{---} \bullet \text{---} &= \frac{3}{2} \sin^2(2\vartheta) \delta (\Phi_1^2 - \Phi_2^2) \\
\text{---} \bullet \text{---} &= \frac{3}{2} \sin(4\vartheta) \delta \Phi_1 \Phi_2
\end{aligned} \tag{26}$$

(4) Cubic vertices (H_3):

$$\begin{aligned}
\begin{array}{c} | \\ \diagdown \quad \diagup \\ | \end{array} &= -(-2g)^{1/2} \sigma^{-\varepsilon/2} \Phi_1^3 \\
\begin{array}{c} | \\ \diagdown \quad \diagup \\ \text{---} \end{array} &= -(-2g)^{1/2} (1 - \delta) \sigma^{-\varepsilon/2} \Phi_1 \Phi_2^2 \\
\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} | \\ \diagdown \quad \diagup \\ \bullet \end{array} &= -(-2g)^{1/2} \delta \frac{\sin^2(2\vartheta)}{2} \sigma^{-\varepsilon/2} (3\Phi_2^2 \Phi_1 - \Phi_1^3) \\
\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} | \\ \diagdown \quad \diagup \\ \bullet \end{array} &= -(-2g)^{1/2} \delta \frac{\sin(4\vartheta)}{4} \sigma^{-\varepsilon/2} (\Phi_2^3 - 3\Phi_2 \Phi_1^2)
\end{aligned}$$

(27)

(5) Quartic vertices (H_4):

$$\begin{aligned}
\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + \begin{array}{c} \text{---} \times \text{---} \\ \text{---} \times \text{---} \end{array} + \begin{array}{c} \diagup \text{---} \\ \text{---} \diagdown \end{array} &= \frac{g}{4} \sigma^{-\varepsilon} (\Phi_1^4 + \Phi_2^4 + 2(1 - \delta) \Phi_1^2 \Phi_2^2) \\
\begin{array}{c} \diagdown \diagup \\ \text{---} \times \text{---} \end{array} + \begin{array}{c} \diagup \diagdown \\ \text{---} \times \text{---} \end{array} &= \frac{g}{4} \delta \sin(4\vartheta) \sigma^{-\varepsilon} (\Phi_1 \Phi_2^3 - \Phi_2 \Phi_1^3) \\
\begin{array}{c} \text{---} \times \text{---} \\ \text{---} \times \text{---} \end{array} + \begin{array}{c} \text{---} \times \text{---} \\ \text{---} \times \text{---} \end{array} + \begin{array}{c} \text{---} \times \text{---} \\ \text{---} \times \text{---} \end{array} &= -\frac{g}{4} \delta \frac{\sin^2(2\vartheta)}{2} \sigma^{-\varepsilon} (\Phi_1^4 + \Phi_2^4 - 6\Phi_1^2 \Phi_2^2)
\end{aligned} \tag{28}$$

In order to obtain the leading corrections we must consider all connected diagrams of order $\mathcal{O}(1/g)$, $\mathcal{O}(\delta/g)$, and $\mathcal{O}(1)$. The contributions of these diagrams are added to H_c in (17). We consider first the contributions to $\mathcal{O}(1/g)$, which arise from the connected tree diagrams

$$\begin{array}{c} \times \text{---} \times \end{array}, \quad \begin{array}{c} \times \\ | \\ \text{---} \times \end{array}, \quad \begin{array}{c} \times \text{---} \times \\ \diagdown \diagup \\ \diagup \diagdown \end{array} \tag{29}$$

and their Φ_1 insertions of zeroth order in g :

$$\begin{array}{c} \times \\ | \\ \text{---} \end{array}, \quad \begin{array}{c} \times \quad \times \\ | \quad | \\ \text{---} \end{array}, \quad \begin{array}{c} \times \quad \times \quad \times \\ | \quad | \quad | \\ \text{---} \end{array} \quad \dots \tag{30}$$

Since each Φ_1 -vertex produces an ε -factor, the smallest order in ε is two. However, terms of order ε^2/g are negligible for our calculation even after one-loop renormalization, since: $1/g \rightarrow 1/g_r + \mathcal{O}[f(\delta_r)/\varepsilon]$ and $\varepsilon \rightarrow 0$. The diagrams of $\mathcal{O}(\delta/g)$ can be generated from those of $\mathcal{O}(1/g)$ by one of the substitutions:

$$\left[\times, \text{---}, \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}, \begin{array}{c} \diagup \text{---} \\ \text{---} \diagdown \end{array} \right] \Rightarrow \left[\bullet, \bullet, \begin{array}{c} \bullet \\ | \\ \text{---} \end{array}, \begin{array}{c} \times \\ \bullet \end{array} \right]. \tag{31}$$

By inspection, we see that the only diagram with an ε -power smaller than two is given by

$$\times \text{---} \bullet \quad (32)$$

which is of order $\varepsilon \delta/g$. After the renormalization, a term of order δ_r appears. But this term can be neglected in comparison with the δ_r/g_r -term from H_c in expression (17). Hence all the tree diagrams would enter only in a higher-order calculation.

Contributions to order g^0 can appear from one-loop diagrams. The only possible candidates are

$$\times \text{---} \bigcirc + \times \text{---} \bigcirc\!\!\!\! \text{---} \quad (33)$$

where the $1/\varepsilon$ -pole from the loop integration and the ε -factor of the Φ_1 -vertex cancel. Following the method of [23], we have found that these diagrams contribute to the coefficient of the imaginary part of $\tilde{G}^{(2M)}$ in (8) a factor

$$c_1^L c_1^T = \exp(-5 + \delta - \delta^2/2). \quad (34)$$

3 Quadratic Fluctuation Determinants

The angle-independent quadratic form is decomposed into a longitudinal and a transverse part. For the longitudinal fluctuations we obtain the determinant

$$\left(\frac{\det V_L}{\det V_{0L}} \right)^{-\frac{1}{2}} = \left[\frac{\det(-V_0 - 6)}{\det(-V_0)} \right]^{-\frac{1}{2}} = \prod_{l=0}^{\infty} \left[\frac{(l + \frac{1}{2}d - 3)(l + \frac{1}{2}d + 2)}{(l + \frac{1}{2}d - 1)(l + \frac{1}{2}d)} \right]^{-\frac{1}{2}\nu_l(d+1)}, \quad (35)$$

which coincides with that appearing in the one-component $g\phi^4$ theory, and is therefore known [23]. It contains a bound state at $l = 0$, this being responsible for the expected imaginary part of $\tilde{G}^{(2M)}$, and $d + 1$ almost-zero eigenmodes of order ε associated with the dilatation and translation degrees of freedom of the instanton in $4 - \varepsilon$ dimensions. Extracting these modes from the product (35) in the framework of collective coordinates [27, 28], we obtain the well-known formal replacement rule for the determinant

$$\left(\frac{\det V_L}{\det V_{0L}} \right)^{-\frac{1}{2}} \implies J^{V_L} (2\pi)^{-\frac{(d+1)}{2}} c_2^L \quad (36)$$

with

$$c_2^L = (2\pi)^{-1/2} 5^{5/2} \exp \left[\frac{3}{\varepsilon} + \frac{3}{4} - \frac{7}{2} \gamma + \frac{3}{\pi^2} \zeta'(2) \right], \quad (37)$$

and the Jacobian of the collective coordinates transformation:

$$J^{V_L} = \lambda^{(d-1)} \left(-\frac{16\pi^2 \lambda^\varepsilon}{15g} \right)^{\frac{d+1}{2}} [1 + \mathcal{O}(\varepsilon, g)]. \quad (38)$$

The expression for c_2^L contains the Euler constant γ , and the derivative of the Riemann-zeta function $\zeta(x)$ at the value $x = 2$. Note the simple pole in ε , which is related to the ultraviolet divergence on the one-loop level.

The transverse fluctuations contain neither negative nor zero modes for δ larger than zero. The corresponding fluctuation determinant is given by

$$\begin{aligned} \left(\frac{\det V_T}{\det V_{0T}} \right)^{-\frac{1}{2}} &= \left\{ \frac{\det [-V_0 - 2(1 - \delta)]}{\det(-V_0)} \right\}^{-\frac{1}{2}} \\ &= \prod_{l=0}^{\infty} \left[\frac{(l + \frac{1}{2}d - 1)(l + \frac{1}{2}d) - 2(1 - \delta)}{(l + \frac{1}{2}d - 1)(l + \frac{1}{2}d)} \right]^{-\frac{1}{2} \nu_l(d+1)}. \end{aligned} \quad (39)$$

Just one zero eigenmode appears for $l = 0$ in the isotropic limit $\delta \rightarrow 0$, due to the rotational invariance in that case. For $\delta > 0$, the numerator in (39) contributes a factor $1/\sqrt{2\delta}$. To avoid this artificial zero-mode divergence for $\delta \rightarrow 0$ we separate, as announced earlier, the angular degree of freedom from the integral measure via the collective-coordinates method. Similar to the longitudinal case, we make the formal substitution

$$\left(\frac{\det V_T}{\det V_{0T}} \right)^{-\frac{1}{2}} \implies J^{V_T} (2\pi)^{-\frac{1}{2}} c_2^T \quad (40)$$

with

$$J^{V_T} = \left(-\frac{16\pi^2 \lambda^\varepsilon}{3g} \right)^{\frac{1}{2}}, \quad (41)$$

which coincides with the Jacobian of the isotropic model. Since the integration interval for the angle ϑ is compact, no singularity appears in the limit $\delta \rightarrow 0$.

