On two complementary approaches aiming at the definition of the determinant of an elliptic partial differential operator

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Abstract

We bring together two apparently disconnected lines of research (of mathematical and of physical nature, respectively) which aim at the definition, through the corresponding zeta function, of the determinant of a differential operator possessing, in general, a complex spectrum. It is shown explicitly how the two lines have in fact converged to a meeting point at which the precise mathematical conditions for the definition of the zeta function and the associated determinant are easy to understand from the considerations coming up from the physical approach, which proceeds by stepwise generalization starting from the most simple cases of physical interest. An explicit formula that establishes the bridge between the two approaches is obtained.

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1 Introduction

It usually happens in different fields of research that the investigations carried out by mathematicians (or by very mathematically minded physicists) on a particular subject, and those done by physicists, on the same matter, appear to be quite disconnected. What is even worse, it turns out too often that the results obtained by any of the two cathegories of researchers remain unknown to the other, even after some years of its publication in international journals. It has recently happended to the author that several original results on the explicit analytic continuation of a number of zeta functions (see [1] and the references therein) have been reobtained independently by mathematical colleagues [2].

Simplifying the description a bit, one could distinguish the mathematical from the physical approach by saying that the first one proceeds from top to bottom, aiming always at the determination of the most general conditions under which a result or a definition is valid. Particular cases and specific applications are usually of secondary importance. The physical approach, on the contrary, proceeds from below to above, stepwise, at the pace demanded by the neeeds to solve specific physical problems of increasing generality and degree of difficulty.

The present investigation has to do with a clear example of this issue, which corresponds to the important problem of the the definition of the concept of determinant of an elliptic partial differential operator (PDO) exhibiting (in general) a complex spectrum. Two different approaches to this problem exist in the literature, possessing respectively the characteristics that we have just described above. Not surprisingly, both lines of research —one of mathematical nature, coming from above, and the other pursued by physicists, coming from below— will be shown here to converge explicity to a meeting point at which the precise mathematical conditions for the definition of the concept of determinant will be easy to understand from the considerations coming from the physical approach, which proceeds by rather straightforward generalization starting from the most simple cases of physical interest. A formula that establishes the bridge between the two approaches will be obtained.

In fact, the definition of the concept of determinant of a general elliptic PDO is still an open problem, in the sense that there are several possible definitions of this concept and that none of them has been proven to extend to the whole class of such operators. In fact, for invertible operators of the form $A = I + K$, with K an operator of trace-class acting on a Hilbert space (of infinite dimension, in general), the most popular definition of determinant is the one due to Fredholm

$$\det{}_{Fr}(I + K) = I + \sum_{n=1}^{\infty} \operatorname{Tr}\left(\wedge^n K\right) \tag{1}$$

This series is absolutely convergent for the operators considered, and the Fredholm determinant has the usual properties of the determinant for finite-dimensional matrices, in particular the fundamental property:

$$\det(AB) = \det(A)\det(B). \tag{2}$$

The Fredholm determinant reduces to the ordinary definition when the dimension of the Hilbert space is finite.

But maybe the most useful definition of determinant for elliptic PDOs is the one proposed by Ray and Singer in the early seventies [3], which makes use of the concept of zeta function of an operator, $\zeta_A(s)$ (a generalization, on its turn, of the Riemann zeta function, $\zeta_R(s) = \sum_{n=1}^{\infty} n^{-s}$, Re $s > 1$). The zeta (or Ray-Singer) determinant is defined as

$$\det_\zeta(A) = \exp\left(-\frac{d}{ds}\zeta_A(s)\Big|_{s=0}\right), \tag{3}$$

and the (formal) definition of $\zeta_A(s)$ is the following

$$\zeta_A(s) = \sum_n \lambda_n^{-s}, \tag{4}$$

being λ_n the eigenvalues of A and s a complex variable. It is clear that there are many problems associated with this definition (for instance, in general the spectrum is unknown, and there can be zero eigenvalues). Being more precise, for a non-negative, selfadjoint operator A one proceeds by analytically continuing in the complex s-plane —from a region Re $s > s_0$, with s_0 some abscissa of convergence, to the region Re $s \leq s_0$— the function given by the series (4) above with the zero eigenvalues excluded from the sum. And this can be carried out explicitly in many different situations, increasingly complicated (see [1] and references therein).

