Gunma-Tech-95-1

Instanton effects and Witten complex in supersymmetric quantum mechanics on $SO(4)$

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We examine supersymmetric quantum mechanics on $SO(4)$ to realize Witten's idea. We find instanton solutions connecting approximate vacuums. We calculate Hessian matrices for these solutions to determine true vacuums. Our result is in agreement with de Rham cohomology of $SO(4)$. We also give a criterion for cancellation of instanton effects for a pair of instanton paths.

1 Introduction

In the last two decades theoretical physics and mathematics have been greatly developed through mutual stimulation [1]. At this present Seiberg-Witten theory[2] has attracted the attention of both theoretical physicists and mathematicians. In this paper we discuss Witten's pioneering work on supersymmetric quantum mechanics and de Rham theory proposed in 1982[3]. He has given a quantum mechanical interpretation about de Rham theory on a Riemannian manifold M. He has considered certain supersymmetric quantum mechanics on the manifold. His idea is to adopt a superpotential which is obtained from the Morse function h of the manifold. For each critical point of h, an approximate vacuum can be identified. By instanton effects some of them found to be false vacuums. True vacuums of the theory correspond to harmonic forms on the manifold. Witten's conjecture is that the number of true vacuums agrees with the dimension of de Rham cohomology. The diminishing of the number of vacuums implies the Morse inequality.

His idea has been established in the operator formalism[4] However, in the Laglangian formalism the Witten's program has been performed only for few cases. The case $M = \mathbf{R}$ has well been investigated in the context of supersymmtry breaking[5]. Recently, Yasui et al.[6] have investigated the case $M = SO(3)$. It has been shown that among four approximate vacuums two survive as true vacuums under instanton effects in agreement with de Rham cohomology of $M = SO(3)$. Our purpose in this paper is to examine a more complicated case $M = SO(4)$ in the Laglangian formalism. We discuss a condition for occurrence of instanton effects and identify true vacuums.

In Sec.2 supersymmetric quantum mechanics on a manifold M is formulated. In Sec.3 $M = SO(2)$ case is discussed for later convenience. In Sec.4 coordinates,metric and critical points for $M = SO(4)$ are treated. In Sec.5 Hessian matrices are calculated and instanton effects are discussed. Section 6 is devoted to summary and discussion.

2 Supersymmetric Quantum Mechanics on a Manifold

In this section the relevant supersymmetric quantum mechanics on a manifold M is formulated. In particular, approximate vacuums, a gradient flow equation and Hessian matrices are described. The supersymmetric quantum mechanics on M is derived from the following Laplacian:

$$\hat{H} = \frac{1}{2}(d_h d_h^\dagger + d_h^\dagger d_h), \tag{2.1}$$

where $d_h = e^{-h} d e^h, d_h^\dagger = e^h d^\dagger e^{-h}$, d is the exterior derivative and $d^\dagger$ is its adjoint operator. Fermion creation and annihilation operators $\hat{\psi}^{*i}$ and $\hat{\psi}^i$ can be identified with the exterior multiplication e_{dx^i} and the interior multiplication $i_{\frac{\partial}{\partial x^i}}$. Subsequently, on a flat metric $d = \hat{\psi}^{*i}\frac{\partial}{\partial x^i}, d^\dagger = \hat{\psi}^i \frac{\partial}{\partial x^i}$ and $\hat{\psi}^{*i_1} \ldots \hat{\psi}^{*i_m}|0>_F$ can be identified with bases of m-forms[3]. An approximate vacuum $|\Omega_l>$ localized near a critical point $P^{(l)}$ satisfies

$$d_h|\Omega_l >= d_h^\dagger|\Omega_l >= 0 \tag{2.2}$$

up to quantum effects. Some of approximate vacuums cease to satisfy (2.2) by quantum effects. If $|\Omega_l>$ is a true vacuum, it remains satisfying (2.2) after taking quantum effects into account. Because $d_h^\dagger$ is the adjoint of d_h , states satisfying (2.2) are elements in $\dfrac{Kerd_h}{Imd_h}$ called the Witten complex. We examine $d_h|\Omega_l>$ quantum mechanically on $M = SO(4)$. If $< \Omega_{l+1}|d_h|\Omega_l >\neq 0$, neither approximate vacuum is a true vacuum. According to [3], the following form is valid:

$$< \Omega_{l+1}|d_h|\Omega_l >= \sum_\gamma n_\gamma e^{-(h(P^{(l+1)})-h(P^{(l)}))}, \tag{2.3}$$

where n_γ is an integer assigned for each instanton path γ.

The supersymmetric hamiltonian derived from (2.1) is

$$2\hat{H} = -g^{-\frac{1}{2}}\nabla_i g^{\frac{1}{2}} g^{ij}\nabla_j + R_{ijkl}\hat{\psi}^k\hat{\psi}^{*l}\hat{\psi}^{*j}\hat{\psi}^i + g^{ij}\frac{\partial h}{\partial x^i}\frac{\partial h}{\partial x^j} + H_{ij}\{\hat{\psi}^{*j}, \hat{\psi}^{*i}\}, \tag{2.4}$$

where g_{ij} and R_{ijkl} are the Riemann metric and tensor, and $\nabla_i = \frac{\partial}{\partial x^i} - \Gamma^l_{ik}\hat{\psi}^{*k}\hat{\psi}_l$ is the covariant derivative; H_{ij} is the Hessian matrix

$$H_{ij} = (\partial_i \partial_j - \Gamma^k_{ij} \partial_k) h. \tag{2.5}$$

Corresponding Laglangian is

$$\mathcal{L} = \frac{1}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} + \frac{1}{2} g^{ij} \frac{\partial h}{\partial x^i} \frac{\partial h}{\partial x^j} + \psi^{*i} (\frac{d}{dt} \psi_i - \Gamma^k_{ij} \psi_k \frac{dx^j}{dt}) + H_{ij} \psi^{*j} \psi^i + \frac{1}{4} R_{ijkl} \psi^i \psi^j \psi^{*k} \psi^{*l}. \tag{2.6}$$

Classical solutions give main contribution to a pathintegral. In a supersymmetric model, we only have to consider quasi classical solutions satisfying equation of motion with fermion disregarded. Quasi classical solutions of (2.6) obey

$$\frac{dx^i}{dt} = \pm g^{ij} \frac{\partial h}{\partial x^j}. \tag{2.7}$$

By the operation of d_h, the Morse index l identical to the number of excited fermions increases by one. We choose one of the sign in (2.7) so that the value of the Morse function h increases.

For $M = SO(3)$, Eq.(2.7) has a pair of instanton solutions connecting two critical points whose Morse indices differ by one. Corresponding action for each instanton solution is written to the one-loop level as[6]

$$S = S_{cl} + \int_{-\infty}^{\infty} dt \{ -\frac{1}{2} \xi^a (\frac{d}{dt} + \lambda_a(t))(\frac{d}{dt} - \lambda_a(t)) \xi^a + \psi^a (\frac{d}{dt} - \lambda_a(t)) \psi^*_a \}, \tag{2.8}$$

where ξ^a are some linear combinations of geodesic coordinates around the instanton path and t is the time parameter of the instanton mode; $\lambda_a(t)$ are eigenvalues of the Hessian matrix $H_i^j = H_{ik} g^{kj}$ at a time t. Corresponding approximate hamiltonian is

$$2\tilde{H} = \sum_a (-(\frac{\partial}{\partial \xi^a})^2 + \lambda_a^2(t)(\xi^a)^2 + \lambda_a(t) \{ \hat{\psi}^*_a, \hat{\psi}^a \}). \tag{2.9}$$

An approximate vacuum near a critical point with the Morse index l is expressed as

$$|\Omega_l >= \pi^{-\frac{3}{2}} \prod_a |\lambda_a|^{\frac{1}{4}} \exp(-\frac{1}{2} \sum_{\mathrm{a}} |\lambda_{\mathrm{a}}| (\xi^{\mathrm{a}})^2) \hat{\psi}^*_{\mathrm{b}_1} \ldots \hat{\psi}^*_{\mathrm{b}_l} |0>_{\mathrm{F}}, \tag{2.10}$$

where $b_1, \ldots, b_l$ correspond to negative eigenvalues of the Hessian matrix H_i^j at the critical point. Like this, approximate vacuums are decided by eigenvalues of H_i^j.