It is useful to illustrate the appearance of a divergent factor $1/\sqrt{2\delta}$ in a careless use of Gaussian integral. Assuming $\delta > 0$, we expand the angle-dependent classical action around $\vartheta = 0$ up to quadratic order. The ensuing Gaussian integral can be evaluated after replacing the finite integration interval $\vartheta \in [-\pi, \pi]$ by the noncompact one $\vartheta \in [-\infty, \infty]$. In this way, the integral

$$J^{V_T}(2\pi)^{-\frac{1}{2}} \int_{-\pi}^{+\pi} d\vartheta \exp \left[-\lambda^\varepsilon \frac{8\pi^2}{3} \frac{\delta}{|g|} \frac{\sin^2(2\vartheta)}{2} \right]$$

is evaluated to

$$\left(\frac{8\pi\lambda^\varepsilon}{3|g|} \right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} d\vartheta \exp \left[-\frac{1}{2} \left(\frac{16\pi^2\lambda^\varepsilon}{3|g|} 2\delta \right) \vartheta^2 \right] = \frac{1}{\sqrt{2\delta}}$$

showing the spurious would-be zero-mode singularity, as the consequence of false rotation-mode treatment.

Excluding the $l = 0$ -mode in the numerator on the right-hand side of (39), we obtain for c_2^T :

$$c_2^T = 2^{1/2} \exp \left\{ -\frac{1}{2} \sum_{l=1}^{\infty} \frac{(2l+d-1)\Gamma(l+d-1)}{\Gamma(d)\Gamma(l+1)} \ln \left[\frac{(l+\frac{d}{2}+1)(l+\frac{d}{2}-2)}{(l+\frac{d}{2}-1)(l+\frac{d}{2})} \right] \right. \\ \left. - \frac{1}{2} \sum_{l=1}^{\infty} \frac{(2l+d-1)\Gamma(l+d-1)}{\Gamma(d)\Gamma(l+1)} \ln \left[1 + \frac{2\delta}{(l+\frac{d}{2}+1)(l+\frac{d}{2}-2)} \right] \right\}. \quad (42)$$

The first sum is the well-known contribution from the isotropic limit. The second sum is to be expanded in powers of δ . For large l , the sum diverges as ε tends to zero. In order to separate off the divergence as a simple $1/\varepsilon$ -pole, we use the zeta-function regularization method described in [23]. A somewhat tedious but straightforward calculation leads to

$$c_2^T = (2\pi)^{-1/6} 3^{1/2} \exp \left[\frac{1}{3\varepsilon} (1-\delta)^2 + \frac{1}{4} - \frac{1}{2} \gamma + \frac{\zeta'(2)}{\pi^2} - \frac{1}{9} \delta + \frac{37}{81} \delta^2 + \mathcal{O}(\delta^3) \right]. \quad (43)$$

The series in front of the ε -singularity ends after the quadratic power of δ , thereby leading to the exact $(1-\delta)^2/3$ -coefficient of the simple ε -pole.

Collecting all contributions to the one-loop expression of $\text{Im}\tilde{G}^{(2M)}$, we obtain the imaginary parts

$$\begin{aligned} & \text{Im}\tilde{G}^{(2M)}(x_1, x_2, \dots, x_{2M})_{i_1, i_2, \dots, i_{2M}} = \\ & - \int d\lambda d^d x_0 \left\{ c_1^L c_1^T c_2^L c_2^T (2\pi)^{-\frac{(d+1)}{2}} (2\pi)^{-1/2} J^{V_L} J^{V_T} \exp\left(\frac{\lambda^\varepsilon A}{g}\right) \right. \\ & \left. \times \prod_{\nu=1}^{2M} \phi_c(x_\nu) \frac{1}{2} \int_0^{2\pi} d\vartheta u_{L i_1}(\vartheta) u_{L i_2}(\vartheta) \cdots u_{L i_{2M}}(\vartheta) \exp[a \sin^2(2\vartheta)] \right\} \quad (44) \end{aligned}$$

with

$$a = \frac{\lambda^\varepsilon 8\pi^2}{3g} \frac{\delta}{2}, \quad A = \frac{8\pi^2}{3} \left[1 - \frac{\varepsilon}{2} (2 + \ln \pi + \gamma) \right] \quad (45)$$

and $i_k = 1, 2$. The expression contains a standard factor 1/2, since for symmetry reasons each saddle point contributes only one half of the Gaussian integral.

It remains to perform the collective coordinates integration over the dilatation (λ), translation (x_0), and the rotation (ϑ) degree of freedom. Having obtained the imaginary parts of the Green functions, we go over to the bare imaginary part of the vertex functions. Taking the Fourier transform and excluding the $(2\pi)^d \delta(\sum_i q_i)$ -factors, an amputation of the external legs of $\tilde{G}^{(2M)}(x_1, x_2, \dots, x_{2M})_{i_1, i_2, \dots, i_{2M}}$ leads to:

$$\begin{aligned} & \text{Im}\Gamma_b^{(2M)}(q_m)_{i_1, i_2, \dots, i_{2M}} = \\ & -c_b \int_0^\infty \frac{d\lambda}{\lambda} \left\{ \lambda^{d-M(d-2)} \left(-\frac{\lambda^\varepsilon 8\pi^2}{3g} \right)^{(d+2+2M)/2} \exp\left(\frac{\lambda^\varepsilon A}{g}\right) \prod_{\nu=1}^{2M} \left(\frac{q_\nu}{\lambda}\right)^2 \tilde{\phi}\left(\frac{q_\nu}{\lambda}\right) \right. \\ & \left. \times \frac{1}{2} \int_0^{2\pi} d\vartheta u_{L i_1}(\vartheta) u_{L i_2}(\vartheta) \cdots u_{L i_{2M}}(\vartheta) \exp[a \sin^2(2\vartheta)] \right\} [1 + \mathcal{O}(\varepsilon, g)] \quad (46) \end{aligned}$$

with

$$\tilde{\phi}(q) = 2^{d/2} \pi^{\frac{(d-2)}{2}} 3^{1/2} |q|^{1-(d/2)} K_{\frac{d}{2}-1}(|q|), \quad (47)$$

where $K_\mu(z)$ is the modified Bessel-function. The coefficient c_b is given by

$$c_b = 2^{-2/3} 3^{1/2} \pi^{-11/3} \exp\left[\frac{1}{3\varepsilon} (10 - 2\delta + \delta^2) + \frac{4\zeta'(2)}{\pi^2} - 4\gamma - 4 + \frac{8}{9}\delta\right]$$

$$-\frac{7}{162}\delta^2 + \mathcal{O}(\delta^3) \Big]. \quad (48)$$

An ε -singularity results from the ultraviolet divergence of one-loop diagrams. As discussed in the following section, this simple pole in ε is removed by a conventional coupling constant renormalization.

The contribution from the imaginary part of the denominator in equation (2) follows for $M = 0$ in (46). We observe that it is of higher order in g in comparison with the imaginary part of the numerator, and can therefore be neglected. So the imaginary part of (8) supplies the desired imaginary part of (2).

4 Renormalization

In the preceding section we have calculated the contribution of the quadratic fluctuations around the isotropic instanton. The almost-zero eigenvalues of translation, dilatation and rotation have been extracted and treated separately. Of course, the resulting expressions are useless, if we are not able to renormalize the theory. This is the most difficult additional problem which arises in the transition from quantum mechanics to higher-dimensional field theories.

A systematic scheme to renormalize both the real and imaginary part of vertex functions for a $g\phi^4$ theory in $4 - \varepsilon$ dimensions was introduced by McKane, Wallace and de Alcantara Bonfim. They calculated the full (real and imaginary) renormalization-group constants using an extended minimal subtraction scheme to one loop (the conventional one is given in [29, 30]). We have extended their method to the case of two coupling constants in view of applications to critical phenomena.