Now we shall summarize the mathematical approach to the question. Presently, the most general conditions for the definition of the zeta function of an elliptic PDO are known to be the following: (i) the order p of A must be real and different from zero; (ii) there must exist a conical neighborhood U of a ray L from the origin in the spectral plane, such that the principal symbol $a_p(x, \xi)$ of the operator A has no eigenvalue in U (for all points x of the manifold M, where A acts, and for any $\xi \in T_x^*M$, with $\xi \neq 0$). Of course in the finite-dimensional case this condition is obviously satisfied. Also, it is immediate that by multiplying the operator A with a convenient constant $c \in \mathbf{C}$, $|c| = 1$, the ray L can be taken to be (without restriction) the negative real axis $\mathbf{R}_-$ (by working then with the operator $A_1 = cA$, instead of A).

However, it turns out that even if $\zeta_A(s)$, $\zeta_B(s)$ and $\zeta_{AB}(s)$ exist, for the elliptic differential operators A and B, in general

$$\det_\zeta(AB) \neq \det_\zeta(A)\det_\zeta(B). \tag{5}$$

This has led Friedlander [4] and Konsevich and Vishik [5] to consider the multiplicative anomaly

$$F(A, B) \equiv \frac{\det(AB)}{\det(A)\det(B)}. \tag{6}$$

For the case when A and B are positive-definite elliptic PDOs of positive order, it has been shown in [4] that the anomaly is equal to one and that the three zeta functions can be defined with the help of a cut in the spectral plane on $\mathbf{R}_-$. In [5], on the other hand, an expression of $F(A, B)$ in terms of the symbols of the operators has been obtained and it has been proven that $F = 1$ for a certain class of PDOs in odd-dimensional manifolds which generalizes the class of elliptic PDOs. Within this class, a new definition of determinant has been introduced, which is valid even for zero-order operators, thus extending non-trivially the definition of zeta determinant. In particular, the definition is valid for invertible pseudo-differential operators close to positive self-adjoint ones, getting then back to the definition of zeta determinant. This much for the mathematical approach.

As starting point for the more physical approaches to the definition of a determinant let us take the papers of Schwarz [6], initiating in the late seventies [7], in which use is made of the concept of Ray-Singer (zeta) determinant for the calculation of the partition function of a gauge field theory. To fix up ideas, in topological field theory, it is given by the expression

$$Z(\beta) = \int_\Gamma \mathcal{D}\omega \, e^{-\beta S(\omega)}, \tag{7}$$

where ω are fields on a manifold M, Γ is some linear space of fields and S is the topological action functional. The constant β was considered in [7] to be real ($\beta = 1$). Later, Witten [8] had to calculate the case when β is purely imaginary and, very recently, Adams and Sen [9] have attacked the situation of arbitrary complex β. Those are in brief the steps of the development in physics. For ulterior uses, it will be enough to consider the case of a compact manifold, M without boundaries and oriented, and a quadratic action S, e.g. $S(\omega) = <\omega, A\omega>$, with respect to some Euclidean metric on M.

In this case, the definition of the zeta function of A, Eq. (3), written under the form of a Mellin transform, is

$$\zeta_A(s) = \sum_{n=1}^{\infty}{}' \lambda_n^{-s} = \frac{1}{\Gamma} \int_0^\infty dt\, t^{s-1} \mathrm{Tr}\left(e^{-tA} - \Pi_{Ker} A\right). \tag{8}$$