Similar situation will hold for $M = SO(4)$. We calculate the Hessian matrix H_i^j to examine approximate vacuums. Moreover, we discuss instanton effects between adjacent approximate vacuums. Transition amplitudes are calculated from eigenvalues of H_i^j up to signs[6]. Owing to the notorious minus signs associated with fermions, it is not easy to determine the signs. It is crucial to determine the signs, because a pair of instanton effects can cancel each other.

3 $SO(2)$ Case

In this section we discuss the case $SO(2)$. We see how instanton effects occur or disappear.

We denote a group element in $SO(n)$ by $A = (a_{ij})$. The Morse function h for $SO(n)$ is given by[7]

$$h = \sum_a c_i a_{ii}, (c_i > 2c_{i+1} > 0). \tag{3.1}$$

For $M = SO(2)$, A is represented as

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{3.2}$$

The Morse function is $h = (c_1 + c_2)\cos\theta$ and the critical points are

$$P^{(0)} = \mathrm{diag}(-1 - 1), \mathrm{P}^{(1)} = \mathrm{diag}(\quad 1 \quad 1), \tag{3.3}$$

with the Morse indices $l = 0$ and $l = 1$ respectively. Since $g_{\theta\theta} = g^{\theta\theta} = 1$, the gradient flow equation (2.7) simplifies to

$$\frac{d\theta}{dt} = -(c_1 + c_2)\sin\theta. \tag{3.4}$$

This equation has the following instanton solution

$$\cos\theta = \tanh((c_1 + c_2)t + \alpha), \tag{3.5}$$

with an arbitrary constant α. Corresponding to $\sin\theta \geq 0$ or $\sin\theta \leq 0$, there are two paths connecting the critical points. For the latter path, we introduce a new coordinate θ' by $\theta' = -\theta$. We call the coordinates θ and θ' odd, because they are opposite in the sign. Thus, we only have to consider the path $0 \leq \theta \leq \pi$ in each coordinate system. We have now two instanton paths

$$\begin{pmatrix} \cos\theta & \mp\sin\theta \\ \pm\sin\theta & \cos\theta \end{pmatrix} \qquad (\sin\theta \geq 0) \tag{3.6}$$

with $\cos\theta$ (3.5).

These instanton paths lead to transitions between approximate vacuums

$$\begin{aligned} |0> &\longrightarrow |1> \equiv \hat{\psi}^*|0> \equiv |\theta> \sim d\theta|0>, \\ |0> &\longrightarrow \hat{\psi}'^*|0> \equiv |\theta'> \sim d\theta'|0>, \end{aligned} \tag{3.7}$$

where the state $|0>$ on the left(right) hand side means a bosonic(bosonic parts of an) approximate vacuum around the critical point $P^{(0)}(P^{(1)})$; the state $|1>$ implies one fermionic mode is excited;moreover, we mean by $|\theta>$ a fermionic mode corresponding to the coordinate θ is excited;the symbol $\sim$ means the identification of a p fermion excited state and a $p-$form[3]. Since $d\theta' = -d\theta$, we see $\hat{\psi}'^*|0> = -\hat{\psi}^*|0>$. The appearance of the relative minus sign for a pair of fermionic excitations is a general feature for an instanton mode. By the operator d_h, the approximate vacuum $|0>$ transforms as

$$d_h|0> = e^{-2(c_1+c_2)}(\hat{\psi}^* + \hat{\psi}'^*)|0> = 0, \tag{3.8}$$

and the matrix element $< 1|d_h|0 >$ vanishes. Thus, there is no instanton effect between the two approximate vacuums $|0>$ and $|1>$. Both of them remain to be true vacuums. We have one true vacuum at each critical point in agreement with the result of de Rham cohomology; $H^0(SO(2)) = H^1(SO(2)) = \mathbf{R}$.

As we have seen, a pair of instanton paths lead to fermionic states with opposite signs and there is no instanton effect. In the present case the original state is $|0>$, which is common to both instanton paths. In general, this is not the case. An approximate vacuum may provide opposite signs for a pair of instanton paths.

We call a state,like $|0>$, common to a pair of instanton paths an even state. We call a state, like $\hat{\psi}^*|0>$, providing different signs for a pair of instanton paths an odd state. We call the evenness and oddness parity. We can summarize the condition instanton effects occur. An approximate vacuum has definite parity for a pair of instanton paths. Instanton effects do not cancel each other for a pair of paths only when the parity of an approximate vacuum does not change along the paths. This judgment will be used later.

4 Coordinates, Metric and Critical Points on $SO(4)$

In this section we first introduce a coordinate system and an invariant metric on $M = SO(4)$. Let us introduce the generalized Euler angles[8]. Consider the following three infinitesimal generators of $SO(4)$:

$$E_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{4.1}$$

A group element A in $SO(4)$ can be parametrized as

$$A = \mathrm{e}^{cE_1} e^{zE_2} e^{yE_3} e^{bE_1} e^{xE_2} e^{aE_1} =$$

$$\begin{pmatrix} \begin{array}{l}(\cos b\cos c - \sin b\sin c\cos z)\cos a \\ +(-(\sin b\cos c + \cos b\sin c\cos z)\cos x \\ + \sin c\sin x\cos y\sin z)\sin a\end{array} & \begin{array}{l}(\cos b\cos c - \sin b\sin c\cos z)\sin a \\ +((\sin b\cos c + \cos b\sin c\cos x)\cos x \\ - \sin c\sin x\cos y\sin z)\cos a\end{array} & \begin{array}{l}(\sin b\cos c + \cos b\sin c\cos z)\sin x \\ + \sin c\cos x\cos y\sin z\end{array} & \sin c\sin y\sin z \\ \begin{array}{l}-(\cos b\sin c + \sin b\cos c\cos z)\cos a \\ +((\sin b\sin c - \cos b\cos c\cos z)\cos x \\ + \cos c\sin x\cos y\sin z)\sin a\end{array} & \begin{array}{l}-(\cos b\sin c + \sin b\cos c\cos z)\sin a \\ -((\sin b\sin c - \cos b\cos c\cos z)\cos x \\ + \cos c\sin x\cos y\sin z)\cos a\end{array} & \begin{array}{l}(-\sin b\sin c + \cos b\cos c\cos z)\sin x \\ + \cos c\cos x\cos y\sin z\end{array} & \cos c\sin y\sin z \\ \begin{array}{l}\sin x(\cos y\cos z + \cos b\cos x\sin z)\sin a \\ + \cos a\sin b\sin z\end{array} & \begin{array}{l}(\sin x\cos y\cos z + \cos b\cos x\sin z)\cos a \\ + \sin a\sin b\sin z\end{array} & \cos x\cos y\cos z - \cos b\sin x\sin z & \sin y\cos z \\ -\sin a\sin x\sin y, & \cos a\sin x\sin y & -\cos x\sin y & \cos y \end{pmatrix}$$

$$\tag{4.2}$$

In appendix A left invariant vector fields are noted. From (A.4) an $SO(4)$-invariant metric is obtained:

$$
(\sin^2 y \quad g^{ij}) =
$$

$$
\begin{array}{c} \\ a \\ b \\ c \\ x \\ z \\ y \end{array}
\begin{array}{c}
\begin{array}{cccccc} a & b & c & x & z & y \end{array} \\
\left(\begin{array}{cccccc}
\frac{1}{\sin^2 x} & -\frac{\cos x}{\sin^2 x} - \frac{\cos b \cos y \cos z}{\sin x \sin z} & \frac{\cos b \cos y}{\sin x \sin z} & 0 & -\frac{\sin b \cos y}{\sin x} & 0 \\
-\frac{\cos x}{\sin^2 x} - \frac{\cos b \cos y \cos z}{\sin x \sin z} & \begin{array}{c} \cot^2 x + \cot^2 z + \sin^2 y \\ +2\frac{\cos b \cos x \cos y \cos z}{\sin x \sin z} \end{array} & -\frac{\cos b \cos x \cos y}{\sin x \sin z} - \frac{\cos z}{\sin^2 z} & \frac{\sin b \cos y \cos z}{\sin z} & \frac{\sin b \cos x \cos y}{\sin x} & 0 \\
\frac{\cos b \cos y}{\sin x \sin z} & -\frac{\cos b \cos x \cos y}{\sin x \sin z} - \frac{\cos z}{\sin^2 z} & \frac{1}{\sin^2 z} & -\frac{\sin b \cos y}{\sin z} & 0 & 0 \\
0 & \frac{\sin b \cos y \cos z}{\sin z} & -\frac{\sin b \cos y}{\sin z} & 1 & -\cos b \cos y & 0 \\
-\frac{\sin b \cos y}{\sin x} & \frac{\sin b \cos x \cos y}{\sin x} & 0 & -\cos b \cos y & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \sin^2 y
\end{array}\right)
\end{array} , \tag{4.3}
$$

$$
(\mathrm{g}_{ij}) =
$$

$$
\begin{array}{c} \\ a \\ b \\ c \\ x \\ z \\ y \end{array}
\begin{array}{c}
\begin{array}{cccccc} a & b & c & x & z & y \end{array} \\
\left(\begin{array}{cccccc}
1 & \cos x & \begin{array}{c} \cos x \cos z \\ -\cos b \sin x \cos y \sin z \end{array} & 0 & \sin b \sin x \cos y & 0 \\
\cos x & 1 & \cos z & 0 & 0 & 0 \\
\begin{array}{c} \cos x \cos z \\ -\cos b \sin x \cos y \sin z \end{array} & \cos z & 1 & \sin b \cos y \sin z & 0 & 0 \\
0 & 0 & \sin b \cos y \sin z & 1 & \cos b \cos y & 0 \\
\sin b \sin x \cos y & 0 & 0 & \cos b \cos y & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
\end{array}
$$

$$ \tag{4.4} $$

In appendix B non-zero Christoffel symbols for this metric are noted.

From (3.1) the Morse function h is given by

$$
\begin{aligned}
h = c_1(&\cos a \cos b \cos c - \cos a \sin b \sin c \cos z - \sin a \sin b \cos c \cos x \\
&+ \sin a \sin c \sin x \cos y \sin z - \sin a \cos b \sin c \cos x \cos z) \\
+c_2(&- \sin a \cos b \sin c - \sin a \sin b \cos c \cos z - \cos a \sin b \sin c \cos x \\
&- \cos a \cos a \sin x \cos y \sin z + \cos a \cos b \cos c \cos x \cos z) \\
&+c_3(\cos x \cos y \cos z - \cos b \sin x \sin z) + c_4 \cos y, \\
&(c_1 > 2c_2 > 4c_3 > 8c_4 > 0).
\end{aligned} \tag{4.5}
$$

At critical points on M the Morse function h takes extremal values. The Morse function h (4.5) has eight critical points, which correspond to diagonal rotations in $SO(4)$;

$$
\begin{aligned}
P^{(0)} &= (-1\ -1\ -1\ -1), P^{(1)} = (-1\ -1\ \ 1\ \ 1), P^{(2)} = (-1\ \ 1\ -1\ \ 1), \\
P^{(3A)} &= (-1\ \ 1\ \ 1\ -1), P^{(3B)} = (1\ -1\ -1\ \ 1), P^{(4)} = (1\ -1\ \ 1\ -1), \\
P^{(5)} &= (1\ \ 1\ -1\ -1), P^{(6)} = (1\ \ 1\ \ 1\ \ 1).
\end{aligned} \tag{4.6}
$$

In Fig.1 instanton paths connecting these critical points are noted.

$$
\begin{array}{ccccccccccccc}
 & & & & & & P^{(3A)} & & & & & & \\
 & & & & & \nearrow & & \searrow & & & & & \\
P^{(0)} & \rightarrow & P^{(1)} & \Rightarrow & P^{(2)} & & & & P^{(4)} & \Rightarrow & P^{(5)} & \rightarrow & P^{(6)} \\
 & & & & & \searrow & & \nearrow & & & & & \\
 & & & & & & P^{(3B)} & & & & & &
\end{array}
$$

FIG.1. Instanton paths connecting critical points. The thick arrows show there exist instanton effects.

Now, the gradient flow equation is consist of simultaneous six differential equations. It will be difficult to find the general solution of the equation. However, we can find easily a pair of instanton solutions connecting adjacent two critical points. In the next section we discuss the instanton solutions and Hessian matrices. Transition amplitudes between approximate vacuums are also computed.

5 Instanton Solutions, Hessian Matrices and Transiton Amplitudes

A. $P^{(0)} \to P^{(1)}$

In this subsection we examine instanton solutions connecting $P^{(0)}$ and $P^{(1)}$. We find a pair of solutions satisfying (2.7). The one is

$$a \equiv c \equiv 0, b \equiv \pi, x = z = 0, y = \cos^{-1} \tanh(c_3 + c_4)(t + \alpha), \tag{5.1}$$

where y is the general solution of $\dfrac{dy}{dt} = -(c_3 + c_4) \sin y$. The symbol $\equiv$ means that we can safely make the substitution in g^{ij}. On the other hand, $x = z = 0$ corresponds to singular points in g^{ij}, but it is meaningful in the form $g^{ij}\partial_j h$. The other is obtained by replacing $x = z = 0$ by $x = z = \pi$ in (5.1). Corresponding instanton paths are

$$\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & \cos y & \pm \sin y \\ & & \mp \sin y & \cos y \end{pmatrix}. \tag{5.2}$$

Since $\sin b = 0$, the 6×6 matrix g^{ij} is reduced to a direct sum of a 3×3 matrix, a 2×2 matrix and a 1×1 matrix. To obtain $H_i^j = H_{ik} g^{kj}$, we only have to calculate relevant elements of H_{ij}. We note them in Appendix C. The Hessian matrix H_i^j for the first path is found to be

$$(\mathrm{H}_i^j) = \begin{array}{c} \\ a \\ b \\ c \\ x \\ z \\ y \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & x & z & y \end{array} \\ \left(\begin{array}{cccccc} c_1+\frac{c_3}{2}-\frac{c_3+c_4}{2}\cos y & \frac{c_3+c_4}{2}(1-\cos y) & \frac{c_4}{2} & & & \\ \frac{c_2+c_3}{2} & c_1-c_3 & \frac{c_2+c_3}{2} & & 0 & \\ \frac{c_2+c_4}{2} & \frac{c_3+c_4}{2}(-1+\cos y) & c_1+\frac{c_2+c_3}{2}-\frac{c_3+c_4}{2}\cos y & & & \\ & & & c_2-\frac{c_3+c_4}{2}\cos y & \frac{c_3-c_4}{2} & \\ & 0 & & \frac{c_3-c_4}{2} & c_2-\frac{c_3+c_4}{2}\cos y & \\ & & & & & -(c_3+c_4)\cos y \end{array}\right) \end{array} \tag{5.3}$$