For an illustration we consider first the four-point vertex function. The result (46) of the one-loop calculation about the instanton reads:

$$\text{Im } \Gamma_{\text{b}}^{(4)}(q_m)_{i,j,k,l} = \sum_{n=0}^{\infty} \left\{ \frac{(-\delta/2)^n}{n!} \frac{1}{2} \int_0^{2\pi} d\vartheta u_{Li}(\vartheta) u_{Lj}(\vartheta) u_{Lk}(\vartheta) u_{Ll}(\vartheta) [\sin(2\vartheta)]^{2n} \right.$$

$$\times \left[-c_b \int_0^\infty \frac{d\lambda}{\lambda} \lambda^\varepsilon \left(-\frac{\lambda^\varepsilon A}{g} \right)^{\frac{d+6+2n}{2}} \exp \left(\frac{\lambda^\varepsilon A}{g} \right) \prod_{\nu=1}^4 \left(\frac{q_\nu}{\lambda} \right)^2 \tilde{\phi} \left(\frac{q_\nu}{\lambda} \right) \right] \quad (49)$$

with $A = \frac{8\pi^2}{3} + \mathcal{O}(\varepsilon)$ and $i, j, k, l = 1, 2$. It remains to evaluate the integral over the parameter λ . This is a relict of the introduction of collective coordinates in the instanton calculation. The integral converges for small λ due to the exponentially decreasing of modified Bessel-function $K_{d/2-1}$. For large λ , the product $(q_\nu/\lambda)^2 \tilde{\phi}(q_\nu/\lambda)$ behaves like

$$\begin{aligned} \left(\frac{q_\nu}{\lambda} \right)^2 \tilde{\phi} \left(\frac{q_\nu}{\lambda} \right) &= 2^2 3^{1/2} \pi \left\{ \left(\frac{q_\nu}{2\lambda} \right)^\varepsilon \left[1 + \frac{1}{2} \varepsilon (\gamma - \ln \pi) + \mathcal{O}(\varepsilon^2) \right] \right. \\ &\quad + \left(\frac{q_\nu}{2\lambda} \right)^2 \left[\left(\frac{q_\nu}{2\lambda} \right)^\varepsilon \left(\frac{2}{\varepsilon} + \gamma - \ln \pi + \mathcal{O}(\varepsilon) \right) \right. \\ &\quad \left. \left. - \left(\frac{2}{\varepsilon} + 1 - \gamma - \ln \pi + \mathcal{O}(\varepsilon) \right) \right] + \mathcal{O} \left(\frac{q_\nu}{2\lambda} \right)^4 \right\}, \quad (50) \end{aligned}$$

so that the integral diverges logarithmically as ε goes to zero. This divergence causes a $1/\varepsilon$ -pole in the imaginary part. Since after the above approximation of $\tilde{\phi}(q_\nu/\lambda)$ the integral diverges for $\lambda \rightarrow 0$, a small λ cutoff μ has to be introduced. In this way, the λ -integral in (49) takes the form

$$-c_b \int_\mu^\infty \frac{d\lambda}{\lambda} \lambda^\varepsilon \left(-\frac{\lambda^\varepsilon A}{g} \right)^{\frac{d+6+2n}{2}} \exp \left(\frac{\lambda^\varepsilon A}{g} \right) (2^2 3^{1/2} \pi)^4 \prod_{\nu=1}^4 \left(\frac{q_\nu}{2\lambda} \right)^\varepsilon \left[1 + \mathcal{O} \left(\frac{q_\nu}{2\lambda} \right) \right]. \quad (51)$$

The coupling constant g is chosen to lie on top of the tip of the left-hand cut in the complex g -plane. Being integrated only near $g \rightarrow 0^-$, the integral can be evaluated perturbatively in g . Using

$$\int_\mu^\infty \frac{d\lambda}{\lambda} \lambda^{a\varepsilon} \lambda^\varepsilon \exp \left[- \left(\frac{\lambda^\varepsilon A}{|g|} \right) \right] = \frac{1}{\varepsilon} \frac{|g|}{A} \mu^{a\varepsilon} \exp \left[- \left(\frac{\mu^\varepsilon A}{|g|} \right) \right] [1 + \mathcal{O}(g)], \quad (52)$$

we get for (51) the expression

$$2^8 3^2 \pi^4 c_b \frac{1}{\varepsilon} \frac{g}{A} \left(-\frac{\mu^\varepsilon A}{g} \right)^{\frac{d+6+2n}{2}} \exp \left(\frac{\mu^\varepsilon A}{g} \right) \prod_{\nu=1}^4 \left(\frac{q_\nu}{\mu} \right)^\varepsilon. \quad (53)$$

After expanding the factor $(q_\nu/\mu)^\varepsilon$ in powers of ε , we obtain the typical finite contribution $\ln(q_\nu/\mu)$. Since this term can be omitted by a minimal subtraction, we are left with

$$\begin{aligned} \text{Im } \Gamma_b^{(4)}(q_m)_{i,j,k,l} = & \\ & 2^8 3^2 \pi^4 c_b \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \left\{ \frac{(-2\delta)^n}{n! \Gamma(2n+3)} \left(\frac{g}{A}\right) \left(-\frac{\mu^\varepsilon A}{g}\right)^{\frac{d+6+2n}{2}} \exp\left(\frac{\mu^\varepsilon A}{g}\right) \right. \\ & \left. \times \left[3\Gamma\left(n + \frac{3}{2}\right)^2 (S_{ijkl} - \delta_{ijkl}) + \Gamma\left(n + \frac{1}{2}\right) \Gamma\left(n + \frac{5}{2}\right) \delta_{ijkl} \right] \right\}, \end{aligned} \quad (54)$$

where

$$S_{ijkl} = \frac{1}{3}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

and

$$\delta_{ijkl} = \begin{cases} 1, & i = j = k = l \\ 0, & \text{otherwise.} \end{cases} \quad (i, j, k, l = 1, 2)$$

The tensor structure arises explicitly upon doing the integrals

$$\int_0^{\pi/2} \sin^{2\alpha+1}(\vartheta) \cos^{2\beta+1}(\vartheta) d\vartheta = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2\Gamma(\alpha+\beta+2)} = \frac{1}{2}B(\alpha+1, \beta+1)$$

for all combinations of indices in $\text{Im } \Gamma_{ijkl}^{(4)}$. The first term in (54) contains only mixed index combinations, whereas the second term has only a contribution from equal indices.

Now we can proceed to renormalize the bare results. The real part of $\Gamma_{ijkl}^{(4)}(q)$ is the perturbative one, and it is easily calculated to one loop. In terms of our coupling constants, the real parts $\text{Re } \Gamma_{ijkl}^{(4)}(q)$ take the form

$$\begin{aligned} \text{Re } \Gamma_b^{(4)}(q_m)_{i,j,k,l} = & \\ & \left\{ -6(1-\delta)g + \left[\frac{3}{4\pi^2\varepsilon} (10 - 14\delta + 4\delta^2) + \mathcal{O}(\varepsilon^0) \right] g^2 \mu^{-\varepsilon} \right\} \{S_{ijkl} - \delta_{ijkl}\} \\ & + \left\{ -6g + \left[\frac{3}{4\pi^2\varepsilon} (10 - 2\delta + \delta^2) + \mathcal{O}(\varepsilon^0) \right] g^2 \mu^{-\varepsilon} \right\} \delta_{ijkl}, \end{aligned} \quad (55)$$

where μ is the arbitrary momentum scale introduced above in (51).

In the absence of an instanton, the wave function renormalization has no one-loop contribution. In the consequence, the expression (55) is rendered finite by a coupling constant renormalization only. The subscript r of the coupling constants indicates renormalized expressions in the conventional perturbative manner, i. e.,

$$\begin{aligned} g_r &= g\mu^{-\varepsilon} - \frac{1}{8\pi^2\varepsilon} (10 - 2\delta + \delta^2) g^2\mu^{-2\varepsilon}, \\ \delta_r &= \delta + \frac{1}{8\pi^2\varepsilon} (-2\delta + \delta^2 + \delta^3) g\mu^{-\varepsilon} \end{aligned} \quad (56)$$

Inserting this into (54) and (55) we find the perturbatively renormalized vertex functions

$$\Gamma_r^{(4)}(q_m)_{i,j,k,l} = -6\mu^\varepsilon [F_1(\delta_r, g_r) (S_{ijkl} - \delta_{ijkl}) + F_2(\delta_r, g_r)\delta_{ijkl}] \quad (57)$$

with

$$\begin{aligned} F_1(\delta_r, g_r) = \\ (1 - \delta_r)g_r + i2^7 3^2 \pi^4 c_r \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \frac{(-2\delta_r)^n \Gamma\left(n + \frac{3}{2}\right)^2}{n! \Gamma(2n + 3)} \left(-\frac{A}{g_r}\right)^{\frac{d+4+2n}{2}} \exp\left(\frac{A}{g_r}\right), \end{aligned} \quad (58)$$

and

$$\begin{aligned} F_2(\delta_r, g_r) = \\ g_r + i2^7 3\pi^4 c_r \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \frac{(-2\delta_r)^n \Gamma\left(n + \frac{1}{2}\right) \Gamma\left(n + \frac{5}{2}\right)}{n! \Gamma(2n + 3)} \left(-\frac{A}{g_r}\right)^{\frac{d+4+2n}{2}} \exp\left(\frac{A}{g_r}\right). \end{aligned} \quad (59)$$

The coefficient c_r is given by

$$c_r = c_b \exp\left[-\frac{A}{8\pi^2\varepsilon} (10 - 2\delta_r + \delta_r^2)\right]$$

$$\begin{aligned}
&= 3^{1/2} 2^{-2/3} \pi^{-2} \exp \left[\frac{1}{3\varepsilon} (10 - 2\delta_r + \delta_r^2) - \frac{1}{3\varepsilon} (10 - 2\delta_r + \delta_r^2) \right. \\
&\quad \left. + \frac{4\zeta'(2)}{\pi^2} - \frac{7}{3}\gamma - \frac{2}{3} + \mathcal{O}(\delta_r) + \mathcal{O}(\delta_r g_r) \right]. \quad (60)
\end{aligned}$$

The $1/\varepsilon$ -pole terms cancel. Thus, c_r remains finite for $\varepsilon \rightarrow 0$. For the leading expansion (4) we can take the perturbative renormalized expression (60) at the position $\delta_r = 0$. The only remaining singularity is the $1/\varepsilon$ -factor in $\text{Im} \Gamma_r^{(4)}(q)_{ijkl}$. It requires a further renormalization. In our special choice of coupling constants, the disastrous $k!$ -divergence of perturbation series appears in the expansions coefficients of g^k . Since δ is a dimensionless anisotropy measure for positive as well as negative g , we have expanded the imaginary part of vertex functions around the isotropic case in a well defined manner. For the derivation of nonperturbative corrections to the renormalization-group functions and for later applications, however, it is convenient to avoid ratios of coupling constants. Therefore we proceed with $v_r = g_r \delta_r$ and g_r instead of δ_r and g_r . Then v_r receive the rule of δ_r .