Moreover, for B an operator of order zero (multiplication by a function) one has the asymptotic expansion

$$\mathrm{Tr}\left(B\, e^{-tA}\right) \sim \sum_k c_k(B|A)\, t^k, \qquad t \to 0^+, \tag{9}$$

where the $c_k(B|A)$ are the heat-kernel coefficients, first calculated by Seeley [10] and subsequently by a long list of authors, in different situations (for very new results and a list of references, see [11]). It can be shown, using these two equations, that the zeta-function of a general non-negative elliptic operator (self-adjoint and of positive order) is meromorphic in the complex s-plane, and analytic at $s = 0$. For $\beta = 1$ in Eq. (7), it was proven in Ref. [7] —by extending the Faddeev-Popov trick for zero modes— that

$$Z(\beta = 1) = (\det{}_\zeta \tilde{A})^{-1/2} \prod_{j=1}^{N} \left(\det(T_j^+ T_j)\right)^{(-1)^{j+1}/2}, \tag{10}$$

being $\widetilde{A} \equiv A - \Pi_{Ker} A$. Here $\{T_j\}$ is a sequence of linear operators which constitute a resolvent of S ($T : \Gamma_j \to \Gamma_{j-1}$, linear spaces, with $S(f + T_1 g) = S(f)$, $T_{j-1} T_j = 0$, $j = 0, 1, \ldots, N$). Notice that when generalizing Eq. (10) to $\beta \neq 1$, all the dependence on β will appear in the first of the determinants on the rhs, and thus we are not going to consider further the other factors, corresponding to the resolvent, in the analysis to follow.

The case $\beta = -i$ was calculated by Witten in 1989 for the case of the Chern-Simons theory [8]. Taking as starting point the previous results, he just had to obtain the additional phase of the determinant. He proceded by directly computing the integral in an orthonormal basis of eigenfunctions for A, x_i, i.e.

$$\prod_n \int_{-\infty}^{\infty} \frac{dx_n}{\sqrt{\pi}} e^{i\lambda_n x_n^2}. \tag{11}$$

Using an ϵ-regularization, one obtains, for each of the factors

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{i\lambda x^2} = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{i\lambda x^2} e^{-\epsilon x^2} = \frac{1}{|\sqrt{\lambda}|} \exp\left(\frac{i\pi}{4} \mathrm{sgn}\, \lambda\right), \tag{12}$$

and for the whole determinant

$$\left(\det_\zeta(-i\widetilde{A})\right)^{-1/2} = \left(\det_\zeta(\widetilde{A})\right)^{-1/2} \exp\left(\frac{i\pi}{4} \eta_A(0)\right), \tag{13}$$

where

$$\eta_A(s) = \sum_n \mathrm{sgn}(\lambda_n) |\lambda_n|^{-s} \tag{14}$$

is the eta function of A (also called Atiyah-Patodi-Singer's eta invariant [12] in the limit $s \to 0$ and affected by a factor $1/2$).

The more general case of β arbitrary complex has been investigated in Ref. [9]. By extending the considerations of Witten, these authors obtain the formula

$$\left(\det(\beta\widetilde{A})\right)^{-1/2} = \exp\left\{-\frac{i\pi}{4}\left[\left(\frac{2\theta}{\pi} \mp 1\right)\zeta \pm \eta\right]\right\} |\beta|^{-\zeta/2} \left(\det |\widetilde{A}|\right)^{-1/2}, \tag{15}$$

where $\zeta \equiv \zeta_{|A|}(0)$, $\eta \equiv \eta_A(0)$, and where the $\pm$ signs correspond, respectively, to the two situations of β belonging to the upper or lower half of the complex plane, respectively, i.e.

$$\beta = |\beta| e^{i\theta} \quad \begin{cases} + : & 0 \leq \theta \leq \pi, \\ - : & \pi \leq \theta \leq 2\pi. \end{cases} \tag{16}$$

As the authors themselves recognize, there is an ambiguity in this expression for β real, and thus the formula only generalizes Witten's case, but *not* Schwarz's one! As anticipated before, the problem is a clear example of the lack of connection between physicists and mathematicians. The authors of [9] seem to be in fact unaware of the strong results that have been summarized above [5].