where the indices $a, \cdots, y$ represent quantum fluctuations $\xi_a, \cdots, \xi_y$ around the classical solution (5.1). Taking some linear combinations of the fluctuation coordinates, the Hessian matrices H_i^j can be diagonalized. Eigenvalues of (5.3) are

$$\begin{aligned} \lambda_{\tilde{a}} &= c_1 + \frac{c_3 - c_4}{2} - \frac{c_3 + c_4}{2}\cos y, \quad \lambda_{\tilde{c}} = c_1 + \frac{c_2}{2} + \frac{c_3 + c_4}{2}(1 - \cos y), \\ \lambda_{\frac{x \pm z}{\sqrt{2}}} &= c_2 \pm \frac{c_3 - c_4}{2} - \frac{c_3 + c_4}{2}\cos y, \quad \lambda_{\tilde{b}} = c_1 - c_3, \quad \lambda_y = -(c_3 + c_4)\cos y, \end{aligned} \tag{5.4}$$

where $\tilde{a}, \tilde{b}$ and $\tilde{c}$ are some linear combinations of a, b and c. The Hessian matrix for the second path is obtained by changing signs of H_a^b, H_b^a, H_b^c and H_c^b in (5.3). Thus, the eigenvalues of H_i^j are common to the two instanton paths. Accordingly, bosonic transition amplitudes of the two instanton solutions are equal. In the following subsections, we will see this situation holds for any pair of instanton paths. We only have to concentrate on fermionic contributions to find whether the two transition amplitudes cancel each other or not. From (5.4) we see any eigenvalue is positive at $t = -\infty$, and no fermionic mode is exited. So, the approximate vacuum is a bosonic one

$$|0>=\pi^{-\frac{3}{2}}\prod_{i=1}^{6}\lambda_i^{\frac{1}{4}}e^{-\frac{1}{2}\lambda_i\xi_i^2}|0>_F, \tag{5.5}$$

where λ_i are eigenvalues of the Hessian matrices at this point. We represent by $|l>$ the approximate vacuum at each critical point $P^{(l)}$. At $t=\infty$ the y mode has the only negative eigenvalue. This implies that the fermionic mode corresponding to y is exited at $t=\infty$ and the following approximate vacuum $|1>=\hat{\psi}_y^*|0>\sim dy|0>$ is induced. The second path induces the following approximate vacuum $\hat{\psi}_{y'}^*|0>\sim dy'|0>$. In this process y is an odd coordinate (See Appendix D.) and we have $dy'=-dy$. The parity of the state $|0>$ does not change and we see

$$d_h|0>\sim(dy+dy')|0>=0. \tag{5.6}$$

The operator d_h increases the fermion number by 1. Since there is no state with fermion number -1, $|0>$ can not be written as $|0>=d_h|-1>$. Hence, $|0>$ is a true vacuum localized around $P^{(0)}$.

B. $P^{(5)}\to P^{(6)}$

In this subsection we examine instanton solutions connecting $P^{(5)}$ and $P^{(6)}$. Owing to the Poincaré duality[9], we can easily perform analysis from the knowledge of the previous subsection. We find a pair of solutions:

$$a\equiv c\equiv b\equiv 0, y=\cos^{-1}\tanh((c_3+c_4)t+\alpha), x=z=0(x=z=\pi). \tag{5.7}$$

These solutions are represented by the following paths

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos y & \pm\sin y \\ & & \mp\sin y & \cos y \end{pmatrix}. \tag{5.8}$$

From (C.1) we see H_{ij} for these instanton solutions are obtained from that of $P^{(0)}\to P^{(1)}$ by changing the signs except for linear terms of $\cos y$. From (4.3) we see the metric g^{ij}

for $P^{(5)} \to P^{(6)}$ is obtained from that of $P^{(0)} \to P^{(1)}$ by changing the signs of linear terms of $\cos y$. Accordingly, H_i^j for $P^{(5)} \to P^{(6)}$ is the opposite sign of (5.3) except for linear terms of $\cos y$. Thus, eigenvalues of H_i^j are

$$\begin{aligned} \lambda_{\tilde{a}} &= -c_1 + \frac{c_4 - c_3}{2} - \frac{c_3 + c_4}{2}\cos y, \quad \lambda_{\tilde{c}} = -c_1 - \frac{c_2}{2} - \frac{c_3 + c_4}{2}(1 + \cos y), \\ \lambda_{\frac{x \pm z}{\sqrt{2}}} &= -c_2 \mp \frac{c_3 - c_4}{2} - \frac{c_3 + c_4}{2}\cos y, \quad \lambda_{\tilde{b}} = c_3 - c_1, \quad \lambda_y = -(c_3 + c_4)\cos y. \end{aligned} \tag{5.9}$$

At $t = -\infty$ any mode except for y has a negative eigenvalue. Therefore, at $t = -\infty$ the following fermionic state is an approximate vacuum $|5>$ around $P^{(5)}$:

$$\begin{aligned} |5> \equiv |\tilde{a}, \tilde{b}, \tilde{c}, \frac{x+z}{\sqrt{2}}, \frac{x-z}{\sqrt{2}}> &\sim d\tilde{a} \wedge d\tilde{b} \wedge d\tilde{c} \wedge d\frac{x+z}{\sqrt{2}} \wedge d\frac{x-z}{\sqrt{2}}|0> \\ &\sim da \wedge db \wedge dc \wedge dx \wedge dz|0> . \end{aligned} \tag{5.10}$$

At $t = \infty$ the mode y has a negative eigenvalue. This means a fermionic mode corresponding to y is excited. Thus, the first instanton path induces an approximate vacuum $|6>$ around $P^{(6)}$:

$$|6> \equiv |\tilde{a}, \tilde{b}, \tilde{c}, \frac{x+z}{\sqrt{2}}, \frac{x-z}{\sqrt{2}}, y> \sim da \wedge db \wedge dc \wedge dx \wedge dz \wedge dy|0> . \tag{5.11}$$

In this process the coordinates a, c, x, z are even and b, y are odd (See Appendix D.). Accordingly, we see $|5>$ is odd and $|6>$ is even. Thus, from the general consideration in section 3 we conclude

$$< 6|d_h|5 >= 0. \tag{5.12}$$

The number of exited fermions is at most 6, so we have $d_h|6>= 0$. Hence, $|6>$ is a true vacuum.

C. $P^{(2)} \to P^{(3A)}$

In this subsection we discuss instanton solutions connecting $P^{(2)}$ and $P^{(3A)}$. We find a pair of solutions:

$$a \equiv c \equiv 0, b \equiv \pi, y = \cos^{-1}\tanh(-((c_3 - c_4)t + \alpha)), x = 0, z = \pi (x = \pi, z = 0), \tag{5.13}$$

where y is the general solution of $\frac{dy}{dt} = (c_3 - c_4)\sin y$. These solutions are represented as

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -\cos y & \mp\sin y \\ & & \mp\sin y & \cos y \end{pmatrix}. \tag{5.14}$$

Since $\sin b = 0$, we can calculate H_i^j as before. For the first solution we find

$$(\mathrm{H}_i^j) = \begin{array}{c} \\ a \\ b \\ c \\ x \\ z \\ y \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & x & z & y \end{array} \\ \begin{pmatrix} c_1 - \frac{c_2}{2} + \frac{c_3}{2}(\cos y - 1) & \frac{c_3}{2}(1 - \cos y) & \frac{c_2}{2} & & & \\ -\frac{c_2+c_3}{2} & c_1 + c_3 & \frac{c_2+c_3}{2} & & \mathbf{0} & \\ \frac{c_2}{2} & \frac{c_3}{2}(\cos y - 1) & c_1 - \frac{c_2}{2} + \frac{c_3}{2}(\cos y - 1) & & & \\ & & & -c_2 - \frac{c_4}{2}\cos y & -c_3 - \frac{c_4}{2} & \\ & \mathbf{0} & & -c_3 - \frac{c_4}{2} & -c_2 - \frac{c_4}{2}\cos y & \\ & & & & & (c_3 - c_4)\cos y \end{pmatrix} \end{array}, \tag{5.15}$$