Now we apply the extended minimal subtraction and perform a second (nonperturbative) renormalization step to eliminate the $1/\varepsilon$ -pole term:

$$\begin{aligned}
g_R &= g_r + i2^7 3\pi^4 c_r \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \left[\frac{v_r^n}{n!} \left(\frac{3}{4\pi^2} \right)^n B \left(n + \frac{1}{2}, n + \frac{5}{2} \right) \right. \\
&\quad \left. \times \left(-\frac{A}{g_r} \right)^{\frac{d+4+4n}{2}} \exp \left(\frac{A}{g_r} \right) \right], \\
v_R &= v_r - i2^7 3\pi^4 c_r \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \left[\frac{v_r^n}{n!} \left(\frac{3}{4\pi^2} \right)^n \left(\frac{4n}{2n+3} \right) B \left(n + \frac{1}{2}, n + \frac{5}{2} \right) \right. \\
&\quad \left. \times \left(-\frac{A}{g_r} \right)^{\frac{d+4+4n}{2}} \exp \left(\frac{A}{g_r} \right) \right]. \quad (61)
\end{aligned}$$

The subscript R denotes the fully renormalized coupling constants. The required wave function renormalization in nonperturbative terms give no leading contributions to the imaginary part (see below). Therefore the $1/\varepsilon$ -singularity in (57) is removed by a nonperturbative coupling-constant renormalization only.

We now proceed to investigate the two-step renormalization for the wave function. There are two additional complications for the $\Gamma^{(2)}$ -functions. First, $\text{Im}\Gamma^{(2)}$ is quadratically divergent. Second, there is a momentum-dependent divergence in the imaginary part of $\Gamma^{(2)}$. For the $g\phi^4$ -theory it was shown in [18] that the undesirable momentum-dependence disappears during the nonperturbative renormalization process. This is comparable to the situation in conventional two-loop perturbation expansion. We continue to show that this non-obvious cancellation still works in the more complex case of two coupling constants and more than one field components.

According to (46) the imaginary part of the unrenormalized two-point vertex function reads

$$\text{Im}\Gamma_b^{(2)}(q)_{i,j} = c_b \int_0^\infty \frac{d\lambda}{\lambda} \left\{ \lambda^2 \left(-\frac{\lambda^\varepsilon A}{g} \right)^{\frac{d+4}{2}} \exp\left(\frac{\lambda^\varepsilon A}{g}\right) \left[\left(\frac{q}{\lambda}\right)^2 \tilde{\phi}\left(\frac{q}{\lambda}\right) \right]^2 \right. \\ \left. \times \frac{1}{2} \int_0^{2\pi} d\vartheta u_{Li}(\vartheta) u_{Lj}(\vartheta) \exp[a \sin^2(2\vartheta)] \right\}. \quad (62)$$

The integration over the angle ϑ can be done:

$$\frac{1}{2} \int_0^{2\pi} d\vartheta u_{Li}(\vartheta) u_{Lj}(\vartheta) \exp[a \sin^2(2\vartheta)] \\ = \delta_{ij} \sum_{n=0}^\infty \frac{a^n}{n!} 4^n \frac{\Gamma\left(n + \frac{1}{2}\right) \Gamma\left(n + \frac{3}{2}\right)}{\Gamma(2n + 2)} = \delta_{ij} \sum_{n=0}^\infty \frac{a^n}{n!} 4^n B\left(n + \frac{1}{2}, n + \frac{3}{2}\right). \quad (63)$$

Thus, we get the usual form: $\text{Im}\Gamma_b^{(2)} = \delta_{ij} \text{Im}\Gamma_b^{(2)}$. The wave function renormalization is contained in $\frac{\partial}{\partial q^2} \Gamma^{(2)}(q)$. Following a similar procedure as in the case of $\Gamma^{(4)}$, we obtain the imaginary part

$$\frac{\partial}{\partial q^2} \text{Im}\Gamma_b^{(2)}(q) = 2^4 \pi^2 3 c_b \frac{1}{\varepsilon} \sum_{n=0}^\infty \left\{ \frac{(-2\delta)^n}{n!} B\left(n + \frac{1}{2}, n + \frac{3}{2}\right) \right. \\ \left. \times \left[\gamma - \frac{1}{2} + \ln\left(\frac{q}{2\mu}\right) \right] \left(-\frac{\mu^\varepsilon A}{g} \right)^{\frac{d+2+2n}{2}} \exp\left(\frac{\mu^\varepsilon A}{g}\right) \right\}. \quad (64)$$

Apart from the appearance of the $(1/\varepsilon) \ln(q/\mu)$ term, a very similar expression is found for $\text{Im} \Gamma_b^{(4)}(q)$. In order to show the cancellation of this term during the second stage of renormalization we need the two-loop expression for the real part of $\frac{\partial}{\partial q^2} \Gamma^{(2)}(q)$

$$\frac{\partial}{\partial q^2} \text{Re} \Gamma_b^{(2)}(q) = 1 + q^{-2\varepsilon} \left(\frac{4}{3} - \frac{2}{3} \delta + \frac{1}{3} \delta^2 \right) g^2 \left[\frac{3}{4(8\pi^2)^2 \varepsilon} + P + \mathcal{O}(\varepsilon) \right], \quad (65)$$

where P denotes some number which is the contribution of order ε^0 . At $q = \mu$, (65) is commonly defined as the inverse wave function renormalization constant $(Z_p^\phi)^{-1}$. In terms of the perturbative renormalized coupling constants Z_p^ϕ is given by

$$Z_p^\phi = 1 - \frac{1}{4(8\pi^2)^2 \varepsilon} (4 - 2\delta_r + \delta_r^2) g_r^2, \quad (66)$$

and together with (64), we find after the first step of renormalization

$$\begin{aligned} \frac{\partial}{\partial q^2} \Gamma_r^{(2)}(q) &= 1 + \left(\frac{4}{3} g_r^2 - \frac{2}{3} v_r g_r + \frac{1}{3} v_r^2 \right) \left[P - \frac{3}{2(8\pi^2)^2} \ln \left(\frac{q}{\mu} \right) + \mathcal{O}(\varepsilon) \right] \\ &+ i 2^4 \pi^2 3 c_r \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \left\{ \frac{v_r^n}{n!} \left(\frac{3}{4\pi^2} \right)^n B \left(n + \frac{1}{2}, n + \frac{3}{2} \right) \right. \\ &\times \left. \left[\gamma - \frac{1}{2} + \ln \left(\frac{q}{2\mu} \right) \right] \left(-\frac{A}{g_r} \right)^{\frac{d+2+4n}{2}} \exp \left(\frac{A}{g_r} \right) \right\}. \quad (67) \end{aligned}$$

We have again introduced the coupling constants v_r and g_r which are more convenient for practical applications. This expression reads in the fully renormalized couplings (61)

$$\begin{aligned} \frac{\partial}{\partial q^2} \Gamma_r^{(2)}(q) &= 1 + \left(\frac{4}{3} g_R^2 - \frac{2}{3} v_R g_R + \frac{1}{3} v_R^2 \right) \left(P - \frac{3}{2(8\pi^2)^2} \ln \left(\frac{q}{\mu} \right) \right) \\ &+ i 2^4 \pi^2 3 c_r \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \left\{ \frac{v_R^n}{n!} \left(\frac{3}{4\pi^2} \right)^n \right. \\ &\times \left. \left[2^4 \pi^2 A B \left(n + \frac{1}{2}, n + \frac{5}{2} \right) \frac{4(n+1)}{2n+3} \left(P - \frac{3}{2(8\pi^2)^2} \ln \left(\frac{q}{\mu} \right) \right) \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + B\left(n + \frac{1}{2}, n + \frac{3}{2}\right) \left(\gamma - \frac{1}{2} - \ln 2 + \ln\left(\frac{q}{\mu}\right) \right) \Big] \\
& \times \left(-\frac{A}{g_R} \right)^{\frac{d+2+4n}{2}} \exp\left(\frac{A}{g_R}\right) \Big\}. \tag{68}
\end{aligned}$$

By using the definition of the B -function in terms of Gamma-functions one can easily read off the cancellation of $(1/\varepsilon) \ln(q/\mu)$ singularities for every power n of the coupling constant v_R . Therefore we are left with

$$\begin{aligned}
\frac{\partial}{\partial q^2} \Gamma_r^{(2)}(q) &= 1 + \left(\frac{4}{3} g_R^2 - \frac{2}{3} v_R g_R + \frac{1}{3} v_R^2 \right) \left[P - \frac{3}{2(8\pi^2)^2} \ln\left(\frac{q}{\mu}\right) \right] \\
&+ i 2^4 \pi^2 3 c_r \frac{1}{\varepsilon} \left(2^4 \pi^2 A P + \gamma - \frac{1}{2} - \ln 2 \right) \\
&\times \sum_{n=0}^{\infty} \left[\frac{v_R^n}{n!} \left(\frac{3}{4\pi^2} \right)^n B\left(n + \frac{1}{2}, n + \frac{3}{2}\right) \left(-\frac{A}{g_R} \right)^{\frac{d+2+4n}{2}} \exp\left(\frac{A}{g_R}\right) \right]. \tag{69}
\end{aligned}$$

Now we can define the nonperturbative minimally subtracted wave function renormalization to remove the $1/\varepsilon$ -pole:

$$\begin{aligned}
Z_{\text{np}}^\phi &= 1 - i 2^4 \pi^2 3 c_r \frac{1}{\varepsilon} \left(2^4 \pi^2 A P + \gamma - \frac{1}{2} - \ln 2 \right) \\
&\times \sum_{n=0}^{\infty} \left[\frac{v_R^n}{n!} \left(\frac{3}{4\pi^2} \right)^n B\left(n + \frac{1}{2}, n + \frac{3}{2}\right) \right. \\
&\quad \left. \times \left(-\frac{A}{g_R} \right)^{\frac{d+2+4n}{2}} \exp\left(\frac{A}{g_R}\right) \right]. \tag{70}
\end{aligned}$$

The full renormalization constant is defined by $Z^\phi := Z_p^\phi Z_{\text{np}}^\phi$ and follows from the product of (66) and (70).