We shall now bring together the above two lines of research, the mathematical one described in the first paragraphs —that tries to extend the definition of the zeta function and of the associated determinant to the most general class of elliptic PDOs possible— and the physical one, that approaches this goal from below, step by step, starting from the simple cases (10) and (13), as the situations in actual physical theories demand (e.g., nowadays, topological field theories). It is certainly clear that the powerful results of the first line of investigation contradict the statement that the zeta function has not been defined in the literature for operators with a negative spectrum [9]. This assertion might only become true if one would add *in the 'physical' literature*. We have seen that a much more general definition of zeta function have been given for operators with a complex spectrum, under the very specific condition of the existence of a conical neighborhood of a certain ray from the origin, in the complex spectral plane, where no eigenvalue is present.

It is quite remarkable to see how the condition of existence of the conical neighborhood over a ray in the spectral plane arises in an absolutely natural way from the physical, stepwise approach to the problem. This is obtained by simply complementing the techniques of Schwarz and Witten with a well-defined analytic continuation on the complex plane of the power function of the constant β. Oddly enough, the final result and, in particular, the definitions themselves of the zeta function and of the associated determinant will turn out to be independent of the position of this ray in the spectral plane.

In fact, the crucial point in the whole procedure is a quite simple one, to be learned in any standard course on complex variable. It consists namely on the proper definition of the analytic continuation of the power function, z^a for $a \in \mathbf{R}$, from $z \in \mathbf{R}_+$ to complex values of z, i.e.,

$$z = |z|\, e^{i\theta} \quad \longrightarrow \quad z^a = |z|^a\, e^{ia\theta}, \qquad a \in \mathbf{R}. \tag{17}$$

The key point here is the appearance of a cut from the origin in the complex z-plane, that can be fixed at will by selecting the interval of variation of the argument, that is,

$$\theta \in [-\gamma, 2\pi - \gamma], \qquad 0 < \gamma < 2\pi. \tag{18}$$

Thus, the cut (in other words, the ray L from the origin) has been here chosen to be

$$L = \{z \in \mathbf{C} \,|\, Arg(z) = \gamma\}. \tag{19}$$

Summing up, the angle γ defines the ray L from the origin on the complex plane which contains the points that are not reached from the specific analytic continuation chosen, in the sense that z^a is not (uniquely) defined for those points z (a double phase appears). Of course, in the books one will find that this ray can be always taken to be $L = \mathbf{R}_-$, and that this is the most 'natural' choice. Also, as we have seen before, by simple multiplication by a complex constant of modulus one, one can reduce the whole discussion for a general operator to this standard case. But we, physicists, should be extremely careful with this

kind of general mathematical considerations. For a given operator A the ray *cannot* always be put on the negative real axis. A different thing is that the general discussion concerning the operator A, with a cut suitably fixed at $L = \{z \in \mathbf{C} \mid Arg(z) = \gamma\}$, can in fact be reduced to an *equivalent* discussion of the operator $A_1 = A\, e^{i(\pi - \gamma)}$ which has the 'standard' cut at $L = \mathbf{R}_-$.

And this is the reason why the formula (15) does not work for the most simple case, when the spectrum of A has a real negative part. Let us now put everything together and derive the desired formula. Recall that now A is an operator with a general real spectrum. Corresponding to the partition of the identity operator into the different parts of the spectrum, $I = \Pi_{Ker} + \Pi_+ + \Pi_-$, let us write $\tilde{A} = A_+ + A_-$ (remember that $\tilde{A} = (I - \Pi_{Ker})A$). β will be a general complex number and we can proceed with the straightforward calculation [9] (we shall from now on drop out the subscript ζ from the det)

$$
\begin{aligned}
\left(\det(\beta\tilde{A})\right)^{-1/2} &= (\det(\beta A_+))^{-1/2}(\det(\beta A_-))^{-1/2} = (\det(\beta A_+))^{-1/2}(\det[(-\beta)(-A_-)])^{-1/2} \\
&= (-1)^{-\frac{1}{2}\zeta_{-A_-}(0)}\beta^{-\frac{1}{2}[\zeta_{A_+}(0)+\zeta_{-A_-}(0)]}\left(\det|\tilde{A}|\right)^{-1/2}. \quad (20)
\end{aligned}
$$