For the second solution, the Hessiam matrix H_i^j is obtained by changing sings of H_a^b, H_b^a, H_b^c and H_c^b in (5.15). Therefore, eigenvalues are common to the two paths. They are

$$\begin{aligned} \lambda_{\tilde{a}} &= c_1 + \frac{c_3}{2}(\cos y - 1), \quad \lambda_{\tilde{c}} = c_1 + \frac{c_3}{2}(1 + \cos y), \\ \lambda_{\frac{x \pm z}{\sqrt{2}}} &= -c_2 \pm c_3 \pm \frac{c_4}{2}(1 - \cos y), \quad \lambda_{\tilde{b}} = c_1 - c_2, \quad \lambda_y = (c_3 - c_4)\cos y. \end{aligned} \tag{5.16}$$

At $t = -\infty$ fermionic modes corresponding to $\frac{x \pm z}{\sqrt{2}}$ are exited. At $t = \infty$ fermionic mode corresponding to y is also exited. Thus, the instantons induce

$$|\frac{x+z}{\sqrt{2}}, \frac{x-z}{\sqrt{2}} > \longrightarrow |\frac{x+z}{\sqrt{2}}, \frac{x-z}{\sqrt{2}}, y > \tag{5.17}$$

Now, the coordinates x, z, a, c are even and b, y are odd(See Appendix D.). So, the original state is odd and the new one is even. From the general consideration we conclude there is no instanton effect in the process $P^{(2)} \to P^{(3A)}$.

D. $P^{(3B)} \to P^{(4)}$

From the duality of $P^{(2)} \to P^{(3A)}$ we find two solutions connecting $P^{(3B)}$ and $P^{(4)}$:

$$a \equiv c \equiv b \equiv 0, y = \cos^{-1}(-\tanh((c_3 - c_4)t + \alpha)), x = 0, z = \pi(x = \pi, z = 0). \qquad (5.18)$$

These solutions are represented by the paths

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -\cos y & \mp\sin y \\ & & \mp\sin y & \cos y \end{pmatrix}. \qquad (5.19)$$

In this process x, z, a, c are even coordinates and b, y are odd ones(See Appendix D.). Eigenvalues of the corresponding Hessian matrix are

$$\begin{aligned} \lambda_{\frac{a+c}{\sqrt{2}}} &= -c_1 + \frac{c_3}{2}(\cos y + 1), \quad \lambda_{\frac{a-c+b}{\sqrt{2}}} = c_1 + \frac{c_3}{2}(\cos y - 1), \\ \lambda_{\frac{x\pm z}{\sqrt{2}}} &= c_2 \mp c_3 - \frac{c_4}{2}(\cos y \pm 1), \quad \lambda_{\tilde{b}} = -c_1 + c_2, \quad \lambda_y = (c_3 - c_4)\cos y, \end{aligned} \qquad (5.20)$$

where $\tilde{b}$ denotes b at $t = -\infty$ and $b + \frac{c_3}{c_2+c_3}(a - c)$ at $t = \infty$. From (5.20) we find the instantons induce the following transition between approximate vacuums:

$$|\frac{a+c}{\sqrt{2}}, b, \frac{a-c+b}{\sqrt{2}} > \to |\frac{a+c}{\sqrt{2}}, b + \frac{c_3}{c_2+c_3}(a-c), \frac{a-c+b}{\sqrt{2}}, y > . \qquad (5.21)$$

The original and the new states are essentially an odd one $da \wedge db \wedge dc|0>$ and an even one $da \wedge db \wedge dc \wedge dy|0>$ respectively. Thus, there is no instanton effect in the process $P^{(3B)}$ and $P^{(4)}$.

E. $P^{(2)} \to P^{(3B)}$

In this subsection we discuss instanton solutions connecting $P^{(2)}$ and $P^{(3B)}$. Setting $b \equiv 0, y = 0, x \equiv z \equiv \pi/2$, the gradient flow equation (2.7) reduces to

$$\begin{aligned} \frac{d(a-c)}{dt} &= -(c_1 - c_2)\sin(a - c), \\ \frac{d(a+c)}{dt} &= -(c_1 + c_2)\sin(a + c). \end{aligned} \qquad (5.22)$$

Setting $a+c\equiv 0$, we get the following solutions:

$$b\equiv 0, y=0, x\equiv z\equiv \pi/2, a+c\equiv 0, a-c=\pm\cos^{-1}\tanh((c_1-c_2)t+\alpha). \tag{5.23}$$

These are represented by the paths

$$\begin{pmatrix} \cos(a-c) & \pm\sin(a-c) & & \\ \pm\sin(a-c) & -\cos(a-c) & & \\ & & -1 & \\ & & & 1 \end{pmatrix}. \tag{5.24}$$

Since $\sin b \equiv \cos x \equiv \cos z \equiv 0$, the metric g^{ij} reduces to a direct sum of two 2×2 matrices and two 1×1 matrices. Relevant H_{ij} to obtain H_i^j are given in (C.2). We get the following Hessian matrix for the both solutions. Non-zero matrix elements are

$$(H_i^j) = \begin{array}{c} \\ a \\ c \end{array}\begin{pmatrix} a & c \\ c_1(\frac{\sin^2 a}{2}-\cos^2 a)+c_2(\frac{\cos^2 a}{2}-\sin^2 a) & -\frac{c_1}{2}\sin^2 a-\frac{c_2}{2}\cos^2 a \\ -\frac{c_1}{2}\sin^2 a-\frac{c_2}{2}\cos^2 a & c_1(\frac{\sin^2 a}{2}-\cos^2 a)+c_2(\frac{\cos^2 a}{2}-\sin^2 a) \end{pmatrix},$$

$$(H_i^j) = \begin{array}{c} \\ x \\ z \end{array}\begin{pmatrix} x & z \\ \frac{c_1}{2}\sin^2 a+\frac{c_2}{2}\cos^2 a+c_3-\frac{c_4}{2} & \frac{c_1}{2}\sin^2 a+\frac{c_2}{2}\cos^2 a+\frac{c_4}{2} \\ \frac{c_1}{2}\sin^2 a+\frac{c_2}{2}\cos^2 a+\frac{c_4}{2} & \frac{c_1}{2}\sin^2 a+\frac{c_2}{2}\cos^2 a+c_3-\frac{c_4}{2} \end{pmatrix},$$

$$H_b^b=-c_1\cos^2 a-c_2\sin^2 a+c_3, \qquad H_y^y=c_1\sin^2 a+c_2\cos^2 a-c_4. \tag{5.25}$$

Eigenvalues of this matrix are

$$\begin{aligned} \lambda_{\frac{a+c}{\sqrt{2}}} &= -c_1\cos^2 a-c_2\sin^2 a, \qquad \lambda_{\frac{a-c}{\sqrt{2}}}=(c_2-c_1)\cos(a-c), \\ \lambda_{\frac{x+z}{\sqrt{2}}} &= c_1\sin^2 a+c_2\cos^2 a, \qquad \lambda_{\frac{x-z}{\sqrt{2}}}=c_3-c_4, \\ \lambda_b &= -c_1\cos^2 a-c_2\sin^2 a+c_3, \qquad \lambda_y=c_1\sin^2 a+c_2\cos^2 a-c_4. \end{aligned} \tag{5.26}$$

From (5.26) we see the instanton solutions cause the following transition:

$$|\frac{a+c}{\sqrt{2}}, b> \to |\frac{a+c}{\sqrt{2}}, b, \frac{a-c}{\sqrt{2}}> . \tag{5.27}$$

In this process a, c are odd coordinates and x, y, z, b are even ones and the original state is odd and the new state is even. Hence, the two instanton effects cancel each other.