A further important Green function is associated with the composite field $(1/2)\phi^2$:

$$G^{(2,1)}(x_1, x_2; x_3)_{i,j} = \left\langle \phi_i(x_1) \phi_j(x_2) \frac{1}{2} \phi_k^2(x_3) \right\rangle. \tag{71}$$

By a straightforward application of the techniques discussed in the preceding sections we obtain for the corresponding vertex function

$$\begin{aligned} \text{Im } \Gamma_{\text{b}}^{(2,1)}(q_1, q_2; q_3)_{i,j} &= -\frac{1}{2} c_{\text{b}} \int_0^\infty \frac{d\lambda}{\lambda} \left\{ \left(-\frac{\lambda^\varepsilon A}{g} \right)^{\frac{d+6}{2}} \exp \left(\frac{\lambda^\varepsilon A}{g} \right) \right. \\ &\times \prod_{\nu=1}^2 \left[\left(\frac{q_\nu}{\lambda} \right)^2 \tilde{\phi} \left(\frac{q_\nu}{\lambda} \right) \right] \hat{\phi} \left(\frac{q_3}{\lambda} \right) \frac{1}{2} \int_0^{2\pi} d\vartheta u_{L_i}(\vartheta) u_{L_j}(\vartheta) \exp [a \sin^2(2\vartheta)] \left. \right\}, \end{aligned} \quad (72)$$

where

$$\hat{\phi}(q) = 2^{(d-2)/2} \pi^{(d-4)/2} 3 |q|^{(4-d)/2} K_{(d-4)/2}(|q|). \quad (73)$$

Now we can proceed as in the case of wave function renormalization. Performing the collective coordinates integration over λ and ϑ we obtain

$$\begin{aligned} \text{Im } \Gamma_{\text{b}}^{(2,1)}(q_1, q_2; q_3)_{i,j} &= \delta_{ij} 2^4 \pi^2 3^2 c_{\text{b}} \frac{1}{\varepsilon} \sum_{n=0}^\infty \left\{ \frac{(-2\delta)^n}{n!} B \left(n + \frac{1}{2}, n + \frac{3}{2} \right) \right. \\ &\times \left[\gamma + \ln \left(\frac{q_3}{2\mu} \right) \right] \left(-\frac{\mu^\varepsilon A}{g} \right)^{\frac{d+4+2n}{2}} \exp \left(\frac{\mu^\varepsilon A}{g} \right) \left. \right\}, \end{aligned} \quad (74)$$

which agrees with the usual form $\Gamma_{\text{b}}^{(2,1)} = \delta_{ij} \Gamma_{\text{b}}^{(2,1)}$. At $q_3 = 0$, $q_1^2 = q_2^2 = \mu^2$, the vertex function $\Gamma^{(2,1)}(q_1, q_2; q_3)$ defines a third renormalization constant, called Z^{ϕ^2} , via

$$\Gamma^{(2,1)} \equiv (Z^\phi Z^{\phi^2})^{-1}. \quad (75)$$

The perturbative Z^{ϕ^2} follows from the real part of $\Gamma^{(2,1)}$:

$$\text{Re} \Gamma_{\text{b}}^{(2,1)} = 1 + g q_3^{-\varepsilon} \left(\frac{4}{3} - \frac{1}{3} \delta \right) \left(-\frac{3}{8\pi^2 \varepsilon} + Q + \mathcal{O}(\varepsilon) \right), \quad (76)$$

where Q is a number of order ε^0 . In terms of the perturbative renormalized coupling constants, the one-loop expression of $Z_{\text{p}}^{\phi^2}$ is given by

$$Z_{\text{p}}^{\phi^2} = 1 + \frac{1}{8\pi^2 \varepsilon} (4 - \delta_{\text{r}}) g_{\text{r}}. \quad (77)$$

Including $\text{Re}\Gamma_b^{(2,1)}$ and $\text{Im}\Gamma_b^{(2,1)}$ we obtain after the perturbative step of renormalization

$$\begin{aligned} \Gamma_r^{(2,1)}(q_1, q_2; q_3) &= 1 + \left(\frac{4}{3}g_r - \frac{1}{3}v_r\right) \left[Q + \frac{3}{8\pi^2} \ln\left(\frac{q_3}{\mu}\right) + \mathcal{O}(\varepsilon) \right] \\ &+ i 2^4 \pi^2 3^2 c_r \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \left\{ \frac{v_r^n}{n!} \left(\frac{3}{4\pi^2}\right)^n B\left(n + \frac{1}{2}, n + \frac{3}{2}\right) \right. \\ &\quad \left. \times \left[\gamma + \ln\left(\frac{q_3}{2\mu}\right) \right] \left(-\frac{A}{g_r}\right)^{\frac{d+4+4n}{2}} \exp\left(\frac{A}{g_r}\right) \right\}. \end{aligned} \quad (78)$$

In the second step of renormalization the $(1/\varepsilon) \ln(q/\mu)$ singularities cancel, and we find

$$\begin{aligned} \Gamma_r^{(2,1)}(q_1, q_2; q_3) &= 1 + \left(\frac{4}{3}g_R - \frac{1}{3}v_R\right) \left[Q + \frac{3}{8\pi^2} \ln\left(\frac{q_3}{\mu}\right) \right] \\ &- i 2^4 \pi^2 3^2 c_r \frac{1}{\varepsilon} \left(\frac{8\pi^2}{3} Q - \gamma + \ln 2 \right) \\ &\times \sum_{n=0}^{\infty} \left[\frac{v_R^n}{n!} \left(\frac{3}{4\pi^2}\right)^n B\left(n + \frac{1}{2}, n + \frac{3}{2}\right) \left(-\frac{A}{g_R}\right)^{\frac{d+4+4n}{2}} \exp\left(\frac{A}{g_R}\right) \right]. \end{aligned} \quad (79)$$

Hence, the nonperturbative Z^{ϕ^2} is

$$\begin{aligned} Z_{\text{np}}^{\phi} Z_{\text{np}}^{\phi^2} &= 1 + i 2^4 \pi^2 3^2 c_r \frac{1}{\varepsilon} \left(\frac{8\pi^2}{3} Q - \gamma + \ln 2 \right) \\ &\times \sum_{n=0}^{\infty} \left[\frac{v_R^n}{n!} \left(\frac{3}{4\pi^2}\right)^n B\left(n + \frac{1}{2}, n + \frac{3}{2}\right) \left(-\frac{A}{g_R}\right)^{\frac{d+4+4n}{2}} \exp\left(\frac{A}{g_R}\right) \right]. \end{aligned} \quad (80)$$

With respect to the leading expansion (4) this is equal to $Z_{\text{np}}^{\phi^2}$.