By using the identities

$$
\zeta_{|A|}(s) = \zeta_{A_+}(s) + \zeta_{-A_-}(s), \qquad \eta_A(s) = \zeta_{A_+}(s) - \zeta_{-A_-}(s), \quad (21)
$$

we get the relations

$$
\zeta_{A_+}(0) = \frac{1}{2}(\zeta + \eta), \qquad \zeta_{-A_-}(0) = \frac{1}{2}(\zeta - \eta), \quad (22)
$$

wherefrom we obtain the final formula

$$
\left(\det(\beta\tilde{A})\right)^{-1/2} = (-1)^{-(\zeta-\eta)/4}\beta^{-\zeta/2}\left(\det|\tilde{A}|\right)^{-1/2}. \quad (23)
$$

In this formula the powers of (-1) and β are to be taken in the γ-analytic continuation chosen for the power function, which is the one appropriate in order that all the powers are uniquely defined. In the cases considered in topological field theory, one is free to choose any value for γ that is not a multiple of $\pi/2$. In a more general case, with βA general elliptic, one must guarantee in advance that this choice exists, in order to be able to define the zeta function. This is the content of the powerful mathematical theorem, as seen from the physical viewpoint. Another important point for the formula (23) to be valid is that $\zeta_{|A|}(s)$ and $\eta_A(s)$ must be analytic at $s = 0$. This is certainly true in the ordinary conditions of a non-negative elliptic operator (see the paragraph before Eq. (10)). That such is also the case for the situations of interest in topological field theories (forms of degree m in a manifold M of odd dimension $2m + 1$) has been beautifully shown in [9]. Again, this is the case for which in the mathematical theory the determinant can be defined: for an automorphism of a vector bundle on an odd-dimensional manifold acting on global sections of this vector bundle.

Also, a natural trace has been introduced in [5] for PDOs of odd class on an odd-dimensional closed M.

To finish, let us consider some particular uses of the formula (22), that connect with results previously obtained in the literature. The most simple case is when β is real and positive. Then only the power of (-1) needs analytical continuation in Eq. (23). On the other hand, for β real and negative the formula reads

$$\left(\det(\beta\widetilde{A})\right)^{-1/2} = (-1)^{(\zeta+\eta)/4}(-\beta)^{-\zeta/2}\left(\det|\widetilde{A}|\right)^{-1/2} \tag{24}$$

where, again, only the power of (-1) needs analytical continuation. For β purely imaginary we reobtain Witten's result

$$\begin{aligned}
\left(\det(\beta\widetilde{A})\right)^{-1/2} &= (-1)^{-(\zeta-\eta)/4+\zeta/2}\, i^{-\zeta/2}\, |\beta|^{-\zeta/2}\left(\det|\widetilde{A}|\right)^{-1/2} \\
&= e^{i\pi\eta/4}\, |\beta|^{-\zeta/2}\left(\det|\widetilde{A}|\right)^{-1/2}.
\end{aligned} \tag{25}$$

It is nice to observe that this result is here just a particular case of the ordinary zeta function procedure for defining the determinant, nothing extra has to be introduced, but the overall consideration of performing analytic continuations properly. Eq. (25) was obtained in [8] using an ϵ regularization factor and for $|\beta| = 1$ (it was already noticed there that any reasonable regularization should yield the same result). Finally, for $L = \mathbf{R}_-$ we recover the formula (15) of Adams and Sen [9], which is certainly unambiguous as a generalization of the Witten case (β pure imaginary) but not of the $\beta = 1$ case of Schwarz to $\beta \in \mathbf{R}_-$.

Summing up, formula (23) encompasses all those situations, extends them to β arbitrary complex, and establishes an explicit connection with the general definition of zeta-function determinant and with the more far reaching considerations in [5].

Acknowledgments

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