F. $P^{(3A)} \to P^{(4)}$

Setting $b \equiv \pi, y = \pi$ and changing $a \to -a, c \to -c$ in (5.23) we obtain two solutions connecting $P^{(3A)}$ and $P^{(4)}$. Corresponding paths are

$$\begin{pmatrix} -\cos(a-c) & \mp\sin(a-c) & & \\ \mp\sin(a-c) & \cos(a-c) & & \\ & & 1 & \\ & & & -1 \end{pmatrix}. \tag{5.28}$$

The Hessian matrix H_i^j and its eigenvalues are given by the opposite sign of (5.25) and (5.26) respectively. The following transition is induced:

$$|\frac{x+z}{\sqrt{2}}, \frac{x-z}{\sqrt{2}}, y > \to |\frac{x+z}{\sqrt{2}}, \frac{x-z}{\sqrt{2}}, y, \frac{a-c}{\sqrt{2}} > . \tag{5.29}$$

The parities of the coordinates are the same as in the previous subsection. These two states have opposite parities. Hence,there is no instanton effect.

G. $P^{(1)} \to P^{(2)}$

As we have seen there is no instanton effect in the previous six processes. In the following two processes we will see there exist instanton effects.

We find two instanton solutions connecting $P^{(1)}$ and $P^{(2)}$:

$$a \equiv -c \equiv \pm\frac{\pi}{2}, x \equiv z \equiv \frac{\pi}{2}, y = 0, b = \cos^{-1}(-\tanh((c_2 - c_3)t + \alpha)). \tag{5.30}$$

Corresponding paths are

$$\begin{pmatrix} -1 & & & \\ & \cos b & \pm\sin b & \\ & \pm\sin b & -\cos b & \\ & & & 1 \end{pmatrix}. \tag{5.31}$$

In this process a, c are even coordinates and x, z, y, b are odd ones. Since $y = 0$ corresponding to a singular metric, we set $y = \epsilon$. Then we have

$(\mathrm{g}^{ij}) =$

$$
\begin{array}{c|cccccc|}
 & a & b & c & x & z & y \\
\hline
a & \frac{1}{\epsilon^2}+\frac{1}{3} & 0 & \left(\frac{1}{\epsilon^2}-\frac{1}{6}\right)\cos b & 0 & \left(-\frac{1}{\epsilon^2}+\frac{1}{6}\right)\sin b & 0 \\
b & 0 & 1 & 0 & 0 & 0 & 0 \\
c & \left(\frac{1}{\epsilon^2}-\frac{1}{6}\right)\cos b & 0 & \frac{1}{\epsilon^2}+\frac{1}{3} & \left(-\frac{1}{\epsilon^2}+\frac{1}{6}\right)\sin b & 0 & 0 \\
x & 0 & 0 & \left(-\frac{1}{\epsilon^2}+\frac{1}{6}\right)\sin b & \frac{1}{\epsilon^2}+\frac{1}{3} & \left(-\frac{1}{\epsilon^2}+\frac{1}{6}\right) & 0 \\
z & \left(-\frac{1}{\epsilon^2}+\frac{1}{6}\right)\sin b & 0 & 0 & \left(-\frac{1}{\epsilon^2}+\frac{1}{6}\right)\cos b & \frac{1}{\epsilon^2}+\frac{1}{3} & 0 \\
y & 0 & 0 & 0 & 0 & 0 & 1
\end{array} . \tag{5.32}
$$

This expression shows that the modes b and y do not combine with another mode. Since g^{ij} and Γ^k_{ij} are independent of a and c, H_{ij} and $H_i^{\,j}$ are common to the two solutions. From (5.32) and (C.3) we get

$(\mathrm{H}_i^{\,j}) =$

$$
\begin{array}{c} \\ a \\ c \\ x \\ z \\ b \\ y \end{array}
\begin{array}{c}
\begin{array}{cccccc} a & c & x & z & b & y \end{array} \\
\left(
\begin{array}{cccccc}
\frac{c_1}{2} - c_2 \cos b & -\frac{c_1}{2}\cos b & \frac{c_2+c_3}{2}\sin b & \frac{c_1}{2}\sin b & & \\
-\frac{c_1}{2}\cos b & \frac{c_1}{2} - c_2\cos b & \frac{c_1}{2}\sin b & \frac{c_2+c_3}{2}\sin b & & \mathbf{0} \\
\frac{c_2+c_3}{2}\sin b & \frac{c_1}{2}\sin b & \frac{c_1}{2} + c_3\cos b & \frac{c_1}{2}\cos b & & \\
\frac{c_1}{2}\sin b & \frac{c_2+c_3}{2}\sin b & \frac{c_1}{2}\cos b & \frac{c_1}{2} + c_3\cos b & & \\
 & & & & (c_3 - c_2)\cos b & 0 \\
 & & \mathbf{0} & & 0 & c_1 - c_4
\end{array}
\right).
\end{array}
\tag{5.33}
$$

Eigenvalues of (5.33) are

$$
\begin{aligned}
2\lambda_1 &= 2c_1 - c_2 - c_3 + (c_3 - c_2)\cos b, \qquad 2\lambda_2 = c_2 + c_3 + (c_3 - c_2)\cos b, \\
2\lambda_3 &= 2c_1 + c_2 + c_3 + (c_3 - c_2)\cos b, \qquad 2\lambda_4 = -c_2 - c_3 + (c_3 - c_2)\cos b, \\
\lambda_b &= (c_3 - c_2)\cos b, \qquad \lambda_y = c_1 - c_4.
\end{aligned}
\tag{5.34}
$$

Linear combinations of modes a, c, x and c correspond to the first four eigenvalues. Only the fourth eigenvalue λ_4 is negative. At $t = -\infty$ the negative eigenvalue is $-c_3$ and the corresponding mode is $\frac{x+z}{\sqrt{2}}$, which is an odd coordinate. At $t = \infty$ the negative eigenvalue has changed to $-c_2$ and the corresponding mode is $\frac{a-c}{\sqrt{2}}$, which is an even coordinate. The excited eigenmode has changed from the odd one to the even one. This is a new feature that can not be seen in the previous six processes. Thus, the instanton solutions cause transition between odd states:

$$|\frac{x+z}{\sqrt{2}}>\rightarrow|\frac{a-c}{\sqrt{2}},b>. \tag{5.35}$$

From the general argument in Sect.3 the two instanton effects do not cancel each other.

H. $P^{(4)}\rightarrow P^{(5)}$

Replacing $y=0$ by $y=\pi$ in (5.30), we obtain two instanton solutions connecting $P^{(4)}$ and $P^{(5)}$. These solutions are represented as

$$\begin{pmatrix} 1 & & & \\ & -\cos b & \mp\sin b & \\ & \pm\sin b & \cos b & \\ & & & -1 \end{pmatrix}. \tag{5.36}$$

Corresponding Hessian matrix H_i^j is obtained by the following replacements $c_1\rightarrow -c_1, c_4\rightarrow -c_4$ in the diagonal part of H_i^j (5.33). Eigenvalues of H_i^j are

$$\begin{aligned} 2\lambda_1 &= c_2+c_3+(c_3-c_2)\cos b, \qquad 2\lambda_2=-2c_1+c_2+c_3+(c_3-c_2)\cos b, \\ 2\lambda_3 &= -c_2-c_3+(c_3-c_2)\cos b, \qquad 2\lambda_4=2c_1-c_2-c_3+(c_3-c_2)\cos b, \\ \lambda_b &= (c_3-c_2)\cos b, \qquad \lambda_y=c_4-c_1. \end{aligned} \tag{5.37}$$

At $t=-\infty$ there are three negative eigenvalues $-c_1+c_2, -c_3$ and $-c_1-c_3$ except for c_4-c_1. Corresponding modes are $\frac{a-c}{\sqrt{2}}$, $\frac{x-z}{\sqrt{2}}$ and $\frac{x+z}{\sqrt{2}}$ respectively. The three eigenvalues gradually change to $-c_1+c_3, -c_2$ and $-c_1-c_2$ respectively at $t=\infty$. Corresponding modes are $\frac{x-z}{\sqrt{2}}$, $\frac{a-c}{\sqrt{2}}$ and $\frac{a+c}{\sqrt{2}}$ respectively. We have now the following transition of states:

$$|\frac{a-c}{\sqrt{2}},\frac{x-z}{\sqrt{2}},\frac{x+z}{\sqrt{2}},y>\rightarrow|\frac{x-z}{\sqrt{2}},\frac{a-c}{\sqrt{2}}\frac{a+c}{\sqrt{2}},y,b>. \tag{5.38}$$

The parities of the coordinates are the same as in the previous subsection and the parities of both states are odd. Hence,there exist instanton effects.