5 Renormalization group functions

After having calculated the imaginary parts of the renormalized couplings and of the renormalization constants Z^{ϕ} and Z^{ϕ^2} , we are in the position to derive the nonperturbative renormalization group functions $\beta(g_R, v_R)$, $\gamma(g_R, v_R)$

and $\gamma_{\phi^2}(g_R, v_R)$ in the usual way. First we derive the β -functions from the expressions (61) for $g_R(\mu)$ and $v_R(\mu)$, respectively, and obtain:

$$\begin{aligned}
\beta_g(g_R, v_R) &= \\
\mu \frac{\partial}{\partial \mu} g_R &= \mu \frac{\partial}{\partial \mu} g_r + i2^7 3\pi^4 c_r \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \mu \frac{\partial}{\partial \mu} \left[\frac{v_r^n}{n!} \left(\frac{3}{4\pi^2} \right)^n B \left(n + \frac{1}{2}, n + \frac{5}{2} \right) \right. \\
&\quad \left. \times \left(-\frac{A}{g_r} \right)^{\frac{d+4+4n}{2}} \exp \left(\frac{A}{g_r} \right) \right], \\
\beta_v(g_R, v_R) &= \\
\mu \frac{\partial}{\partial \mu} v_R &= \mu \frac{\partial}{\partial \mu} v_r - i2^7 3\pi^4 c_r \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \mu \frac{\partial}{\partial \mu} \left[\frac{v_r^n}{n!} \left(\frac{3}{4\pi^2} \right)^n B \left(n + \frac{1}{2}, n + \frac{5}{2} \right) \right. \\
&\quad \left. \times \left(\frac{4n}{2n+3} \right) \left(-\frac{A}{g_r} \right)^{\frac{d+4+4n}{2}} \exp \left(\frac{A}{g_r} \right) \right]. \quad (81)
\end{aligned}$$

The derivatives are taken at fixed bare couplings g and v . These functions will be finite as $\varepsilon \rightarrow 0$. Furthermore, they will have a left-hand cut in the complex g -plane. On top of the tip of this cut we find

$$\begin{aligned}
\beta_g(g_R, v_R) &= \beta_g^p(g_r, v_r) - i2^7 3\pi^4 c_r \sum_{n=0}^{\infty} \left[\frac{v_r^n}{n!} \left(\frac{3}{4\pi^2} \right)^n B \left(n + \frac{1}{2}, n + \frac{5}{2} \right) \right. \\
&\quad \left. \times \left(-\frac{A}{g_r} \right)^{\frac{d+6+4n}{2}} \exp \left(\frac{A}{g_r} \right) \right], \\
\beta_v(g_R, v_R) &= \beta_v^p(g_r, v_r) + i2^7 3\pi^4 c_r \sum_{n=0}^{\infty} \left[\frac{v_r^n}{n!} \left(\frac{3}{4\pi^2} \right)^n B \left(n + \frac{1}{2}, n + \frac{5}{2} \right) \right. \\
&\quad \left. \times \left(\frac{4n}{2n+3} \right) \left(-\frac{A}{g_r} \right)^{\frac{d+6+4n}{2}} \exp \left(\frac{A}{g_r} \right) \right]. \quad (82)
\end{aligned}$$

The real parts are the familiar perturbative β -functions

$$\begin{aligned}
\beta_g^p(g_r, v_r) &= -\varepsilon g_r + \frac{1}{8\pi^2} \left(-2g_r v_r + v_r^2 + 10g_r^2 \right), \\
\beta_v^p(g_r, v_r) &= -\varepsilon v_r + \frac{3}{8\pi^2} \left(4g_r v_r - v_r^2 \right). \quad (83)
\end{aligned}$$

In the leading contributions the pair of coupling constants (g_r, v_r) may be replaced by (g_R, v_R) in (82). This can be proven using (61). The result is

$$\begin{aligned}\beta_g(g_R, v_R) &= \beta_g^P(g_R, v_R) - i2^7 3\pi^4 c_r \sum_{n=0}^{\infty} \left[\frac{v_R^n}{n!} \left(\frac{3}{4\pi^2} \right)^n B \left(n + \frac{1}{2}, n + \frac{5}{2} \right) \right. \\ &\quad \left. \times \left(-\frac{A}{g_R} \right)^{\frac{d+6+4n}{2}} \exp \left(\frac{A}{g_R} \right) \right], \\ \beta_v(g_R, v_R) &= \beta_v^P(g_R, v_R) + i2^7 3\pi^4 c_r \sum_{n=0}^{\infty} \left[\frac{v_R^n}{n!} \left(\frac{3}{4\pi^2} \right)^n B \left(n + \frac{1}{2}, n + \frac{5}{2} \right) \right. \\ &\quad \left. \times \left(\frac{4n}{2n+3} \right) \left(-\frac{A}{g_R} \right)^{\frac{d+6+4n}{2}} \exp \left(\frac{A}{g_R} \right) \right]. \quad (84)\end{aligned}$$

Note that the β -functions have a form expected within a minimal subtracted scheme: $\beta_g(g_R, v_R) = -\varepsilon g_R + \beta_g^4(g_R, v_R)$ and $\beta_v(g_R, v_R) = -\varepsilon v_R + \beta_v^4(g_R, v_R)$, where the β^4 denotes the β -functions in four dimensions. The critical large-distance behavior of the correlation functions is defined by the anomalous dimension of the field ϕ , which is defined by

$$\gamma(g_R, v_R) = \mu \frac{\partial}{\partial \mu} \ln Z^\phi = \mu \frac{\partial}{\partial \mu} \ln Z_p^\phi + \mu \frac{\partial}{\partial \mu} \ln Z_{np}^\phi. \quad (85)$$

With (66) for Z_p^ϕ and (70) for Z_{np}^ϕ we get the ε -independent expression

$$\begin{aligned}\gamma(g_R, v_R) &= \\ &= \frac{1}{2(8\pi^2)^2} (4g_R^2 - 2v_R g_R + v_R^2) + i2^4 \pi^2 3 c_r \left(2^7 \pi^4 3^{-1} P + \gamma - \frac{1}{2} - \ln 2 \right) \\ &\quad \times \sum_{n=0}^{\infty} \left[\frac{v_R^n}{n!} \left(\frac{3}{4\pi^2} \right)^n B \left(n + \frac{1}{2}, n + \frac{3}{2} \right) \left(-\frac{A}{g_R} \right)^{\frac{d+4+4n}{2}} \exp \left(\frac{A}{g_R} \right) \right]. \quad (86)\end{aligned}$$

The divergence of the length scale at critical temperature is governed by the anomalous dimension of the composite field ϕ^2 :

$$\gamma_{\phi^2}(g_R, v_R) = -\mu \frac{\partial}{\partial \mu} \ln Z^{\phi^2} = -\mu \frac{\partial}{\partial \mu} \ln Z_p^{\phi^2} - \mu \frac{\partial}{\partial \mu} \ln Z_{np}^{\phi^2}. \quad (87)$$

Using (77) and (80), we obtain

$$\begin{aligned} \gamma_{\phi^2}(g_R, v_R) &= \frac{1}{8\pi^2} (4g_R - v_R) + i 2^4 \pi^2 3^2 c_r \left(\frac{8\pi^2}{3} Q - \gamma + \ln 2 \right) \\ &\times \sum_{n=0}^{\infty} \left[\frac{v_R^n}{n!} \left(\frac{3}{4\pi^2} \right)^n B \left(n + \frac{1}{2}, n + \frac{3}{2} \right) \left(-\frac{A}{g_R} \right)^{\frac{d+6+4n}{2}} \exp \left(\frac{A}{g_R} \right) \right]. \end{aligned} \quad (88)$$

For the leading contributions to the imaginary part, it was possible to replace (g_r, v_r) by (g_R, v_R) . We remark that the expansions (84),(86) and (88) have the correct isotropic limit for $v_R \rightarrow 0$.

6 Generalization to the case $N = 3$

Generalizing (9) to $N = 3$ we write:

$$\begin{aligned} \vec{\phi} &= \vec{u}_L \phi_c + \vec{u}_L \xi + \vec{u}_{T_1} \eta + \vec{u}_{T_2} \chi \\ &= \begin{pmatrix} \sin \theta \cos \vartheta \\ \sin \theta \sin \vartheta \\ \cos \theta \end{pmatrix} (\phi_c + \xi) + \begin{pmatrix} -\sin \vartheta \\ \cos \vartheta \\ 0 \end{pmatrix} \eta + \begin{pmatrix} \cos \theta \cos \vartheta \\ \cos \theta \sin \vartheta \\ -\sin \theta \end{pmatrix} \chi, \end{aligned} \quad (89)$$

where θ and ϑ are the rotation angles of the isotropic instanton. This change of variables yields the following expansion of the energy functional:

$$\begin{aligned} H(\vec{\phi}) &= H_c + \frac{1}{2} \int d^d x \xi M_L \xi + \frac{1}{2} \int d^d x \eta M_{T_1} \eta + \frac{1}{2} \int d^d x \chi M_{T_2} \chi \\ &- \frac{\varepsilon 4(2)^{1/2}}{(-g)^{1/2}} \int d^d x \frac{\lambda^3}{[1 + \lambda^2(x - x_0)^2]^2} \xi \\ &- (-8g)^{1/2} \int d^d x \frac{\lambda}{1 + \lambda^2(x - x_0)^2} [\xi^3 + (1 - \delta)(\xi \eta^2 + \xi \chi^2)] \\ &+ \delta F(g, \vartheta, \theta, \xi, \eta, \chi) \end{aligned} \quad (90)$$

with M_L and $M_{T_1} = M_{T_2} = M_T$ of equation (14). For brevity, we have written down explicitly only the terms responsible for the leading contributions in the expansion (4). The remaining terms are denoted collectively by

$\delta F(g, \vartheta, \theta, \xi, \eta, \chi)$. The classical contribution to the energy functional is

$$H(\phi_{1c}, \phi_{2c}, \phi_{3c}) = -\frac{\lambda^\varepsilon 8\pi^2}{g} \left[1 - \frac{\varepsilon}{2} (2 + \ln \pi + \gamma) \right] - \frac{\lambda^\varepsilon \delta}{2g} \frac{8\pi^2}{3} \left[\sin^4 \theta \sin^2(2\vartheta) + \sin^2(2\theta) \right] + \mathcal{O}\left(\frac{\delta}{g} \varepsilon\right). \quad (91)$$