6 Summary and Discussion

We have investigated instanton solutions connecting the approximate vacuums on $SO(4)$. For any pair of adjacent approximate vacuums we have found two instanton paths and have discussed the transition amplitude. We have found in the two processes instanton effects exist. We have given the criterion for the presence of instanton effects; if the parities of the two approximate vacuums are identical, there exist instanton effects. If they are opposite, the two instanton effects cancel each other. Two approximate vacuums coupled by instanton effects are not true vacuums. The approximate vacuums around $P^{(1)}, P^{(2)}, P^{(4)}$ and $P^{(5)}$ are not true vacuums. We have left the four true vacuums around $P^{(0)}, P^{(3A)}, P^{(3B)}$ and $P^{(6)}$. This result is in agreement with the de Rham cohomology of $SO(4)$.

In the $SO(3)$ case, two time dependent harmonic oscillator modes accompany with one instanton mode[6]. One definite Euler angle corresponds to the instanton mode in every process. In the $SO(4)$ case,a particular coordinate corresponds to the instanton mode in each process. The other five coordinates except for the instanton mode play the role of time dependent harmonic oscillators.

We have found a pair of instanton solutions connecting two critical points with Morse index differs by one. We do not have discussed the general solution of the gradient flow equation (2.7). However, I believe there are not any other solutions connecting the critical points. To make this point clear, further investigation will be needed.

Acknowledgments

The author thanks for Dr.K.Kuribayashi for useful discussion. He is indebted to Dr.Y.Yasui for valuable discussion.

Appendix A: Left Invariant Vector Fields and an Invariant Metric

From the expression of the group element A (4.2) left invariant vector fields e_{ij} corresponding to E_{ij} can be read as

$$\begin{aligned}
e_{12} &= \frac{\partial}{\partial a}, \qquad e_{23} = -\frac{\sin a\cos x}{\sin x}\frac{\partial}{\partial a} + \frac{\sin a}{\sin x}\frac{\partial}{\partial b} + \cos a\frac{\partial}{\partial x}, \\
e_{34} &= -\frac{\sin b\sin x\cos z}{\sin y\sin z}\frac{\partial}{\partial b} + \frac{\sin b\sin x}{\sin y\sin z}\frac{\partial}{\partial c} - \frac{\sin x\cos y}{\sin y}\frac{\partial}{\partial x} + \cos x\frac{\partial}{\partial y} + \frac{\cos b\sin x}{\sin y}\frac{\partial}{\partial z}, \\
e_{13} &= -\frac{\cos a\cos x}{\sin x}\frac{\partial}{\partial a} + \frac{\cos a}{\sin x}\frac{\partial}{\partial b} - \sin a\frac{\partial}{\partial x}, \\
e_{14} &= \frac{\cos a\cos y}{\sin x\sin y}\frac{\partial}{\partial a} - (\frac{\cos a\cos x\cos y}{\sin x\sin y} + \frac{\cos a\cos b\cos z - \sin a\sin b\cos x\cos z}{\sin y\sin z})\frac{\partial}{\partial b} \\
&+ \frac{\cos a\cos b - \sin a\sin b\cos x}{\sin y\sin z}\frac{\partial}{\partial c} + \sin a\cos x\frac{\cos y}{\sin y}\frac{\partial}{\partial x} + \sin a\sin x\frac{\partial}{\partial y} \\
&- \frac{\cos a\sin b + \sin a\cos b\cos x}{\sin y}\frac{\partial}{\partial z}, \\
e_{24} &= \frac{\sin a\cos y}{\sin x\sin y}\frac{\partial}{\partial a} - \cos a\cos x\frac{\cos y}{\sin y}\frac{\partial}{\partial x} - \cos a\sin x\frac{\partial}{\partial y} \\
&+ \frac{\cos a\cos b\cos x - \sin a\sin b}{\sin y}\frac{\partial}{\partial z}. \qquad \text{(A.1)}
\end{aligned}$$

An $SO(4)$ invariant metric is now given by

$$g^{ij} = \sum_{(\alpha\beta)} e^i_{\alpha\beta} e^j_{\alpha\beta}. \qquad \text{(A.2)}$$

Appendix B: Christoffel Symbols

In this appendix, we summarize 38 non-zero Christoffel symbols under the metric (4.4). We use them in computing the Hessian H_{ij}.

$$\begin{aligned}
\Gamma^z_{xy} &= -\Gamma^y_{xz} = -\frac{1}{2}\cos b\sin y, \qquad \Gamma^x_{xy} = \Gamma^z_{yz} = \Gamma^a_{ya} = \Gamma^c_{yc} = \frac{\cos y}{2\sin y}, \\
\Gamma^b_{xy} &= \frac{\sin b\cos z}{2\sin y\sin z}, \quad \Gamma^b_{xy} = \frac{\sin b}{2\sin y\sin z}, \quad \Gamma^b_{xz} = \frac{1}{2}\sin b\cos y, \quad \Gamma^b_{xa} = \Gamma^a_{xb} = -\frac{1}{2\sin x}, \\
\Gamma^a_{xa} &= \Gamma^b_{xb} = \frac{\cos x}{2\sin x}, \qquad \Gamma^y_{xc} = \frac{1}{2}\sin b\sin y\sin z, \qquad \Gamma^b_{xc} = -\frac{1}{2}\cos b\cos y\sin z + \frac{\cos x}{2\sin x},
\end{aligned}$$

$$
\begin{aligned}
\Gamma^a_{xc} &= \frac{\cos x\cos z}{2\sin x}, \qquad \Gamma^x_{yz} = -\frac{\cos b}{2\sin y}, \qquad \Gamma^a_{yz} = -\frac{\sin b}{2\sin x\sin y}, \qquad \Gamma^b_{yz} = \frac{\sin b\cos x}{2\sin x\sin y},\\
\Gamma^z_{ya} &= -\frac{\sin b\sin x}{2\sin y}, \qquad \Gamma^b_{ya} = -\frac{\cos x\cos y}{2\sin y} - \frac{\cos b\sin y\cos z}{2\sin y\sin z}, \qquad \Gamma^c_{ya} = \frac{\cos b\sin x}{2\sin y\sin z},\\
\Gamma^x_{yc} &= -\frac{\sin b\sin z}{2\sin y}, \qquad \Gamma^b_{yc} = -\frac{\cos y\cos z}{2\sin y} - \frac{\cos b\cos y\sin z}{2\sin x\sin y}, \qquad \Gamma^a_{yc} = \frac{\cos b\sin z}{2\sin x\sin y},\\
\Gamma^y_{za} &= \frac{1}{2}\sin b\sin x\sin y, \qquad \Gamma^b_{xc} = -\frac{1}{2}\cos b\sin x\cos y + \frac{\cos x}{2\sin z}, \qquad \Gamma^c_{za} = -\frac{\cos x}{2\sin z},\\
\Gamma^b_{zb} &= \Gamma^c_{zc} = \frac{\cos z}{2\sin z}, \qquad \Gamma^c_{zb} = \Gamma^b_{zc} = -\frac{1}{2\sin z}, \qquad \Gamma^x_{ab} = \frac{1}{2}\sin x,\\
\Gamma^x_{ac} &= \frac{1}{2}\sin x\cos z, \qquad \Gamma^y_{ac} = -\frac{1}{2}\cos b\sin x\sin y\sin z, \qquad \Gamma^z_{ac} = \frac{1}{2}\cos x\sin z,\\
\Gamma^b_{ac} &= -\frac{1}{2}\sin b\sin x\cos y\sin z, \qquad \Gamma^z_{bc} = \frac{1}{2}\sin z.
\end{aligned} \tag{B.1}
$$

Appendix C: Relevant Hessian Matrices

In this appendix, we note relevant Hessian matrices H_{ij}.