With similar steps as for $N = 2$ we obtain the bare imaginary parts of the vertex functions:

$$\begin{aligned} \text{Im } \Gamma_b^{(2M)}(q_m)_{i_1, i_2, \dots, i_{2M}} = & -c_b^{(3)} \int_0^\infty \frac{d\lambda}{\lambda} \left\{ \lambda^{d-M(d-2)} \left(-\frac{\lambda^\varepsilon 8\pi^2}{3g} \right)^{(d+3+2M)/2} \exp\left(\frac{\lambda^\varepsilon A}{g}\right) \prod_{\nu=1}^{2M} \left(\frac{q_\nu}{\lambda}\right)^2 \tilde{\phi}\left(\frac{q_\nu}{\lambda}\right) \right. \\ & \times \frac{1}{2} \int_0^{2\pi} \int_0^\pi \sin \theta d\theta d\vartheta u_{L i_1}(\vartheta, \theta) u_{L i_2}(\vartheta, \theta) \cdots u_{L i_{2M}}(\vartheta, \theta) \\ & \left. \times \exp\left[a \sin^4 \theta \sin^2(2\vartheta) + a \sin^2(2\theta)\right] \right\} [1 + \mathcal{O}(\varepsilon, g)], \quad (92) \end{aligned}$$

where

$$\begin{aligned} c_b^{(3)} = & 2^{-5/6} 3\pi^{-13/3} \exp\left[\frac{1}{3\varepsilon} (11 - 4\delta + 2\delta^2) + \frac{5\zeta'(2)}{\pi^2} - \frac{9}{2} \gamma - \frac{17}{4} \right. \\ & \left. + \frac{16}{9} \delta - \frac{7}{81} \delta^2 + \mathcal{O}(\delta^3)\right], \quad (i_k = 1, 2, 3), \quad (93) \end{aligned}$$

with a, A as in (45). After evaluating the integrals over the collective coordinates λ, θ , and ϑ we obtain for the imaginary part of the four-point vertex function

$$\begin{aligned} \text{Im } \Gamma_b^{(4)}(q_m)_{i,j,k,l} = & 2^8 3^2 \pi^4 c_b^{(3)} \frac{1}{\varepsilon} \sum_{n=0}^\infty \left\{ \frac{(-2\delta)^n}{\Gamma(2n + 7/2)} \left(\frac{g}{A}\right) \left(-\frac{\mu^\varepsilon A}{g}\right)^{\frac{d+7+2n}{2}} \exp\left(\frac{\mu^\varepsilon A}{g}\right) \right\} \end{aligned}$$

$$\begin{aligned}
& \times \sum_{p=0}^n \frac{\Gamma(3+2n-p)\Gamma\left(p+\frac{1}{2}\right)}{(n-p)!p!\Gamma(2(n-p)+3)} \left[3\Gamma\left(n-p+\frac{3}{2}\right)^2 (S_{ijkl} - \delta_{ijkl}) \right. \\
& \quad \left. + \Gamma\left(n-p+\frac{1}{2}\right)\Gamma\left(n-p+\frac{5}{2}\right) \delta_{ijkl} \right]. \tag{94}
\end{aligned}$$

Similarly the bare imaginary parts of $\frac{\partial}{\partial q^2}\Gamma^{(2)}(q)$ and $\Gamma^{(2,1)}(q_1, q_2; q_3)$ are found to be

$$\begin{aligned}
\frac{\partial}{\partial q^2}\text{Im}\Gamma_{\text{b}}^{(2)}(q)_{i,j} &= \delta_{ij} 2^4\pi^2 3 c_{\text{b}}^{(3)} \left[\gamma - \frac{1}{2} + \ln\left(\frac{q}{2\mu}\right) \right] \\
& \times \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \left[\frac{(-2\delta)^n}{n!} I(n) \left(-\frac{\mu^\varepsilon A}{g}\right)^{\frac{d+3+2n}{2}} \exp\left(\frac{\mu^\varepsilon A}{g}\right) \right], \tag{95}
\end{aligned}$$

and

$$\begin{aligned}
\text{Im}\Gamma_{\text{b}}^{(2,1)}(q_1, q_2; q_3)_{i,j} &= \delta_{ij} 2^4\pi^2 3^2 c_{\text{b}}^{(3)} \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \left\{ \frac{(-2\delta)^n}{n!} I(n) \right. \\
& \quad \left. \times \left[\gamma + \ln\left(\frac{q_3}{2\mu}\right) \right] \left(-\frac{\mu^\varepsilon A}{g}\right)^{\frac{d+5+2n}{2}} \exp\left(\frac{\mu^\varepsilon A}{g}\right) \right\}, \tag{96}
\end{aligned}$$

with

$$I(n) \equiv \sum_{p=0}^n \binom{n}{p} B\left(2+2n-p, p+\frac{1}{2}\right) B\left(n-p+\frac{1}{2}, n-p+\frac{3}{2}\right). \tag{97}$$

Now we apply the extended renormalization scheme. Using the coupling constants v and g , we obtain for the nonperturbative renormalized couplings

$$\begin{aligned}
g_{\text{R}} &= g_{\text{r}} + i2^7 3\pi^4 c_{\text{r}}^{(3)} \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \left[\frac{v_{\text{r}}^n}{n!} \left(\frac{3}{4\pi^2}\right)^n J_g(n) \left(-\frac{A}{g_{\text{r}}}\right)^{\frac{d+5+4n}{2}} \exp\left(\frac{A}{g_{\text{r}}}\right) \right], \\
v_{\text{R}} &= v_{\text{r}} - i2^7 3\pi^4 c_{\text{r}}^{(3)} \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \left[\frac{v_{\text{r}}^n}{n!} \left(\frac{3}{4\pi^2}\right)^n J_v(n) \left(-\frac{A}{g_{\text{r}}}\right)^{\frac{d+5+4n}{2}} \exp\left(\frac{A}{g_{\text{r}}}\right) \right], \tag{98}
\end{aligned}$$

with

$$\begin{aligned}
J_g(n) &\equiv \sum_{p=0}^n \binom{n}{p} B\left(3+2n-p, p+\frac{1}{2}\right) B\left(n-p+\frac{1}{2}, n-p+\frac{5}{2}\right), \\
J_v(n) &\equiv \sum_{p=0}^n \left\{ \binom{n}{p} \left[\frac{4(n-p)}{2(n-p)+3} \right] B\left(3+2n-p, p+\frac{1}{2}\right) \right. \\
&\quad \left. \times B\left(n-p+\frac{1}{2}, n-p+\frac{5}{2}\right) \right\}, \tag{99}
\end{aligned}$$

and

$$c_r^{(3)} = 3 \, 2^{-5/6} \pi^{-5/2} \exp\left[\frac{5\zeta'(2)}{\pi^2} - \frac{8}{3}\gamma - \frac{7}{12} + \mathcal{O}(\delta_r) + \mathcal{O}(\delta_r g_r)\right]. \tag{100}$$

For the nonperturbative renormalization constants Z^ϕ and Z^{ϕ^2} we find

$$\begin{aligned}
Z_{\text{np}}^\phi &= 1 - i \, 2^4 \pi^2 3 c_r^{(3)} \frac{1}{\varepsilon} \left(2^4 \pi^2 A P + \gamma - \frac{1}{2} - \ln 2 \right) \\
&\quad \times \sum_{n=0}^{\infty} \left[\frac{v_{\text{R}}^n}{n!} \left(\frac{3}{4\pi^2} \right)^n I(n) \left(-\frac{A}{g_{\text{R}}} \right)^{\frac{d+3+4n}{2}} \exp\left(\frac{A}{g_{\text{R}}}\right) \right] \tag{101}
\end{aligned}$$

and

$$\begin{aligned}
Z_{\text{np}}^{\phi^2} &= 1 + i \, 2^4 \pi^2 3^2 c_r^{(3)} \frac{1}{\varepsilon} \left(\frac{8\pi^2}{3} Q - \gamma + \ln 2 \right) \\
&\quad \times \sum_{n=0}^{\infty} \left[\frac{v_{\text{R}}^n}{n!} \left(\frac{3}{4\pi^2} \right)^n I(n) \left(-\frac{A}{g_{\text{R}}} \right)^{\frac{d+5+4n}{2}} \exp\left(\frac{A}{g_{\text{R}}}\right) \right]. \tag{102}
\end{aligned}$$

To calculate the nonperturbative renormalization group functions $\beta(g_{\text{R}}, v_{\text{R}})$, $\gamma(g_{\text{R}}, v_{\text{R}})$ and $\gamma_{\phi^2}(g_{\text{R}}, v_{\text{R}})$, we simply repeat the calculations of the case $N = 2$. Using the expressions (98) for g_{R} and v_{R} yields the β -functions

$$\begin{aligned}
\beta_g(g_{\text{R}}, v_{\text{R}}) &= \\
&\beta_g^{\text{p}}(g_{\text{R}}, v_{\text{R}}) - i \, 2^7 3 \pi^4 c_r^{(3)} \sum_{n=0}^{\infty} \left[\frac{v_{\text{R}}^n}{n!} \left(\frac{3}{4\pi^2} \right)^n J_g(n) \left(-\frac{A}{g_{\text{R}}} \right)^{\frac{d+7+4n}{2}} \exp\left(\frac{A}{g_{\text{R}}}\right) \right],
\end{aligned}$$

$$\beta_v(g_R, v_R) = \beta_v^p(g_R, v_R) + i2^7 3\pi^4 c_r^{(3)} \sum_{n=0}^{\infty} \left[\frac{v_R^n}{n!} \left(\frac{3}{4\pi^2} \right)^n J_v(n) \left(-\frac{A}{g_R} \right)^{\frac{d+7+4n}{2}} \exp\left(\frac{A}{g_R} \right) \right]. \quad (103)$$

The expressions for the nonperturbative anomalous dimension of the field ϕ and the composite field ϕ^2 are given by

$$\begin{aligned} \gamma(g_R, v_R) = & \gamma^p(g_R, v_R) + i2^4 \pi^2 3 c_r^{(3)} \left(2^7 \pi^4 3^{-1} P + \gamma - \frac{1}{2} - \ln 2 \right) \\ & \times \sum_{n=0}^{\infty} \left[\frac{v_R^n}{n!} \left(\frac{3}{4\pi^2} \right)^n I(n) \left(-\frac{A}{g_R} \right)^{\frac{d+5+4n}{2}} \exp\left(\frac{A}{g_R} \right) \right], \end{aligned} \quad (104)$$

and

$$\begin{aligned} \gamma_{\phi^2}(g_R, v_R) = & \gamma_{\phi^2}^p(g_R, v_R) + i2^4 \pi^2 3^2 c_r^{(3)} \left(\frac{8\pi^2}{3} Q - \gamma + \ln 2 \right) \\ & \times \sum_{n=0}^{\infty} \left[\frac{v_R^n}{n!} \left(\frac{3}{4\pi^2} \right)^n I(n) \left(-\frac{A}{g_R} \right)^{\frac{d+7+4n}{2}} \exp\left(\frac{A}{g_R} \right) \right], \end{aligned} \quad (105)$$

respectively.

7 Discussion and Conclusions

Our main results are the real and imaginary parts (103)–(105) of the renormalization-group functions β , γ and γ_{ϕ^2} in terms of the nonperturbative renormalized couplings g_R and v_R . Via the dispersion relation (5), we obtain the large-order behavior (7). It is useful to check the expansion terms proportional to powers of $1/|g|$ accompanying the v^n in the imaginary part of each renormalization-group function. If the power of $1/|g|$ is denoted by $p(n)$, we note that $p(n)$ is the same as for the corresponding vertex function before the integration over the dilatation parameter λ . This integration with the measure $d \ln \lambda$ reduces $p(n)$ by 1 [see (52)]. However this effect is compensated by the differentiation with respect to the logarithm of the scale

parameter μ in the definition of the renormalization group functions. The power $p(n)$ in the vertex functions can simply be explained: For a $2M$ -point vertex function with k ϕ^2 -insertions there is first a power $(1/|g|)^{(M+k)}$ from the fields. This is multiplied by a factor $(1/|g|)^{(d+(N-1)+1)/2}$ from the d translational, $(N-1)$ rotational, and one dilatational would-be zero modes. The classical contribution to the energy functional yields a factor $(1/|g|)^2$ for each power of the anisotropic constant v . Hence, for the renormalization group function which follows at the one-loop level from the vertex function $\Gamma^{(2M,k)}$, the value of $p(n)$ is given by

$$p(n) = \frac{d + N + 2(M + k) + 4n}{2}. \quad (106)$$

Inserting the imaginary parts into a dispersion relation (5), we can estimate the large-order coefficients of the corresponding series expansions

$$f(g, v) = \sum_{k,n=0}^{\infty} f_{kn} g^k v^n, \quad (107)$$

where $n \ll k$. The function f stands for β_g , β_v , γ and γ_{ϕ^2} , respectively. The rules how to go from the imaginary parts of f to f_{kn} are known from (7). For instance, the large-order coefficients of the β -functions for $N = 3$ are given by

$$\begin{aligned} \beta_{g, kn} &= c_r^{(3)} 2^7 3\pi^3 \left(\frac{3}{4\pi^2}\right)^n \frac{J_g(n)}{n!} \\ &\quad \times (-1)^k \left(\frac{3}{8\pi^2}\right)^k k! k^{2n+(d+5)/2} [1 + \mathcal{O}(1/k)], \\ \beta_{v, kn} &= -c_r^{(3)} 2^7 3\pi^3 \left(\frac{3}{4\pi^2}\right)^n \frac{J_v(n)}{n!} \\ &\quad \times (-1)^k \left(\frac{3}{8\pi^2}\right)^k k! k^{2n+(d+5)/2} [1 + \mathcal{O}(1/k)], \end{aligned} \quad (108)$$

with J , $c_r^{(3)}$ from (99) and (100), respectively.

The leading large- k behavior can be checked by means of a simple combinatorial analysis. We start with a series in terms of the standard couplings

u and v [see (1)]. For illustration, we consider the contribution of the pure u -powers. It is known from a theory with only one coupling constant, that the coefficients have the large-order behavior

$$f_l u^l \longrightarrow \gamma (-\alpha)^l \Gamma(l + b + 1) u^l \quad (l \gg 1). \quad (109)$$

Changing over to our couplings ($u = g - v$), the right-hand side of (109) contributes

$$\gamma (-1)^{l-k} (-\alpha)^l \Gamma(l + b + 1) \binom{l}{k} g^k v^{l-k}, \quad (110)$$

with

$$\binom{l}{k} = \frac{\Gamma(l + 1)}{k! (l - k)!}.$$

Replacing l by $l = n + k$, the contribution of $g^k v^n$ has the form

$$f_{kn} g^k v^n \longrightarrow \gamma (\alpha)^n (-\alpha)^k \frac{\Gamma(k + n + b + 1) \Gamma(k + n + 1)}{k! n!} g^k v^n. \quad (111)$$

Now, for large k , the Γ -functions can be approximated by

$$\Gamma(k + \delta + 1) \longrightarrow k! k^\delta (1 + \mathcal{O}(1/k)). \quad (112)$$

Thus in the region $n \ll k$ we obtain

$$f_{kn} \sim (-\alpha)^k k! k^{2n+b}, \quad (113)$$

in agreement with (108).

As stated in the introduction, the stable cubic fixed point is expected to lie in the vicinity of the isotropic fixed point. Since v becomes very small in this region, reasonable results should be obtained by resumming the g -series accompanying each power v^n . This will be done in a forthcoming publication.

References

- [1] E. Brezin, J. C. Le Guillou and J. Zinn-Justin,
Phys. Rev. B 10 (1974) 893.
- [2] H. Kleinert, V. Schulte-Frohlinde,
Phys. Lett. B 342 (1995) 284.
- [3] W. Janke and H. Kleinert,
Resummation of divergent Perturbation Series, World Scientific, Singa-
pore (to be published).
- [4] J. Zinn-Justin,
Quantum Field Theory and Critical Phenomena,
Clarendon Press, Oxford (1989).
- [5] J. C. Le Guillou and J. Zinn-Justin,
Phys. Rev. Lett. 39 (1977) 95.
- [6] J. C. Le Guillou and J. Zinn-Justin,
Phys. Rev. B 21 (1980) 3976.
- [7] J. C. Le Guillou and J. Zinn-Justin,
J. Physique Lett. 46 (1985) L137.
- [8] J. C. Le Guillou and J. Zinn-Justin,
J. Physique 48 (1987) 19.
- [9] J. C. Le Guillou and J. Zinn-Justin,
J. Physique 50 (1989) 1365.
- [10] L. N. Lipatov, JETP Lett. 25 (1977) 104.
- [11] L. N. Lipatov, Sov. Phys. JETP 45 (1977) 216.
- [12] E. Brezin, J. C. Le Guillou and J. Zinn-Justin,
Phys. Rev. D 15 (1977) 1544.
- [13] E. B. Bogomolny,
Phys. Lett. B 67 (1977) 193.

- [14] G. Parisi,
Phys. Lett. B 66 (1977) 167.
- [15] G. Parisi,
Phys. Lett. B 68 (1977) 361.
- [16] J. S. Langer, Ann. Phys. 41 (1967) 108.
- [17] J. C. Collins and D. E. Soper, Ann. Phys. 112 (1978) 209.
- [18] A. J. McKane, D. J. Wallace and O. F. de Alcantara Bonfim,
J. Phys. A 17 (1984) 1861.
- [19] T. Banks, C. M. Bender and T. T. Wu, Phys. Rev. D 8 (1973) 3346.
- [20] T. Banks and C. M. Bender, Phys. Rev. D 8 (1973) 3366.
- [21] W. Janke, Phys. Lett. A 143 (1990) 107.
- [22] H. Kleinert,
PATH INTEGRALS in Quantum Mechanics, Statistics
and Polymer Physics,
(World Scientific, Singapore, 1995)
- [23] A. J. McKane and D. J. Wallace, J. Phys. A 11 (1978) 2285.
- [24] I. T. Drummond, Nucl. Phys. B 94 (1975) 115.
- [25] I. T. Drummond and G. M. Shore, Ann. Phys. 121 (1979) 204.
- [26] Higher transcendental functions, Bateman manuscript project,
ed. A. Erdélyi (McGraw-Hill, New York, 1953) Vol. II.
- [27] J. Zittartz and J. S. Langer Phys. Rev. 148 (1966) 741.
- [28] J.-L. Gervais and B. Sakita, Phys. Rev. D 11 (1975) 2943.
- [29] G. 't Hooft and M. Veltman, Nucl. Phys. B 44 (1972) 189.
- [30] G. 't Hooft, Nucl. Phys. B 61 (1973) 455.