The following H_{ij} are necessary in computing H^j_i for the processes $P^{(0)} \to P^{(1)}, P^{(5)} \to P^{(6)}, P^{(2)} \to P^{(3A)}$ and $P^{(3B)} \to P^{(4)}$:

$$
\begin{aligned}
H_{aa} &= H_{cc} = -c_1\cos b + c_2(\sin x\sin z\cos y - \cos b\cos x\cos z),\\
H_{ab} &= -c_1\cos b\cos x + \frac{c_2}{2}(\cos b\sin^2 x\cos z + \sin x\sin z\cos x\cos y - 2\cos b\cos z)\\
&+ \frac{c_3}{2}(\sin^2 x\cos z\cos y + \cos b\sin x\sin z\cos x),\\
H_{ac} &= c_1(\sin x\sin z\cos y - \cos b\cos x\cos z) + \frac{c_2}{2}(\cos b(\sin^2 x\cos^2 z + \cos^2 x\sin^2 z - 2)\\
&+ 2\sin x\sin z\cos x\cos z\cos y) - \frac{1}{2}(c_3\cos x\cos z + c_4)\cos b\sin x\sin z\sin^2 y\\
&- \frac{c_3}{2}(\cos y(\sin^2 x\cos^2 z + \cos^2 x\sin^2 z) + \cos b\sin x\sin z\cos x\cos z),\\
H_{bb} &= -c_1\cos b - c_2\cos b\cos x\cos z + c_3\cos b\sin x\sin z,\\
H_{bc} &= -c_1\cos b\cos z + \frac{c_2}{2}(\cos b\cos x\sin^2 z + \sin x\sin z\cos z\cos y - 2\cos b\cos x),\\
H_{xx} &= H_{zz} = -\cos x\cos z(c_2\cos b + c_3\cos y),\\
H_{xz} &= -\cos x\cos z(c_2\cos y + c_3\cos b) - \frac{1}{2}(c_3\cos x\cos z + c_4)\cos b\sin^2 y,\\
H_{yy} &= -(c_3\cos x\cos z + c_4)\cos y.
\end{aligned} \tag{C.1}
$$

Note that we have not discarded the terms containing $\sin x$ or $\sin z$ which can survive combined with the metric g^{ij} .

We use the following H_{ij} in the processes $P^{(2)} \to P^{(3B)}$ and $P^{(3A)} \to P^{(4)}$:

$$
\begin{aligned}
H_{aa} &= H_{cc} = c_1(\sin^2 a \cos y - \cos^2 a \cos b) + c_2(\cos^2 a \cos y - \sin^2 a \cos b), \\
H_{ac} &= c_1(\cos^2 a \cos y - \sin^2 a \cos b) + c_2(\sin^2 a \cos y - \cos^2 a \cos b) \\
&+ \frac{1}{2} \cos b \sin^2 y(c_1 \sin^2 a + c_2 \cos^2 a - c_4), \\
H_{bb} &= -c_1 \cos a \cos a \cos c + c_2 \sin a \cos b \sin c + c_3 \cos b, \\
H_{xx} &= H_{zz} = c_1 \sin^2 a \cos y + c_2 \cos^2 a \cos y + c_3 \cos b, \\
H_{xz} &= \frac{1}{2} \cos b \sin^2 y(-c_1 \sin^2 a - c_2 \cos^2 a + c_4), \\
H_{yy} &= c_1 \sin^2 a \cos y + c_2 \cos^2 a \cos y - c_4 \cos y, \qquad \text{(C.2)}
\end{aligned}
$$

where we have used $c \equiv -a$.

In the processes $P^{(1)} \to P^{(2)}$ and $P^{(4)} \to P^{(5)}$ the following H_{ij} are necessary:

$$
\begin{aligned}
H_{aa} &= H_{cc} = c_1 \cos y - c_2 \cos b, \\
H_{ac} &= c_1(\frac{1}{2} \sin^2 y - 1) \cos b + c_2(1 - \frac{1}{2} \sin^2 b) \cos y + \frac{c_3}{2} \cos y \sin^2 b, \\
H_{ax} &= H_{cz} = \frac{c_2 + c_3}{2} \sin b, \\
H_{az} &= H_{cx} = c_1(1 - \frac{1}{2} \sin^2 y) \sin b + \frac{c_3 - c_2}{2} \cos y \cos b \sin b, \\
H_{xx} &= H_{zz} = c_1 \cos y + c_3 \cos b, \\
H_{xz} &= (1 - \frac{1}{2} \sin^2 y)(c_1 \cos b + c_3 \cos y) + \frac{c_2}{2} \cos y \sin^2 b, \\
H_{bb} &= (c_3 - c_2) \cos b, \qquad H_{yy} = (c_1 - c_4) \cos y. \qquad \text{(C.3)}
\end{aligned}
$$

Appendix D: Parities of Coordinates

In this appendix, we identify parities of coordinates.

We discuss the process $P^{(0)} \to P^{(1)}$ as an example. Around the first instanton solution (5.1) connecting $P^{(0)}$ and $P^{(1)}$,the group element A is of the form up to second order of the fluctuation coordinates:

$$\begin{pmatrix} -1 & -a-b-c & 0 & 0 \\ a+b+c & -1 & -x+z\cos y_0 & 0 \\ 0 & z-x\cos y_0 & \cos y_0 & y\cos y_0 \\ 0 & 0 & -y\cos y_0 & \cos y_0 \end{pmatrix}, \tag{D.1}$$

where y_0 denotes the classical solution and $a, b, \ldots, y$ are abbreviations of the fluctuation coordinates. On the other hand, around the second solution of $P^{(0)} \to P^{(1)}$ A is of the form:

$$\begin{pmatrix} -1 & -a+b-c & 0 & 0 \\ a-b+c & -1 & -x+z\cos y_0 & 0 \\ 0 & z-x\cos y_0 & \cos y_0 & -y\cos y_0 \\ 0 & 0 & y\cos y_0 & \cos y_0 \end{pmatrix}. \tag{D.2}$$

Comparing (D.1) and (D.2), we find a, c, x and z are even coordinates and b and y are odd ones in this process.

In the same way, we can identify parities of the coordinates in each process. Results are summarized in TABLE 1.

TABLE 1: Parities of coordinates.

Processes	even coordinates	odd coordinates
$P^{(0)} \to P^{(1)}, P^{(5)} \to P^{(6)}$ $P^{(2)} \to P^{(3A)}, P^{(3B)} \to P^{(4)}$	a,c,x,z	b,y
$P^{(2)} \to P^{(3B)}, P^{(3A)} \to P^{(4)}$	b,x,z,y	a,c
$P^{(1)} \to P^{(2)}, P^{(4)} \to P^{(5)}$	a,c	b,x,z,y

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